XMUT220 Signals and Systems

Final Exam Revision Questions (Second Half) – Solution

A. Time and Frequency Characteristics

1. A continuous-time LTI system is described by the differential equation given below.

$$\frac{dy(t)}{dt} + 2y(t) = x(t) + \frac{dx(t)}{dt}$$

a. Determine its transfer function equation. [4 marks]

b. Find the impulse response $h(t)$ of the system. [2 marks]

Solution

a. Taking the Fourier transforms of the differential equation given above, we get:

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega) + j\omega X(\omega)$$

or

$$(j\omega + 2)Y(\omega) = (1 + j\omega)X(\omega)$$

Hence, rearranging the equation given above, the frequency response $H(\omega)$ is described in terms of the following transfer function equation of the system:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1 + j\omega}{2 + j\omega} = \frac{2 + j\omega - 1}{2 + j\omega} = 1 - \frac{1}{2 + j\omega}$$

b. Considering the transfer function equation of the system:

$$H(\omega) = 1 - \frac{1}{2 + j\omega}$$

Taking the inverse Fourier transform of $H(\omega)$, the impulse response $h(t)$ is:

$$h(t) = \delta(t) - e^{-2t}u(t)$$

2. Consider a continuous-time LTI system described by the following differential equation:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

Using the Fourier transform, find the output $y(t)$ to each of the following input signals:

a. $x(t) = e^{-1}u(t)$. [12 marks]

b. $x(t) = u(t)$. [10 marks]

Note:

Name	Function	Fourier Transform
Step function	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
Exponential function	$e^{-at}u(t)$	$\frac{1}{a + j\omega}$

Solution

a. Taking the Fourier transforms of the differential equation of the system, we have:

$$j\omega Y(\omega) + 2Y(\omega) = X(\omega)$$

Hence, the transfer function equation of the system is:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{2 + j\omega}$$

From the Fourier transform table, the Fourier transform of the exponential function is:

$$e^{-at}u(t) \leftrightarrow \frac{1}{a + j\omega}$$

Thus, the Fourier transform of the input signal is:

$$X(\omega) = \frac{1}{1 + j\omega}$$

And

$$Y(\omega) = X(\omega)H(\omega) = \left(\frac{1}{1 + j\omega}\right)\left(\frac{1}{2 + j\omega}\right) = \left(\frac{1}{1 + j\omega}\right) - \left(\frac{1}{2 + j\omega}\right)$$

Therefore, taking the inverse Fourier transform, the output of the system is:

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

b. From the Fourier transform table, the following pairs of Fourier transform are:

$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

Thus,

$$X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

As a result,

$$Y(\omega) = X(\omega)H(\omega) = \left[\pi\delta(\omega) + \frac{1}{j\omega}\right]\left(\frac{1}{2+j\omega}\right) = \pi\delta(\omega)\left(\frac{1}{2+j\omega}\right) + \frac{1}{j\omega(2+j\omega)}$$

Using the partial-fraction expansion technique, we have:

$$Y(\omega) = \left(\frac{\pi}{2}\right)\delta(\omega) + 0.5\left(\frac{1}{j\omega}\right) - 0.5\left(\frac{1}{2+j\omega}\right) = 0.5\left[\pi\delta(\omega) + \frac{1}{j\omega}\right] - 0.5\left(\frac{1}{2+j\omega}\right)$$

As we used the fact that $f(\omega)\delta(\omega) = f(0)\delta(\omega)$, thus:

$$y(t) = 0.5u(t) - 0.5e^{-2t}u(t) = 0.5(1 - e^{-2t})u(t)$$

3. For its frequency response, determine the magnitude and phase shift at low, high and cut-off frequencies for the following systems:

   a. $H(\omega) = 1 + j\omega/10.$ [24 marks]

   b. $H(\omega) = 1/(1 + j\omega/100).$ [24 marks]

   c. $H(\omega) = [10^4(1 + j\omega)]/[(10 + j\omega)(100 + j\omega)].$ [24 marks]

Solution

a. For the first system, the magnitude and phase shift of the frequency response are determined as follows.

Magnitude of Frequency Response:

The equation for the magnitude of the frequency response of the system is:

$$|H(\omega)|_{\text{dB}} = 20\log_{10}|H(\omega)| = 20\log_{10}\left|1 + j\frac{\omega}{10}\right|$$

For $\omega \ll 10$, as $\omega \to 0$:

$$|H(\omega)|_{\text{dB}} = 20\log_{10}\left|1 + j\frac{\omega}{10}\right| \to 20\log_{10}(1) = 0$$

For $\omega \gg 10$, as $\omega \to \infty$:

$$|H(\omega)|_{\text{dB}} = 20\log_{10}\left|1 + j\frac{\omega}{10}\right| \to 20\log_{10}\left(\frac{\omega}{10}\right)$$

At the corner frequency ω = 10.

$$H(10)|_{\text{dB}} = 20\log_{10}|1 + j1| = 20\log_{10}\sqrt{2} \approx 3\text{ dB}$$

Phase Shift of Frequency Response:

Next, the equation for the phase-shift of the frequency response of the system is:

$$\theta_H(\omega) = \tan^{-1}\left(\frac{\omega}{10}\right)$$

As $\omega \to 0$,

$$\theta_H(\omega) = \tan^{-1}\left(\frac{\omega}{10}\right) \to 0$$

As $\omega \to \infty$,

$$\theta_H(\omega) = \tan^{-1}\left(\frac{\omega}{10}\right) \to \frac{\pi}{2}$$

At ω = 10,

$$\theta_H(10) = \tan^{-1}(1) = \frac{\pi}{4}$$

b. For the second system, the magnitude and phase shift of the frequency response are determined as follows.

Magnitude of Frequency Response:

The equation for the magnitude of the frequency response of the system is:

$$|H(\omega)|_{\text{dB}} = 20\log_{10}\left|\frac{1}{1+\frac{j\omega}{100}}\right| = -20\log_{10}\left|1+j\frac{\omega}{100}\right|$$

For $\omega \ll 100$, as $\omega \to 0$:

$$|H(\omega)|_{\text{dB}} = -20\log_{10}\left|1+j\frac{\omega}{100}\right| \to -20\log_{10}(1) = 0$$

For $\omega \gg 100$, as $\omega \to +\infty$:

$$|H(\omega)|_{\text{dB}} = -20\log_{10}\left|1+j\frac{\omega}{100}\right| \to -20\log_{10}\left(\frac{\omega}{100}\right)$$

On a log frequency scale $-20\log_{10}(\omega/100)$ is a straight line with a slope of - 20 dB/decade.

As shown in the figure given below, this straight line intersects the 0-dB axis at the comer frequency ω = 100.

At the corner frequency ω = 100,

$$H(100)|_{\text{dB}} = -20\log\sqrt{2} = -3\ \text{dB}$$

Phase Shift of Frequency Response:

Next, the phase shift of the frequency response of the system is calculated from:

$$\theta_H(\omega) = -\tan^{-1}\left(\frac{\omega}{100}\right)$$

As $\omega \to 0$,

$$\theta_H(\omega) = -\tan^{-1}\left(\frac{\omega}{100}\right) \to 0$$

As $\omega \to \infty$,

$$\theta_H(\omega) = -\tan^{-1}\left(\frac{\omega}{100}\right) \to -\frac{\pi}{2}$$

At $\omega = 100$,

$$\Theta_H(100) = -\tan^{-1}(1) = -\frac{\pi}{4}$$

c. For the third system, the magnitude and phase shift of the frequency response are determined as follows.

Magnitude of Frequency Response:

First, we rewrite H(w) in standard form as:

$$H(\omega) = \frac{10(1+j\omega)}{\left(1+\frac{j\omega}{10}\right)\left(1+\frac{j\omega}{100}\right)}$$

Then

$$|H(\omega)|_{\text{dB}} = 20\log_{10}10 + 20\log_{10}|1+j\omega| - 20\log_{10}\left|1+j\frac{\omega}{10}\right| - 20\log_{10}\left|1+j\frac{\omega}{100}\right|$$

Note that there are three corner frequencies, ω = 1, ω = 10, and ω = 100.

At corner frequency ω = 1,

$$H(1)|_{\text{dB}} = 20 + 20\log_{10}\sqrt{2} - 20\log_{10}\sqrt{1.01} - 20\log_{10}\sqrt{1.0001} \approx 23\ \text{dB}$$

At corner frequency ω = 10,

$$H(10)|_{\text{dB}} = 20 + 20\log_{10}\sqrt{101} - 20\log_{10}\sqrt{2} - 20\log_{10}\sqrt{1.01} \approx 37\ \text{dB}$$

At corner frequency ω = 100,

$$H(100)|_{\text{dB}} = 20 + 20\log_{10}\sqrt{10{,}001} - 20\log_{10}\sqrt{101} - 20\log_{10}\sqrt{2} \approx 37\ \text{dB}$$

Phase Shift of Frequency Response:

Next, the equation for the phase shift of the frequency response of the system is:

$$\theta_H(\omega) = \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}\left(\frac{\omega}{100}\right)$$

As $\omega \to 0$,

$$\theta_H(\omega) = 0 - 0 - 0 = 0$$

As $\omega \to \infty$,

$$\theta_H(\omega) = \frac{\pi}{2} - \frac{\pi}{2} - \frac{\pi}{2} = -\frac{\pi}{2}$$

The phase shifts of the frequency response of the system at the cut-off frequencies are outlined as follows.

At corner frequency ω = 1,

$$\theta_H(1) = \tan^{-1}(1) - \tan^{-1}(0.1) - \tan^{-1}(0.01) = 0.676 \text{ rad}$$

At corner frequency ω = 10,

$$\theta_H(10) = \tan^{-1}(10) - \tan^{-1}(1) - \tan^{-1}(0.1) = 0.586 \text{ rad}$$

At corner frequency ω = 100,

$$\theta_H(100) = \tan^{-1}(100) - \tan^{-1}(10) - \tan^{-1}(1) = -0.696 \text{ rad}$$

4. The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

a. Knowing the transfer function of the system is given below, determine the frequency response of the system and sketch its Bode plot. [8 marks]

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

b. Specify, as a function of frequency, the group delay associated with this system. [4 marks]

c. If $x(t) = e^{-t}u(t)$, determine $Y(j\omega)$, the Fourier transform of the output. [4 marks]

d. Using the technique of partial-fraction expansion, determine the output $y(t)$ for the input $x(t)$ in part (c). [4 marks]

e. Repeat parts (c) and (d), if the input has as its Fourier transform:

i. $X(j\omega) = \frac{1+j\omega}{2+j\omega}$ [4 marks]

ii. $X(j\omega) = \frac{2+j\omega}{1+j\omega}$ [4 marks]

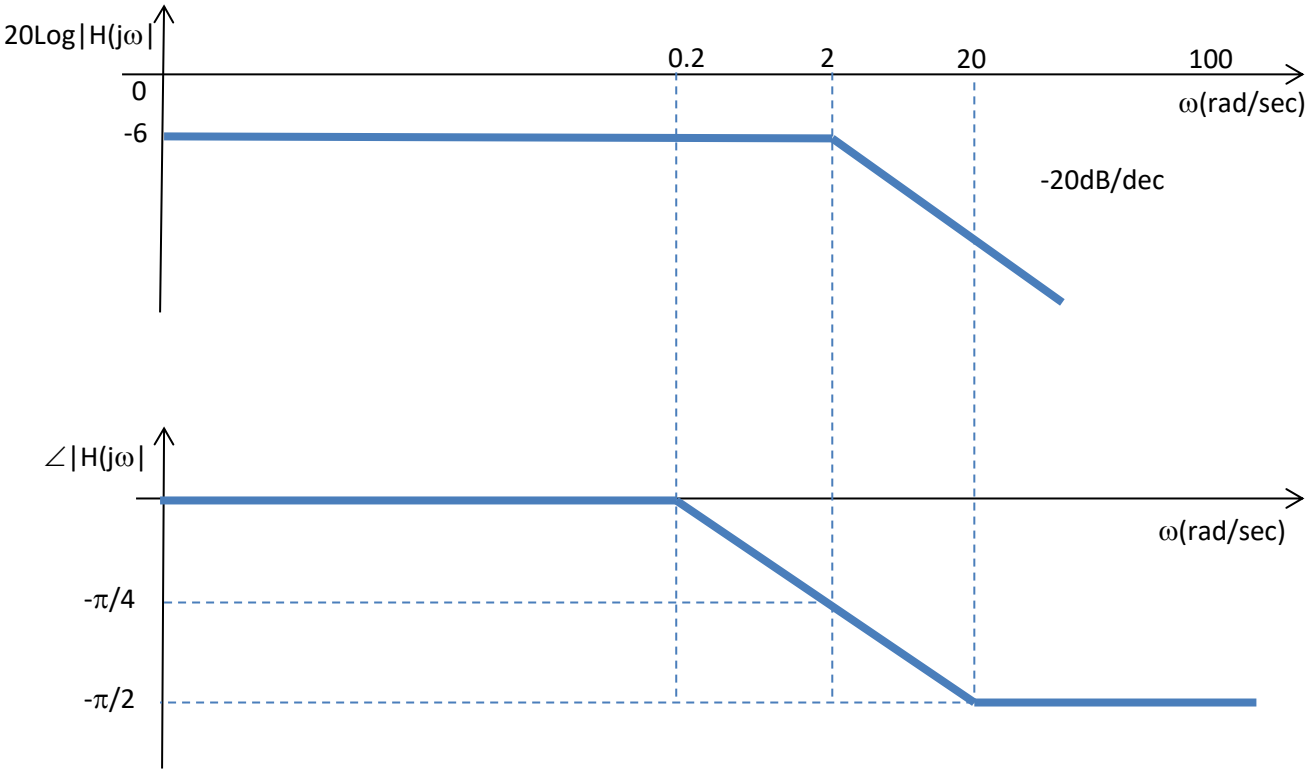
iii. $X(j\omega) = \frac{1}{(2+j\omega)(1+j\omega)}$ [4 marks]

