XMUT220 Signals and Systems

Tutorial 5b: Time and Frequency Characteristics (Time-Domain Analysis) – Solution

A. Introduction to Time-Domain Analysis

1. Performing time domain analysis for a given first-order system is different from performing the same thing for a second-order system.

   a. Describe the parameters in the time response analysis which are common for first- and second-order systems. [2 marks]

   b. List the parameters in the time response analysis which are specific for second-order system. [2 marks]

   c. Why is it not accurate to use the rise time equation for first-order system for calculating the rise time of second-order system? [2 marks]

Solution

a. Parameters in the time response analysis which are common for first and second order systems are typically the time constant, rise time, settling time, and steady state errors.

b. Parameters in the time response analysis which are specific for second order system are typically natural frequency, damping ratio, the time-to-peak and percentage overshoot. These phenomena only exist in the second order system.

c. It not accurate to approximate the rise time equation for first order system for calculating the rise time of second order system as first order, as the assumption behind the nature of the response of the system is different in the first order system from the second order system.

   This is true especially for those underdamped responses of the second order system. The errors rising from the inaccuracies are prominent in this type of responses. This approximate is less inaccurate when we have overdamped responses of the second order system which in nature its response is very similar to the response of first order system.

2. The transient response of a second-order system can be determined from its transfer function equation. Depending on the type of roots in the equation, the response can be categorised as underdamped, critically damped and overdamped.

   a. Prove the roots of the equation for second order system are: [6 marks]

$$j\omega = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \quad \text{or} \quad j\omega = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$

   b. Prove the time domain equation of the underdamped response of a second-order system when it is given a step input is as shown below. [12 marks]

$$c(t) = 1 - \frac{e^{-\zeta\omega_t}}{\sqrt{1-\zeta^2}}\sin[\omega_d t + \phi]$$

   Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)$

Solution

a. Given the general transfer function equation of second order system is given as:

$$\frac{C(\omega)}{R(\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2}$$

If the denominator (e.g. characteristic equation of the second order system) is equated to zero, then the equation above becomes.

$$(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2 = 0$$

Rearrange the equation

$$(j\omega)^2 + 2\zeta\omega_n j\omega + \zeta^2\omega_n^2 - \zeta^2\omega_n^2 + \omega_n^2 = 0$$

Then

$$(j\omega + \zeta\omega_n)^2 - \zeta^2\omega_n^2 + \omega_n^2$$

So

$$(j\omega + \zeta\omega_n)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2 = 0$$

The roots of the equation for second order system as shown above are:

$$j\omega = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \quad \text{or} \quad j\omega = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$

b. For underdamped case ($0 < \zeta < 1$), the transfer function of the system $C(j\omega)/R(j\omega)$ can be written as follows:

$$\frac{C(\omega)}{R(\omega)} = \frac{\omega_n^2}{(j\omega + \zeta\omega_n + j\omega_d)(j\omega + \zeta\omega_n - j\omega_d)}$$

Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$. The frequency ω_d is called the damped natural frequency.

For a step input, $R(\omega) = 1/j\omega$ and $C(\omega)$ can be written:

$$C(\omega) = \frac{\omega_n^2}{j\omega(j\omega + \zeta\omega_n + j\omega_d)(j\omega + \zeta\omega_n - j\omega_d)}$$

Applying partial fraction to the equation given above:

$$C(\omega) = \frac{1}{j\omega} - \frac{j\omega + \zeta\omega_n}{(j\omega + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(j\omega + \zeta\omega_n)^2 + \omega_d^2}$$

Hence the inverse transform of the equation above is:

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

This equation can be simplified into:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\left(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\sin\omega_d t\right)$$

Say, $\zeta = \cos\phi$, hence $\sqrt{1-\zeta^2} = \sin\phi$ (e.g. for underdamped response $0 < \zeta < 1$):

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)(\sin\phi\cos\omega_d t + \cos\phi\sin\omega_d t)$$

The expression that outlines the underdamped response of the second-order system when it is given a step input function is:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_t}}{\sqrt{1-\zeta^2}}\right)\sin[\omega_d t + \phi]$$

Where:

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

And

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

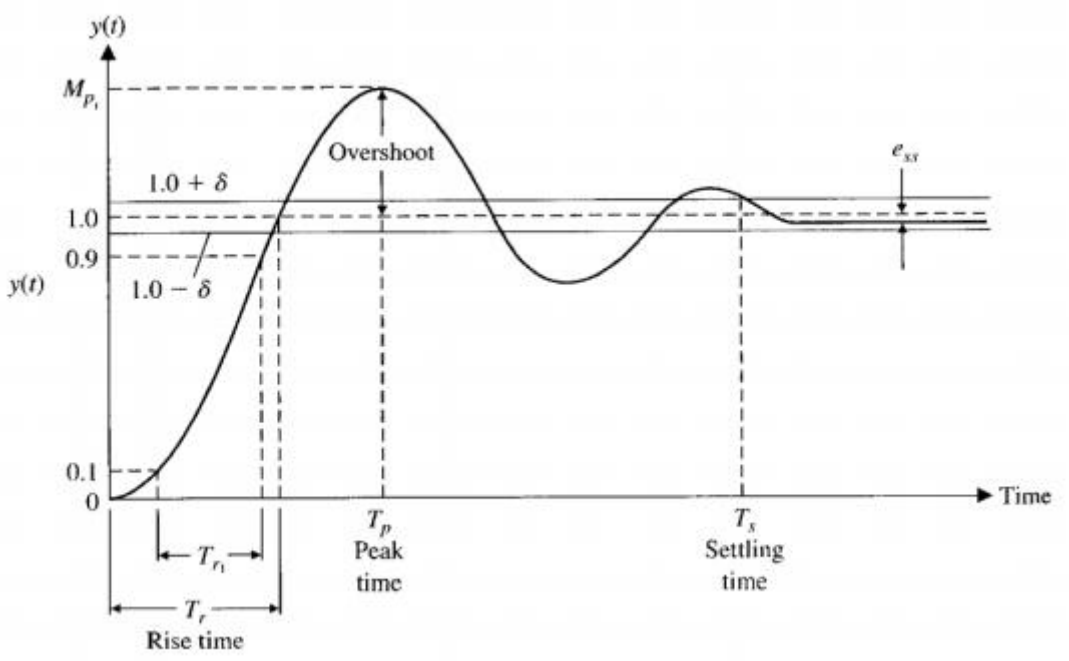
3. The rise time of the transient response of a given second-order system cannot be determined very accurately from the rise time equation for a first-order system.

