Useful Formulae (Second Half)

Time and Frequency Characteristics

First-Order Time-Domain Analysis

For a standard transfer function of the first-order system:

$$G(j\omega) = \frac{K}{(j\omega + a)}$$

Time constant:

$$T = \frac{1}{a}$$

Rise time:

$$T_r = \frac{2.2}{a}$$

Settling time:

$$T_s = \frac{4}{a}$$

Second-Order Time-Domain Analysis

For a standard transfer function of the second order system:

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\omega_n\zeta(j\omega) + \omega_n^2}$$

Rise time (for underdamped response):

$$T_r = \frac{(1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)}{\omega_n}$$

Time-to-peak (n = #peak):

$$T_p = \frac{\pi n}{\omega_n\sqrt{1-\zeta^2}}$$

Percent overshoot:

$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

Damping factor:

$$\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln\left(\frac{\%OS}{100}\right)^2}}$$

Settling time:

$$T_s = \frac{4}{\zeta \omega_n}$$

Sampling and Reconstruction

Sampling equation:

$$x_s(t) = x(t) \cdot \mathrm{comb}(t) = x(t) \cdot \sum_{m=-\infty}^{\infty} \delta(t - mt_0) = \sum_{m=-\infty}^{\infty} x[m]\delta(t - mt_0)$$

Discrete time samples:

$$X_s(j\omega) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega t} = \sum_{m=-\infty}^{\infty} x[m]e^{-j2\pi f m t_0}$$

Z-Transforms

No	Sequence	Transform	ROC
1	$\delta[n]$	1	All z
2	$u[n]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert > 1$
3	$-u[n-1]$	$\frac{1}{1 - z^{-1}}$	$\lvert z\rvert < 1$
4	$\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5	$a^n u[n]$	$\frac{1}{1 - az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6	$-a^n u[-n-1]$	$\frac{1}{1 - az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7	$na^n u[n]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8	$-na^n u[-n-1]$	$\frac{az^{-1}}{(1 - az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9	$\begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{1 - a^N z^{-N}}{1 - az^{-1}}$	$\lvert z\rvert > 0$
10	$\sin(\omega_0 n)u[n]$	$\frac{\sin \omega_0 \, z^{-1}}{1 - (2\cos \omega_0)z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
11	$\cos(\omega_0 n)u[n]$	$\frac{1 - \cos \omega_0 \, z^{-1}}{1 - (2\cos \omega_0)z^{-1} + z^{-2}}$	$\lvert z\rvert > 1$
12	$[a^n \sin \omega_0 n]u[n]$	$\frac{(a \sin \omega_0)z^{-1}}{1 - (2a\cos \omega_0)z^{-1} + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
13	$[a^n \cos \omega_0 n]u[n]$	$\frac{1 - (a\cos \omega_0)z^{-1}}{1 - (2a\cos \omega_0)z^{-1} + a^2 z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Z-transform Properties

No	Property	Sequence	Transform	ROC
1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2.
2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the region.
3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X\left(e^{-j\omega_0}z\right)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e. the set of points $\{\|a\|z\}$for z in R.
4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R(i.e. R^{-1} is equal to the set of point z^{-1}, where z is in R).
5	Time expansion	$x_{(k)}[n]\begin{cases}x[r], n = rk \\ 0, n \neq rk\end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$(i.e. the set of point $z^{1/k}$, where z is in R).
6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
7	Convolution	$x_1[n] * x_2[n]$	$X_1(z) \cdot X_2(z)$	At least the intersection of R_1 and R_2.
8	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$.
9	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\left(\frac{1}{1-z^{-1}}\right)X(z)$	At least the intersection of R and $\|z\| > 1$.
10	Differentiation in the z-domain	$nx[n]$	$-z\left[\frac{dX(z)}{dz}\right]$	R
11	Initial value theorem	If $x[n] = 0$ for $x < 0$, then $x[0] = \lim_{z\to\infty} X(z)$		

Note:

- For a sequence $x[n]$, its z-transform is $X(z)$ and its ROC is R.
- For a sequence $x_1[n]$, its z-transform is $X_1(z)$ and its ROC is R_1.
- For a sequence $x_2[n]$, its z-transform is $X_2(z)$ and its ROC is R_2.