Solution

a. Taking the Fourier transform of both sides of the given differential equation, we obtain:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{(2+j\omega)}$$

Based on the transfer function equation given above, the Bode plot is as shown in the following figure (e.g. magnitude and phase shift graphs):



b. From the expression for $H(j\omega)$, we obtain the phase shift of the system is:

$$\measuredangle H(j\omega) = -\tan^{-1}(\omega/2)$$

Therefore,

$$\tau(\omega) = \frac{d\measuredangle H(j\omega)}{d\omega} = \frac{2}{4+\omega^2}$$

c. Since $x(t) = e^{-t}u(t)$, then its transform is:

$$X(j\omega) = \frac{1}{(1+j\omega)}$$

Therefore, substituting $X(j\omega)$ and $H(j\omega)$ in the equation from part (a) for $Y(j\omega)$:

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(1+j\omega)(2+j\omega)}$$

d. Taking the inverse Fourier transform of the partial fraction expansion of $Y(j\omega)$ we obtain:

$$y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

e. For the three case studies given in this question:

i. For the first case, the equation becomes:

$$Y(j\omega) = \frac{1 + j\omega}{(2 + j\omega)^2}$$

Taking the inverse Fourier transform of the partial fraction expansion of $Y(j\omega)$, we obtain:

$$y(t) = e^{-2t}u(t) - te^{-2t}u(t)$$

ii. For the second case, the equation becomes:

$$Y(j\omega) = \frac{1}{(1 + j\omega)}$$

Taking the inverse Fourier transform of $Y(j\omega)$, we obtain:

$$y(t) = e^{-t}u(t)$$

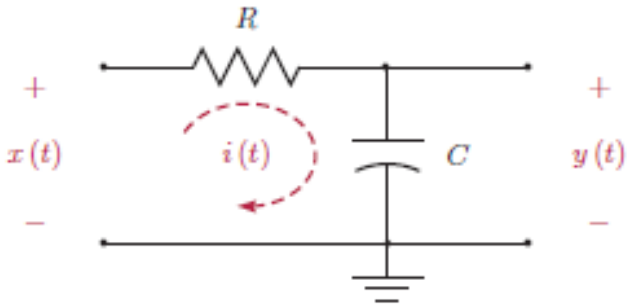
iii. For the third case, the equation becomes:

$$Y(j\omega) = \frac{1}{(1 + j\omega)(2 + j\omega)^2}$$

Taking the inverse Fourier transform of the partial fraction expansion of $Y(j\omega)$, we obtain:

$$y(t) = e^{-t}u(t) + e^{-2t}u(t) - te^{-2t}u(t)$$

5. Consider the continuous-time LTI system implemented as the RC circuit shown in the figure below. The voltage source $x(t)$ is considered the input to this system. The voltage $y(t)$ across the capacitor is considered the system output.



a. Determine the transfer function of the circuit. [6 marks]

b. If the values of resistor and capacitor are R = 1 Ω and C = 1 F, Determine the time-domain transient response of the system (e.g. time constant, rise time, and settling time). [8 marks]

c. Is it possible for the step response of the system to have an oscillatory behaviour? [2 marks]

Solution

a. Let find first the differential equation governing the input and output of this circuit.

The current through the resistor is the current through capacitor. Thus:

$$i(t) = C\left[\frac{dy(t)}{dt}\right]$$

The voltage across the resistor is:

$$V_R = Ri(t) = RC\left[\frac{dy(t)}{dt}\right]$$

Applying Kirchhoff's voltage law (KVL), the total input voltage across the circuit is the voltage across resistor and the voltage across capacitor.

$$V_T = V_R + V_C$$

Therefore, with $V_T = x(t)$ and $V_C = y(t)$, as a result:

$$x(t) = RC\left[\frac{dy(t)}{dt}\right] + y(t)$$

Thus, rearranging the equation above, it is now:

$$\frac{y(t)}{x(t)} = \frac{1}{RC\left[\frac{dy(t)}{dt}\right] + 1}$$

Applying the Fourier transform, the frequency response of this circuit is:

$$H(j\omega) = \frac{1}{RCj\omega + 1}$$

b. If R = 1 Ω and C = 1 F, determine the time-domain transient response of the system (e.g. time constant, rise time, and settling time). For the given first-order system:

$$H(j\omega) = \frac{1}{j\omega + a}$$

If R = 1 Ω and C = 1 F, then the transfer function equation of the circuit is given as:

$$H(j\omega) = \frac{1}{j\omega + 1}$$

Time constant (τ):

$$\tau = \frac{1}{a} = \frac{1}{1} = 1 \text{ second}$$

Rise time (T_r):

$$T_r = \frac{2.2}{a} = \frac{2.2}{1} = 2.2 \text{ seconds}$$

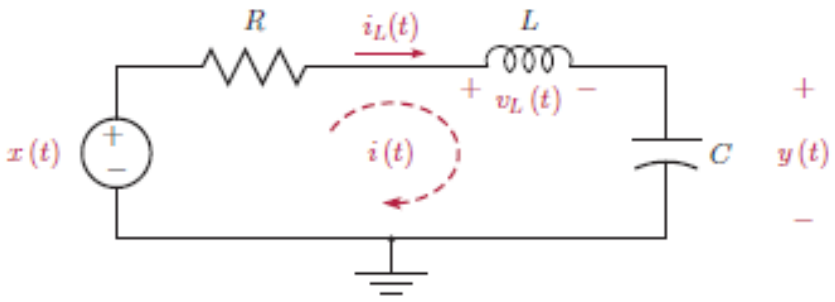
Settling time (T_s):

$$T_s = \frac{4}{a} = \frac{4}{1} = 4 \text{ seconds}$$

c. Is it possible for the step response of the system to have an oscillatory behaviour?

Since this is a continuous time first-order system, the step response of the system has to be non-oscillatory.

6. Consider the continuous-time LTI system implemented as the RLC circuit shown in the following figure. The voltage source $x(t)$ is considered the input to this system. The voltage $y(t)$ across the capacitor is considered the system output.



a. Determine the transfer function of the circuit. [8 marks]

b. If R = 2 Ω, L = 1 H, and C = 1 F, determine the transient response of the system (e.g. natural frequency, damping ratio, time-to-peak, and percentage overshoot). [20 marks]

c. How should R, L, and C be related so that there is no oscillation in the step response? [4 marks]

Solution

a. Let us first find the differential equation governing the input and output of this circuit.

The current through resistor is the current through the inductor and is also the current through capacitor.

$$i(t) = C\left[\frac{dy(t)}{dt}\right]$$

The voltage across resistor is:

$$V_R = Ri(t) = RC\left[\frac{dy(t)}{dt}\right]$$

The voltage across inductor is:

$$V_L = L\frac{di(t)}{dt} = LC\left[\frac{d^2y(t)}{dt^2}\right]$$

Total input voltage is equal to sum of voltage across inductor, voltage across resistor, and voltage across capacitor:

$$V_T = V_L + V_R + V_C$$

Therefore, with the input and voltage across the circuit, $V_T = x(t)$ and the voltage at the output is the voltage across the capacitor, $V_C = y(t)$, the differential equation of the given circuit is:

$$x(t) = LC\left[\frac{d^2y(t)}{dt^2}\right] + RC\left[\frac{dy(t)}{dt}\right] + y(t)$$

Thus

$$\frac{y(t)}{x(t)} = \frac{1}{LC\left[\frac{d^2y(t)}{dt^2}\right] + RC\left[\frac{dy(t)}{dt}\right] + 1}$$

Applying the Fourier transform, the transfer function of this circuit is therefore:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{LC(j\omega)^2 + RCj\omega + 1}$$

We may rewrite this to be:

$$H(j\omega) = \frac{\left(1/\sqrt{LC}\right)^2}{(j\omega)^2 + (R/L)j\omega + \left(1/\sqrt{LC}\right)^2} \qquad (Eq.1)$$

b. If R = 2 Ω, L = 1 H, and C = 1 F, determine the transient response of the system (e.g. natural frequency, damping ratio, time-to-peak, and percentage overshoot).

Giving a standardised second order transfer function:

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\omega_n\zeta(j\omega) + \omega_n^2} \qquad (Eq.2)$$

If R = 2 Ω, L = 1 H, and C = 1 F, the transfer function of the system becomes:

$$G(j\omega) = \frac{1}{(j\omega)^2 + 2(j\omega) + 1}$$

The natural frequency (ω_n) is:

$$\omega_n = \sqrt{1} = 1 \text{ rad/s}$$

The damping ratio (ζ) is:

$$\zeta = \frac{1}{2\omega_n} = \frac{1}{2(1)} = 0.5$$

The time-to-peak (with $n = 1$) to the first peak (T_p) is:

$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{1\sqrt{1-(0.5)^2}} = \frac{\pi}{\sqrt{0.75}} = 3.63 \text{ seconds}$$

The percentage overshoot ($\%OS$) is:

$$\%OS = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \times 100\% = e^{-\left(\frac{0.5\times3.14}{\sqrt{1-(0.5)^2}}\right)} \times 100\% = 90.5\%$$

c. By equating the equations (1) with (2), thus:

$$2\omega_n\zeta = \frac{R}{L}$$

With $\omega_n = 1/\sqrt{LC}$, rearranging the equation given above, the damping ratio is:

$$\zeta = \frac{(R/L)}{2\omega_n} = \frac{(R/L)}{2\left(1/\sqrt{LC}\right)}$$

In order for the step response to have no oscillations, we must have $\zeta \geq 1$.

$$\frac{R}{2}\sqrt{\frac{C}{L}} \geq 1$$

Thus

$$R \geq 2\sqrt{\frac{L}{C}}$$

7. A causal discrete-time LTI system is described by:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$$

where $x[n]$ and $y[n]$ are the input and output of the system, respectively.

a. Determine the equation for the frequency response $H(\Omega)$ of the system. [4 marks]

b. Find the impulse response $h[n]$ of the system. [4 marks]

Solution

a. Taking the Fourier transform of the transfer function equation of the system, we obtain:

$$Y(\Omega) - \frac{3}{4}e^{-j\Omega}Y(\Omega) + \frac{1}{8}e^{-j2\Omega}Y(\Omega) = X(\Omega)$$

or

$$\left(1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}\right)Y(\Omega) = X(\Omega)$$

Thus,

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1}{1 - \frac{3}{4}e^{-j\Omega} + \frac{1}{8}e^{-j2\Omega}} = \frac{1}{\left(1 - \frac{1}{2}e^{-j\Omega}\right)\left(1 - \frac{1}{4}e^{-j\Omega}\right)}$$

b. Using partial-fraction expansions, we have:

$$H(\Omega) = \frac{1}{\left(1-\frac{1}{2}e^{-j\Omega}\right)\left(1-\frac{1}{4}e^{-j\Omega}\right)} = \frac{2}{1-\frac{1}{2}e^{-j\Omega}} - \frac{1}{1-\frac{1}{4}e^{-j\Omega}}$$

Taking the inverse Fourier transform of $H(\Omega)$, we obtain:

$$h[n] = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u[n]$$

8. Consider a discrete-time LTI system described by:

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{2}x[n-1]$$

a. Determine the transfer function equation, $H(\Omega) = Y(\Omega)/X(\Omega)$ of the system. [6 marks]

b. Find the impulse response $h[n]$ of the system. [4 marks]

c. Determine its response $y[n]$ to the input: [8 marks]

$$x[n] = \cos\left[\left(\frac{\pi}{2}\right)n\right]$$

Note:

Function	Fourier Transform	Condition
$e^{j\Omega_0 n}$	$2\pi\delta(\Omega-\Omega_0)$	$\lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
$\cos\Omega_0 n$	$\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$	$\lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$

Solution

a. Taking the Fourier transform of the transfer function equation of the system, we obtain:

$$Y(\Omega) - \frac{1}{2}e^{-j\Omega}Y(\Omega) = X(\Omega) + \frac{1}{2}e^{-j\Omega}X(\Omega)$$

Rearranging the equation above, thus:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1+\frac{1}{2}e^{-j\Omega}}{1-\frac{1}{2}e^{-j\Omega}}$$

b. Expanding the transfer function equation of the system, it is now:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1+\frac{1}{2}e^{-j\Omega}}{1-\frac{1}{2}e^{-j\Omega}} = \frac{1}{\left(1-\frac{1}{2}e^{-j\Omega}\right)} + \frac{1}{2}\left(\frac{e^{-j\Omega}}{1-\frac{1}{2}e^{-j\Omega}}\right)$$

Taking the inverse Fourier transform of $H(\Omega)$, we obtain:

$$h(n) = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$$= \begin{cases} 1 & n = 0 \\ \left(\frac{1}{2}\right)^{n-1} & n \geq 1 \end{cases}$$

c. From the Fourier transform pairs of:

$$\cos \Omega_0 n \leftrightarrow \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)] \qquad |\Omega|, [\Omega_0] \leq \pi$$

As a result,

$$X(\Omega) = \pi\left[\delta\left(\Omega - \frac{\pi}{2}\right) + \delta\left(\Omega + \frac{\pi}{2}\right)\right] \qquad |\Omega| \leq \pi$$

Then

$$Y(\Omega) = X(\Omega)H(\Omega)$$

$$= \pi\left[\delta\left(\Omega - \frac{\pi}{2}\right) + \delta\left(\Omega + \frac{\pi}{2}\right)\right]\left(\frac{1 + \frac{1}{2}e^{-j\Omega}}{1 - \frac{1}{2}e^{-j\Omega}}\right)$$

$$= \pi\left(\frac{1 + \frac{1}{2}e^{-j\Omega}}{1 - \frac{1}{2}e^{-j\Omega}}\right)\delta\left(\Omega - \frac{\pi}{2}\right) + \pi\left(\frac{1 + \frac{1}{2}e^{j\Omega}}{1 - \frac{1}{2}e^{j\Omega}}\right)\delta\left(\Omega + \frac{\pi}{2}\right)$$

$$= \pi\left(\frac{1 - j\frac{1}{2}}{1 + j\frac{1}{2}}\right)\delta\left(\Omega - \frac{\pi}{2}\right) + \pi\left(\frac{1 + j\frac{1}{2}}{1 - j\frac{1}{2}}\right)\delta\left(\Omega + \frac{\pi}{2}\right)$$

$$= \pi\delta\left(\Omega - \frac{\pi}{2}\right)e^{-j2\tan^{-1}\left(\frac{1}{2}\right)} + \pi\delta\left(\Omega + \frac{\pi}{2}\right)e^{j2\tan^{-1}\left(\frac{1}{2}\right)}$$