   a. Derive the rise time equation for a second-order system as shown below. [8 marks]

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$$

   Where:

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$



b. Derive the graph of normalised rise time versus damping ratio for a second-order underdamped response. [8 marks]

Solution

a. Using the expression of underdamped second order system with a step input as shown below:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right]$$

Where:

$$\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

To find the rise time of the response of the system, we have to equate the expression for $c(t) = 1$. From the above we have:

$$1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)T_r + \phi\right] = 1$$

Hence

$$\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)T_r + \phi\right] = 0$$

So

$$\left(\omega_n\sqrt{1-\zeta^2}\right)T_r + \phi = \pi$$

This equation can be simplified as shown below:

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$$

Where:

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

b. The graph is actually approximated by the following polynomials for maximum error less than 0.5 % for $0 < \zeta < 0.9$:

$$\omega_n T_r = 1.76\zeta^3 + 0.417\zeta^2 + 1.039\zeta + 1$$

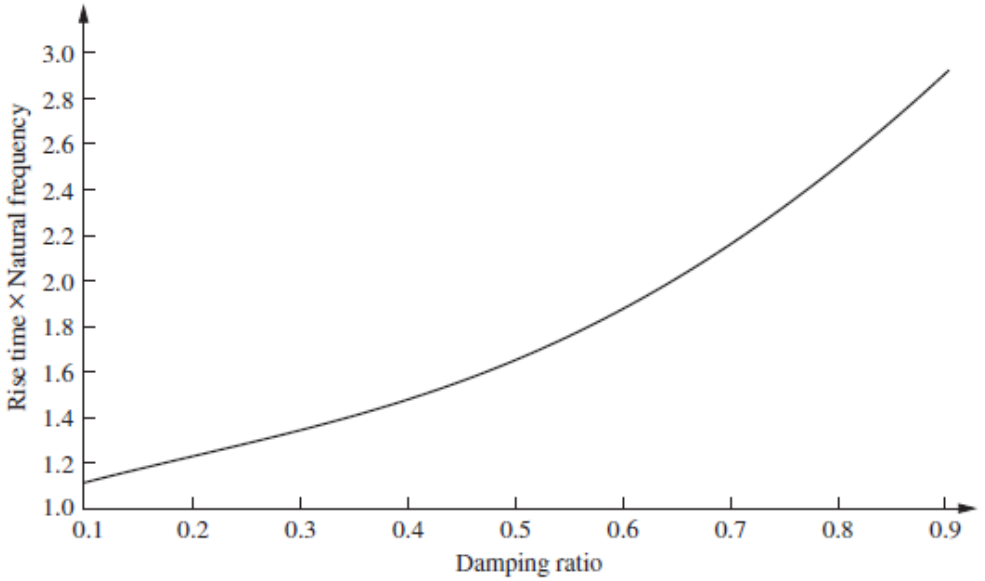
And, for maximum error less than 5 % for $0.1 < \zeta < 0.9$:

$$\zeta = 0.115(\omega_n T_r)^3 - 0.883(\omega_n T_r)^2 + 2.504\omega_n T_r - 1.738$$

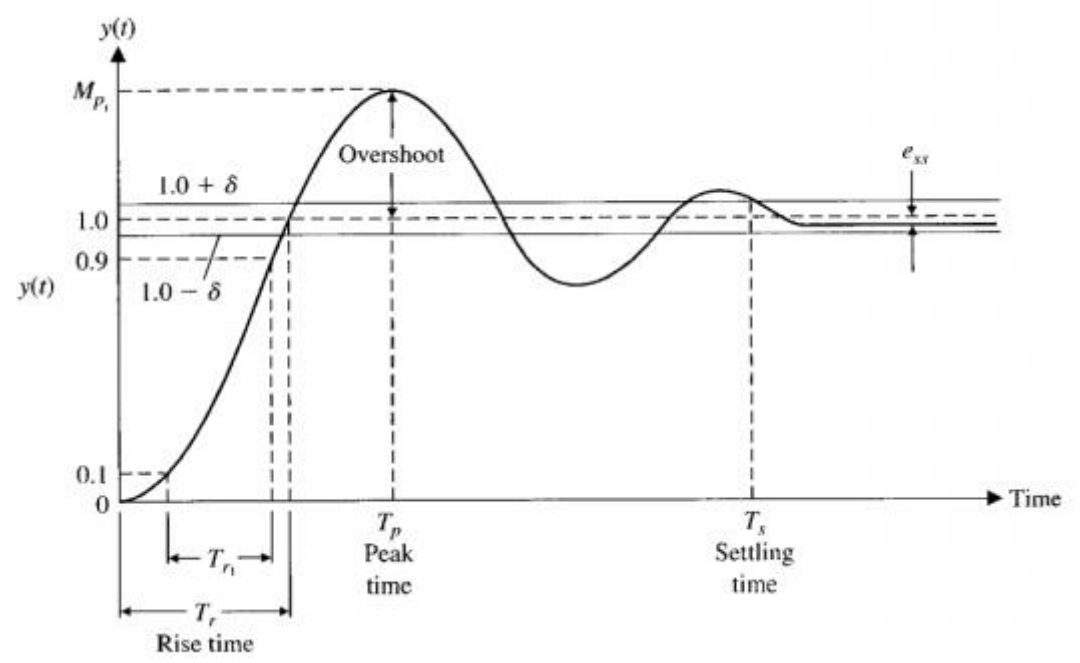
The polynomials above were obtained using the `polyfit()` function in MATLAB and the data used for deriving this equation are from the table as shown below.

Damping Ratio	Normalised Rise Time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

The graph of normalized rise time versus damping ratio for a second-order underdamped response is as shown in the figure below.



