

Final Exam Review Topic (Second Half)

XMUT220 Signals and Systems

Final Exam

- Closed book, 2 hours, 4-5 questions, and 100 marks.
- ALL topics on the second half of the semester. No content from the first-half.
- English-Chinese dictionary and non-programmable calculator are allowed in the exam.
- Selected formulas in the systems and signals for the second half of the semester are provided.

Final Exam

Expect four types of questions in the final exam:

- Explain…

  Describe the frequency response characteristics of the system given below: [5 marks]

$$G(j\omega) = \frac{(j\omega + 1)}{(j\omega + 3)(j\omega + 7)}$$

- Calculate…

  Find the z-transform the following signal: [5 marks]

$$x[n] = u[n-2] + u[n+1]$$

Final Exam

- Design …

  Using decimation and interpolation methods, design a digital signal processing system that performs a 15/7 ratio signal conversion. [15 marks]

- Sketch …

  For the given signal $x(t) = \cos 10\pi t$, sketch its frequency spectrum after the sampling when it is sampled at:

  a. $f_s = 20$ Hz. [10 marks]

  b. $f_s = 5$ Hz. [10 marks]

Topics of Final Exam (Second Half)

Topics in the final exam will cover:

a. Time and frequency characteristics (40%).

b. Sampling and reconstruction (40%).

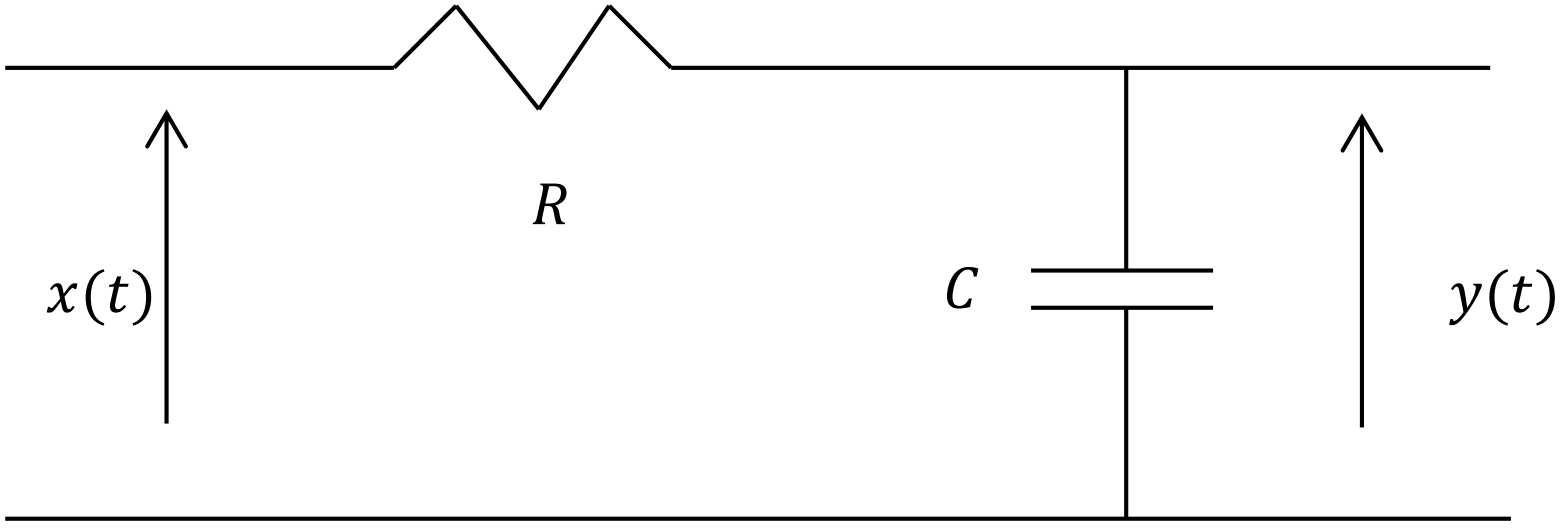
c. z-transform (20%).

Topics in Time and Frequency Characteristics

Topics in time and frequency characteristics:

- Transfer function of the physical system.
- Impulse and step response inputs.
- Frequency domain analysis (e.g. Bode plot and frequency spectrum + overview of other methods).
- Transient response of a first order system (e.g. τ, t_r, and t_s).
- Transient response of a second order system (e.g. ω_n, ζ, t_p, and $\%OS$).
- Frequency response analysis (e.g. magnitude and phase of Bode plots

First-Order System Time Response



- Consider the continuous-time LTI system implemented as the RC circuit shown above.
- The voltage source $x(t)$ is considered the input to this system.
- The voltage $y(t)$ across the capacitor is considered the system output