Taking the inverse Fourier transform of $Y(\Omega)$ and using the pair of discrete Fourier transform given below,

$$e^{j\Omega_0 n} \leftrightarrow 2\pi\delta(\Omega - \Omega_0) \qquad |\Omega|, |\Omega_0| \leq \pi$$

we get:

$$y[n] = \frac{1}{2}e^{j\left(\frac{\pi}{2}\right)n}e^{-j2\tan^{-1}\left(\frac{1}{2}\right)} + \frac{1}{2}e^{-j\left(\frac{\pi}{2}\right)}e^{j2\tan^{-1}\left(\frac{1}{2}\right)}$$

$$= \cos\left[\left(\frac{\pi}{2}\right)n - 2\tan^{-1}\left(\frac{1}{2}\right)\right]$$

9. Consider the discrete-time LTI system shown as differential equation given below.

$$y[n] = x[n] + x[n-1]$$

a. Find the equation for the frequency response $H(\Omega) = Y(\Omega)/X(\Omega)$ of the system. [4 marks]

b. Find the impulse response $h[n]$ of the system. [2 marks]

c. Sketch the magnitude response $|H(\Omega)|$ and the phase response $\theta(\Omega)$. [8 marks]

d. Find the cut-off frequency (3-dB bandwidth) of the system. [6 marks]

Solution

a. Taking the discrete-time Fourier transform of the differential equation, we have:

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = 1 + e^{-j\Omega}$$
$$= e^{-\frac{j\Omega}{2}}\left(e^{\frac{j\Omega}{2}} + e^{-\frac{j\Omega}{2}}\right)$$
$$= 2e^{-j\Omega/2}\cos\left(\frac{\Omega}{2}\right) \qquad |\Omega| \le \pi$$

b. By the definition of $h[n]$ and taking into account the differential equation of the system, we obtain:

$$h[n] = \delta[n] + \delta[n-1]$$

or

$$h[n] = \begin{cases} 1 & 0 \le n \le 1 \\ 0 & \text{otherwise} \end{cases}$$

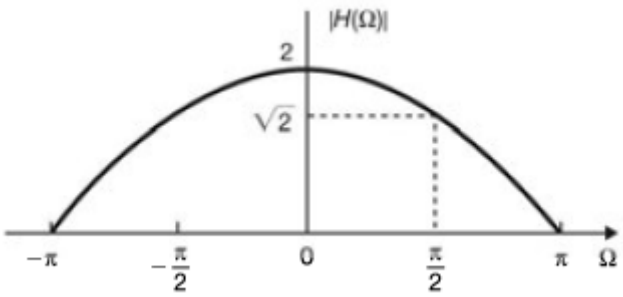
c. The equations for the magnitude and phase shift for the frequency response of the system are:

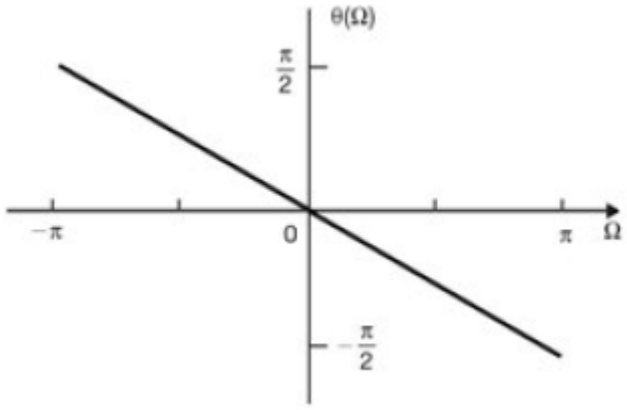
$$|H(\Omega)| = 2\cos\left(\frac{\Omega}{2}\right) \qquad |\Omega| \le \pi$$

And

$$\theta(\Omega) = -\frac{\Omega}{2} \qquad |\Omega| \le \pi$$

The magnitude and phase shift plots of the frequency response are sketched in the figures given below.





d. Let $\Omega_{3\text{dB}}$ be the 3-dB bandwidth of the system. Then, by definition, we obtain:

$$|H(\Omega_{3\text{dB}})| = \frac{1}{\sqrt{2}}|H(\Omega)|_{\max}$$

We obtain:

$$\cos\left(\frac{\Omega_{3\text{dB}}}{2}\right) = \frac{1}{\sqrt{2}}$$

Thus

$$\Omega_{3\text{dB}} = \frac{\pi}{2}$$

We see that the system is actually a discrete-time wideband low-pass finite impulse response (FIR) filter.

B. Sampling and Reconstruction

1. Determine the following Nyquist frequency (maximum frequency component) and Nyquist rate (sampling frequency) of the following signals:

   a. $x(t) = 3\sin(t/10)$. [6 marks]

   b. $x(t) = 0.5\cos(\pi t/5) + 2\sin(8\pi t)$. [6 marks]

Solution

a. For the signal $x(t) = 3\sin(t/10)$, taking the Fourier transform of the given signal:

$$X(j\omega) = \frac{3\pi}{j}\left[\delta\left(\omega - \frac{1}{10}\right) - \delta\left(\omega + \frac{1}{10}\right)\right]$$

As a result, its Nyquist frequency = 1/20π Hz and its sampling rate = 2 x 1/20π = 1/10π Hz.

b. For the signal $x(t) = 0.5\cos(\pi t/5) + 2\sin(8\pi t)$, taking the Fourier transform of the given signal:

$$X(j\omega) = 0.5\pi\left[\delta\left(\omega - \frac{\pi}{5}\right) + \delta\left(\omega + \frac{\pi}{5}\right)\right] + \frac{2\pi}{j}[\delta(\omega - 8\pi) - \delta(\omega + 8\pi)]$$

From the two frequency components of the signal, the highest is 8/2 = 4 Hz.

As a result, its Nyquist frequency = 4 Hz and sampling rate = 8 Hz.

2. Wheel, rotating at 6 Hz, is seen in a dark room by means of a strobe light flashing at a rate of 8 Hz.

a. Determine the apparent rotational speed and sense of rotation of the wheel. [4 marks]

b. Repeat the question if the flashes occur at 12 Hz, 16 Hz, or 24 Hz. [4 marks]

Solution

a. For a given sampling system, the Nyquist interval is:

$$\left[-\frac{f_s}{2}, -\frac{f_s}{2}\right] = [-4, 4]\ \text{Hz}$$

The 6 Hz frequency of the wheel lies outside the interval. Therefore, it will be aliased with:

$$f - f_s = 6 - 8 = -2\ \text{Hz}$$

Thus, the wheel will appear to be turning at 2 Hz in the opposite direction.

b. If sampling frequency f_s = 12 Hz, the Nyquist interval is:

$$\left[-\frac{f_s}{2}, -\frac{f_s}{2}\right] = [-6, 6]\ \text{Hz}$$

Points on the wheel will appear to be moving up and down, with either positive or negative sense of rotation.

For the other two sampling frequencies 16 Hz and 24 Hz, the Nyquist interval is:

$$\left[-\frac{f_s}{2}, -\frac{f_s}{2}\right] = [-8, 8]\ \text{Hz}$$

or

$$\left[-\frac{f_s}{2}, -\frac{f_s}{2}\right] = [-12, 12]\ \text{Hz}$$

Therefore, the original 6 Hz frequency component lies in the interval, and no aliasing will be perceived.

3. For $-\infty < t < +\infty$, consider the signal given in the equation below:

$$x(t) = 2\cos\left(2\pi t + \frac{\pi}{4}\right)$$

a. Determine if the signal is bandlimited. [4 marks]

b. Using the sampling periods listed below, find out whether the Nyquist sampling rate condition is satisfied and if the sampled signal looks like the original signal or not.

i. $T_s = 0.4$ second/sample. [6 marks]

ii. $T_s = 0.5$ second/sample. [6 marks]

iii. $T_s = 1$ second/sample. [6 marks]

Solution

a. Since $x(t)$ only has the frequency 2π, it is band limited with Ω_{max} = 2π (rad/s). For any T_s the sampled signal is given as:

$$x_s(t) = \sum_{n=-\infty}^{\infty} 2\cos\left(2\pi Ts + \frac{\pi}{4}\right)\delta(t - nTs) \qquad (Eq.1)$$

b. For the given sampling rates

i. $T_s = 0.4$ second/sample:

Using T_s = 0.4 s/sample the sampling frequency in rad/s is $\Omega_s = 2\pi/T_s = 5\pi > 2\Omega_{max} = 4\pi$, satisfying the Nyquist sampling rate condition.

The samples in equation (1) with $-\infty < n < \infty$ are then:

$$x(nT_s) = 2\cos\left[2\pi(0.4)n + \frac{\pi}{4}\right]$$

$$= 2\cos\left[\left(\frac{4\pi}{5}\right)n + \frac{\pi}{4}\right] \qquad (Eq.2)$$

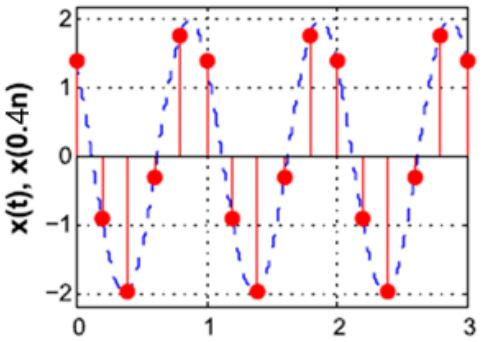
The corresponding sampled signal $x_s(t)$ repeats periodically every five samples.

Indeed, letting $m = n - 5$,

$$x_s(t + 5T_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t - (n-5)T_s)$$

Since $x((m+5)T_s) = x(mT_s)$, thus:

$$x_s(t + 5T_s) = \sum_{n=-\infty}^{\infty} x\big((m+5)T_s\big)\,\delta(t - mT_s) = x_s(t)$$



ii. $T_s = 0.5$ second/sample:

When T_s = 0.5, the sampling frequency is:

$$\Omega_s = \frac{2\pi}{T_s} = 4\pi = 2\Omega_{max}$$

The result given above is barely satisfying the Nyquist sampling rate condition.

With $-\infty < n < +\infty$, the samples in the equation (1) are now:

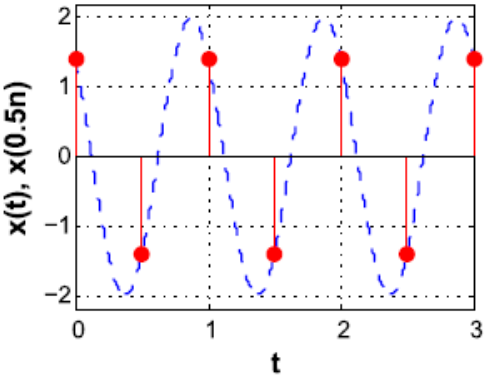
$$x(nT_s) = 2\cos\left[2\pi(0.5)n + \frac{\pi}{4}\right]$$

$$= 2\cos\left[\left(\frac{2\pi}{2}\right)n + \frac{\pi}{4}\right]$$

In this case, the sampled signal repeats periodically every 2 samples; as can easily be checked:

$$x((n+2)T_s) = x(nT_s)$$

According to the Nyquist sampling rate condition, this is the minimum number of samples per period allowed before we start having aliasing.



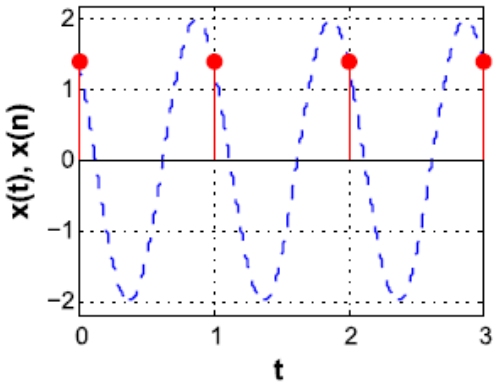
iii. $T_s = 1$ second/sample:

If we let $\Omega_s = \Omega_{max} = 2\pi$ corresponding to the sampling period T_s = 1, the samples in equation (1) are:

$$x(nT_s) = 2\cos\left[2\pi(1)n + \frac{\pi}{4}\right]$$

$$= 2\cos\left(2\pi n + \frac{\pi}{4}\right)$$

$$= 2\cos\left(\frac{\pi}{4}\right) = \sqrt{2}$$



As a result, the sampled signal is $\sqrt{2}\delta T_s(t)$. In this case, the sampled signal cannot be possibly converted back into the continuous-time sinusoid. Thus, we have lost the information provided by the sinusoid.