4. The transient response behaviour of second order systems is typically determined from the time domain analysis parameters which are specific for second-order system.



a. Prove that the time-to-peak (T_p) of the second order system is determined from the following equation. [12 marks]

$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad \text{where } n = \text{nth peak}$$

b. Derive the percentage overshoot ($\%OS$) of the second order system which is as shown in the equation below. [14 marks]

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100\%$$

Solution

a. Derivation of the time-to-peak (T_p) of the second order system. Consider the expression of underdamped second order system with a step input as shown below:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right]$$

As per definition of the time-to-peak (T_p), the response curve reaches to its maximum value. Hence at this point

$$\frac{dc(t)}{dt} = 0$$

So, for the first derivative of the above equation (e.g. $z' = uv' + u'v$)

$$\frac{dc(t)}{dt} = -\left[\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right]\omega_n\sqrt{1-\zeta^2}\cos\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right] - \left[\frac{(\zeta\omega_n)e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right]\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right]$$

Putting $dc(t)/dt = 0$, the equation given above becomes:

$$-\left[\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right]\omega_n\sqrt{1-\zeta^2}\cos\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right] - \left[\frac{(\zeta\omega_n)e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right]\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right] = 0$$

Rearranging the equation above

$$\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}\cos\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right] = \left[\frac{(\zeta\omega_n)e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right]\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right]$$

The equation becomes:

$$\tan\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right] = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

Knowing $\phi = \tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)$, hence $\tan\phi = \sqrt{1-\zeta^2}/\zeta$, the equation above becomes:

$$\tan\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right] = \tan\phi$$

Hence

$$\left(\omega_n\sqrt{1-\zeta^2}\right)t = n\pi$$

Where: $n = 1, 2, 3, \ldots$

So, considering the maximum overshoot occurs at $n = 1$, the equation for time-to-peak is given below:

$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad \text{where } n = \text{nth peak}$$

b. Derivation of the percentage overshoot ($\%OS$) of the second order system. Using the expression for underdamped second order system.

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t + \phi\right]$$

To find the percentage overshoot, we determine the percentage overshoot ($\%OS$) of the second order system from the maximum value of the transient response.

$$c(t)_{max} = 1 - \left(\frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\right)\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)T_p + \phi\right]$$

Substitute T_p in the equation above with the expression given below for $n = 1$ (for maximum value).

$$T_p = \frac{n\pi}{\left(\omega_n\sqrt{1-\zeta^2}\right)}$$

The equation for the time-to-peak is:

$$c(t)_{max} = 1 - \frac{e^{-\zeta\omega_n\left(\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\right)}}{\sqrt{1-\zeta^2}}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)\left(\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\right) + \phi\right]$$

Thus

$$c(t)_{max} = 1 - \left(\frac{e^{\frac{-\zeta\pi}{\omega_n\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\right)\sin(\pi + \phi)$$

Knowing that $\sin(\pi + \phi) = -\sin\phi$, the equation above becomes:

$$c(t)_{max} = 1 - \left(\frac{e^{\frac{-\zeta\pi}{\omega_n\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\right)(-\sin\phi) = 1 + \left(\frac{e^{\frac{-\zeta\pi}{\omega_n\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\right)\sqrt{1-\zeta^2} = 1 + e^{\frac{-\zeta\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

The maximum percentage overshoot is:

$$\%OS = c(t)_{max} - 1 = \left(1 + e^{\frac{-\zeta\pi}{\omega_n\sqrt{1-\zeta^2}}}\right) - 1$$

As a result, the percentage overshoot is:

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100\%$$

B. Time-Domain Analysis of Systems

5. You are given the following first order systems:

   i. System 1:

$$G_1(\omega) = \frac{5}{j\omega + 5}$$

   ii. System 2:

$$G_2(\omega) = \frac{20}{j\omega + 20}$$

   a. For both systems, calculate the time constant, rise time, and settling time for each system. [6 marks]

   b. Based on the results in part (d), comment on the differences in transient and steady-state responses of the first system with the second system. [4 marks]

   c. Simulate and describe the transient responses of the systems when each of the systems is subjected to a step input. [6 marks]

   d. Simulate and describe the transient responses of the systems when each of the systems is subjected to an impulse input. [6 marks]

Solution

a. The time constant, rise time, and settling time for each system are calculated as follows:

For System (i):

$$G_1(\omega) = \frac{5}{(j\omega + 5)} = \frac{1}{j\omega/5 + 1}$$

Time constant:

$$\tau = \frac{1}{5} = 0.2 \text{ second}$$

Rise time:

$$T_r = 2.2\tau = \frac{2.2}{a} = \frac{2.2}{5} = 0.44 \text{ second}$$

Settling time:

$$T_s = 4\tau = \frac{4}{a} = \frac{4}{5} = 0.8 \text{ second}$$

For System (ii):

$$G_2(\omega) = \frac{20}{(j\omega + 20)} = \frac{1}{j\omega/20 + 1}$$

Time constant:

$$\tau = \frac{1}{20} = 0.05 \text{ second}$$

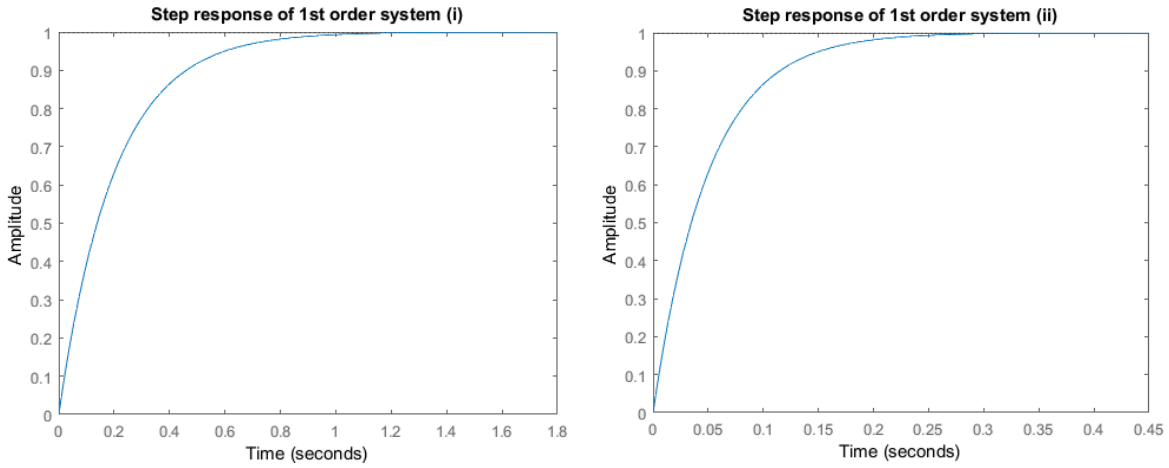
Rise time:

$$T_r = 2.2\tau = \frac{2.2}{a} = \frac{2.2}{20} = 0.11 \text{ second}$$

Settling time:

$$T_s = 4\tau = \frac{4}{a} = \frac{4}{20} = 0.2 \text{ second}$$

b. For transient response of the system, it seems System (i) has considerable longer time constant, rise time and settling time compared with System (ii) e.g. τ = 0.2 second, T_r = 0.44 second and T_s = 0.8 second compared with τ = 0.05 second, T_r = 0.11 second and T_s = 0.2 second respectively.



