XMUT220 Signals and Systems

Tutorial 5c: Time and Frequency Characteristics (Frequency Domain Analysis) – Solution

A. Frequency Response of Systems

1. Frequency response analysis has its advantages and disadvantages compared with transient response analysis.
   a. Describe the frequency response analysis.
   b. List and describe types of input used for evaluating the frequency response of the system.
   c. Give an example of frequency response analysis.

Solution

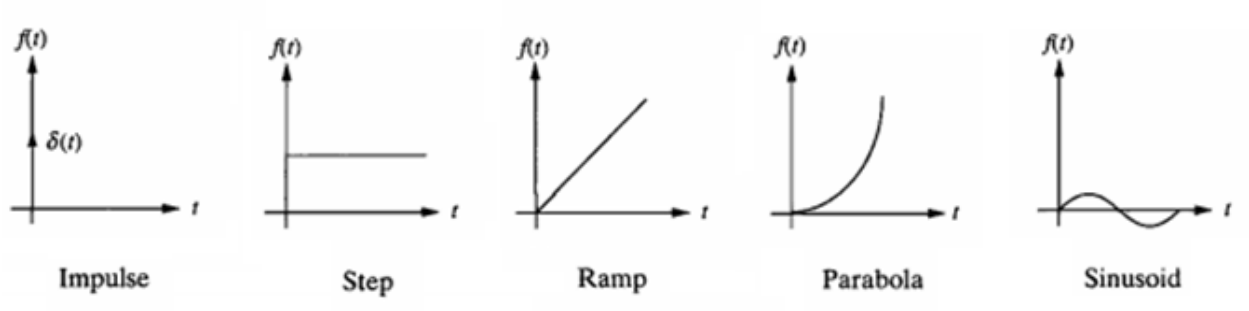
a. Frequency response analysis is:
   - Analysis of response of the system in the frequency domain.
   - Response of the system observed in the output in relation to the input injected to the system.

b. Types of input for evaluating the frequency response of the system:
   - Standard: impulse, step, ramp, parabola and sinusoidal.
   - Others: triangle wave, square wave, exponential, harmonics, etc.

   The following table outlines the details of these test signals.

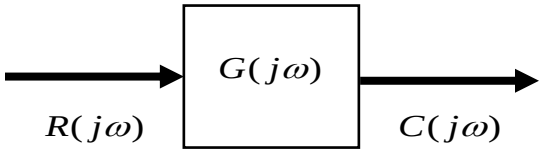
Input	Function	Description	Use
Impulse	$\delta(t)$	$\delta(t) = \infty,\ \text{for } (-)0 < t < (+)0$ $\delta(t) = 0,\quad \text{elsewhere}$	Transient response analysis

Step	$u(t)$	$u(t) = 1,\ \text{for } t > 0$ $u(t) = 0,\ \text{for } t < 0$	Transient response and steady state analysis
Ramp	$tu(t)$	$tu(t) = 1,\ for\ t \geq 0$ $tu(t) = 0,\ \text{elsewhere}$	Steady-state analysis
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2,\ \text{for } t \geq 0$ $\frac{1}{2}t^2u(t) = 0,\ \text{elsewhere}$	Steady-state analysis
Sinusoid	$\sin \omega t$		Transient response and steady state analysis

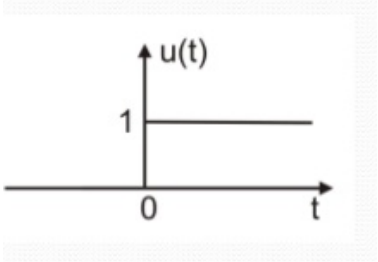


c. Example of a frequency response analysis:

A system ($G(j\omega)$) as shown below, with its input ($R(j\omega)$) and its output ($C(j\omega)$).

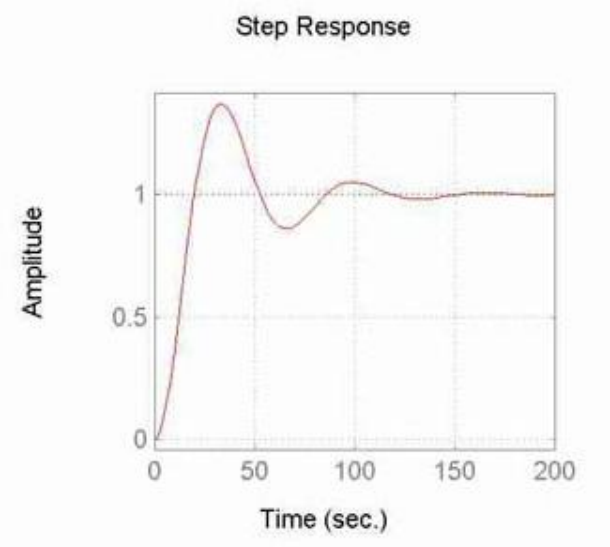


Test input (e.g. a step input $u(t)$) for evaluating the characteristics and behaviour of the system.