4. Consider the two sinusoids when $-\infty \leq t \leq +\infty$ and where: $\Omega_1 > 2\Omega_0$:

$$x_1(t) = \cos(\Omega_0 t) \qquad \text{and} \qquad x_2(t) = \cos[((\Omega_0 + \Omega_1)t)]$$

a. Show that if we sample these signals using $T_s = 2\pi/\Omega_1$, we cannot differentiate the sampled signals, i.e. $x_1(nT_s) = x_2(nT_s)$. [6 marks]

b. Explain the significance of this. [2 marks]

Solution

a. Sampling the two signals using $T_s = 2\pi/\Omega_1$, we have with $-\infty \leq n \leq +\infty$:

$$x_1(nT_s) = \cos(\Omega_0 n T_s)$$

And

$$x_2(nT_s) = \cos[(\Omega_0 + \Omega_1)nT_s]$$

But since $\Omega_1 T_s = 2\pi$, the sinusoid $x_2(nT_s)$ can be written:

$$x_2(nT_s) = \cos[(\Omega_0 T_s + \pi)n]$$

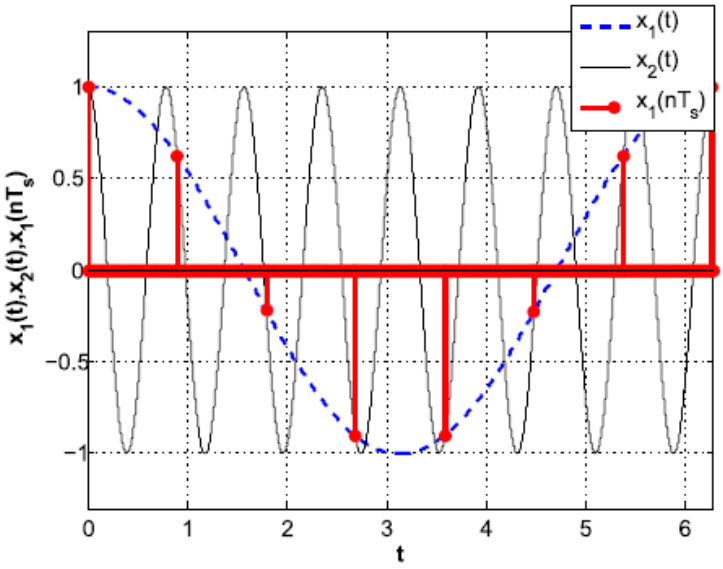
$$= \cos(\Omega_0 T_s n)$$

$$= x_1(nT_s)$$

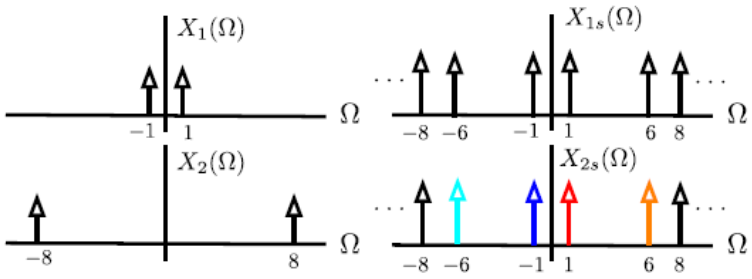
It is making it impossible to differentiate the two sampled signals.

Since the sampling frequency Ω_1 is smaller than twice the maximum frequency of $x_2(t)$ or $2(\Omega_0 + \Omega_1)$ the sampled version of this signal is aliased.

See the following figure for $\Omega_0 = 1$ and $\Omega_0 + \Omega_1 = 8$.



b. The spectrum of the signal in the frequency domain is shown in the figure below. The spectra of the two sinusoids are different, but the spectra of the sampled signals are identical.

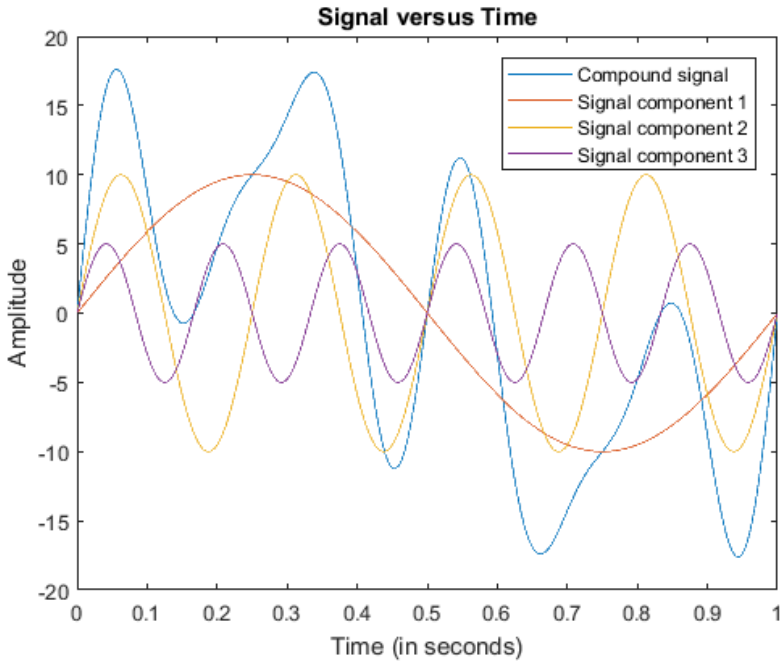


5. The analogue signal $x(t) = 10\sin(2\pi t) + 10\sin(8\pi t) + 5\sin(12\pi t)$ as shown below, where t is in seconds, is sampled at a rate of f_s = 5 Hz.

   a. Determine and sketch the signal $x_a(t)$ aliased with $x(t)$. [12 marks]

   b. Show that the original signal and its alias have the same sample values, that is, show that $x(nT) = x_a(nT)$. [8 marks]

   c. Repeat part (a) if the sampling rate is f_s = 10 Hz. [12 marks]



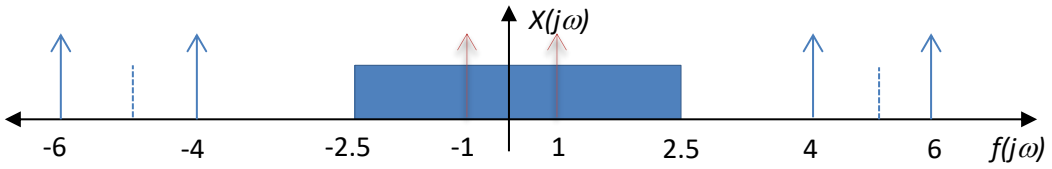
Solution

a. The three frequency components of the signal correspond to frequencies f_1 = 1 Hz, f_2 = 4 Hz, and f_3 = 6 Hz. Of these, f_2 and f_3 lie outside the Nyquist interval [−2.5, 2.5] Hz.

Therefore, they will be aliased with $f_{2a} = f_2 - f_s$ = 4 − 5 = −1 Hz and $f_{3a} = f_3 - f_s$ = 6 − 5 = 1 Hz and the aliased signal will be:

$$x_a(t) = 10\sin(2\pi t) + 10\sin(-2\pi t) + 5\sin(2\pi t)$$
$$= 5\sin(2\pi t)$$

The sketch of the frequency spectrum of the signal is given below.



Notice from the frequency spectrum above, since 4 Hz and 6 Hz signal components are more than 5 Hz sampling rate, the alias signal is 1 Hz or -1 Hz.

b. To show that they have the same sample values, we set $t = nT$, with $T = 1/f_s = 1/5$ sec. Then,

$$x(nT) = 10\sin\left(\frac{2\pi n}{5}\right) + 10\sin\left(\frac{8\pi n}{5}\right) + 5\sin\left(\frac{12\pi n}{5}\right)$$

but,

$$\sin\left(\frac{8\pi n}{5}\right) = \sin\left(2\pi n - \frac{2\pi n}{5}\right)$$
$$= -\sin\left(\frac{2\pi n}{5}\right)$$

and

$$\sin\left(\frac{12\pi n}{5}\right) = \sin\left(2\pi n + \frac{2\pi n}{5}\right)$$
$$= \sin\left(\frac{2\pi n}{5}\right)$$

thus,

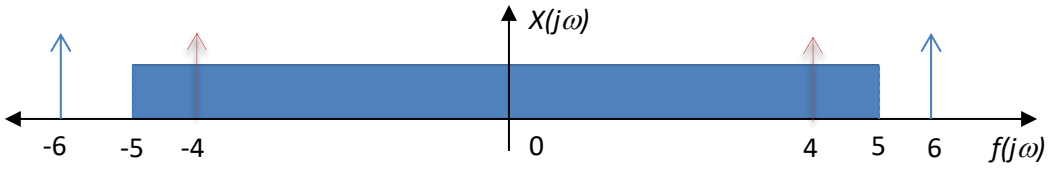
$$x(nT) = 10\sin\left(\frac{2\pi n}{5}\right) - 10\sin\left(\frac{2\pi n}{5}\right) + 5\sin\frac{2\pi n}{5})$$
$$= 5\sin\left(\frac{2\pi n}{5}\right)$$
$$= x_a(nT)$$

c. If f_s = 10 Hz, then the Nyquist interval is [−5, 5] Hz and only f_3 lies outside it.

It will be aliased with $f_3 - f_s$ = 6 − 10 = −4 Hz resulting in the aliased signal:

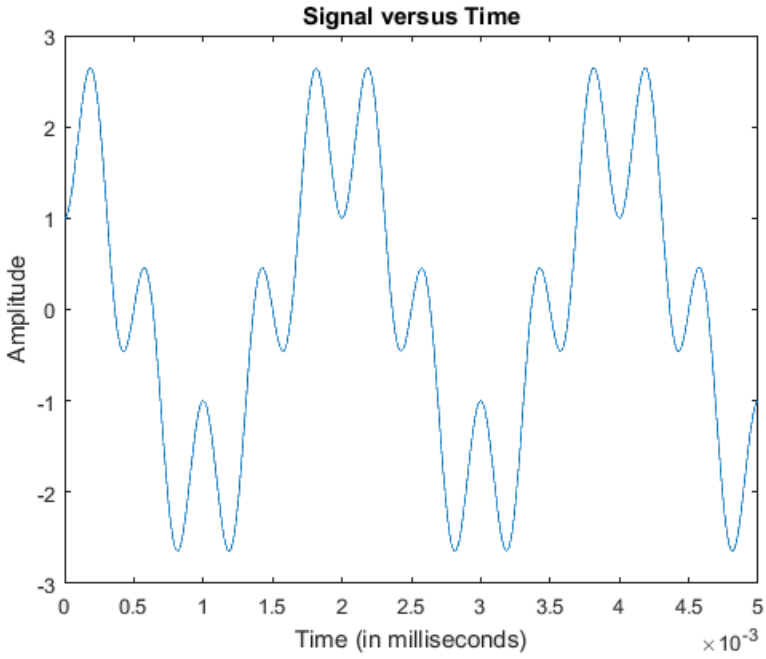
$$x_a(t) = 10\sin(2\pi t) + 10\sin(8\pi t) + 5\sin(2\pi(-4)t)$$
$$= 10\sin(2\pi t) + 5\sin(8\pi t)$$

The sketch of the frequency spectrum of the signal is given below.



Notice from the frequency spectrum above, since 6 Hz signal component is more than 10 Hz sampling rate, the alias signal is 4 Hz or -4 Hz.

6. The signal $x(t) = \cos(5\pi t) + 4\sin(2\pi t)\sin(3\pi t)$ as shown below, where t is in milliseconds, is sampled at a rate of 3 kHz.



a. Determine the signal $x_a(t)$ aliased with $x(t)$. [6 marks]

b. Determine and sketch two other signals $x_1(t)$ and $x_2(t)$ that are different from each other and from $x(t)$, yet they are aliased with the same $x_a(t)$ that you found. [16 marks]

Solution

a. Using the trigonometry identity: $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$, we find:

$$x(t) = \cos(5\pi t) + 4\sin(2\pi t)\sin(3\pi t)$$
$$= \cos(5\pi t) + 2[\cos(\pi t) - \cos(5\pi t)]$$
$$= 2\cos(\pi t) - \cos(5\pi t)$$

From the equation given above, the frequency components in the signal are f_1 = 0.5 kHz and f_2 = 2.5 kHz.

b. Since the frequency components in the signal are f_1 = 0.5 kHz and f_2 = 2.5 kHz and the Nyquist interval is [−1.5, 1.5] kHz, f_1 lies inside the interval and f_2 is outside it.

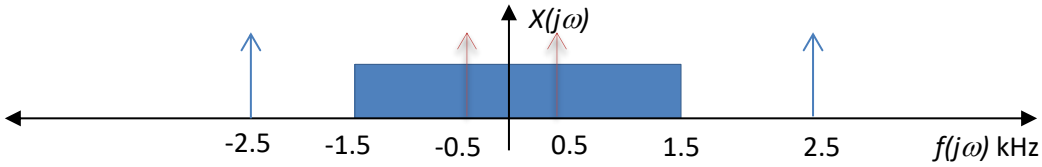
Thus, the second frequency component f_2 will be aliased with f_{2a} = 2.5 − 3 = −0.5 kHz giving rise to the signal:

$$x_a(t) = 2\cos(2\pi f_1 t) - \cos(2\pi f_2 a t)$$
$$= 2\cos(\pi t) - \cos(-\pi t) = \cos(\pi t)$$

A class of signals aliased with $x(t)$ and $xa(t)$ is obtained by replacing f_1 and f_2 by their shifted versions: $f_1 + mf_s$, $f_2 + nf_s$ resulting in:

$$x_{mn}(t) = 2\cos(\pi t + 6\pi mt) - \cos(\pi t - 6\pi nt)$$

The sketch of the frequency spectrum of the signal is given below.



Notice from the frequency spectrum above, since 2.5 Hz signal component is more than 3 Hz sampling rate, the alias signal is 0.5 Hz or -0.5 Hz.

7. Let a signal $x(t) = \cos(8\pi t) + \cos(4\pi t)\cos(6\pi t)$, where t is in seconds.

   a. Determine the frequency components of the signal. [4 marks]

   b. Determine and sketch the signal $x_a(t)$ aliased with $x(t)$, if the sampling rate is 5 Hz.

   [12 marks]

Solution

a. Using a trigonometry identity: $2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$, we write:

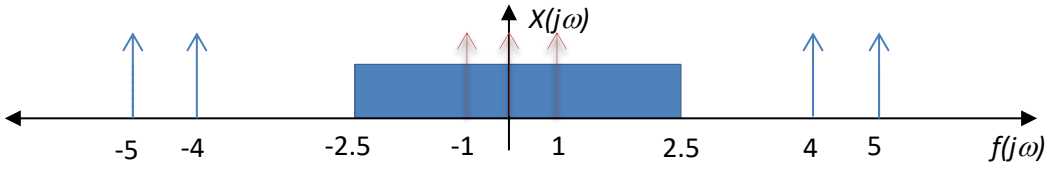
$$\begin{aligned} x(t) &= \cos(8\pi t) + \cos(4\pi t)\cos(6\pi t) \\ &= \cos(8\pi t) + 0.5\cos(6+4)\pi t + 0.5\cos(6-4)\pi t \\ &= \cos(8\pi t) + 0.5\cos(10\pi t) + 0.5\cos(2\pi t). \end{aligned}$$

The frequencies contained in $x(t)$ are thus, 1 Hz, 4 Hz, and 5 Hz.

b. If the sampling rate is 5 Hz, then the Nyquist interval is [−2.5, 2.5] Hz, and therefore f_1 = 1 Hz frequency component is inside the interval and experiences no aliasing.