First-Order System Time Response

For the given first-order system, its transfer-function equation is:

$$G(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{RCj\omega + 1}$$

If R = 1 Ω and C = 1 F, then as a result the transfer function of the circuit is given as:

$$G(j\omega) = \frac{1}{j\omega + 1}$$

- Time constant (τ):

$$T = \frac{1}{a} = \frac{1}{1} = 1 \text{ second}$$

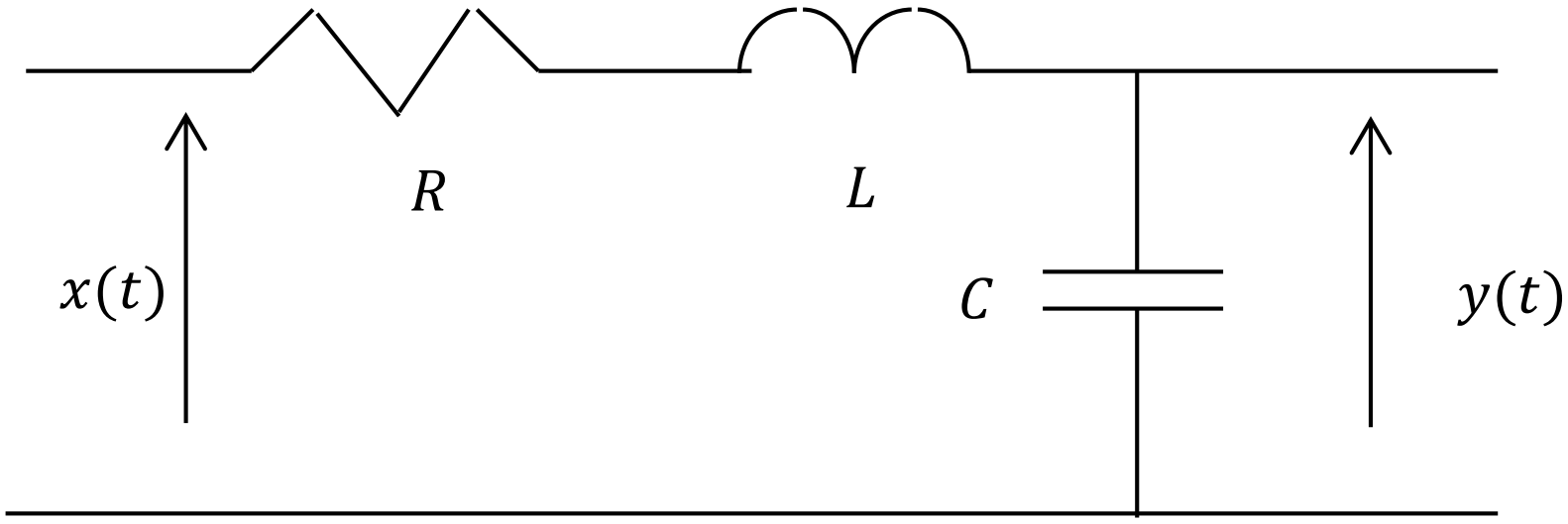
- Rise time (T_r):

$$T_r = \frac{2.2}{a} = \frac{2.2}{1} = 2.2 \text{ seconds}$$

- Settling time (T_s):

$$T_s = \frac{4}{a} = \frac{4}{1} = 4 \text{ seconds}$$

Second-Order System Time Response



- Consider the continuous-time LTI system implemented as the RLC circuit shown in the figure above.
- The voltage source $x(t)$ is considered the input to this system.
- The voltage $y(t)$ across the capacitor is considered the system output.

Second-Order System Time Response

- Giving a standardised second-order transfer function equation:

$$H(\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\omega_n \xi (j\omega) + \omega_n^2}$$

- If R = 1 Ω, L = 1 H, and C = 1 F, as a result the transfer function equation of the system becomes:

$$G(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{LC(j\omega)^2 + RCj\omega + 1} = \frac{1}{(j\omega)^2 + (j\omega) + 1}$$

- Natural frequency:

$$\omega_n = \sqrt{1} = 1 \text{ rad/s}$$

Second-Order System Time Response

- Damping ratio:

$$\zeta = \frac{1}{2\omega_n} = \frac{1}{2} = 0.5$$

- Time-to-peak (with $n = 1$, to the first peak):

$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$= \frac{\pi}{1\sqrt{1-(0.5)^2}} = \frac{\pi}{\sqrt{0.75}} = 3.63 \text{ seconds}$$

- Percentage overshoot:

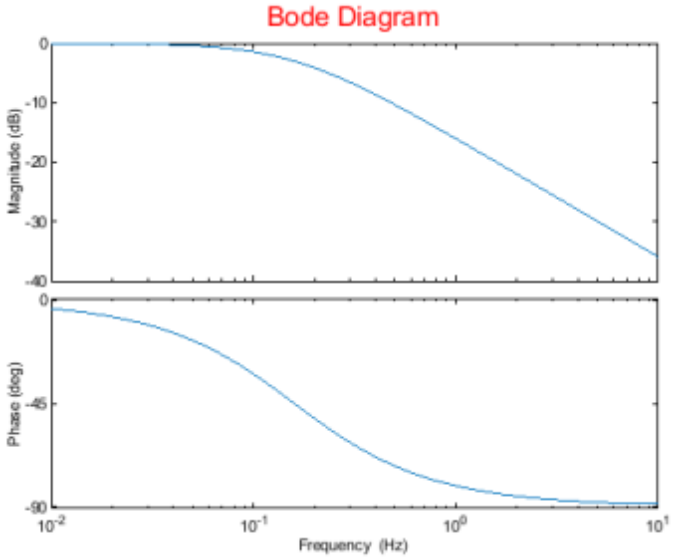
$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100\%$$