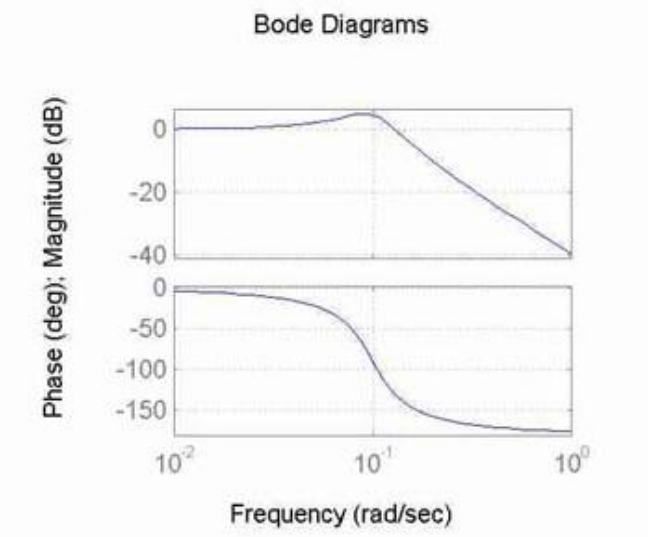


After a test input (e.g. a step input) is given to the input of the system, we measure the transient response (time domain analysis) and frequency responses (Bode plot) are observed at the output of the system.

For time-domain analysis of the system, measure or simulate the transient response of the system.



For frequency-domain analysis of the system, measure or simulate frequency response (magnitude and phase analysis) of the system.



Finally, perform analysis and evaluation of the system based on the measurement or simulation results. The characteristics and behaviour of the system can be determined from the result of the analysis and evaluation e.g. in terms of its transient response, steady-state condition, gain/magnitude, phase shift, etc.

2. Frequency response analysis has its advantages and disadvantages compared with transient response analysis.

   a. List and describe a simple heuristic for performing the frequency response analysis using Bode plots.

   b. Describe briefly at least three examples of application of frequency response analysis.

Solution

a. The following steps are a simple heuristic to perform frequency response analysis (i.e. Bode plots):

   i. Derivation of the response of the system from the physical system.

   ii. Derivation of transfer function equation of the system.

   iii. Determining the equations of the frequency response of the system (magnitude and phase equations).

   iv. Plotting for frequency response of the system:

   - Magnitude plot.
   - Phase plot.

   v. Description of the characteristics of the frequency response of the system.

b. Brief descriptions of application of frequency response analysis:

- Evaluation of performance of a receiver in a digital communication system – the performance of the device can be measured through its frequency response, bandwidth, gain and phase shift.
- Analysis of filter circuit – the characteristics and behaviour of the filter circuit can be done through analysis of relevant parameters of the filter circuit in the frequency domain e.g. cut-off frequency, operating bandwidth, quality factor, etc.
- Design of control system – measurement of the performance of the system in frequency domain and its subsequent system and controller design for optimal application of the control system.

3. Given a first-order system with the following differential equation given below.

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

a. Derive the equations for determining the magnitude and phase of the frequency response of the system.

b. What is break frequency? For given first-order system, determine its break frequency.

Solution

a. Derivation of the equations for determining the magnitude and phase of the frequency response of the first-order system is as follows.

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

Taking the Fourier transform of the differential equation given above:

$$(j\omega)Y(j\omega) + aY(j\omega) = X(j\omega)$$

The equation for frequency response of the first-order system is:

$$H(j\omega) = \frac{X(j\omega)}{Y(j\omega)} = \frac{1}{(j\omega) + a}$$

The equation for gain of the frequency response of the first-order system is:

$$|H(j\omega)| = \frac{1}{\sqrt{(j\omega)^2 + a^2}}$$

The equation for phase of the frequency response of the first-order system is:

$$\angle H(j\omega) = -\tan^{-1}\left(\frac{j\omega}{a}\right)$$

b. Break frequency is frequency of the system when the gain of the system is equal to $1/\sqrt{2}$.

The break frequency of the given first-order system is:

$$|H(j\omega)| = \frac{1}{\sqrt{(j\omega)^2 + a^2}} = \frac{1}{\sqrt{2}}$$

Thus, the equation for break frequency of first-order system is:

$$j\omega = \sqrt{2 - a^2}$$

4. Given a second-order system with the following differential equation given below.

$$\frac{d^2y(t)}{dt^2} + (2\zeta\omega_n)\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

a. Derive the equations for determining the magnitude and phase of the frequency response of the system.

b. Describe the different characteristics of the break frequency in a second-order system.

Solution

a. Derivation of the equations for determining the magnitude and phase of the frequency response of the second-order system is as follows.