For steady-state response, the overall response of System (ii) is more responsive e.g. take less time to settle down compared with System (i).

c. Simulate and describe the transient responses of the systems when each of these systems subjected to a step input.

For System (i) and a step input of $1/j\omega$:

$$G_1(\omega) = \frac{5}{j\omega(j\omega + 5)}$$

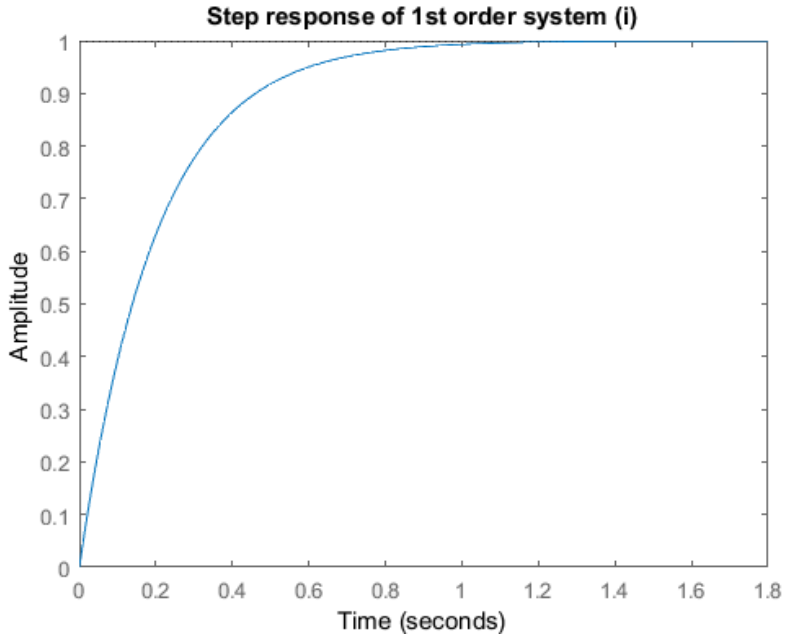
Take partial fraction of the equation given above

$$G_1(\omega) = \frac{1}{j\omega} - \frac{1}{j\omega + 5}$$

The inverse transform of the equation given above is:

$$g_1(t) = 1 - e^{-5t}$$

The transient response of the System (i) is an increasing exponential response with a time constant of 5 seconds.



For System (ii) and a step input of $1/j\omega$:

$$G_2(\omega) = \frac{20}{j\omega(j\omega + 20)}$$

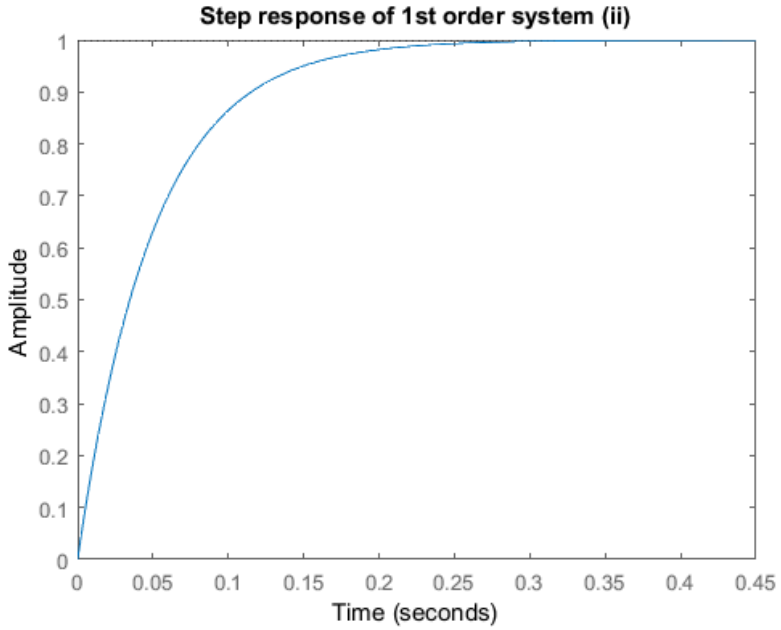
Take partial fraction of the equation above:

$$G_2(\omega) = \frac{1}{j\omega} - \frac{1}{j\omega + 20}$$

Therefore, the inverse transform of the equation given above is:

$$g_2(t) = 1 - e^{-20t}$$

The transient response of the System (ii) is an increasing exponential response with a time constant of 20 seconds.



d. Simulate and describe the transient responses of the systems when each of these systems subjected to an impulse input.

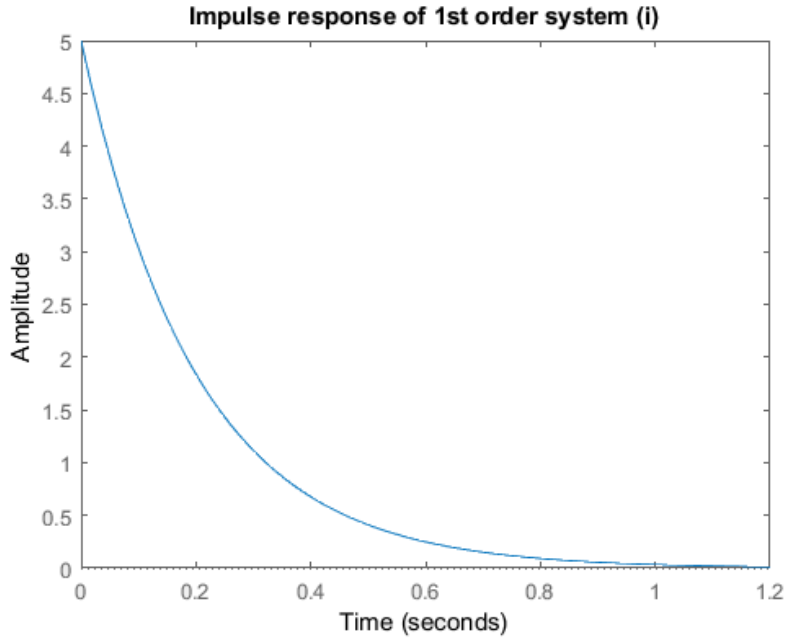
For System (i) and impulse input:

$$G_1(\omega) = \frac{5}{(j\omega + 5)}$$

The inverse transform of the equation given above is:

$$g_1(t) = 5e^{-5t}$$

The transient response of the System (i) is a decreasing exponential response with a time constant of 5 seconds.