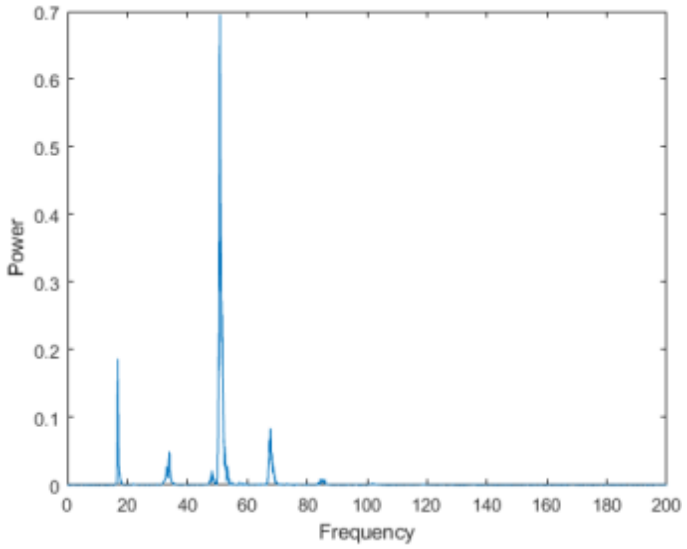
$$= e^{-\left(0.5\times 3.14/\sqrt{1-(0.5)^2}\right)} \times 100\% = 90.5\%$$

Frequency-Domain Analysis

Methods commonly used for frequency-domain analysis:

- Bode plot – frequency domain characteristics of the system.
- Frequency spectrum – components of signal.



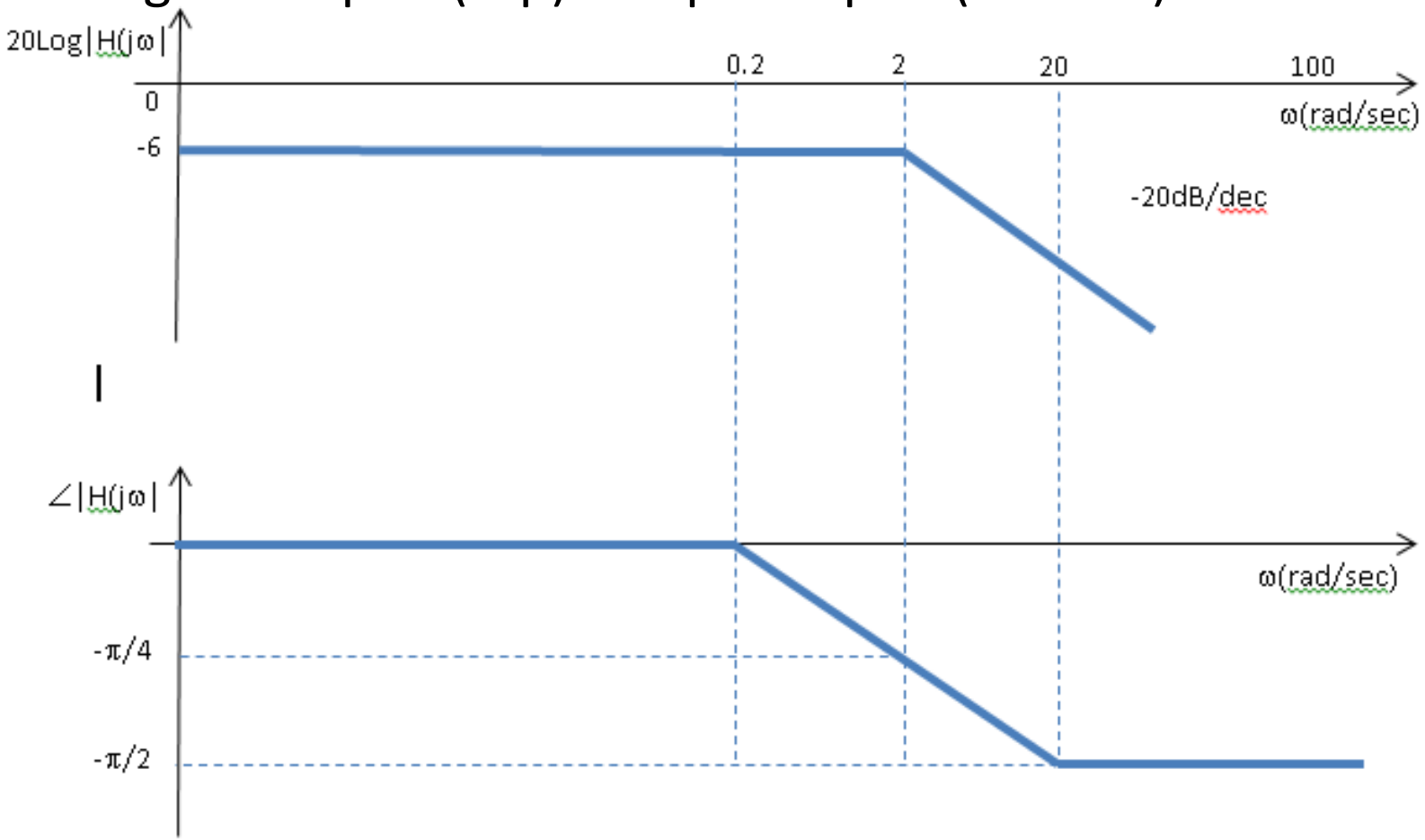


Frequency Response Plot

- Transfer function of the system:

$$G(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{(2 + j\omega)}$$

- Magnitude plot (top) and phase plot (bottom):