Knowing the standardised form of second-order differential equation is given as:

$$\frac{d^2y(t)}{dt^2} + (2\zeta\omega_n)\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

Taking the Fourier transform of the differential equation given above:

$$(j\omega)^2 Y(j\omega) + 2\zeta\omega_n Y(j\omega) + \omega_n^2 Y(j\omega) = \omega_n^2 X(j\omega)$$

The equation for frequency response of the second-order system is:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}$$

Thus

$$H(j\omega) = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n(\omega) + \omega_n^2}$$

Normalise the equation with ω_n^2

$$H(j\omega) = \frac{1}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + j\left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]}$$

The equation for gain of the frequency response of the second-order system is:

$$|H(j\omega)| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2}}$$

The equation for phase of the frequency response of the second-order system is:

$$\angle H(j\omega) = -\tan^{-1}\left[\frac{\left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}\right]$$

Or

$$\angle H(j\omega) = -\tan^{-1}\left[\frac{2\zeta(\omega)\omega_n}{\omega_n^2-\omega^2}\right]$$

b. Break frequency in a second order system depends on the nature of its damping condition. Because of this reason, there are various break frequencies of a second-order system:

- Overdamped second-order system: There are two real break frequencies due to unequal and real roots of overdamped second-order system.
- Critically damped second-order system: There is one break frequency due to equal and real roots of critically second-order system.
- Underdamped second-order system: There are two complex pair of break frequencies due to complex pair of roots of underdamped second-order system. The break frequency is actually the modulus of these complex pair.

5. For discrete-time systems, their frequency response characteristics are very similar to their continuous system counterparts.

a. Derive the equation determining the frequency response for discrete first-order system given below. Determine also the equations for its impulse and step responses.

$$y[n] - ay[n-1] = x[n]$$

Where: IaI < 1.

b. Derive the equation for determining a discrete second-order system as shown below. Determine also the equations for its impulse and step responses.

$$y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$$

Where: $0 < r < 1$ and $0 \leq \theta \leq \pi$.

Solution

a. The equations for the given discrete first-order system are as follows:

$$y[n] - ay[n-1] = x[n]$$

Applying z-transform on the sequence given above:

$$Y[e^{j\omega}] - ae^{-j\omega}Y[e^{j\omega}] = X[e^{j\omega}]$$

Rearranging the equation above

$$\frac{Y[e^{j\omega}]}{X[e^{j\omega}]} = \frac{1}{1 - ae^{-j\omega}}$$

The equations for the frequency response of the discrete first-order system:

- The equation for the frequency response of the discrete first-order system is:

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

- The equation for an impulse response of the system is:

$$h[n] = a^n u[n]$$

- The equation for a step response of the system is:

$$s[n] = h[n] * u[n] = \left[\frac{1 - a^{n+1}}{1 - a}\right] u[n]$$

b. The equations for the given discrete second-order system are determined as follows.

$$y[n] - 2r\cos\theta\, y[n-1] + r^2 y[n-2] = x[n]$$

Applying z-transform on the equation given above:

$$Y[e^{j\omega}] - 2r\cos\theta\, e^{-j\omega}Y[e^{j\omega}] + r^2 e^{-j2\omega}Y[e^{j\omega}] = X[e^{j\omega}]$$

Rearranging the equation given above:

$$\frac{Y[e^{j\omega}]}{X[e^{j\omega}]} = \frac{1}{1 - 2r\cos\theta\, e^{-j\omega} + r^2 e^{-j2\omega}}$$

The equations for the frequency response of the discrete second-order system:

- The frequency response for this system is:

$$H(e^{j\omega}) = \frac{1}{1 - 2r\cos\theta e^{-j\omega} + r^2 e^{-j2\omega}}$$

- The denominator of $H(e^{j\omega})$ can be factored to obtain (note: Euler law of $\cos\theta = (e^{j\theta} + e^{-j\theta})/2$:

$$H(e^{j\omega}) = \frac{1}{[1 - (re^{j\theta})e^{-j\omega}][1 - (re^{-j\theta})e^{-j\omega}]}$$

- For $\theta \neq 0$ or π, the two factors in the denominator of $H(e^{j\omega})$ are different.

- By applying a partial-fraction expansion, this yields:

$$H(e^{j\omega}) = \frac{A}{1-(re^{j\theta})e^{-j\omega}} + \frac{B}{1-(re^{-j\theta})e^{-j\omega}}$$

Where:

$$A = \frac{e^{j\theta}}{2j\sin\theta}, \qquad B = \frac{e^{-j\theta}}{2j\sin\theta}$$

The equation for an impulse response of the discrete second-order system:

- In this case, the impulse response of the system is:

$$h[n] = \left[A(re^{j\theta})^n + B(re^{-j\theta})^n\right]u[n]$$

Where:

$$A = \frac{e^{j\theta}}{2j\sin\theta}, \qquad B = \frac{e^{-j\theta}}{2j\sin\theta}$$