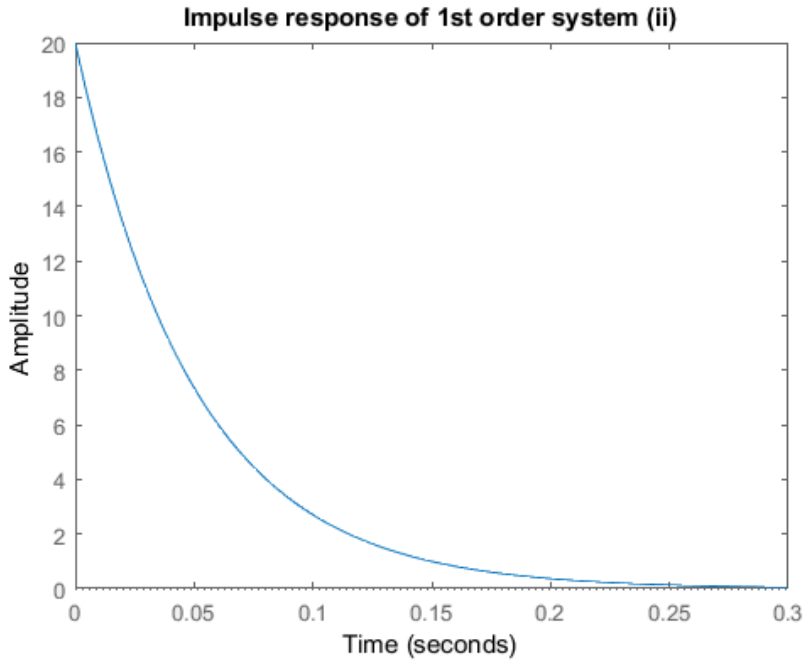
For System (ii) and impulse input:

$$G_2(\omega) = \frac{20}{(j\omega + 20)}$$

Therefore

$$g_2(t) = 20e^{-20t}$$

The transient response of the System (ii) is a decreasing exponential response with a time constant of 20 seconds.



6. You are given the following second-order systems:

   i. System 1:

$$G_1(\omega) = \frac{16}{(j\omega)^2 + 3j\omega + 16}$$

   ii. System 2:

$$G_2(\omega) = \frac{0.04}{(j\omega)^2 + 0.02j\omega + 0.04}$$

   iii. System 3:

$$G_3(\omega) = \frac{1.05 \times 10^7}{(j\omega)^2 + 1.6 \times 10^3 j\omega + 1.05 \times 10^7}$$

   a. Determine the transient responses of the systems based on the roots of equations. [12 marks]

   b. For each of the second order systems given above, simulate and find the time response characteristics of the systems in terms of: [36 marks]

   - Damping Ratio (ζ),
   - Natural Frequency (ω_n),
   - Settling Time (T_s),
   - Peak Time (T_p),
   - Rise Time (T_r), and
   - Percentage Overshoot ($\%OS$).

   c. Comment on the transient response results of the three systems given in (a) and (b). [6 marks]

Solution

a. Determine the transient responses of the systems. By considering a standard second order equation as follows:

$$G(\omega) = \frac{(\omega_n)^2}{(j\omega)^2 + 2\zeta\omega_n j\omega + (\omega_n)^2}$$

- For System (i), with the following transfer function:

$$G_1(\omega) = \frac{16}{(j\omega)^2 + 3j\omega + 16}$$

  Characteristic equation of System (i) is $(j\omega)^2 + 3j\omega + 16$

Roots of the System (i)'s characteristic equation are:

$$\text{root}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{-3 \pm \sqrt{(3)^2 - 4(1)(16)}}{(2)(1)} = -1.5 \pm j\frac{\sqrt{55}}{2}$$

Since the roots are a pair of complex numbers, the transient response of the System (i) is underdamped response.

- For System (ii), with the following transfer function:

$$G_2(\omega) = \frac{0.04}{(j\omega)^2 + 0.02j\omega + 0.04}$$

Characteristic equation of System (ii) is $(j\omega)^2 + 0.02j\omega + 0.04$

Roots of the System (ii)'s characteristic equation are:

$$\text{root}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{-0.02 \pm \sqrt{(0.02)^2 - 4(1)(0.04)}}{(2)(1)} = -0.01 \pm j0.63$$

Since the roots are a pair of complex numbers, the transient response of the System (ii) is also underdamped response.

- For System (iii), with the following transfer function:

$$G_3(\omega) = \frac{1.05 \times 10^7}{(j\omega)^2 + 1.6 \times 10^3 j\omega + 1.05 \times 10^7}$$

Characteristic equation of System (iii) is $(j\omega)^2 + 10j\omega + 16$

Roots of the System (iii)'s characteristic equation are:

$$\text{root}_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{-1.6 \times 10^3 \pm \sqrt{(1.6 \times 10^3)^2 - 4(1)(1.05 \times 10^7)}}{(2)(1)}$$

$$= -800 \pm j3{,}140$$

Since the roots are a pair of complex numbers, the transient response of the System (iii) is also underdamped response.

