XMUT220 Signals and Systems

Final Exam Revision Questions (Second Half)

A. Time and Frequency Characteristics

- Obtain transfer function from the given physical system.
- Derive, calculate and sketch the impulse and step inputs response.
- Derive, calculate and sketch frequency response plot (magnitude and phase).
- Solve transient response of a first order system (e.g. time constant, rise time, and settling time).
- Solve transient response of a second order system (e.g. natural frequency, damping ratio, time-to-peak, and percentage overshoot).

1. A continuous-time LTI system is described by:

$$\frac{dy(t)}{dt} + 2y(t) = x(t) + \frac{dx(t)}{dt}$$

   a. Determine its transfer function equation. [4 marks]

   b. Find the impulse response $h(t)$ of the system. [2 marks]

2. Consider a continuous-time LTI system described by:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

   Using the Fourier transform, find the output $y(t)$ to each of the following input signals:

   a. $x(t) = e^{-1}u(t)$. [12 marks]

   b. $x(t) = u(t)$. [10 marks]

   Note:

Name	Function	Fourier Transform
Step function	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$

Exponential function	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$

3. For its frequency response, determine the magnitude and phase shift at low, high and cut-off frequencies for the following systems:

   a. $H(\omega) = 1 + j\omega/10.$ [24 marks]

   b. $H(\omega) = 1/(1 + j\omega/100).$ [24 marks]

   c. $H(\omega) = [10^4(1 + j\omega)]/[(10 + j\omega)(100 + j\omega)].$ [24 marks]

4. The output $y(t)$ of a causal LTI system is related to the input $x(t)$ by the differential equation:

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

   a. Knowing the transfer function equation of the system is given below, determine the frequency response of the system and sketch its Bode plot. [8 marks]

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

   b. Specify, as a function of frequency, the group delay associated with this system. [4 marks]

   c. If $x(t) = e^{-t}u(t)$, determine $Y(j\omega)$, the Fourier transform of the output. [4 marks]

   d. Using the technique of partial-fraction expansion, determine the output $y(t)$ for the input $x(t)$ in part (c). [4 marks]

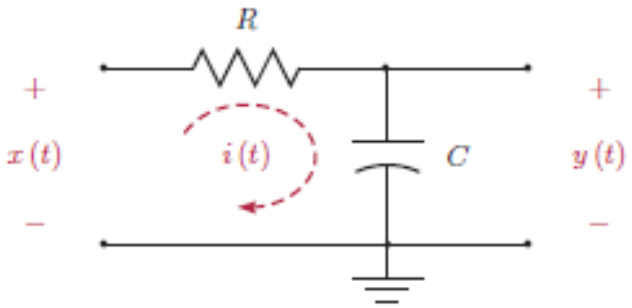
   e. Repeat parts (c) and (d), if the input has as its Fourier transform:

      i. $X(j\omega) = \frac{1+j\omega}{2+j\omega}$ [4 marks]

      ii. $X(j\omega) = \frac{2+j\omega}{1+j\omega}$ [4 marks]

      iii. $X(j\omega) = \frac{1}{(2+j\omega)(1+j\omega)}$ [4 marks]

5. Consider the continuous-time LTI system implemented as the RC circuit shown in the figure below. The voltage source $x(t)$ is considered the input to this system. The voltage $y(t)$ across the capacitor is considered the system output.



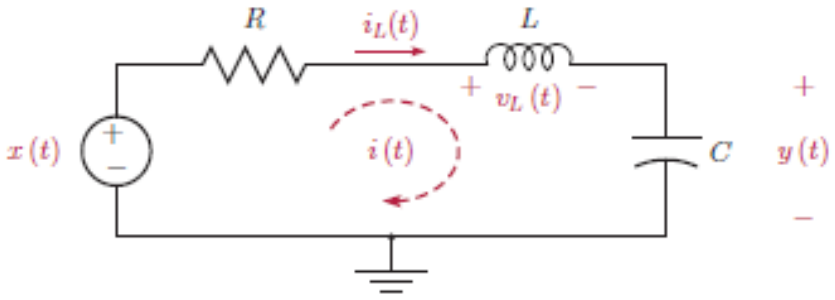
a. Determine the transfer function of the circuit. [6 marks]

b. If R = 1 Ω and C = 1 F, Determine the time-domain transient response of the system (e.g. time constant, rise time, and settling time). [8 marks]

c. Is it possible for the step response of the system to have an oscillatory behaviour?