The second and third components f_2 = 4 Hz and f_3 = 5 Hz lie outside this interval and will be aliased with the frequencies f_{2a} = 4 − 5 = −1 Hz and f_{3a} = 5 − 5 = 0 Hz, etc.

The sketch of the frequency spectrum of the signal is given below.



8. Consider the following signals listed below, determine if they are band-limited or not. If not, determine the frequency for which the energy of the non-band-limited signal corresponds to 99% of its total energy and use this result to approximate its maximum frequency.

   a. $x_1(t) = u(t+0.5) - u(t-0.5)$. [8 marks]

b. $x_2(t) = e^{-t}u(t)$. [8 marks]

Solution

a. The signal $x_1(t) = u(t + 0.5) - u(t - 0.5)$ is a unit pulse signal. Clearly this signal can easily be sampled by choosing any value of T_s << 1.

For instance, T_s = 0.01 second would be a good value, giving a discrete-time signal $x_1(nT_s) = 1$, for $0 \le nT_s = 0.01n \le 1$ or $0 \le n \le 100$. There seems to be no problem in sampling this signal; however, we see that the Fourier transform of $x_1(t)$,

$$X_1(\Omega) = \frac{\left(e^{j0.5\Omega} - e^{-j0.5\Omega}\right)}{j\Omega}$$

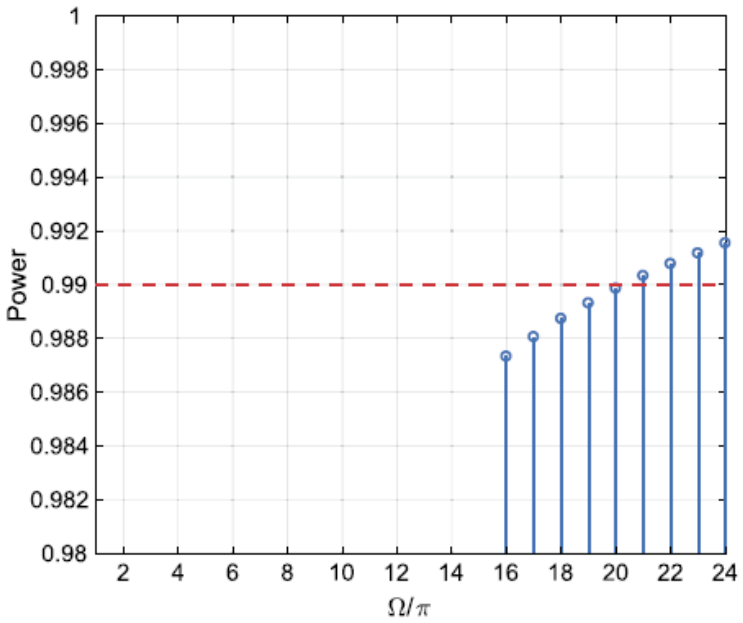
$$= \frac{\sin(0.5\Omega)}{0.5\Omega}$$

does not have a maximum frequency and so $x_1(t)$ is not band limited. Thus, any chosen value of T_s will cause aliasing. Fortunately, the values of the sinc function go fast to zero, so that one could compute an approximate maximum frequency that would include 99% of the energy of the signal.

Using Parseval's energy relation, we see that the energy of $x_1(t)$ (the area under $x_1^2(t)$ is 1 and if we wish to find a value Ω_M, such that 99% of this energy is in the frequency band $[-\Omega_M, +\Omega_M]$, we need to look for the limits of the following integral, so it equals 0.99:

$$0.99 = \frac{1}{2\pi}\int_{-\Omega_M}^{+\Omega_M}\left[\frac{\sin(0.5\Omega)}{0.5\Omega}\right]^2 d\Omega$$

By looking at the figure below, we find that for $\Omega_M = 20\pi$ (rad/s), 98.9% of the energy of the signal is included, and thus it could be used to approximate the actual maximum frequency and we could choose $T_s < \pi/\Omega_M = 0.05$ second/sample as the sampling period.



b. For the causal exponential $x(t) = e^{-t}u(t)$, its Fourier transform is:

$$X(\Omega) = \frac{1}{1 + j\Omega}$$

so that:

$$|X(\Omega)| = \frac{1}{\sqrt{(1 + \Omega^2)}}$$

which does not go to zero for any finite Ω, then $x(t)$ is not band limited.

To find a frequency $+\Omega_M$ so that 99% of the energy is in $-\Omega_M \leq \Omega \leq +\Omega_M$ we let:

$$\frac{1}{2\pi}\int_{-\Omega_M}^{+\Omega_M} |X(\Omega)|^2 \, d\Omega = \frac{0.99}{2\pi}\int_{-\infty}^{+\infty} |X(\Omega)|^2 \, d\Omega$$

which gives:

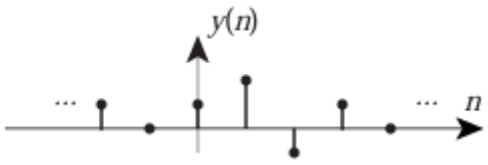
$$2\tan^{-1}(\Omega)|_0^{\Omega_M} = 2 \times 0.99\tan^{-1}(\Omega)|_0^{\infty}$$

or

$$\Omega_M = \tan\left(\frac{0.99\pi}{2}\right)$$
$$= 63.66 \text{ rad/s}$$

If we choose $\Omega_s = 2\pi/T_s = 5\Omega_M$ or $T_s = 2\pi/(5 \times 63.66) \approx 0.02$ second/sample, there will be hardly any aliasing or loss of information.

9. Given the following waveform of discrete-time signal as shown in the figure below.



Sketch the waveform after the reconstruction process using the following method:

a. Zero-order hold. [5 marks]

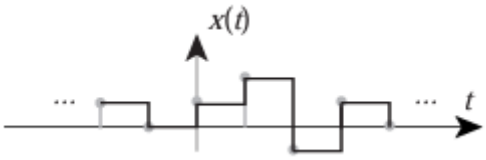
b. First-order hold (linear interpolation). [5 marks]

c. Ideal interpolation. [5 marks]

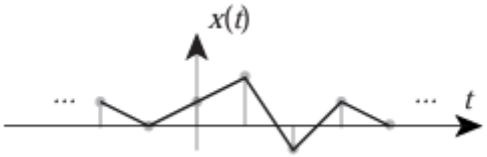
d. Weighted Dirac delta. [5 marks]

Solution

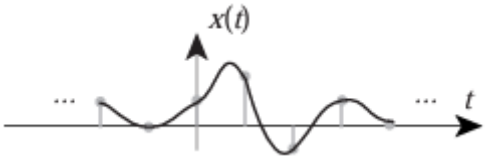
a. The waveform sketch of the signal using the zero-order hold method is as shown in the figure below.



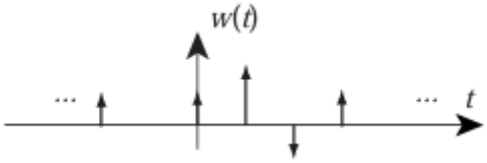
b. The waveform sketch of the signal using first-order hold (linear interpolation) is as shown in the figure below.



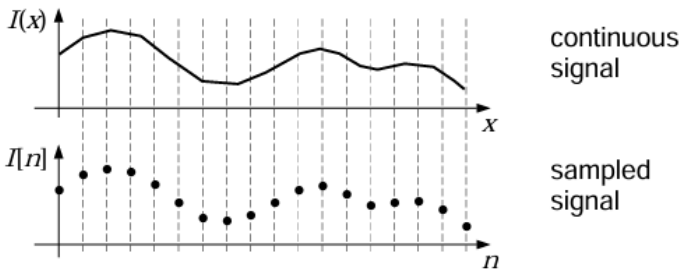
c. The waveform sketch of the signal using ideal interpolation is as shown in the figure below.



d. The waveform sketch of the signal using weighted Dirac delta is as shown in the figure below.



10. Given a continuous-time signal which is sampled by impulse sampling signal given in the figure below.



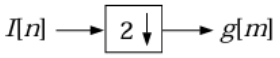
Sketch the result after the discrete-time sampling of the signal.

a. Down sampling by a factor of 2. [5 marks]

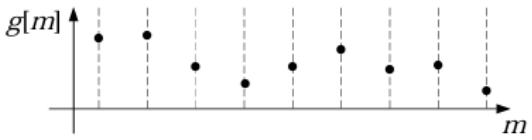
b. Up sampling by a factor of 2. [5 marks]

Solution

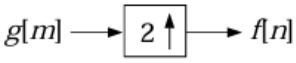
a. Down sampling by a factor of $M = 2$.



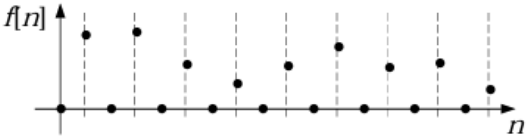
The result after the down sampling process is as shown in the figure below.



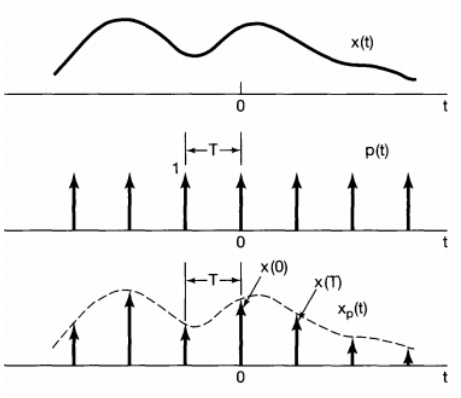
b. Up-sampling by a factor of $L = 2$.



The result after the up-sampling process is as shown in the figure below.



11. In a digital signal processing system (DSP) in a discotheque, the following discrete-time pulse is sampled at 40 kHz from a continuous-time signal (e.g. cut off frequency of 20 kHz).



The DSP system is connected to a subwoofer pre-amplifier (slower unit, with 4 kHz operating frequency) and to a digital signal conversion processor for laser light animation and effect (faster unit, with sampling frequency of 50 kHz).

a. By ensuring the no-aliasing requirement of the sampling process, reconstruct through relevant decimation and interpolation processes so the discrete-time signal can fit the performance requirements of the other units. [16 marks]

b. Construct a diagram that shows the processes in the system from sampling to reconstruction. Include in your diagram any additional components and parts for successful operation of the processes. [6 marks]

Solution

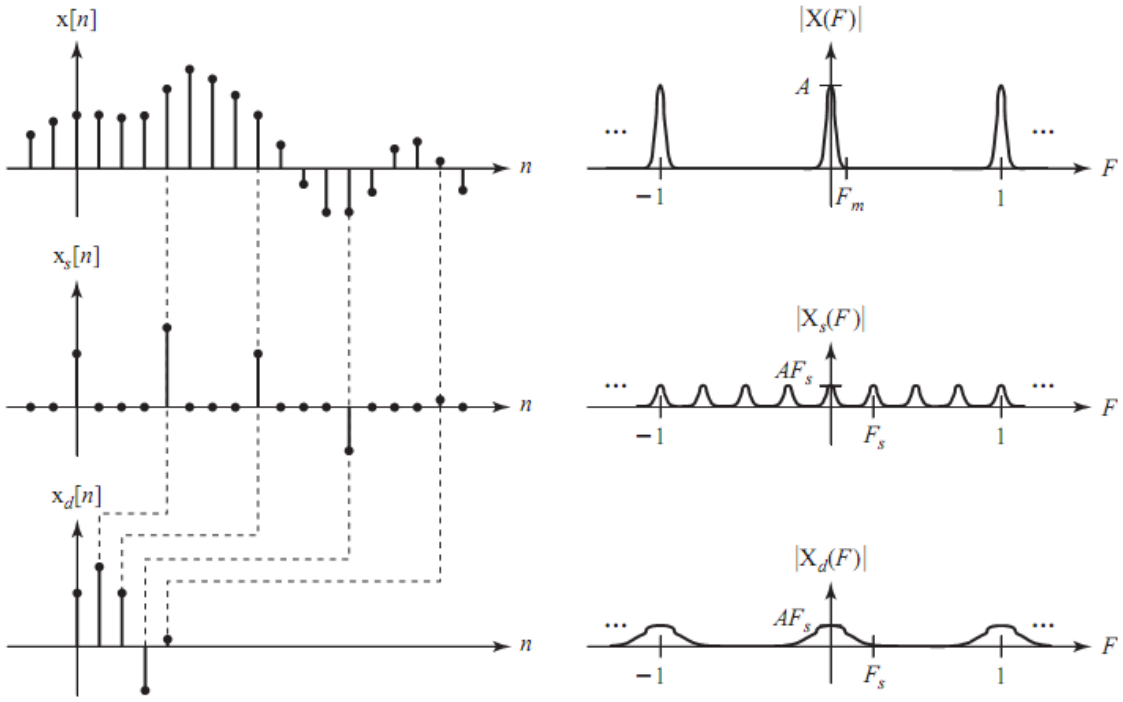
a. Reconstruct the given discrete-time signals using (a) decimation (b) interpolation.

i. Decimation process for the amplifier unit:

Interfacing with the pre-amplifier requires decimation process with ratio of:

$$M = 4 \text{ kHz}/20 \text{ kHz} = 1/5 \text{ times}$$

Shown in the diagram below is decimation process (with a ratio of 1/5).