Frequency Response Plot

- Transfer function of the system:

$$G(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{(2 + j\omega)}$$

- Gain and phase of the system:
  - f_0 = 0 rad/s, gain =-6 dB and phase = 0.
  - f_c = 2 rad/s, gain = -6 dB and phase = $\pi/4$.
  - f_∞ = ∞ rad/s, gain = -20 dB/decade and phase = $\pi/2$.
- Analysis of the system:

  The system allow low frequency components of the signal to pass through i.e. low pass filter.

Topics in Sampling and Reconstruction

Topics in sampling and reconstruction:

- Sampling theorem of Nyquist.
- Sampling of signals with impulse sampling signal.
- Aliasing effects.
- Reconstruction of sampled signals (i.e. zero-order hold and first-order hold methods).
- Decimation (down sampling) and extrapolation (up sampling).

Sampling of Signals

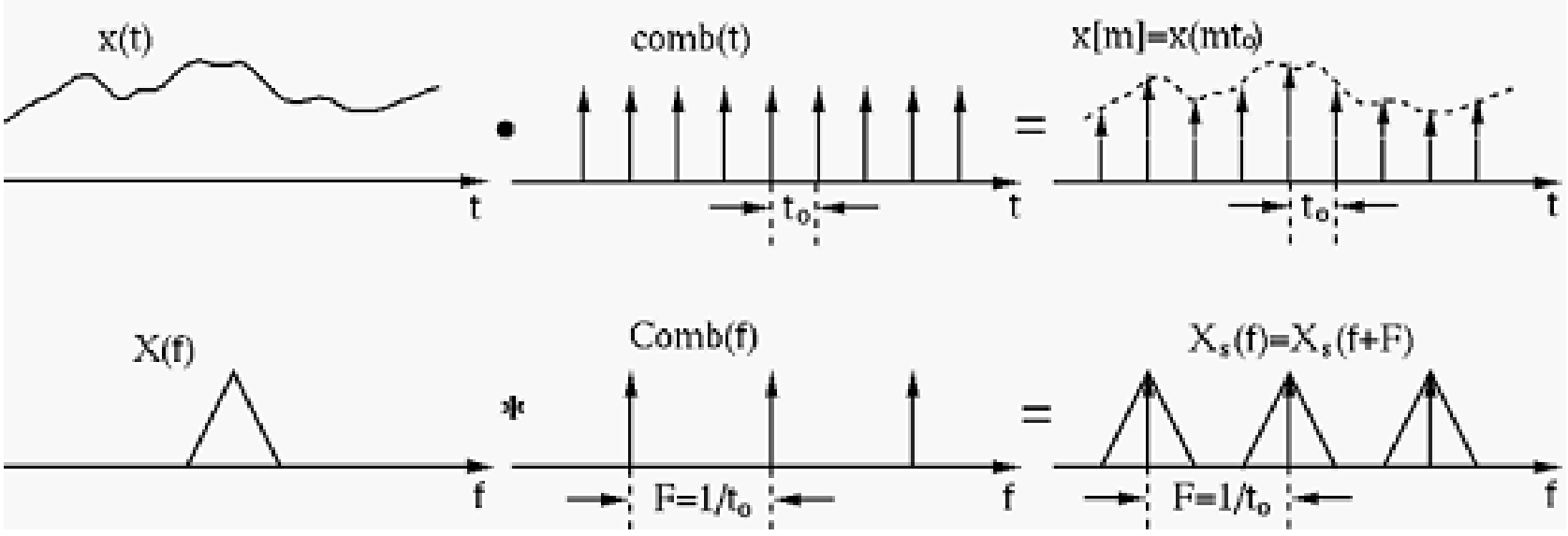
- Sampling of signal $x(t)$:

$$x|m| = x(mt_0) = x(m/F) \quad \text{with} \quad m = \cdots, -1, 0, \ 1, \ldots$$

- Thus:

$$x_s(t) = x(t) \cdot \mathrm{comb}(t)$$

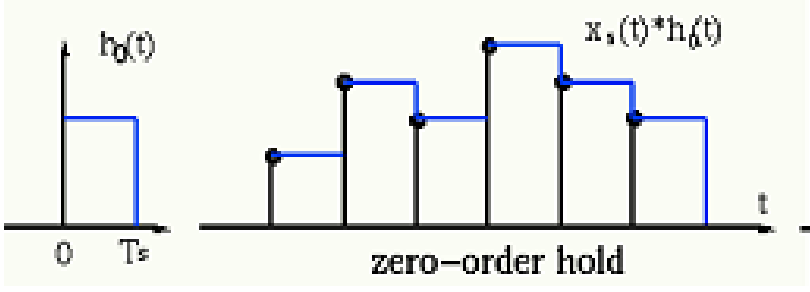
$$= x(t) \sum_{m=-\infty}^{\infty} \delta(t - mt_0) = \sum_{m=-\infty}^{\infty} x[m]\delta(t - mt_0)$$