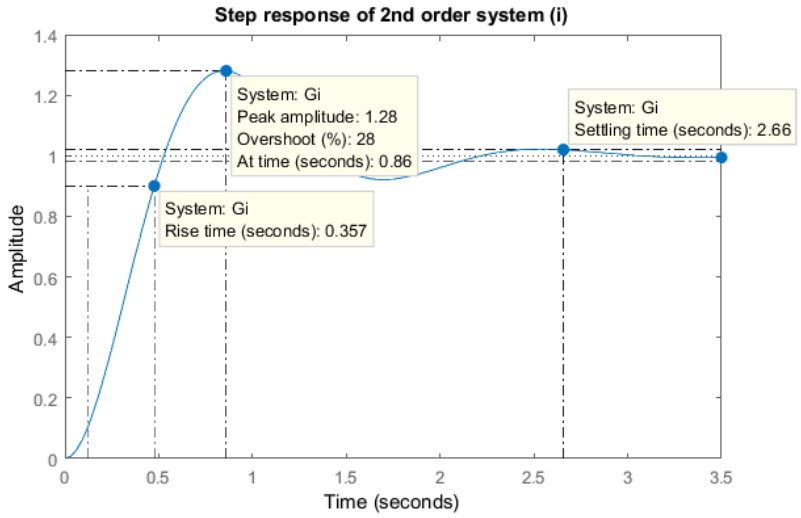
b. Simulation and calculation of the time response characteristics of the systems:

By considering a standard second order equation as follows:

$$G(\omega) = \frac{(\omega_n)^2}{(j\omega)^2 + 2\zeta\omega_n j\omega + (\omega_n)^2}$$

System (i):

The result of simulation of the time response of System (i) in MATLAB is shown in the figure below.



Transfer function equation of System (i):

$$G_1(\omega) = \frac{16}{(j\omega)^2 + 3j\omega + 16}$$

Natural frequency:

$$\omega_n^2 = 16 \text{ rad/sec}$$

$$\omega_n = 4$$

Damping ratio:

$$2\zeta\omega_n = 3, \text{ therefore } \zeta = 3/2\omega_n = 3/2(4) = 0.375$$

Peak time (first peak):

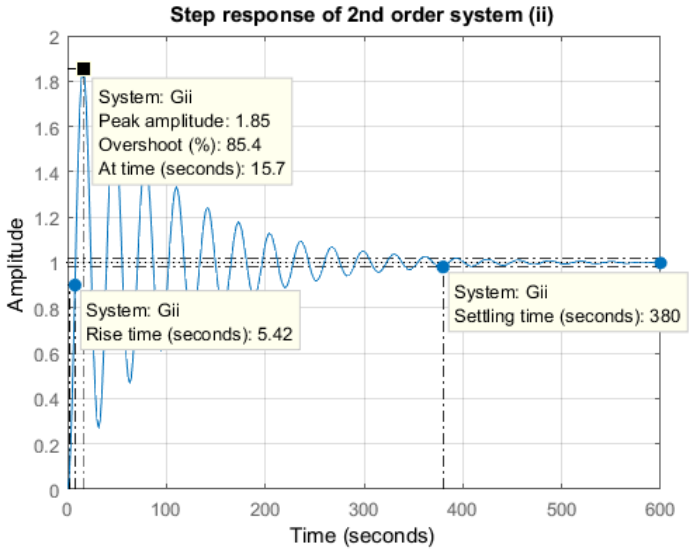
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.8472 \text{ second}$$

Percentage overshoot:

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 28.06\ \%$$

System (ii):

The result of simulation of the time response of System (ii) in MATLAB is shown in the figure below.



Transfer function equation of System (ii):

$$G_2(\omega) = \frac{0.04}{(j\omega)^2 + 0.02j\omega + 0.04}$$

Natural frequency:

$$\omega_n^2 = 0.04 \text{ rad/sec}$$

$$\omega_n = 0.2$$

Damping ratio:

$$2\zeta\omega_n = 0.02, \text{ therefore } \zeta = 0.02/2\omega_n = 0.02/2(0.2) = 0.05$$

Peak time (first peak):

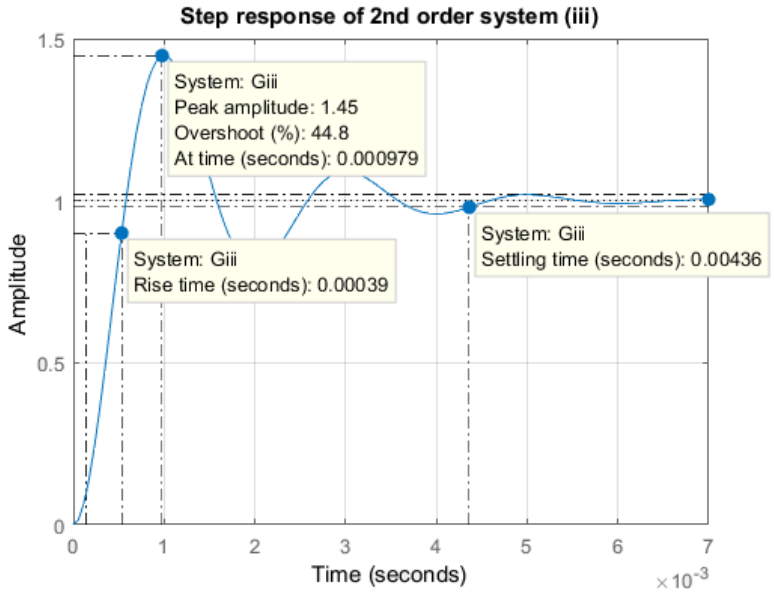
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 15.73 \text{ seconds}$$

Percentage overshoot:

$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 85.45\ \%$$

System (iii):

The result of simulation of the time response of System (iii) in MATLAB is shown in the figure below.



Transfer function equation of system (iii):

$$G_3(\omega) = \frac{1.05 \times 10^7}{(j\omega)^2 + 1.6 \times 10^3 j\omega + 1.05 \times 10^7}$$

Natural frequency:

$$\omega_n^2 = 1.05 \times 10^7 \text{ rad/sec}$$

$$\omega_n = 3{,}240$$

Damping ratio:

$$2\zeta\omega_n = 1.6 \times 10^3, \text{ therefore } \zeta = 1.6 \times 10^3 / 2\omega_n = 1.6 \times 10^3 / 2(3{,}240) = 0.247$$

Peak time (first peak):

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.001 \text{ second}$$

Percentage overshoot:

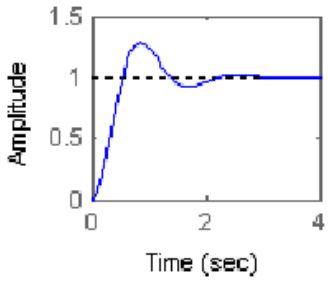
$$\%OS = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100 = 44.92\ \%$$

c. Comparing the transient responses of the three given systems:

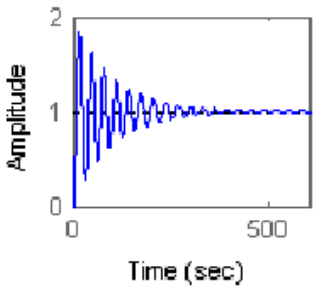
It seems that the fastest settling time, peak and rise time are recorded by System (iii) with the slowest are recorded by System (ii).