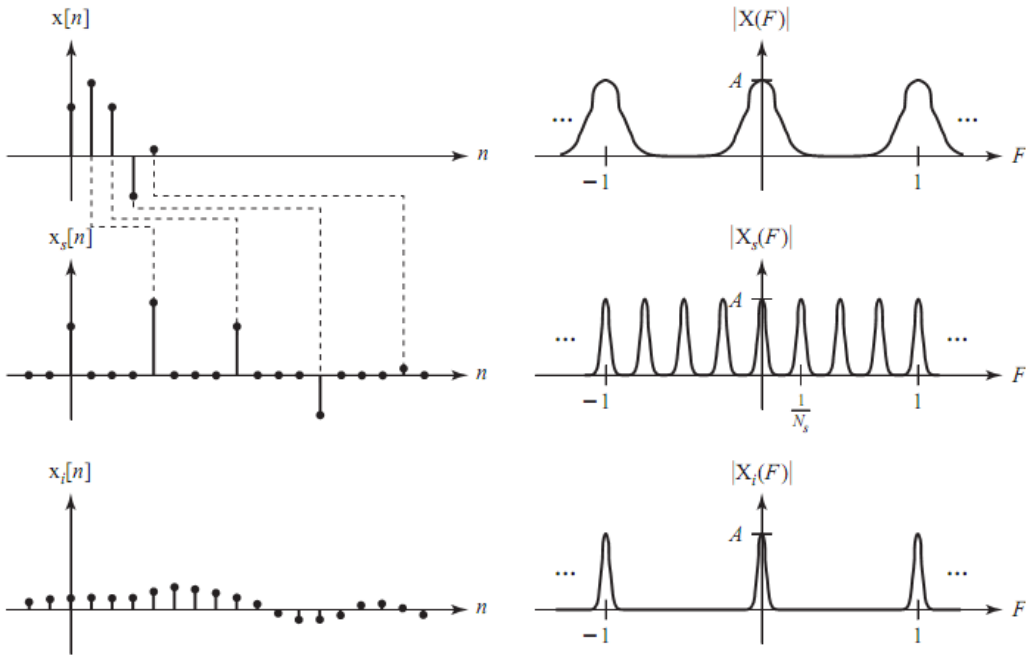
ii. Interpolation process for the digital signal conversion processor:

To ensure non-aliasing requirement, the interpolation process is performed at twice the maximum frequency of the processor i.e. twice its operating frequency.

Interfacing with the processor requires interpolation process with ratio of:

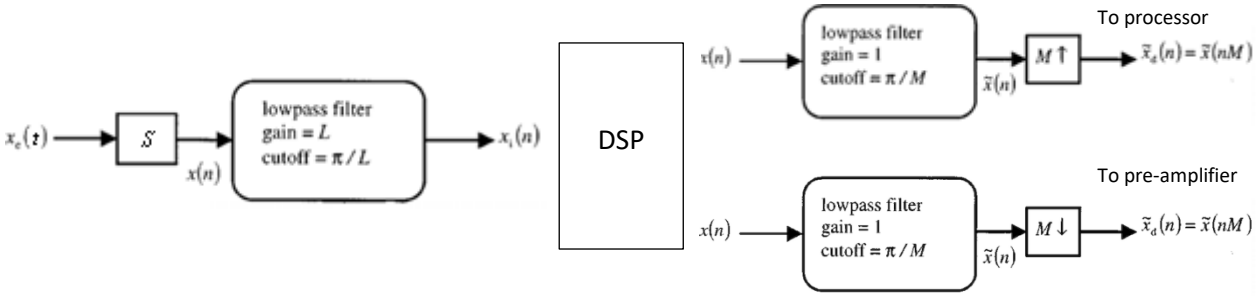
$$L = 100 \text{ kHz}/20 \text{ kHz} = 5 \text{ times}$$

Shown in the diagram below is interpolation process (with a ratio of 5).



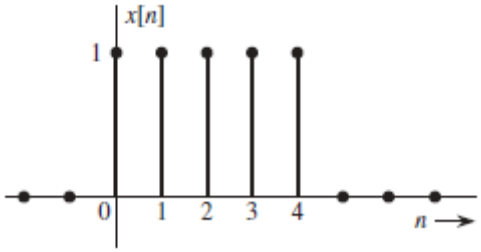
b. Construct a diagram that shows the processes of the system from sampling to reconstruction.

Shown in the diagram below are parts of the digital signal processing systems: sampling unit (s), low pass filter at the input (L_i), two low pass filter at the output (l_{o1} and l_{o2}) and signal decimator ($M <$) and signal interpolator ($M >$).



C. Z-Transform

1. Find the z-transforms of:
   a. $\delta[n]$. [2 marks]
   b. $u[n]$. [6 marks]
   c. $\cos(\beta n)\, u[n]$. [6 marks]
   d. The signal shown in the figure below. [4 marks]



Recall that by definition

$$X[z] = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \frac{x[3]}{z^3} + \cdots \qquad (Eq.1)$$

Solution

a. For $x[n] = \delta[n]$, $x[0] = 1$, and $x[2] = x[3] = x[4] = \cdots = 0$.

Therefore,

$$\delta[n] \leftrightarrow 1 \qquad \text{for all } z$$

b. For $x[n] = u[n]$, $x[0] = x[1] = x[3] = \cdots = 1$.

Therefore,

$$X[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

This geometric sum simplifies to:

$$X[z] = \frac{1}{1 - \frac{1}{z}} \qquad \left|\frac{1}{z}\right| < 1$$

Or

$$X[z] = \frac{z}{z - 1} \qquad |z| > 1$$

Therefore,

$$u[n] \leftrightarrow \frac{z}{z - 1} \qquad |z| > 1$$

c. Recall that $\cos(\beta n) = (e^{j\beta n} + e^{-j\beta n})/2$. Moreover, according to the pair of Fourier transform below,

$$e^{\pm j\beta n}u[n] \leftrightarrow \frac{z}{z - e^{\pm j\beta}}$$

Thus

$$|z| > |e^{\pm j\beta}| = 1$$

Therefore,

$$X[z] = \frac{1}{2}\left[\frac{z}{z - e^{j\beta}} + \frac{z}{z - e^{-j\beta}}\right]$$
$$= \frac{z(z - \cos\beta)}{z^2 - 2z\cos\beta + 1} \qquad |z| > 1$$

d. Here, $x[0] = x[1] = x[2] = x[3] = x[4] = 1$ and $x[5] = x[6] = \cdots = 0$.

Therefore, according to the equation (1),

$$X[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}$$
$$= \frac{(z^4 + z^3 + z^2 + z + 1)}{z^4} \qquad \text{for all } z \neq 0$$

We can also express this result in a more compact form by summing the geometric progression on the right-hand side of the foregoing equation. From the result with $r = 1/z$, $m = 0$, and $n = 4$, we obtain:

$$X[z] = \frac{\left(\frac{1}{z}\right)^5 - \left(\frac{1}{z}\right)^0}{\frac{1}{z} - 1}$$
$$= \left(\frac{z}{z-1}\right)(1 - z^{-5})$$

2. For the given discrete-time series, their z-transforms are:

a. $x[n] = 3^n u[n]$. [6 marks]

b. $x[n] = u[n] - u[n-2]$. [4 marks]

c. $x[n] = \cos(\pi/3)u[n]$. [8 marks]

d. $x[n] = \{3, 1, 4\} + u[n]$. [4 marks]

Solution

For the given discrete time series, their z-transforms are:

a. For a series $x[n] = 3^n u[n]$, its z-transform is:

$$\mathrm{Z}\{a^n u[n]\} = 1 + az^{-1} + a^2 z^{-2} + \cdots$$
$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$
$$= \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

As a result, the above z-transform becomes:

$$\begin{aligned} \mathrm{Z}\{3^n u[n]\} &= \frac{1}{1-3z^{-1}} \\ &= \frac{z}{z-3} \end{aligned}$$

b. The z-transform of series $x[n] = u[n] - u[n-2]$ is:

$$\begin{aligned} \mathrm{Z}\{u[n]-u[n-2]\} &= \left(\frac{z}{z-1}\frac{z}{z}\right) - \left(z^{-2}\frac{z}{z-1}\right) \\ &= \frac{z^2-1}{z(z-1)} = \frac{z+1}{z} \\ &= 1+z^{-1} \end{aligned}$$

c. For the series $x[n] = \cos(\pi/3)u[n]$, its z-transform is:

$$\begin{aligned} \mathrm{Z}\{\cos(\omega n)u[n]\} &= \mathrm{Z}\left\{\frac{1}{2}\left[e^{j\omega n} + e^{-j\omega n}\right]u[n]\right\} \\ &= \frac{1}{2}\mathrm{Z}\{e^{j\omega n}u[n]\} + \frac{1}{2}\mathrm{Z}\{e^{-j\omega n}u[n]\} \\ &= \frac{1}{2}\left(\frac{z}{z-e^{j\omega}}\right) + \frac{1}{2}\left(\frac{z}{z-e^{-j\omega}}\right) \end{aligned}$$

Put the two terms over a common denominator:

$$\begin{aligned} X(z) &= \frac{1}{2}\left(\frac{z}{z-e^{j\omega}}\right)\left(\frac{z-e^{-j\omega}}{z-e^{-j\omega}}\right) + \frac{1}{2}\left(\frac{z}{z-e^{-j\omega}}\right)\left(\frac{z-e^{-j\omega}}{z-e^{-j\omega}}\right) \\ &= \frac{1}{2}\left(\frac{2z^2 - ze^{-j\omega} - ze^{j\omega}}{z^2 - ze^{-j\omega} - ze^{j\omega} + 1}\right) \\ &= \frac{2z^2 - z\cos(\omega)}{z^2 - 2z\cos(\omega) + 1} \end{aligned}$$

As a result, the above z-transform becomes:

$$\begin{aligned} \mathrm{Z}\left\{\cos\left(\frac{\pi}{3}\right)u[n]\right\} &= \frac{z^2 - z\cos\left(\frac{\pi}{3}\right)}{z^2 - 2z\cos\left(\frac{\pi}{3}\right) + 1} \\ &= \frac{z^2 - 0.5z}{z^2 - z + 1} \end{aligned}$$

d. For the series $x[n] = \{3,1,4\} + u[n]$, its z-transform is:

$$\mathrm{Z}\{\{3,1,4\} + u[n]\} = \left(\frac{3z^2+z+4}{z^2}\right)\left(\frac{z-1}{z-1}\right) + \left(\frac{z}{z-1}\right)\left(\frac{z^2}{z^2}\right)$$

$$= \frac{4z^3 - 2z^2 + 3z - 4}{z^3 - z^2}$$

3. Find the inverse *z*-transforms of the following functions:

a. Real poles: [10 marks]

$$X(z) = \frac{8z - 19}{(z-2)(z-3)}$$

b. Repeated poles: [20 marks]

$$X(z) = \frac{z(2z^2 - 11z + 12)}{(z-1)(z-2)^3}$$

c. Complex poles: [10 marks]

$$X(z) = \frac{2z(3z + 17)}{(z-1)(z^2 - 6z + 25)}$$

Solution

a. Expanding $X[z]$ into partial fractions yields:

$$X[z] = \frac{8z - 19}{(z-2)(z-3)} = \frac{3}{z-2} + \frac{5}{z-3}$$

Thus

$$x[n] = [3(2)^{n-1} + 5(3)^{n-1}]u[n-1]$$

This will always produce the equation with $u[n-1]$ which is not convenient. So, the equation is modified a little bit as follows.

$$\frac{X[z]}{z} = \frac{8z - 19}{z(z-2)(z-3)} = \frac{(-19/6)}{z} + \frac{(3/2)}{z-2} + \frac{(5/3)}{z-3}$$

Multiplying both sides by z yields:

$$X[z] = -\frac{19}{6} + \frac{3}{2}\left(\frac{z}{z-2}\right) + \frac{5}{3}\left(\frac{z}{z-3}\right)$$

From the Fourier transform table, it follows that:

$$x[n] = -\left(\frac{19}{6}\right)\delta[n] + \left[\left(\frac{3}{2}\right)(2)^n + \left(\frac{5}{3}\right)(3)^n\right]u[n]$$

b. Expanding $X[z]$ into partial fraction yields:

$$\frac{X[z]}{z} = \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

Where:

$$k = \left.\frac{2z^2 - 11z + 12}{(z-1)(z-2)^3}\right|_{z=1} = -3$$

$$a_0 = \left.\frac{2z^2 - 11z + 12}{(z-1)(z-2)^3}\right|_{z=2} = -2$$

Therefore,

$$\frac{X[z]}{z} = \frac{2z^2 - 11z + 12}{(z-1)(z-2)^3} = -\frac{3}{z-1} - \frac{2}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

Multiply both sides of equation above by z and let $z \to \infty$. This yields:

$$0 = -3 - 0 + 0 + a_2 \Rightarrow a_2 = 3$$

Letting z take any convenient value, say, $z = 0$, on both sides of the equation above. This produces:

$$\frac{12}{8} = 3 + \frac{1}{4} + \frac{a_1}{4} - \frac{3}{2}$$

which yields $a_1 = -1$.

Therefore,

$$\frac{X[z]}{z} = -\frac{3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{(z-2)}$$

and

$$X[z] = -3\left(\frac{z}{z-1}\right) - 2\left[\frac{z}{(z-2)^3}\right] - \frac{z}{(z-2)^2} + 3\left(\frac{z}{z-2}\right)$$

Now, using the Fourier transform table, this yields:

$$x[n] = \left[-3 - 2\left(\frac{n(n-1)}{8}\right)(2)^n - \frac{n}{2}(2)^n + 3(2)n\right]u[n]$$

$$= -\left[3 + \frac{1}{4}(n^2 + n - 12)2^n\right]u[n]$$

c. For the given function with complex poles:

$$X[z] = \frac{2z(3z+17)}{(z-1)(z^2 - 6z + 25)} = \frac{2z(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

The equation above could be solved using the first-order factors method or quadratic factors method.