- Knowing that Euler law of $\sin\theta = (e^{j\theta} - e^{-j\theta})/2j$:

$$h[n] = r^n\left\{\frac{\sin[(n+1)\theta]}{\sin\theta}\right\}u[n]$$

- For $\theta = 0$ or π, the two factors in the denominator of transfer function equation of the system are the same.
- When $\theta = 0$,

$$H(e^{j\omega}) = \frac{1}{(1-re^{-j\omega})^2}$$

Or

$$h[n] = (n+1)r^n u[n]$$

- When $\theta = \pi$,

$$H(e^{j\omega}) = \frac{1}{(1+re^{-j\omega})^2}$$

Or

$$h[n] = (n+1)(-r)^n u[n]$$

The equation for a step response of the discrete second-order system:

- For $\theta \neq 0$ or π,

$$s[n] = h[n] * u[n] = \left[\left(\frac{1-\left(re^{j\theta}\right)^{n+1}}{1-re^{j\theta}}\right)+\left(\frac{1-\left(re^{-j\theta}\right)^{n+1}}{1-re^{-j\theta}}\right)\right]u[n]$$

- For $\theta = 0$,

$$s[n] = \left[\frac{1}{(r-1)^2} - \frac{r}{(r-1)^2}(r)^n + \frac{r}{r-1}(n+1)(r)^n\right]u[n]$$

- For $\theta = \pi$,

$$s[n] = \left[\frac{1}{(r+1)^2} + \frac{r}{(r+1)^2}(-r)^n + \frac{r}{r+1}(n+1)(-r)^n\right]u[n]$$

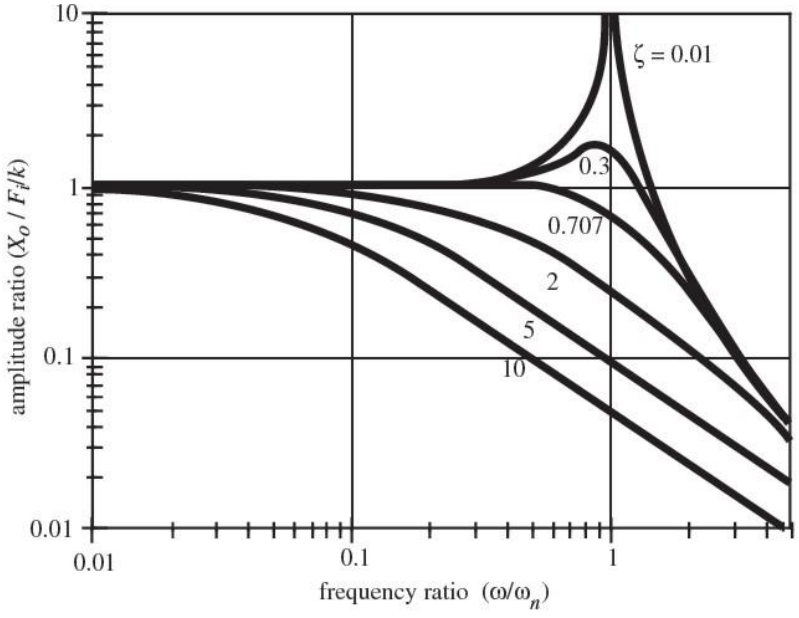
6. Peak is a phenomenon when an underdamped second order system experiences a resonance condition occurring at its peak frequency. See MATLAB file in the Appendix for its simulation.

   a. Derive the equation for determining the resonance frequency of the system at the peak of its frequency response.

   b. Derive the equations for determining the gain of the system given at the peak of its frequency response.

   c. Derive the equations for determining the phase of the system given at the peak of its frequency response.

   d. Describe bandwidth of the given system. Derive the equation for the bandwidth of the system.



Solution

a. Derivation of the equations for determining the resonance frequency of the second-order system at the peak of its frequency response is as follows.

Knowing the equation for gain of the frequency response of the second-order system is:

$$|H(j\omega)| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2}}$$

Simplifying the equation above for gain at the peak of frequency response with $u = \omega/\omega_n$, the equation becomes:

$$M = |H(j\omega)| = \frac{1}{\sqrt{[1-(u)^2]^2 + \left(2\zeta(u)\right)^2}} \quad \text{or} \quad \left\{[1-(u)^2]^2 + \left(2\zeta(u)\right)^2\right\}^{-1/2}$$

Where: $u = \omega/\omega_n$

At peak of the frequency response ($\omega = \omega_r$), the system experiences a resonance condition e.g. a maximum (peak) magnitude.