[2 marks]

6. Consider the continuous-time LTI system implemented as the RLC circuit shown in the following figure. The voltage source $x(t)$ is considered the input to this system. The voltage $y(t)$ across the capacitor is considered the system output.



a. Determine the transfer function of the circuit. [8 marks]

b. If R = 2 Ω, L = 1 H, and C = 1 F, determine the transient response of the system (e.g. natural frequency, damping ratio, time-to-peak, and percentage overshoot). [20 marks]

c. How should R, L, and C be related so that there is no oscillation in the step response?

[4 marks]

7. A causal discrete-time LTI system is described by:

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n]$$

where $x[n]$ and $y[n]$ are the input and output of the system, respectively.

a. Determine the equation for the frequency response $H(\Omega)$ of the system. [4 marks]

b. Find the impulse response $h[n]$ of the system. [4 marks]

8. Consider a discrete-time LTI system described by:

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{2}x[n-1]$$

a. Determine the transfer function equation, $H(\Omega) = Y(\Omega)/X(\Omega)$ of the system. [6 marks]

b. Find the impulse response $h[n]$ of the system. [4 marks]

c. Determine its response $y[n]$ to the input: [8 marks]

$$x[n] = \cos\left[\left(\frac{\pi}{2}\right)n\right]$$

Note:

Function	Fourier Transform	Condition
$e^{j\Omega_0 n}$	$2\pi\delta(\Omega - \Omega_0)$	$\lvert\Omega\rvert, \lvert\Omega_0\rvert \leq \pi$
$\cos\Omega_0 n$	$\pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$	$\lvert\Omega\rvert, \lvert\Omega_0\rvert \leq \pi$

9. Consider the discrete-time LTI system shown as differential equation given below.

$$y[n] = x[n] + x[n-1]$$

a. Find the equation for the frequency response $H(\Omega) = Y(\Omega)/X(\Omega)$ of the system. [4 marks]

b. Find the impulse response $h[n]$ of the system. [2 marks]

c. Sketch the magnitude response $|H(\Omega)|$ and the phase response $\theta(\Omega)$. [8 marks]

d. Find the cut-off frequency (3-dB bandwidth) of the system. [6 marks]

B. Sampling and Reconstruction

- Understand sampling theorem of Nyquist.
- Derive, calculate and sketch of sampling of signals.
- Derive, calculate and sketch of aliasing effects.
- Derive, calculate and sketch of reconstruction of sampled signals.
- Understand and apply decimation and extrapolation.

1. Determine the following Nyquist frequency (maximum frequency component) and Nyquist rate (sampling frequency) of the following signals:

a. $x(t) = 3\sin(t/10)$. [6 marks]

b. $x(t) = 0.5\cos(\pi t/5) + 2\sin(8\pi t)$. [6 marks]

2. Wheel, rotating at 6 Hz, is seen in a dark room using a strobe light flashing at a rate of 8 Hz.
   a. Determine the apparent rotational speed and sense of rotation of the wheel. [4 marks]
   b. Repeat the question if the flashes occur at 12 Hz, 16 Hz, or 24 Hz. [4 marks]