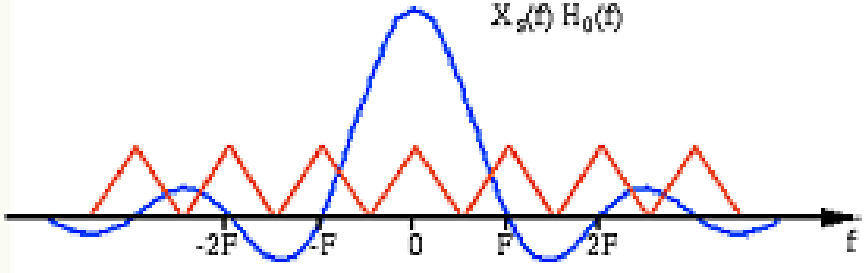


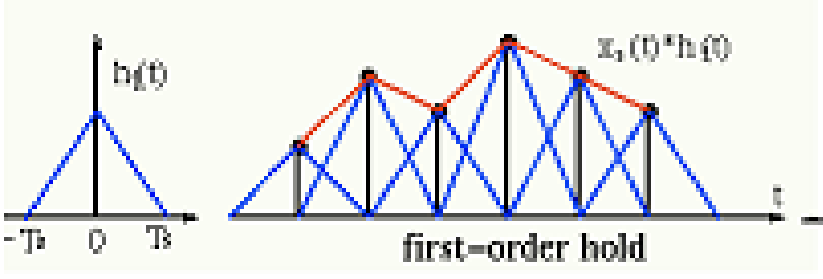
Reconstruction of Signals

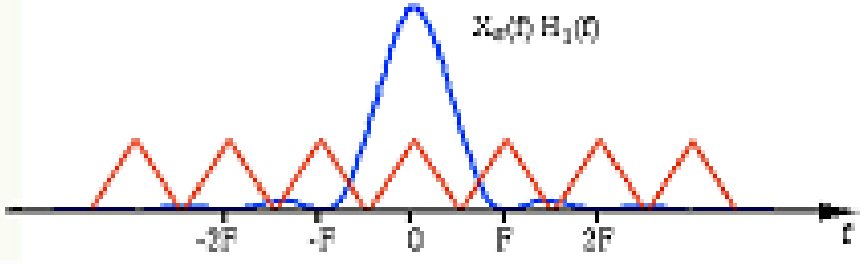
- Zero-Order Hold and First-Order Hold methods:

$$h_0(t) = \begin{cases} 1 & 0 \le t < t_0 \\ 0 & \text{otherwise} \end{cases} \qquad h_1(t) = \begin{cases} 1 - |t|/t_0 & |t| < t_0 \\ 0 & \text{otherwise} \end{cases}$$