It looks like System (i) and (iii) have lightly damped responses whereas System (ii) has a decaying oscillation response.

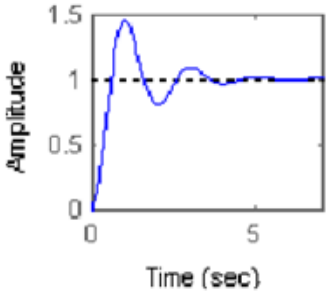
The biggest percentage overshoot is experienced by System (ii).



System (i)



System (ii)



System (iii)

7. The steady-state error of a particular system depends on the type of input given to that system.

   a. Derive the steady-state error of the given system for an underdamped second-order system with step input is as shown below. [10 marks]

$$C(\infty) = \lim_{\omega \to 0} j\omega R(\omega) G(\omega)$$

   b. Determine and simulate in MATLAB the steady-state error of the systems given below when given a step input: [20 marks]

      i. System 1:

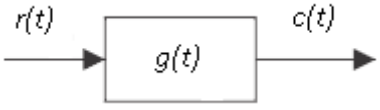
$$G_1(\omega) = \frac{1}{j\omega + 0.3}$$

      ii. System 2:

$$G_2(\omega) = \frac{1}{[0.2(j\omega)^2 + 5j\omega + 5]}$$

Solution

a. For a system in the time domain as shown below, the input and output of a given system are determined as follows.



Where:

- $g(t)$ = System function in time domain
- $r(t)$ = Reference signal function in time domain
- $c(t)$ = Output signal function in time domain

Response (output) of the given system is determined from the following expression:

Output = Reference x System

Or

$$c(t) = r(t)g(t)$$

Apply the final value theorem as shown below to equation above.

$$\lim_{t\to\infty} f(t) = \lim_{\omega\to 0} j\omega\, F(\omega)$$

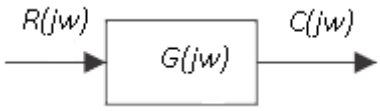
Let t approach infinity for steady-state condition, the response of the system becomes.

$$c(\infty) = \lim_{t\to\infty} r(t)g(t)$$

Taking the Fourier transform on the equation given above.

$$C(\infty) = \lim_{\omega\to 0} j\omega R(\omega)G(\omega)$$

When t approach infinity in time domain for steady-state condition, as a result ω is approaching zero in the frequency domain.



Thus, the response of the system ($c(t)$) when the system in its steady-state condition is:

$$c(\infty) = \lim_{t\to\infty} r(t)g(t)$$

Or

$$C(\infty) = \lim_{\omega\to 0} j\omega R(\omega)G(\omega)$$

Where:

- $G(\omega)$ = System function in frequency domain
- $R(\omega)$ = Reference signal function in frequency domain
- $C(\omega)$ = Output signal function in frequency domain

b. Determine and simulate in MATLAB the steady-state error of the systems given below when the systems are given a step input:

i. For $G_1(\omega) = 1/(j\omega + 0.3)$, knowing the equation for response of the system at steady-state condition is given as:

$$C_1(\infty) = \lim_{\omega\to 0} j\omega R(\omega)G_1(\omega)$$

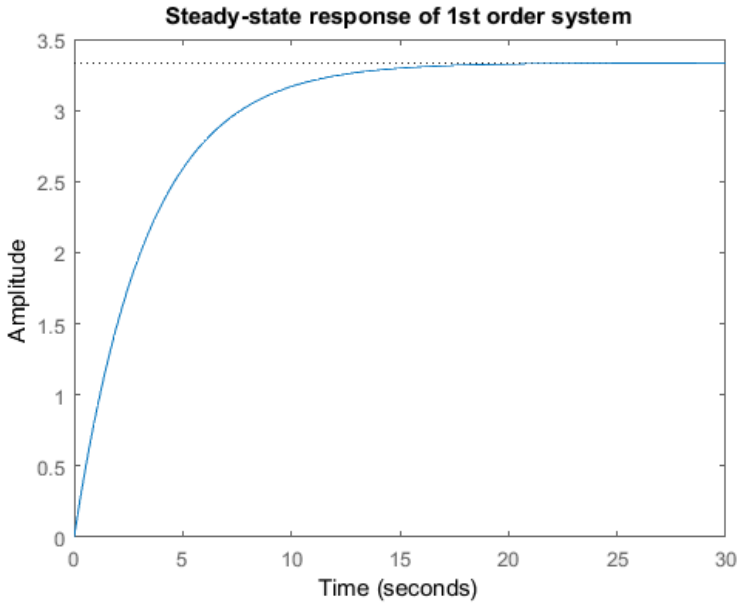
For a step input of $1/j\omega$, put the values and variables into the equation:

$$C_1(\infty) = \lim_{\omega \to 0} (j\omega)\left(\frac{1}{j\omega}\right)\left[\frac{1}{(j\omega + 0.3)}\right]$$

Or

$$C_1(\infty) = \lim_{\omega \to 0} \frac{1}{(j\omega + 0.3)} = \frac{1}{0.3} = 3.33$$

The steady-state response simulation result of the given first-order system in MATLAB is shown in the figure given below.



As shown in the plot, when given a step input, the first-order system settles to a final value of 3.33 instead of 1.