3. For $-\infty < t < +\infty$, consider the signal given in the equation below:

$$x(t) = 2\cos\left(2\pi t + \frac{\pi}{4}\right)$$

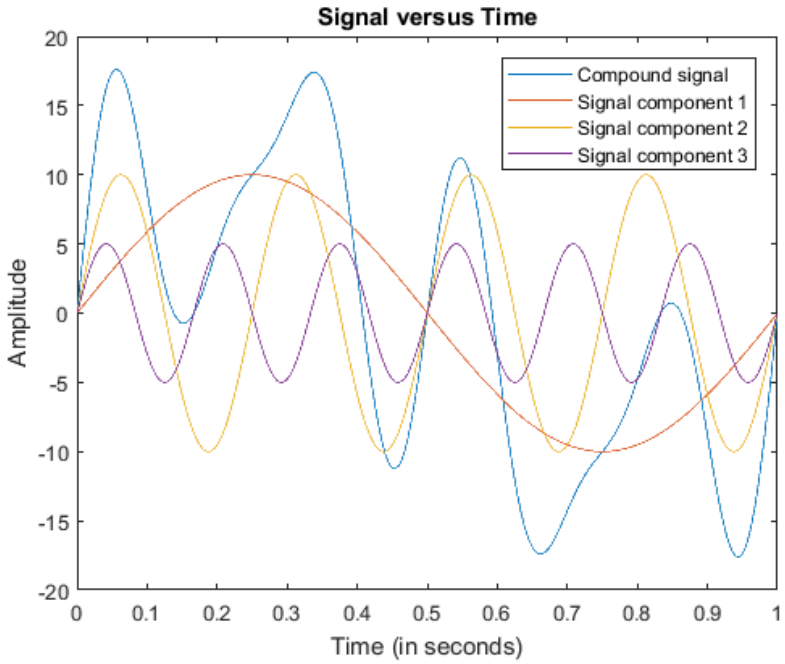
   a. Determine if the signal is bandlimited. [4 marks]
   b. Using the sampling periods listed below, find out whether the Nyquist sampling rate condition is satisfied and if the sampled signal looks like the original signal or not.
      i. $T_s = 0.4$ second/sample. [6 marks]
      ii. $T_s = 0.5$ second/sample. [6 marks]
      iii. $T_s = 1$ second/sample. [6 marks]

4. Consider the two sinusoids when $-\infty \leq t \leq +\infty$ and where: $\Omega_1 > 2\Omega_0$:

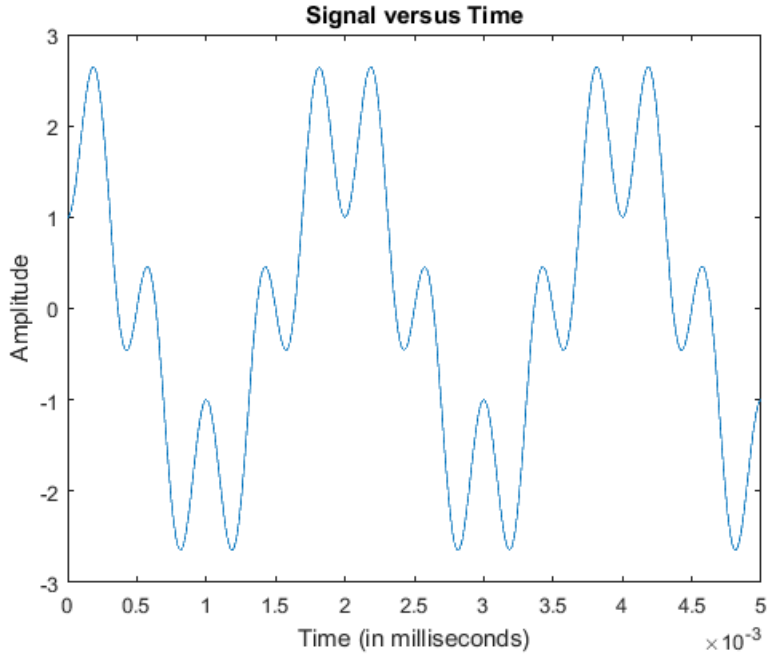
$$x_1(t) = \cos(\Omega_0 t) \qquad \text{and} \qquad x_2(t) = \cos[((\Omega_0 + \Omega_1)t)]$$

   a. Show that if we sample these signals using $T_s = 2\pi/\Omega_1$, we cannot differentiate the sampled signals, i.e. $x_1(nT_s) = x_2(nT_s)$. [6 marks]
   b. Explain the significance of this. [2 marks]

5. The analogue signal $(t) = 10\sin(2\pi t) + 10\sin(8\pi t) + 5\sin(12\pi t)$ as shown below, where t is in seconds, is sampled at a rate of f_s = 5 Hz.
   a. Determine and sketch the signal $xa(t)$ aliased with $x(t)$. [12 marks]
   b. Show that the original signal and its alias have the same sample values, that is, show that $x(nT) = xa(nT)$. [8 marks]
   c. Repeat part (a) if the sampling rate is f_s = 10 Hz. [12 marks]



6. The signal $x(t) = \cos(5\pi t) + 4\sin(2\pi t)\sin(3\pi t)$ as shown below, where t is in milliseconds, is sampled at a rate of 3 kHz.



a. Determine the signal $xa(t)$ aliased with $x(t)$. [6 marks]

b. Determine and sketch two other signals $x_1(t)$ and $x_2(t)$ that are different from each other and from $x(t)$, yet they are aliased with the same $xa(t)$ that you found. [16 marks]

7. Let a signal $x(t) = \cos(8\pi t) + \cos(4\pi t)\cos(6\pi t)$, where t is in seconds.

   a. Determine the frequency components of the signal. [4 marks]

   b. Determine and sketch the signal $xa(t)$ aliased with $x(t)$, if the sampling rate is 5 Hz.