So, at ($\omega = \omega_r$), the first derivate of the magnitude of M is zero e.g. $dM/du = 0$. Then, differentiate M with respect to u:

$$\frac{dM}{du} = -\frac{1}{2}[(1-u^2)^2 + (2\zeta u)^2]^{-3/2}[2(1-u^2)(-2u) + 2(2\zeta u)(2\zeta)]$$

Simplifying the equation given above:

$$\frac{dM}{du} = -\frac{1}{2}\{[1-u^2]^2 + (2\zeta u)^2\}^{-3/2}[4u(u^2 - 1 + 2\zeta^2)]$$

At resonance condition, substitute $u = u_r$ and $dM/du = 0$ in the above equation:

$$-\frac{1}{2}[(1-u_r^2)^2 + (2\zeta u_r)^2]^{-\frac{3}{2}}[4u_r(u_r^2 - 1 + 2\zeta^2)] = 0$$

Thus

$$4u_r(u_r^2 - 1 + 2\zeta^2) = 0$$

Then

$$u_r^2 - 1 + 2\zeta^2 = 0$$

So, at resonance frequency

$$u_r = \sqrt{1 - 2\zeta^2}$$

Since $u_r = \omega_r/\omega_n$, the resonance frequency (ω_r) is:

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

Where: $0 \leq \zeta \leq 1/\sqrt{2}$

b. Derivation of the equations for determining the gain of the second-order system at the peak of its frequency response is as follows.

The resonant peak is the peak (maximum) value of the magnitude of $H(j\omega)$. It is denoted by M_r. At $u = u_r$, the magnitude of $H(j\omega)$ is:

$$M_r = \frac{1}{\sqrt{(1-u_r^2)^2 + (2\zeta u_r)^2}}$$

Substitute $u_r = \sqrt{1-2\zeta^2}$ and $1 - u_r^2 = 2\zeta^2$ in the above equation

$$M_r = \frac{1}{\sqrt{(2\zeta^2)^2 + \left(2\zeta\sqrt{1-2\zeta^2}\right)^2}}$$

Thus, the equation for gain at the peak of frequency response ($\omega = \omega_r$) is:

$$M_r = \frac{1}{2\zeta\sqrt{1-2\zeta^2}}$$

The equation for phase at the peak of frequency response ($\omega = \omega_r$) is:

$$\angle H(j\omega) = -\tan^{-1}\left[\frac{2\zeta(\omega_r)\omega_n}{\omega_n^2 - \omega_r^2}\right]$$

Where: ω_r = resonance frequency.

c. Derivation of the equations for determining the phase of the second-order system at the peak of its frequency response is as follows.

The equation for phase of frequency response of the system is:

$$\angle H(j\omega) = -\tan^{-1}\left[\frac{\left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)}\right]$$

During resonance condition, at peak frequency $\omega = \omega_r$, the phase shift of the system (Φ) is:

$$\Phi = -\tan^{-1}\left[\frac{\left(2\zeta\left(\frac{\omega_r}{\omega_n}\right)\right)}{\left(1-\left(\frac{\omega_r}{\omega_n}\right)^2\right)}\right]$$

d. Bandwidth of the second-order system is the range of frequency over which the magnitude of the frequency response of the system drops to 0.707 from its zero frequency value.

Knowing the equation for determining the magnitude of the peak of the frequency response of the second-order system is:

$$M_r = \frac{1}{\sqrt{(1-u_r^2)^2 + (2\zeta u_r)^2}}$$

At $\omega = 0$, the value of u will be zero. Substitute $u = 0$, in M:

$$M = \frac{1}{\sqrt{(1-(0)^2)^2 + (2\zeta(0))^2}} = 1$$

Therefore, the magnitude of frequency response of the system is one at $\omega = 0$.

At 3-dB frequency, the magnitude of the frequency response of the system will be 0.707 of the magnitude of the frequency response of the system at $\omega = 0$. For example, at $\omega = \omega_B$, $M = 0.707(1) = 1/\sqrt{2}$.

$$M = \frac{1}{\sqrt{2}} = M_r = \frac{1}{\sqrt{(1-u_b^2)^2 + (2\zeta u_b)^2}}$$

Thus

$$2 = \left(1-u_b^2\right)^2 + (2\zeta u_b)^2$$

Let $u_b^2 = x$

$$2 = (1-x)^2 + (2\zeta)^2 x$$

Thus

$$x^2 + (4\zeta^2 - 2)x - 1 = 0$$

Then

$$x_{1,2} = \frac{-(4\zeta^2 - 2) \pm \sqrt{(4\zeta^2 - 2)^2 + 4}}{2}$$

Consider only the positive value of x:

$$x = 1 - 2\zeta^2 + \sqrt{(2\zeta^2 - 1)^2 + 1}$$

So

$$x = 1 - 2\zeta^2 + \sqrt{(2 - 4\zeta^2 + 4\zeta^4)}$$

Substitute $x = u_b^2 = \omega_b^2/\omega_n^2$

$$\omega_b^2/\omega_n^2 = 1 - 2\zeta^2 + \sqrt{(2 - 4\zeta^2 + 4\zeta^4)}$$

Thus

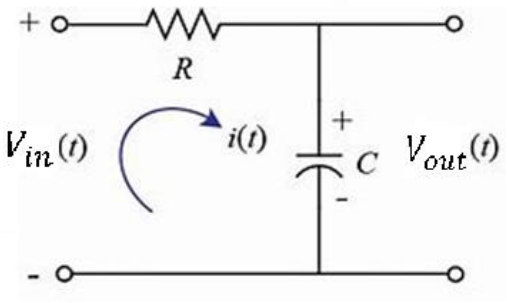
$$\omega_b = \omega_n\sqrt{1 - 2\zeta^2 + \sqrt{(2 - 4\zeta^2 + 4\zeta^4)}}$$