As a result, the steady-state response of the system has a steady state error of:

$$e_1(\infty) = \left(\frac{3.33 - 1}{1}\right) \times 100\% = 233\%$$

ii. For $G_2(\omega) = 1/[0.2(j\omega)^2 + j\omega + 5]$, first, normalise the transfer function equation of the given second order system to unity coefficient format:

$$G_2(\omega) = \frac{1}{[0.2(j\omega)^2 + j\omega + 5]} = \frac{5}{[(j\omega)^2 + 5j\omega + 25]}$$

Knowing the equation for the response of the system at steady-state condition is given as:

$$C_2(\infty) = \lim_{\omega \to 0} j\omega R(\omega) G_2(\omega)$$

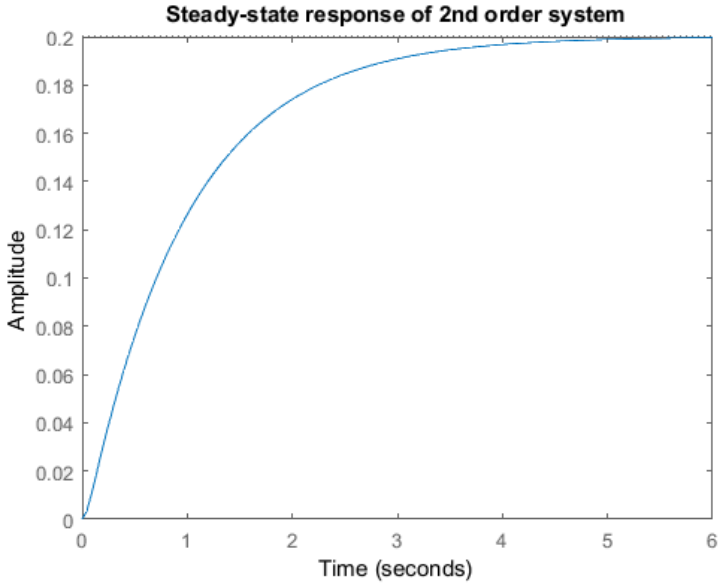
For a step input of $1/j\omega$, put the values and variables into the equation

$$C_2(\infty) = \lim_{\omega \to 0} (j\omega)\left(\frac{1}{j\omega}\right)\left[\frac{5}{(j\omega)^2 + 5j\omega + 25}\right]$$

Or

$$C_2(\infty) = \lim_{\omega \to 0} \frac{5}{(j\omega)^2 + 5j\omega + 25} = \frac{5}{25} = 0.2$$

The steady-state response simulation result of the given second-order system in MATLAB is shown in the figure given below.



As shown in the plot, when given a step input, the first-order system settles to a final value of 0.2 instead of 1.

As a result, the steady-state response of the system has a steady state error of:

$$e_2(\infty) = \left(\frac{0.2 - 1}{1}\right) \times 100\% = -80\%$$

EXTRA

Prove that the time-domain equations of the transient responses of the given second-order system when it is given a step input are as described below.

a. For the critically damped response second-order system, the time-domain equation is: [12 marks]

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \quad \text{for} \quad t \geq 0$$

b. For the overdamped response second-order system, the time-domain equation is: [14 marks]

$$c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-j\omega_1 t}}{j\omega_1} - \frac{e^{-j\omega_2 t}}{j\omega_2}\right) \quad \text{for} \quad t \geq 0$$

Where: $\omega_1 = \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$ and $\omega_2 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$

Solution

a. We know that a second-order system can be represented with the following standardised transfer-function equation.

$$\frac{C(\omega)}{R(\omega)} = \frac{\omega_n^2}{(j\omega + \zeta\omega_n + j\omega_d)(j\omega + \zeta\omega_n - j\omega_d)}$$

Where:

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Thus, in a critically damped second-order system ($\zeta = 1$), the two roots of the $C(\omega)/R(\omega)$ are nearly equal.

$$\omega_d = \omega_n\sqrt{1-(1)^2} = 0$$

As a result, the standardised equation above becomes:

$$\frac{C(\omega)}{R(\omega)} = \frac{\omega_n^2}{(j\omega + \omega_n)(j\omega + \omega_n)}$$

Furthermore, the system may be approximated by a critical damped one:

$$C(\omega) = \left[\frac{\omega_n^2}{(j\omega + \omega_n)^2}\right] R(\omega)$$

For a unit-step input ($R(\omega) = 1/j\omega$) and $C(\omega)$ can be written as:

$$C(\omega) = \frac{\omega_n^2}{(j\omega + \omega_n)^2}\left(\frac{1}{j\omega}\right)$$

Taking the fractional function of the equation given above

$$C(\omega) = \left(\frac{1}{j\omega}\right) - \frac{\omega_n^2}{(j\omega + \omega_n)^2}$$

The inverse transform of the equation given above is:

$$c(t) = 1 - e^{\omega_n t}(1 + \omega_n t)$$

Thus, the transient response ($c(t)$) of the critically damped second-order system is an increase exponential function which is modulated by a time-related function.

b. Knowing that a second-order system can be represented with the following standardised transfer-function equation.

$$\frac{C(\omega)}{R(\omega)} = \frac{\omega_n^2}{(j\omega + \zeta\omega_n + j\omega_d)(j\omega + \zeta\omega_n - j\omega_d)}$$

Where:

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Thus, in overdamped case ($\zeta > 1$), the two roots of $C(\omega)/R(\omega)$ are negative real and unequal. So, the equation above becomes:

$$\frac{C(\omega)}{R(\omega)} = \frac{\omega_n^2}{\left(j\omega + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)\left(j\omega + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)}$$

We know that $j = \sqrt{-1}$, as a result the equation above becomes:

$$C(\omega) = \left[\frac{\omega_n^2}{\left(j\omega + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)\left(j\omega + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)}\right] R(\omega)$$

For a unit-step input $R(\omega) = 1/j\omega$, and $C(\omega)$ can be written as:

$$C(\omega) = \left[\frac{\omega_n^2}{\left(j\omega + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)\left(j\omega + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)}\right]\left(\frac{1}{j\omega}\right)$$

Applying partial fraction to the equation given above:

$$C(\omega) = \left(\frac{1}{j\omega}\right) + \frac{\omega_n^2}{2\left(j\omega + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)} - \frac{\omega_n^2}{2\left(j\omega + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)}$$

The inverse Fourier transform of the equation above is:

$$c(t) = 1 + \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta + \sqrt{\zeta^2-1}\right)} e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}$$
$$- + \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta + \sqrt{\zeta^2-1}\right)} e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

Alternatively, the equation given above can be represented as:

$$c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-j\omega_1 t}}{j\omega_1} - \frac{e^{-j\omega_2 t}}{j\omega_2}\right) \qquad \text{for} \quad t \geq 0$$

Where:

$$\omega_1 = \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n \qquad \text{and} \qquad \omega_2 = \left(\zeta - \sqrt{\zeta^2-1}\right)\omega_n$$

Thus, the transient response of the second-order system ($c(t)$) for an overdamped case ($\zeta > 1$) has two decaying exponential terms.