First-order method:

Using the Heaviside "cover-up" method:

$$\frac{X[z]}{z} = \frac{2z(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

Thus

$$\frac{X[z]}{z} = \frac{2}{z-1} + \frac{1.6e^{-j2.246}}{z-3-j4} + \frac{1.6e^{j2.246}}{z-3+j4}$$

And

$$X[z] = 2\left(\frac{z}{z-1}\right) + \left(1.6e^{-j2.246}\right)\left(\frac{z}{z-3-j4}\right) + \left(1.6e^{j2.246}\right)\left(\frac{z}{z-3+j4}\right)$$

The inverse transform is:

$$x[n] = [2 + 3.2(5)^n \cos(0.927n - 2.246)]u[n]$$

Quadratic factors method:

Considering the complex part of the given equation as a quadratic factor:

$$\frac{X[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{Az+B}{z^2-6z+25}$$

Multiplying both sides by z and letting $z \to \infty$, we find:

$$0 = 2 + A \Rightarrow A = -2$$

And

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

To find B, we let z take any convenient value, say, $z = 0$. This step yields:

$$\frac{-34}{25} = -2 + \frac{B}{25} \Rightarrow B = 16$$

Therefore,

$$\frac{X[z]}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2-6z+25}$$

And

$$X[z] = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$$

Using the Fourier transform table, where we identify $A = -2$, $B = 16$, $|\gamma| = 5$, and $a = -3$. Therefore,

$$r = \frac{100 + 256 - 192}{25 - 9} = 3.2$$

And

$$\beta = \cos^{-1}\left(\frac{3}{5}\right) = 0.927 \text{ rad}$$

And

$$\theta = \tan^{-1}\left(\frac{-10}{-8}\right) = -2.246 \text{ rad}$$

So that:

$$x[n] = [2 + 3.2(5)^n \cos(0.927n - 2.246)]u[n]$$

4. For the given z-transforms, their inverse z-transforms are:

a. $X(z) = \frac{2z^4+7z^3+z^2+8z+3}{z^4}$. [4 marks]

b. $X(z) = \frac{4z+5}{z^2} + \frac{6}{1-3z^{-1}}$. [4 marks]

c. $X(z) = \frac{7z}{z-1} + \frac{8z}{z-2}$. [4 marks]

d. $X(z) = \frac{z-3}{z^2-3z+2}$. [12 marks]

e. $X(z) = \frac{2z}{z^2-2z+2}$. [12 marks]

f. $X(z) = \frac{z-1}{z^3+z^2+8z+8}$. [12 marks]

Solution

For the given z-transforms, their inverse z-transforms are:

a. Given the z-transform is:

$$X(z) = \frac{2z^4 + 7z^3 + z^2 + 8z + 3}{z^4}$$

Thus, its inverse z-transform is:

$$\mathrm{Z}^{-1}\left\{\frac{2z^4 + 7z^3 + z^2 + 8z + 3}{z^4}\right\} = \mathrm{Z}^{-1}\{2 + 7z^{-1} + z^{-2} + 8z^{-3} + 3z^{-4}\}$$

$$= \{2, 7, 1, 8, 3\}$$

b. Given the z-transform is:

$$X(z) = \frac{4z + 5}{z^2} + \frac{6}{1 - 3z^{-1}}$$

Thus, its inverse z-transform is:

$$\mathrm{Z}^{-1}\{4z^{-1} + 5z^{-2}\} + 6\mathrm{Z}^{-1}\left\{\frac{1}{1 - 3z^{-1}}\right\} = \{0, 4, 5\} + 6(3)^n u[n]$$

c. Given the z-transform is:

$$X(z) = \frac{7z}{z-1} + \frac{8z}{z-2}$$

Thus, its inverse z-transform is:

$$\mathrm{Z}^{-1}\left\{\frac{7z}{z-1} + \frac{8z}{z-2}\right\} = 7\mathrm{Z}^{-1}\left\{\frac{z}{z-1}\right\} + 8\mathrm{Z}^{-1}\left\{\frac{z}{z-2}\right\}$$
$$= 7u[n] + 8(2)^n u[n]$$

d. Given the z-transform is:

$$X(z) = \frac{z-3}{z^2 - 3z + 2}$$

Compute poles:

$$z^2 - 3z + 2 = (z-1)(z-2) = 0$$

Thus, the poles in the system are:

$$p_1 = 1 \quad \text{and} \quad p_2 = 2$$

Write expansion for $X(z)/z$:

$$\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_2}{z-2}$$

Compute residues:

$$A_0 = \frac{(0-3)}{(0-1)(0-2)} = -\frac{3}{2}$$

$$A_1 = \frac{(1-3)}{(1-0)(1-2)} = 2$$

$$A_2 = \frac{(2-3)}{(2-0)(2-1)} = -\frac{1}{2}$$

Multiplied by z:

$$X(z) = -\frac{3}{2} + 2\frac{z}{z-1} - \frac{1}{2}\frac{z}{z-2}$$

Inverting by inspection, its inverse z-transform is:

$$x[n] = -\frac{3}{2}\delta[n] + 2u[n] - \frac{1}{2}(2)^n u[n]$$

e. Given the z-transform is:

$$X(z) = \frac{2z}{z^2 - 2z + 2}$$

Compute poles of the characteristic equation of the z-transform:

$$z^2 - 2z + 2 = 0$$

Thus, the poles are:

$$p_1 = 1 + j \quad \text{and} \quad p_2 = 1 - j$$

Write expansion for $X(z)/z$:

$$\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z-(1+j)} + \frac{A_2}{z-(1-j)}$$

Compute residues:

$$A_0 = \frac{2(0)}{(0-(1+j))(0-(1-j))} = 0$$

$$A_1 = \frac{2(1+j)}{(1+j)((1+j)-(1-j))} = -j$$

$$A_2 = \frac{2(1-j)}{(1-j)((1-j)-(1+j))} = j$$

Multiplied by z:

$$X(z) = \frac{-j}{z-(1+j)} + \frac{j}{z-(1-j)}$$

Inverting by inspection, its inverse z-transform is:

$$x[n] = -j(1+j)^n u[n] + j(1-j)^n u[n]$$

Where: $p = 1 + j = \sqrt{2}e^{\frac{j\pi}{4}}$ and $A = -j = e^{-\frac{j\pi}{2}}$, knowing that $z = a + jb$, its trigonometric form is $z = r(\cos\theta + j\sin\theta)$ where r is the modulus of z (i.e. $r = |z|$) and θ is argument of z. The sequence becomes:

$$x[n] = 2\left(\sqrt{2}\right)^2 \sin\left(\frac{\pi n}{4}\right) u[n]$$

f. Given the z-transform is:

$$X(z) = \frac{z-1}{z^3 + z^2 + 8z + 8}$$

Compute poles from the characteristic equation given above:

$$z^3 + 4z^2 + 8z + 8 = 0$$

Thus, the poles are:

$$p_1 = -2\ ;\ p_2 = 2e^{j2.09}; p_3 = 2e^{-j2.09}$$

Write expansion for $X(z)/z$:

$$\frac{X(z)}{z} = \frac{A_0}{z} + \frac{A_1}{z+2} + \frac{A_2}{z-2e^{j2.09}} + \frac{A_3}{z-2e^{-j2.09}}$$

Compute residues:

$$A_0 = (z-0)\frac{X(z)}{z}\Big|_{z=0} = \frac{(0-1)}{(0+2)(0-2e^{j2.09})(0-2e^{-j2.09})} = -\frac{1}{8}$$

$$A_1 = (z+2)\frac{X(z)}{z}\Big|_{z=-2} = \frac{(0-1)}{(0+2)(0-2e^{j2.09})(0-2e^{-j2.09})} = \frac{3}{8}$$

$$A_3 = \left(z-2e^{j2.09}\right)\frac{X(z)}{z}\Big|_{z=2e^{j2.09}} = \frac{(2e^{j2.09}-1)/2e^{j2.09}}{(2e^{j2.09}+2)(2e^{j2.09}-2e^{-j2.09})} = 0.19e^{-j2.29}$$

Multiplied by z:

$$X(z) = -\frac{1}{8} + \frac{3}{8}\frac{z}{z+2} + \frac{0.19e^{-j2.09}}{z-2e^{j2.09}} + \frac{0.19e^{j2.09}}{z-2e^{-j2.09}}$$

Inverting by inspection, its inverse z-transform is:

$$x[n] = -\frac{1}{8}\delta[n] + \frac{3}{8}(-2)^n u[n] + (0.19)2^n e^{j(2.09-2.29)}u[n] + (0.19)2^n e^{-j(2.09-2.29)}u[n]$$

Simplify the complex equation:

$$x[n] = -\frac{1}{8}\delta[n] + \frac{3}{8}(-2)^n u[n] + (0.38)2^n \cos(2.09n - 2.29)u[n]$$

5. Find the poles and zeros of the following z-transforms:

   a. $X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$. [6 marks]

   b. $X_2(z) = [(z^{-1}-1)(z^{-1}+2)^2]/\left[z^{-1}\left(z^{-2}+\sqrt{2}z^{-1}+1\right)\right]$. [6 marks]

Solution

a. To determine the poles and zeros more, let us express the z-transform $X_1(z)$ as a function of positive powers of z:

$$X_1(z) = \frac{N_1(z)}{D_1(z)} = \frac{z^3(1+2z^{-1}+3z^{-2}+4z^{-3})}{z^3} = \frac{z^3+2z^2+3z+4}{z^3}$$

The poles of $X_1(z)$ are three poles at $z = 0$, as the roots of $D_1(z) = 0$.

The zeros of $X_2(z)$ are the roots of $N_1(z) = z^3 + 2z^2 + 3z + 4 = 0$, which its roots are $z_1 = -1.65$, $z_2 = -0.175 + j1.547$, $z_3 = -0.175 - j1.547$.

Notice that the coefficients of $N_1(z)$ are real and as such the complex roots are a complex conjugate pair.

b. Likewise, expressing z-transform $X_2(z)$ as a function of positive powers of z,

$$X_2(z) = \frac{N_2(z)}{D_2(z)}$$
$$= \frac{z^3(z^{-1}-1)(z^{-1}+2)^2}{z^3\left[z^{-1}\left(z^{-2}+\sqrt{2}z^{-1}+1\right)\right]}$$
$$= \frac{(1-z)(1+2z)^2}{1+\sqrt{2}z+z^2}$$

The poles of $X_2(z)$ are the roots of $D_2(z) = 1+\sqrt{2}z+z^2 = 0$, or $p_{1,2} = -0.707 \pm j0.707$.

The zeros of $X_2(z)$ are the roots of $N_2(z) = (1-z)(1+2z)^2 = 0$, or $z_1 = 1$ and $z_{2,3} = -0.5$ (double zeros).

6. Find the *z*-transform and the corresponding ROC for the signal $\gamma^n u[n]$. [10 marks]

Solution

The z-transform of the signal is given as:

$$X[z] = \sum_{n=0}^{\infty} \gamma^n u[n] z^{-n}$$

Since $u[n] = 1$ for all $n \geq 0$, the z-transform is:

$$X[z] = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^n = 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots$$

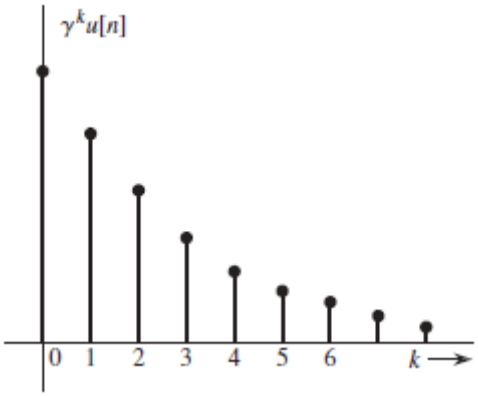
It is helpful to remember the geometric progression and its sum:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \text{if } |x| < 1$$

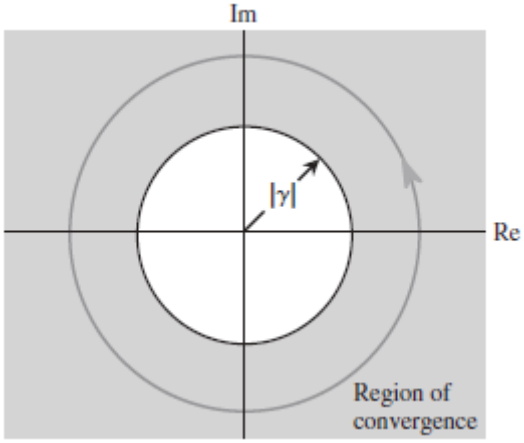
Applying this relationship to the given equation yields:

$$X[z] = \frac{1}{1-\frac{\gamma}{z}} \left|\frac{\gamma}{z}\right| < 1$$
$$= \frac{z}{z-\gamma} \quad |z| > |\gamma|$$

Observe that $X[z]$ exists only for $|z| > |\gamma|$ as shown in the figure below. For $|z| < |\gamma|$, the sum in the given signal does not converge i.e. it goes to infinity.



Therefore, the ROC of $X[z]$ is the shaded region outside the circle of radius $|\gamma|$, centered at the origin, in the *z*-plane, as depicted in the figure below.



7. Find the z-transform of a discrete-time pulse $x[n] = u[n] - u[n-10]$. Determine the region of convergence of $X(z)$. [8 marks]

Solution

The Z-transform of the sequence $x[n]$ is:

$$X(z) = \sum_{n=0}^{9} 1 - z^{-n}$$

$$= \frac{1 - z^{-10}}{1 - z^{-1}}$$

$$= \frac{z^{10} - 1}{z^9(z-1)} \qquad (Eq.1)$$

That this sum equals the term on the right can be verified by multiplying the left term by the denominator $(1 - z^{-1})$ and showing the result is the same as the numerator. In fact,

$$(1-z^{-1})\sum_{n=0}^{9}1-z^{-n}=\sum_{n=0}^{9}1-z^{-n}-\sum_{n=0}^{9}1-z^{-n-1}$$
$$=(1+z^{-1}+\cdots+z^{-9})-(z^{-1}+\cdots+z^{-9}+z^{-10})$$

Since $x[n]$ is a finite sequence there is no problem with the convergence of the sum, although $X(z)$ in equation (1) seems to indicate the need for $z \neq 1$ (z = 1 makes the numerator and denominator go to zero).