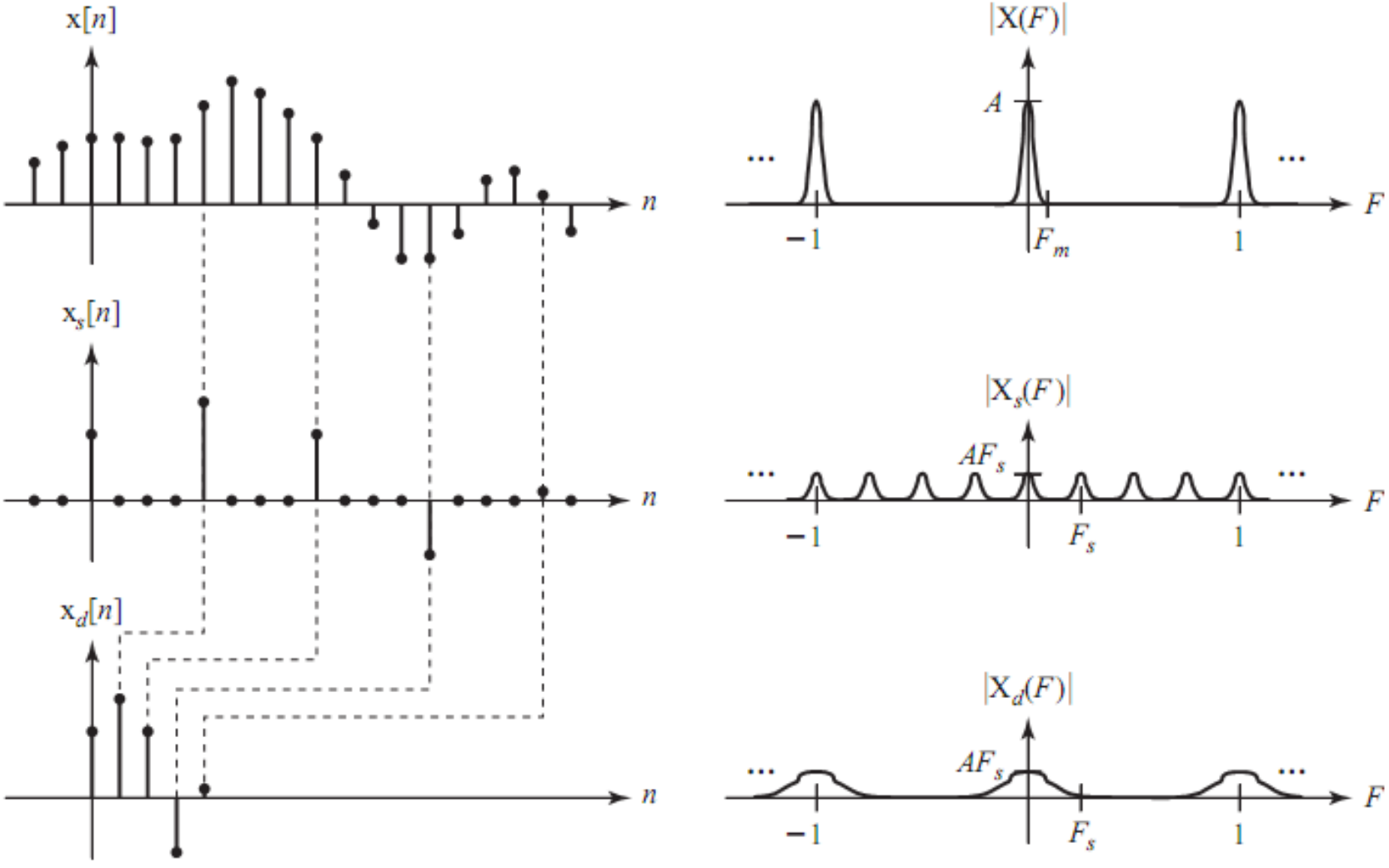






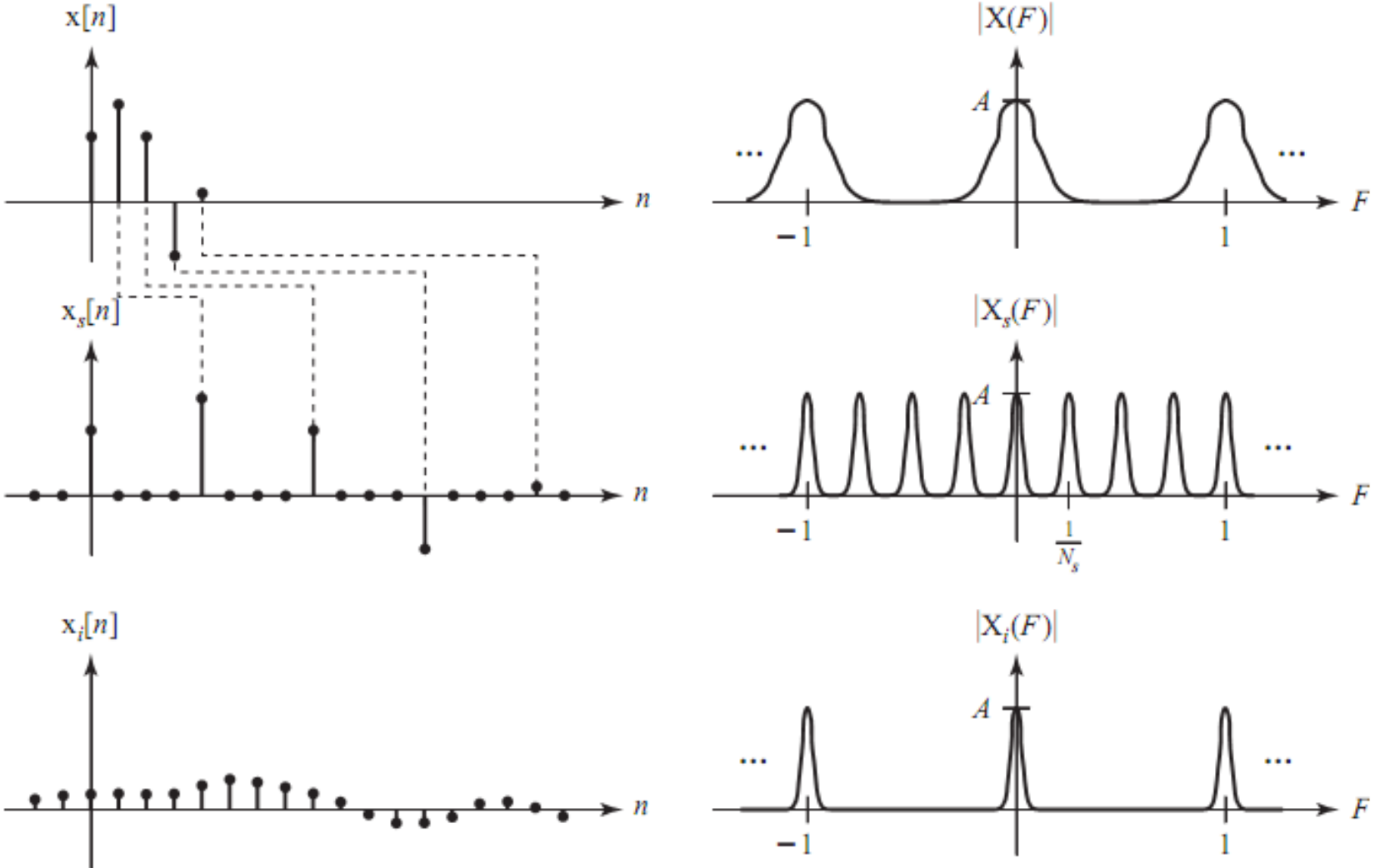
Decimation and Interpolation

- Decimation by a factor of 1/5 times ($M = 5$). Sampled decimated signal ($x(d)$) appears at 5 x higher frequency. Decimated frequency spectrum = $A \times F(s) * \text{original}$.