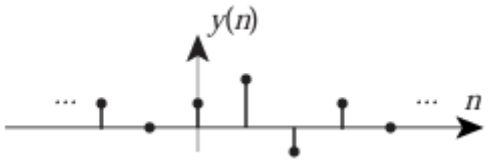
   [12 marks]

8. Consider the following signals listed below, determine if they are band-limited or not. If not, determine the frequency for which the energy of the non-band-limited signal corresponds to 99% of its total energy and use this result to approximate its maximum frequency.

   a. $x_1(t) = u(t + 0.5) - u(t - 0.5)$. [8 marks]

   b. $x_2(t) = e^{-t}u(t)$. [8 marks]

9. Given the following waveform of discrete-time signal as shown in the figure below.



   Sketch the waveform after the reconstruction process using the following method:

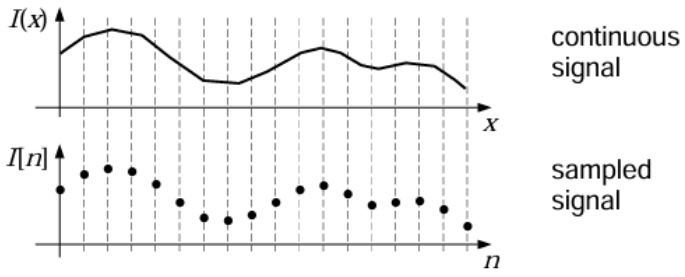
   a. Zero-order hold. [5 marks]

   b. First-order hold (linear interpolation). [5 marks]

   c. Ideal interpolation. [5 marks]

   d. Weighted Dirac delta. [5 marks]

10. Given a continuous-time signal which is sampled by impulse sampling signal given in the figure below.

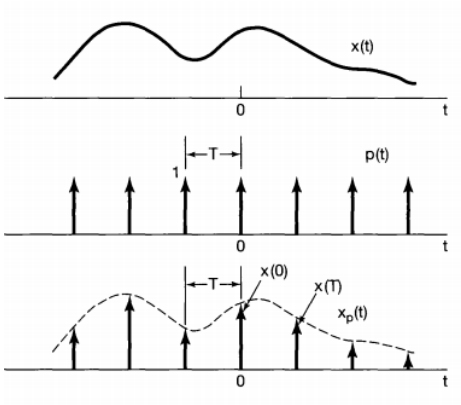


    Sketch the result after the discrete-time sampling of the signal.

    a. Down sampling by a factor of 2. [5 marks]

    b. Up sampling by a factor of 2. [5 marks]

11. In a digital signal processing system (DSP) in a discotheque, the following discrete-time pulse is sampled at 40 kHz from a continuous-time signal (e.g. cut off frequency of 20 kHz).



The DSP system is connected to a subwoofer pre-amplifier (slower unit, with 4 kHz operating frequency) and to a digital signal conversion processor for laser light animation and effect (faster unit, with sampling frequency of 50 kHz).

a. By ensuring the no-aliasing requirement of the sampling process, reconstruct through relevant decimation and interpolation processes so the discrete-time signal can fit the performance requirements of the other units. [16 marks]

b. Construct a diagram that shows the processes in the system from sampling to reconstruction. Include in your diagram any additional components and parts for successful operation of the processes. [6 marks]

C. Z-Transform

- Derive and calculate of Z-transform.
- Derive and calculate of inverse Z-transform.
- Understand, prove and apply properties of Z-transform.
- Apply z-transform in the signal and systems.