However, from the sum if we let z = 1 then $X(1)$ = 10, so there is no need to restrict z to be different from 1. This is caused by the pole at z = 1 being cancelled by a zero at z = 1.

Indeed, the zeros z_k of $X(z)$ (see equation (1) are the roots of $z^{10}-1=0$, which are $z_k = e^{j2\pi k/10}$ for k = 0…9. Therefore, the zero when k = 0, or z_0 = 1, cancels the pole at 1 so that:

$$X(z)=\frac{\prod_{k=1}^{9}\left(z-e^{j\pi k/5}\right)}{z^9}$$

In this example, $X(z)$ has nine poles at the origin of the z-plane, and nine zeros around the unit circle except at z = 1. Thus, the whole z-plane excluding the origin is the region of convergence of $X(z)$. Notice that $X(z)$ can also be written as a polynomial:

$$X(z)=1+z^{-1}+z^{-2}+z^{-3}+z^{-4}+z^{-5}+z^{-6}+z^{-7}+z^{-8}+z^{-9}$$

which only tends to infinity when z = 0, so the region of convergence of $X(z)$ is the whole z-plane excluding the origin.

8. The poles of $X(z)$ are z = 0.5 and z = 2. Find all the possible signals that can be associated with $X(z)$ according to different regions of convergence. [6 marks]

Solution

Possible regions of convergence are:

- R_1: $|z| > 2\}$, the outside of a circle of radius 2, we associate with $X(z)$ a causal signal $x_1[n]$.
- R_2: $|z| < 0.5\}$, the inside of a circle of radius 0.5, an anti-causal signal $x_2[n]$ can be associated with $X(z)$.
- R_3: $0.5 < |z| < 2\}$, a torus of radii 0.5 and 2, a noncausal signal $x_3[n]$ can be associated with $X(z)$.

Three different signals can be connected with $X(z)$ by considering three different regions of convergence.

9. Given the following signals described as sequences listed below:

$$x_1[n]=\left(\frac{1}{2}\right)^2 u[n] \qquad \text{and} \qquad x_2[n]=-\left(\frac{1}{2}\right)^n u[-n-1]$$

a. Describe the characteristics of $x_1[n]$ and $x_2[n]$. [2 marks]

b. Determine the z-transform of $x_1[n]$ and its ROC. [4 marks]

c. Determine the z-transform of $x_2[n]$ and its ROC. [4 marks]

d. Compare the difference and similarity between $x_1[n]$ and $x_2[n]$. [2 marks]

e. Determine the z-transform of $x_1[n] + x_2[n]$ and its ROC. [4 marks]

Solution

a. The signal $x_1[n]$ is causal, while $x_2[n]$ is anti-causal.

b. The Z-transform of $x_1[n]$ is:

$$X_1(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^2 z^{-n}$$
$$= \frac{1}{1 - 0.5z^{-1}}$$
$$= \frac{z}{z - 0.5}$$

It is provided that $|0.5z - 1| < 1$ or that its region of convergence be R_1: $|z| > 0.5$. The region R_1 is the outside of a circle of radius 0.5.

c. The signal $x_2[n]$ grows as n decreases from -1 to $-\infty$, and the rest of its values are zero. Its Z-transform is found to be:

$$X_2(z) = -\sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^2 z^{-n}$$
$$= -\sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^{-m} z^m + 1$$
$$= -\sum_{m=0}^{\infty} 2^m z^m + 1$$
$$= \frac{-1}{1 - 2z} + 1$$
$$= \frac{z}{z - 0.5}$$

The signal is with a region of convergence R_2: $|z| < 0.5$.

d. It is very important to notice that although the signals are clearly different, their Z-transforms are identical. It is their regions of convergence that differentiate them.

e. The Z-transform of $x_1[n] + x_2[n]$ does not exist given that the intersection of R_1 and R_2 is empty.

10. For the z-transform properties listed below, describe their significances in the transformation and how describe how to use them in simplifying the transformation process.

a. Linearity property. [4 marks]

b. Delay property. [4 marks]

c. Convolution property. [4 marks]

Solution

a. Linearity property of z-transform:

$$Z\{ax[n] + by[n]\} = aX(z) + bY(z)$$

Where:

$$X(z) = Z\{x[n]\} \quad \text{and} \quad Y(z) = Z\{y[n]\}$$

Use this to break up computation of z-transform (and their inverses).

b. Delays (time shift) property of z-transform:

$$Z\{x[n - D]\} = z^{-D}Z\{x[n]\} = z^{-D}X(z)$$

Delaying a signal by $D > 0$ multiplies its z-transform by z^{-D}. But the z-transform is only defined for causal signals.

c. Convolution property of z-transform:

$$Z\{x[n] * y[n]\} = X(z).Y(z)$$

Where:

$$X(z) = Z\{x[n]\} \quad \text{and} \quad Y(z) = Z\{y[n]\}$$

This follows since convolution is polynomial multiplication. No more convolutions to compute.

11. For each of the following difference equations and associated input and initial conditions, determine the output $y[n]$.

a. Knowing that input, $x[n] = (1/3)^n$ and initial condition $y[-1] = 1$. [10 marks]

$$y[n] - \left(\frac{1}{2}\right) y[n-1] = x[n]$$

b. Knowing that input, $x[n] = (1/2)^n$ and initial conditions $y[-1] = 1$ and $y[-2] = 2$. [10 marks]

$$3y[n] - 4y[n-1] + y[n-2] = x[n]$$

Solution

a. Knowing from the z-transform table, the z-transform of input sequence $x[n] = (1/3)^n$ is:

$$x[n] \leftrightarrow X(z) = \frac{z}{z - \frac{1}{3}} \qquad |z| > \left|\frac{1}{3}\right|$$

Taking the (unilateral) z-transform of the given difference equation, we get:

$$Y(z) - \frac{1}{2}\{z^{-1}Y(z) + y[-1]\} = X(z)$$

Substituting the input $X(z)$ and initial condition $y[-1] = 1$ into the expression given above, we get:

$$\left(1 - \frac{1}{2}z^{-1}\right) Y(z) = \frac{1}{2} + \frac{z}{z - \frac{1}{3}}$$

Or

$$\left(\frac{z - \frac{1}{2}}{z}\right) Y(z) = \frac{1}{2} + \frac{z}{z - \frac{1}{3}}$$

Thus, rearranging the equation above, it becomes:

$$Y(z) = \frac{1}{2}\left(\frac{z}{z - \frac{1}{2}}\right) + \frac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$$

$$= \frac{7}{2}\left(\frac{z}{z - \frac{1}{2}}\right) - 2\left(\frac{z}{z - \frac{1}{3}}\right)$$

Hence, taking the inverse z-transform, the equation for the sequence is:

$$y[n] = 7\left(\frac{1}{2}\right)^{n+1} - 2\left(\frac{1}{3}\right)^n \qquad n \geq -1$$

b. Knowing from the z-transform table, the z-transform of input sequence $x[n] = (1/2)^n$ is:

$$x[n] \leftrightarrow X(z) = \frac{z}{z-\frac{1}{2}} \qquad |z| > \left|\frac{1}{2}\right|$$

Taking the (unilateral) z-transform of the given difference equation, we obtain:

$$3Y(z) - 4\{z^{-1}Y(z) + y[-1]\} + \{z^{-2}Y(z) + z^{-1}y[-1] + y[-2]\} = X(z)$$

Substituting input $X(z)$ and initial conditions $y[-1] = 1, y[-2] = 2$ into the expression given above, we get:

$$(3 - 4z^{-1} + z^{-2})Y(z) = 2 - z^{-1} + \frac{z}{z-\frac{1}{2}}$$

Or

$$\frac{3(z-1)\left(z-\frac{1}{3}\right)}{z^2}Y(z) = \frac{3z^2 - 2z + \frac{1}{2}}{z\left(2-\frac{1}{2}\right)}$$

Thus, rearranging the equation given above, it is now:

$$Y(z) = \frac{z\left(3z^2 - 2z - \frac{1}{2}\right)}{3(z-1)\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$= \frac{3}{2}\left(\frac{z}{z-1}\right) - \frac{z}{z-\frac{1}{2}} + \frac{1}{2}\left(\frac{z}{z-\frac{1}{3}}\right)$$

Hence, applying the inverse z-transform, the sequence is:

$$y[n] = \frac{3}{2} - \left(\frac{1}{2}\right)^n + \frac{1}{2}\left(\frac{1}{3}\right)^n \qquad n \geq -2$$

12. A causal discrete-time LTI system is described by:

$$y[n] - \left(\frac{3}{4}\right)y[n-1] + \left(\frac{1}{8}\right)y[n-2] = x[n]$$

where $x[n]$ and $y[n]$ are the input and output of the system, respectively.

a. Determine the system function $H(z)$. [6 marks]

b. Find the impulse response $h[n]$ of the system. [10 marks]

c. Find the step response $s[n]$ of the system. [16 marks]

Solution

a. Taking the z-transform of the differential equation of the system, we obtain:

$$Y(z) - \left(\frac{3}{4}\right)z^{-1}Y(z) + \left(\frac{1}{8}\right)z^{-2}Y(z) = X(z)$$

Or

$$\left[1 - \left(\frac{3}{4}\right)z^{-1} + \left(\frac{1}{8}\right)z^{-2}\right]Y(z) = X(z)$$

Thus

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{1}{1 - \left(\frac{3}{4}\right)z^{-1} + \left(\frac{1}{8}\right)z^{-2}}$$

$$= \frac{z^2}{z^2 - \left(\frac{3}{4}\right)z + \frac{1}{8}} = \frac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)} \qquad |z| > \frac{1}{2}$$

b. Using partial-fraction expansion, we have:

$$\frac{H(z)}{z} = \frac{z}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)} = \frac{c_1}{z - \frac{1}{2}} + \frac{c_2}{z - \frac{1}{4}}$$

Where:

$$c_1 = \left.\frac{z}{z - \frac{1}{4}}\right|_{z=1/2} = 2$$

$$c_2 = \left.\frac{z}{z - \frac{1}{2}}\right|_{z=1/4} = -1$$

Thus

$$H(z) = 2\left(\frac{z}{z - \frac{1}{2}}\right) - \frac{z}{z - \frac{1}{4}} \qquad |z| > \frac{1}{2}$$

Taking the inverse z-transform of $H(z)$, we get:

$$h[n] = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u[n]$$

c. Given the following transform pairs from the z-transform table

$$x[n] = u[n] \leftrightarrow X(z) = \frac{z}{z-1} \qquad |z| > 1$$

Then

$$Y(z) = X(z)H(z) = \frac{z^3}{(z-1)\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} \qquad |z| > 1$$

Again, using partial-fraction expansion, we have:

$$\frac{Y(z)}{z} = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)} = \frac{c_1}{(z-1)} + \frac{c_2}{\left(z-\frac{1}{2}\right)} + \frac{c_3}{\left(z-\frac{1}{4}\right)}$$

Where:

$$c_1 = \left.\frac{z^2}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}\right|_{z=1} = \frac{8}{3}$$

$$c_2 = \left.\frac{z^2}{(z-1)\left(z-\frac{1}{4}\right)}\right|_{z=\frac{1}{2}} = -2$$

$$c_3 = \left.\frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}\right|_{z=\frac{1}{4}} = \frac{1}{3}$$

Thus

$$Y(z) = \frac{8}{3}\left(\frac{z}{z-1}\right) - 2\left(\frac{z}{z-\frac{1}{2}}\right) + \frac{1}{3}\left(\frac{z}{z-\frac{1}{4}}\right) \qquad |z| > 1$$

Taking the inverse z-transformation of $Y(z)$, we obtain:

$$y[n] = s[n] = \left[\frac{8}{3} - 2\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{4}\right)^n\right]u[n]$$

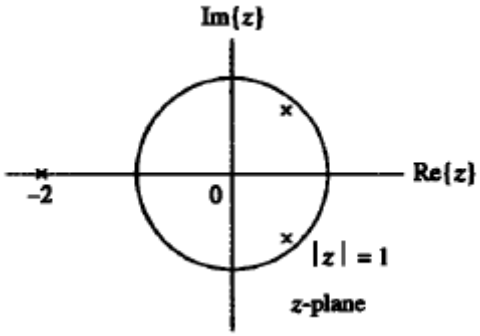
13. An LTI system has the transfer function given below:

$$H(z) = \frac{2}{1 - 0.9e^{j\frac{\pi}{4}}z^{-1}} + \frac{2}{1 - 0.9e^{-j\frac{\pi}{4}}z^{-1}} + \frac{3}{1 + 2z^{-1}}$$

Find the impulse response, assuming that the system is (a) stable or (b) causal. Can the system be both stable and causal? [8 marks]

Solution

The system has poles at $z = 0.9e^{j\frac{\pi}{4}}$, $z = 0.9e^{-j\frac{\pi}{4}}$, and $z = -2$ as shown in the figure below.



If the system is stable, then the ROC of the system includes the unit circle.

The two poles inside the unit circle contributes to the right-sided terms to the impulse response, while the pole outside the unit circle contributes to a left-sided term. Hence, for case (a),

$$h[n] = 2\left(0.9e^{j\frac{\pi}{4}}\right)^n u[n] + 2\left(0.9e^{-j\frac{\pi}{4}}\right)^n u[n] - 3(-2)^n u[n-1]$$

$$= 4(0.9)^n \cos\left(\frac{\pi}{4}n\right) u[n] - 3(-2)^n u[n-1]$$

If the system is assumed to be causal, then all poles contribute to the right-sided terms to impulse response, so for case (b), we have:

$$h[n] = 2\left(0.9e^{j\frac{\pi}{4}}\right)^n u[n] + 2\left(0.9e^{-j\frac{\pi}{4}}\right)^n u[n] + 3(-2)^n u[n]$$

$$= 4(0.9)^n \cos\left(\frac{\pi}{4}n\right) u[n] + 3(-2)^n u[n]$$

Note that this LTI system cannot be stable and causal, since there is a pole outside the unit circle.