Bandwidth ω_b in the frequency response is inversely proportional to the rise time (T_r) in the time domain transient response.

$$T_r = \frac{1}{\omega_n\sqrt{1 - 2\zeta^2 + \sqrt{(2 - 4\zeta^2 + 4\zeta^4)}}}$$

B. Frequency Response Analysis of First Order System

7. For the following continuous time systems, perform the tasks listed below. Suppose the continuous-time system is represented as follows:
   - A resistor and capacitor in series, with the output voltage across *C* defined as $V_{out}(t)$.
   - Input voltage is a voltage source defined as $V_{in}(t)$.
   - Current ($i(t)$) flowing in the circuit.



   a. Derive equation for transient response of the system.
   b. Derive equations for the magnitude and phase of the frequency response of the system.
   c. Determine the frequency response of the system when R = 100 kΩ and C = 10 uF.
   d. Determine the points for drawing the Bode plot of magnitude (gain) response of the system.
   e. Drawing the Bode plot of magnitude (gain) response of the system.
   f. Simulate in MATLAB the Bode plot of magnitude (gain) response of the system.
   g. Describe the frequency response characteristics (magnitude) of the system at the critical frequencies.

h. Determine the points for drawing the Bode plot of phase response of the system.

i. Drawing the Bode plot of phase response of the given system.

j. Simulate in MATLAB the Bode plot of phase response of the system.

k. Describe the frequency response characteristics (phase) of the system at the critical frequencies.

Solution

a. First of all, for the given circuit, determine the impedances in the circuit:

$$Z_R = R$$

$$Z_c = \left(\frac{1}{j\omega C}\right)$$

Derive the response of the system e.g. for a series circuit the voltage across the voltage source is the sum of the voltage across the capacitor and the resistor

$$V_{out}(t) = \left(\frac{Z_c}{Z_R + Z_c}\right) V_{in}(t)$$

Derive transfer function of the system. Rearranging the equation given above, taking into account $Z_c = 1/j\omega C$ and $Z_R = R$, as a result:

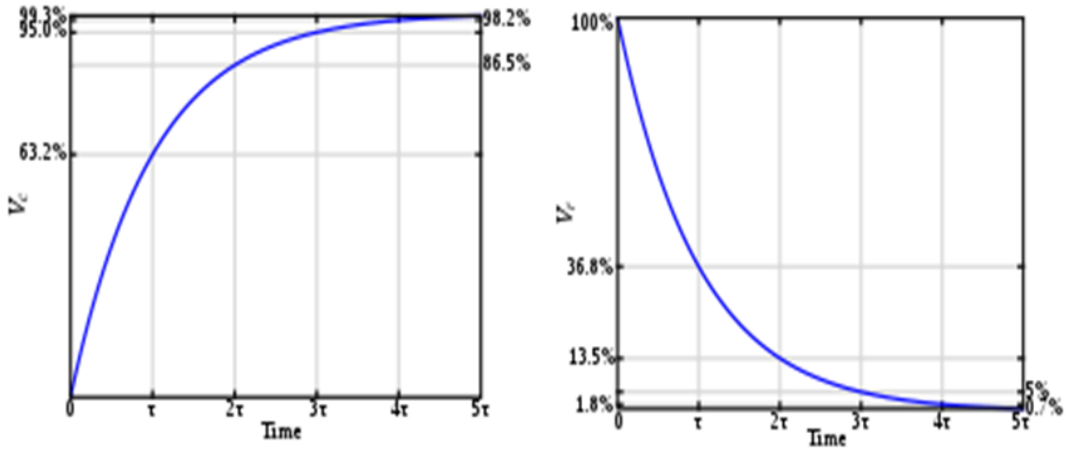
$$V_{out}(j\omega) = \left[\frac{\frac{1}{j\omega C}}{R + \left(\frac{1}{j\omega C}\right)}\right] V_{in}(j\omega)$$

Hence the transfer function of the system is:

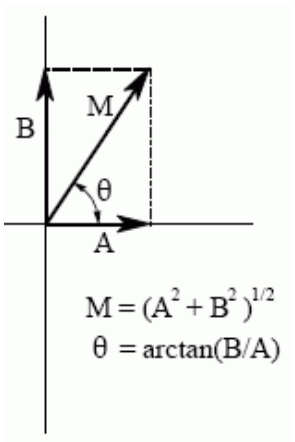
$$G(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{1}{1 + j\omega RC}$$