Decimation and Interpolation

- Interpolation by a factor of 5 times ($L = 5$). Sampled interpolated signal ($x(i)$) appears at 5 x lower frequency. Interpolated frequency spectrum = $A \times F(s) * \text{original}$.



Sampling and Aliasing

- The analogue signal $x(t)$ is sampled at a rate of f_s = 5 Hz.

$$x(t) = 10\sin(2\pi t) + 10\sin(8\pi t) + 5\sin(12\pi t)$$

Where: t is in seconds.

- For the signal given above:

a. Determine the signal $x_a(t)$ aliased with $x(t)$.

b. Determine the equation that describe the sampling process.

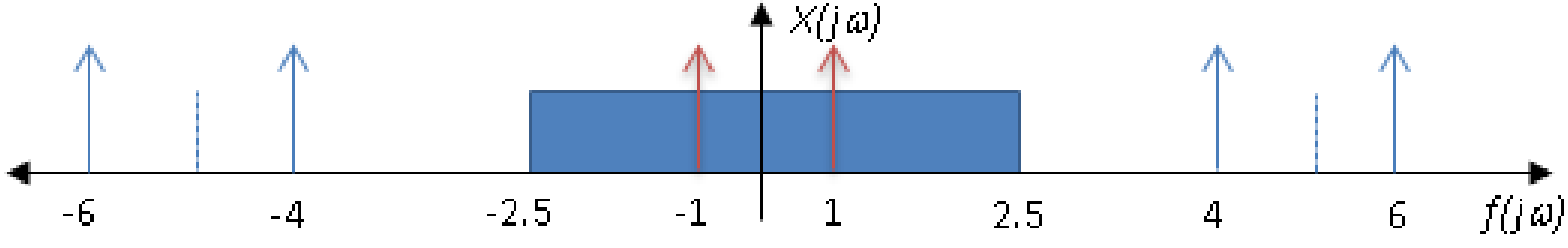
c. Plot the frequency spectrum of the sampled signals.

Sampling and Aliasing

- The three terms of the signal correspond to frequencies f_1 = 1 Hz, f_2 = 4 Hz, and f_3 = 6 Hz.
- Of these, f_2 and f_3 lie outside the Nyquist interval [−2.5, 2.5].
- Therefore, they will be aliased with $f_2 - f_s$ = 4 − 5 = −1 Hz and $f_3 - f_s$ = 6 − 5 = 1 Hz and the aliased signal will be:

$$x_a(t) = 10\sin(2\pi t) + 10\sin(2\pi(-1)t) + 5\sin(2\pi t)$$

$$= 5\sin(2\pi t)$$



Topics in z-Transform

Topics in z-transforms:

- z-transforms of common signals and series (e.g. sequence, power series, exponential, delay, etc.).
- Inverse z-transforms (e.g. inspection, partial fractions and power series methods).
- Properties of z-transforms (e.g. linearity, time shifting, conjugation/conjugate symmetry, differentiation and integration, time and frequency scaling, duality, Parseval's relation, convolution, multiplication and duality).
- Applications of z-transforms (e.g. solving of systems with differential equations, time and frequency responses, and analysis of DTLTI signals and systems.

z-Transform

- Sequence:

$$x[n]$$

- Power series:

$$x[n] = a^n u[n]$$

- Power series with delay:

$$x[n] = -a^n u[-n-1]$$

- Arbitrary function:

$$f(x) = \begin{cases} a^n, & 0 \le n \le N-1 \\ x, & \text{otherwise} \end{cases}$$

z-Transform

- Find z-transform of $x[n] = 3^n u[n]$

$$\mathrm{Z}\{a^n u[n]\} = 1 + az^{-1} + a^2 z^{-2} + \cdots$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \frac{1}{1 - az^{-1}}$$

$$= \frac{z}{z - a}$$

- As a result the above z-transform becomes:

$$\mathrm{Z}\{3^n u[n]\} = \frac{1}{1 - 3z^{-1}} = \frac{z}{z - 3}$$

z-Transform

- Find z-transform of $x[n] = -a^n u[-n-1]$. This sequence is non-zero only for $n \leq -1$.