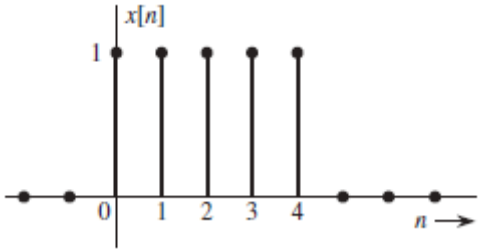
1. Find the *z*-transforms of:

a. $\delta[n]$. [2 marks]

b. $u[n]$. [6 marks]

c. $\cos(\beta n)\, u[n]$. [6 marks]

d. The signal shown in the figure below. [4 marks]



Recall that by definition

$$X[z] = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \frac{x[3]}{z^3} + \cdots \qquad (Eq.1)$$

2. For the given discrete time series, their z-transforms are:

   a. $x[n] = 3^n u[n]$. [6 marks]

   b. $x[n] = u[n] - u[n-2]$. [4 marks]

   c. $x[n] = \cos(\pi/3)u[n]$. [8 marks]

   d. $x[n] = \{3, 1, 4\} + u[n]$. [4 marks]

3. Find the inverse *z*-transforms of the following functions:

   a. Real poles: [10 marks]

$$X(z) = \frac{8z - 19}{(z-2)(z-3)}$$

   b. Repeated poles: [20 marks]

$$X(z) = \frac{z(2z^2 - 11z + 12)}{(z-1)(z-2)^3}$$

   c. Complex poles: [10 marks]

$$X(z) = \frac{2z(3z + 17)}{(z-1)(z^2 - 6z + 25)}$$

4. For the given z-transforms, their inverse z-transforms are:

   a. $X(z) = \frac{2z^4+7z^3+z^2+8z+3}{z^4}$. [4 marks]

   b. $X(z) = \frac{4z+5}{z^2} + \frac{6}{1-3z^{-1}}$. [4 marks]

   c. $X(z) = \frac{7z}{z-1} + \frac{8z}{z-2}$. [4 marks]

   d. $X(z) = \frac{z-3}{z^2-3z+2}$. [12 marks]

e. $X(z) = \frac{2z}{z^2-2z+2}$. [12 marks]

f. $X(z) = \frac{z-1}{z^3+z^2+8z+8}$. [12 marks]

5. Find the poles and zeros of the following Z-transforms:

a. $X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$. [6 marks]

b. $X_2(z) = [(z^{-1} - 1)(z^{-1} + 2)^2]/\left[z^{-1}\left(z^{-2} + \sqrt{2}z^{-1} + 1\right)\right]$. [6 marks]

6. Find the *z*-transform and the corresponding ROC for the signal $\gamma^n u[n]$. [10 marks]

7. Find the Z-transform of a discrete-time pulse $x[n] = u[n] - u[n-10]$. Determine the region of convergence of $X(z)$. [8 marks]

8. The poles of $X(z)$ are z = 0.5 and z = 2. Find all the possible signals that can be associated with $X(z)$ according to different regions of convergence. [6 marks]

9. Given the following signals described as sequences listed below:

$$x_1[n] = \left(\frac{1}{2}\right)^2 u[n] \qquad \text{and} \qquad x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$$

a. Describe the characteristics of $x_1[n]$ and $x_2[n]$. [2 marks]

b. Determine the z-transform of $x_1[n]$ and its ROC. [4 marks]

c. Determine the z-transform of $x_2[n]$ and its ROC. [4 marks]

d. Compare the difference and similarity between $x_1[n]$ and $x_2[n]$. [2 marks]

e. Determine the z-transform of $x_1[n] + x_2[n]$ and its ROC. [4 marks]

10. For the z-transform properties listed below, describe their significances in the transformation and how describe how to use them in simplifying the transformation process.

a. Linearity property. [4 marks]

b. Delay property. [4 marks]

c. Convolution property. [4 marks]

11. For each of the following difference equations and associated input and initial conditions, determine the output $y[n]$.

a. Knowing that input, $x[n] = (1/3)^n$ and initial condition $y[-1] = 1$. [10 marks]

$$y[n] - \left(\frac{1}{2}\right) y[n-1] = x[n]$$

b. Knowing that input, $x[n] = (1/2)^n$ and initial conditions $y[-1] = 1$ and $y[-2] = 2$. [10 marks]

$$3y[n] - 4y[n-1] + y[n-2] = x[n]$$

12. A causal discrete-time LTI system is described by:

$$y[n] - \left(\frac{3}{4}\right) y[n-1] + \left(\frac{1}{8}\right) y[n-2] = x[n]$$

where $x[n]$ and $y[n]$ are the input and output of the system, respectively.