- The z-transform is:

$$X(z) = \sum_{n=-\infty}^{\infty} -a^n[-n-1]z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

- For $|a^{-1}z| < 1$, or $|z| < |a|$, the series converges to:

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \qquad |z| < |a|$$

Inverse z-Transform

- By inspection from pairs of transform in the z-transform table:

$$a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \qquad \text{for } |z| > |a|$$

- By applying partial fraction:

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1 - d_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{(1 - d_k z^{-1})^m}$$

Inverse z-Transform

- By applying power series:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-1}$$

$$= \cdots + x[-2]z^{-1} + x[-1]z^{-1} + x[0] + x[1]z^{-1}$$

$$+x[2]z^{-2} + \cdots$$

Inverse z-Transform

- Find the sequence $x[n]$ with its z-transform given as:

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}$$

- Since $M = N = 2$ this can be expressed as:

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

- The value B_0 can be found by long division:

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})}$$

Inverse z-Transform

- The coefficients A_1 and A_2 can be found using:

$$A_k = (1 - d_k z^{-1}) X(z)\big|_{z = d_k}$$

- So

$$A_1 = \left.\frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1}}\right|_{z^{-1} = 2} = \frac{1 + 4 + 4}{1 - 2} = -9$$

- And

$$A_2 = \left.\frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{1}{2}z^{-1}}\right|_{z^{-1} = 1} = \frac{1 + 2 + 1}{\frac{1}{2}} = 8$$

Inverse z-Transform

- So, the given equation becomes:

$$X(z) = 2 + \frac{9}{\left(1 - \frac{1}{2}z^{-1}\right)} + \frac{8}{(1 - z^{-1})}$$

- Using the fact that the ROC is $|z| > 1$, the terms can be inverted one at a time by inspection to give:

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^{n} u[n] + 8u[n]$$

Properties of z-Transform

- Linearity property:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z) \quad \text{ROC } R_{x1} \cap R_{x2}$$

- Time shifting property:

$$x[n - n_0] \leftrightarrow z^{-no}X(z) \qquad \text{ROC} = R_x$$

- Exponential multiplication property:

$$Z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \qquad \text{ROC} = |z_0|R_x$$

- Differentiation property:

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz}, \qquad \text{ROC} = R_x$$

Properties of z-Transform

- Conjugation property:

$$x^*[n] \leftrightarrow X^*(z^*), \qquad \text{ROC} = R_x$$

- Time reversal property:

$$x^*[-n] \leftrightarrow X^*\left(\frac{1}{z^*}\right) \qquad \text{ROC} = \frac{1}{R_x}$$

- Convolution:

$$x_1[n] * x_2[n] \leftrightarrow X_1(z) \cdot X_2(z) \qquad \text{ROC } R_{x1} \cap R_{x2}$$

Properties of z-Transform

- Initial and Final Value Theorem:

  If $x[n]$ is zero for $n < 0$, then initial value theorem says:

$$x[0] = \lim_{z \to \infty} X(z)$$

  For a causal signal $x[n]$, the final value theorem states that:

$$x[\infty] = \lim_{z \to 1} [z - 1]\, X(z)$$

Applications of z-Transform

Solve discrete-time LTI (DTLTI) systems of differential equation:

- One way of looking at the z transform is that it bears a relationship to difference equations analogous to the relationship of the Fourier or Laplace transform to differential equations.
- A linear difference equation with initial conditions can be converted by the z transform into an algebraic equation.
- Then, it is solved and the solution in the time domain is found by an inverse z transform.

Applications of z-Transform

Time response of the discrete-time LTI systems:

- Exponential input response of DTLTI systems.
  - The response of the DTLTI system to the input signal $x[n] = z_0^n$ is $y[n] = z_0^n H(z_0)$.
  - The response of a DTLTI system to the exponential signal $x[n] = z_0^n$ is a scaled version of the input signal.
  - The scale factor is the value of the system function at the point $z = z_0$.

Applications of z-Transform

- Exponentially damped sinusoid input response of discrete-time LTI systems.
  - The amplitude of the signal is multiplied by the magnitude of the DTLTI system function evaluated at the point $z = r_0 e^{j\Omega}$.
  - The phase of the cosine function is incremented by an amount equal to the phase of the DTLTI system function evaluated at the point $z = r_0 e^{j\Omega}$.

Applications of z-Transform

Frequency response of the discrete-time LTI systems:

- Performed through the transformation $z \to e^{j\Omega}$.
- Frequency response is determined (to within a multiplicative constant) by the vectors from the finite poles and finite zeros of the transfer function to the *z*-plane point $z_0 = e^{j\Omega_0}$.
- The magnitude of the frequency response:

$$\left|H\left(e^{j\Omega}\right)\right| = \frac{\left|e^{j\Omega}\right|}{|e^{j\Omega} - p_1||e^{j\Omega} - p_2|}$$

- The phase of the frequency response:

$$\measuredangle H\left(e^{j\Omega}\right) = \measuredangle e^{j\Omega} - \measuredangle\left(e^{j\Omega} - p_1\right) - \measuredangle\left(e^{j\Omega} - p_2\right)$$

Applications of z-Transform

Analysis of discrete-time LTI systems:

- The DTLTI systems are causal when:

$$\lim_{z\to\infty}|H(z)| = \lim_{z\to\infty}\left(\frac{b_M}{a_N}\right)z^{M-N} < \infty$$

- The DTLTI systems are stable when:

$$\sum_{n=-\infty}^{\infty}|h[n]| < \infty$$

- Causal and anti-causal DTLTI systems:
  - Causal: $|z| > r_1$ and $r_1 < 1$.
  - Anti causal: $|z| < r_2$ and $r_2 > 1$.
  - Neither causal or anti causal: $r_1 < |z| < r_2$.