a. Determine the system function $H(z)$. [6 marks]

b. Find the impulse response $h[n]$ of the system. [10 marks]

c. Find the step response $s[n]$ of the system. [16 marks]

13. An LTI system has the transfer function given below:

$$H(z) = \frac{2}{1 - 0.9e^{j\frac{\pi}{4}}z^{-1}} + \frac{2}{1 - 0.9e^{-j\frac{\pi}{4}}z^{-1}} + \frac{3}{1 + 2z^{-1}}$$

Find the impulse response, assuming that the system is (a) stable or (b) causal. Can the system be both stable and causal? [8 marks]

USEFUL FORMULAE

Time and Frequency Characteristics

First-Order Time-Domain Analysis

For a standard transfer function of the first-order system:

$$G(j\omega) = \frac{K}{(j\omega + a)}$$

Time constant:

$$T = \frac{1}{a}$$

Rise time:

$$T_r = \frac{2.2}{a}$$

Settling time:

$$T_s = \frac{4}{a}$$

Second-Order Time-Domain Analysis

For a standard transfer function of the second order system:

$$G(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\omega_n\zeta(j\omega) + \omega_n^2}$$

Rise time:

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} \qquad \text{Where: } \phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Rise time (for underdamped response):

$$T_r = \frac{(1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)}{\omega_n}$$

Time-to-peak (n = #peak):

$$T_p = \frac{\pi n}{\omega_n\sqrt{1-\zeta^2}}$$

Percent overshoot:

$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100$$

Damping factor:

$$\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln\left(\frac{\%OS}{100}\right)^2}}$$

Settling time:

$$T_s = \frac{4}{\zeta \omega_n}$$

Sampling and Reconstruction

Sampling equation:

$$x_s(t) = x(t) \cdot \text{comb}(t) = x(t) \cdot \sum_{m=-\infty}^{\infty} \delta(t - mt_0) = \sum_{m=-\infty}^{\infty} x[m]\delta(t - mt_0)$$

Discrete time samples:

$$X_s(j\omega) = \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega t} = \sum_{m=-\infty}^{\infty} x[m]e^{-j2\pi f m t_0}$$

Z-Transform Table

No	Sequence	Transform	ROC
1	$\delta[n]$	1	All z
2	$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$-u[n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
4	$\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
6	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
7	$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
8	$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
9	$\begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
10	$\sin(\omega_0 n)u[n]$	$\frac{\sin\omega_0 z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
11	$\cos(\omega_0 n)u[n]$	$\frac{1-\cos\omega_0 z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
12	$[a^n \sin\omega_0 n]u[n]$	$\frac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\| > \|a\|$
13	$[a^n \cos\omega_0 n]u[n]$	$\frac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\| > \|a\|$

Z-transform Properties Table

No	Property	Sequence	Transform	ROC
1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2.
2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the region.
3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e. the set of points $\{\|a\|z\}$for z in R.
4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R(i.e. R^{-1} is equal to the set of point z^{-1}, where z is in R).
5	Time expansion	$x_{(k)}[n]\begin{cases}x[r], n = rk \\ 0, n \neq rk\end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$(i.e. the set of point $z^{1/k}$, where z is in R).
6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
7	Convolution	$x_1[n] * x_2[n]$	$X_1(z) \cdot X_2(z)$	At least the intersection of R_1 and R_2.
8	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$.
9	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\left(\frac{1}{1 - z^{-1}}\right)X(z)$	At least the intersection of R and $\|z\| > 1$.
10	Differentiation in the z-domain	$nx[n]$	$-z\left[\frac{dX(z)}{dz}\right]$	R
11	Initial value theorem	If $x[n] = 0$ for $x < 0$, then $x[0] = \lim_{z\to\infty} X(z)$		

Note:

- For a sequence $x[n]$, its z-transform is $X(z)$ and its ROC is R.
- For a sequence $x_1[n]$, its z-transform is $X_1(z)$ and its ROC is R_1.
- For a sequence $x_2[n]$, its z-transform is $X_2(z)$ and its ROC is R_2.