Taking inverse transform of the equation above will give the transient response of the system:

$$V_{out}(t) = V_{in} e^{-\frac{t}{RC}}$$



b. As a result, the magnitude and phase of frequency response of the system are:



Magnitude of the frequency response:

$$|G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}}$$

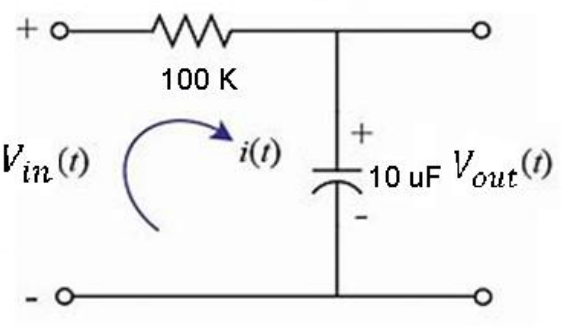
Thus

$$\angle G(j\omega) = -\tan^{-1}(\omega RC/1)$$

So, the phase shift of the response of the system:

$$\theta = -\tan^{-1}(\omega RC)$$

c. Determine the frequency response of the given first order system when R = 100 kΩ and C = 10 uF.



Thus, the transfer function equation of the given first order system becomes:

$$G(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega}$$

Determine the magnitude of frequency response of the system (gain).

$$|G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2 R^2 C^2}} = \frac{1}{\sqrt{1 + \omega^2}}$$

Determine the phase of the frequency response of the system.

$$\theta = -\tan^{-1}(\omega RC) = -\tan^{-1}(\omega)$$

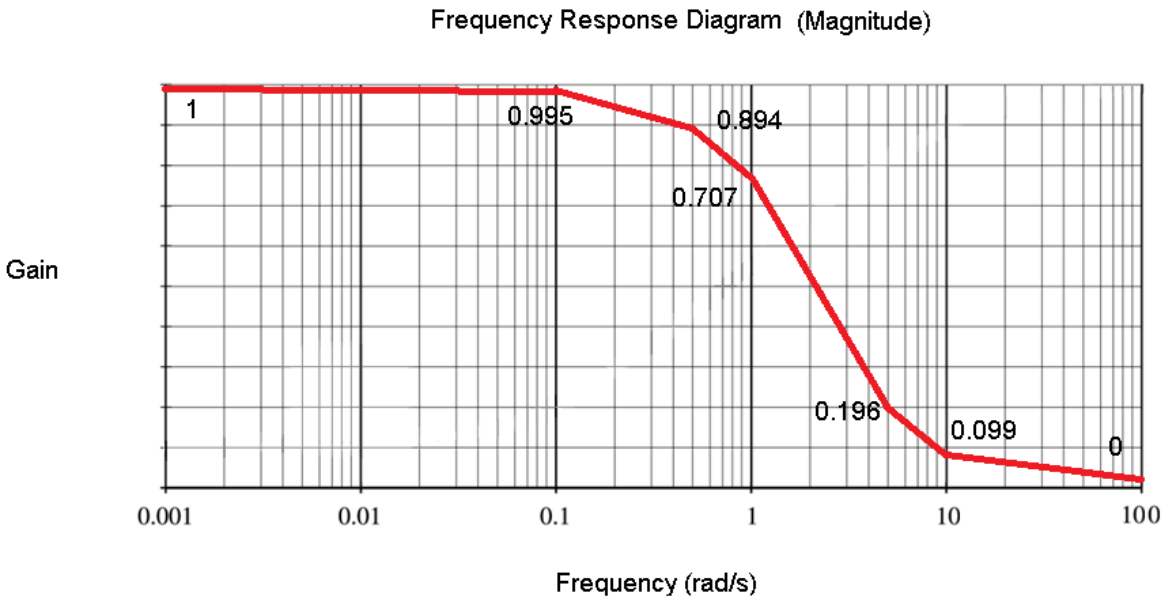
d. Knowing the magnitude (gain) of frequency response of the system is calculated from:

$$|G(j\omega)| = \frac{1}{\sqrt{1 + \omega^2}}$$

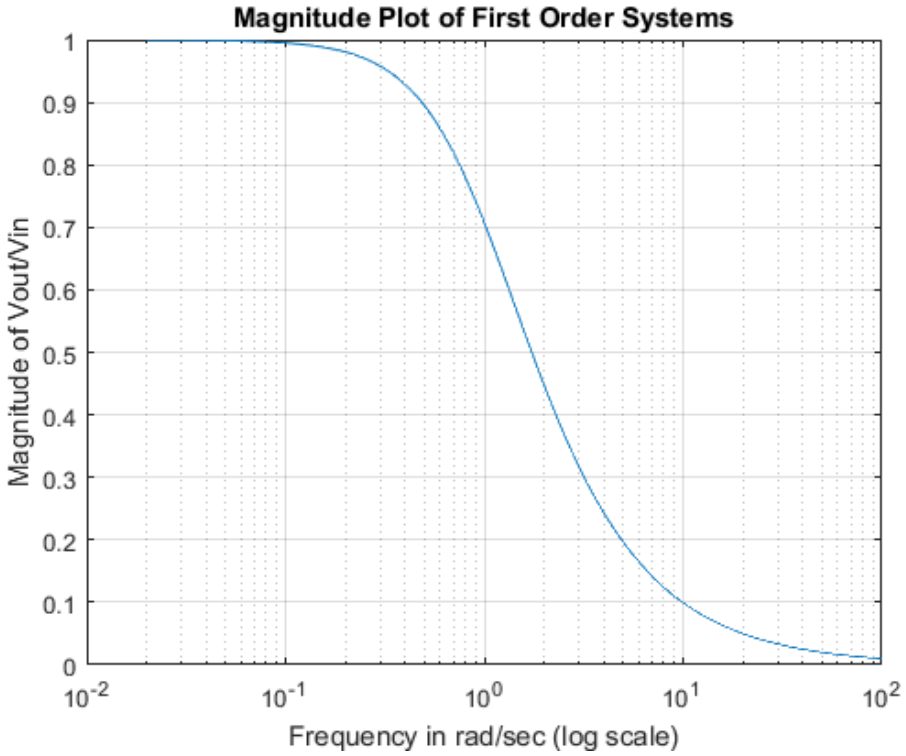
Determine the points for drawing the Bode plot of magnitude (gain) response of the given first order system.

$\boldsymbol{\omega}$ $\mathbf{(rad/s)}$	$\boldsymbol{\lvert G(j\omega)\rvert}$	$\mathbf{20 \log \lvert G(j\omega)\rvert}$ $\mathbf{(dB)}$
0	1	0
0.1	0.995	-0.044
0.5	0.894	-0.973
1	0.707	-3
5	0.196	-14.155
10	0.099	-20

e. Drawing the Bode plot of magnitude (gain) response of the given first order system.



f. MATLAB simulation of the Bode plot of magnitude (gain) response of the given first order system.



g. Describe the frequency response characteristics (magnitude) of the system at the critical frequencies.

- As ω -> 0, the magnitude of the frequency response is

$$|G(0)| \cong 1$$

- For ω=1/RC, the magnitude of the frequency response is

$$|G(1/RC)| \cong \frac{1}{\sqrt{2}} = 0.707$$

- As ω -> ∞, the magnitude of the frequency response is

$$|G(\infty)| \cong 0$$

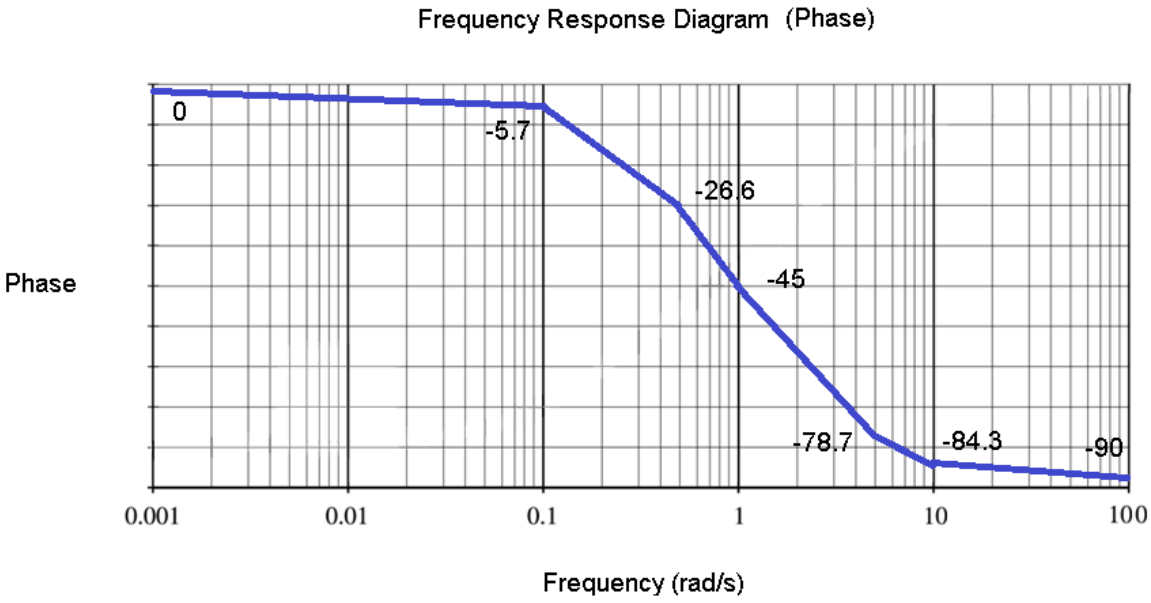
h. Knowing the phase shift of the frequency response of the system is calculated from:

$$\theta = -\tan^{-1}(\omega)$$

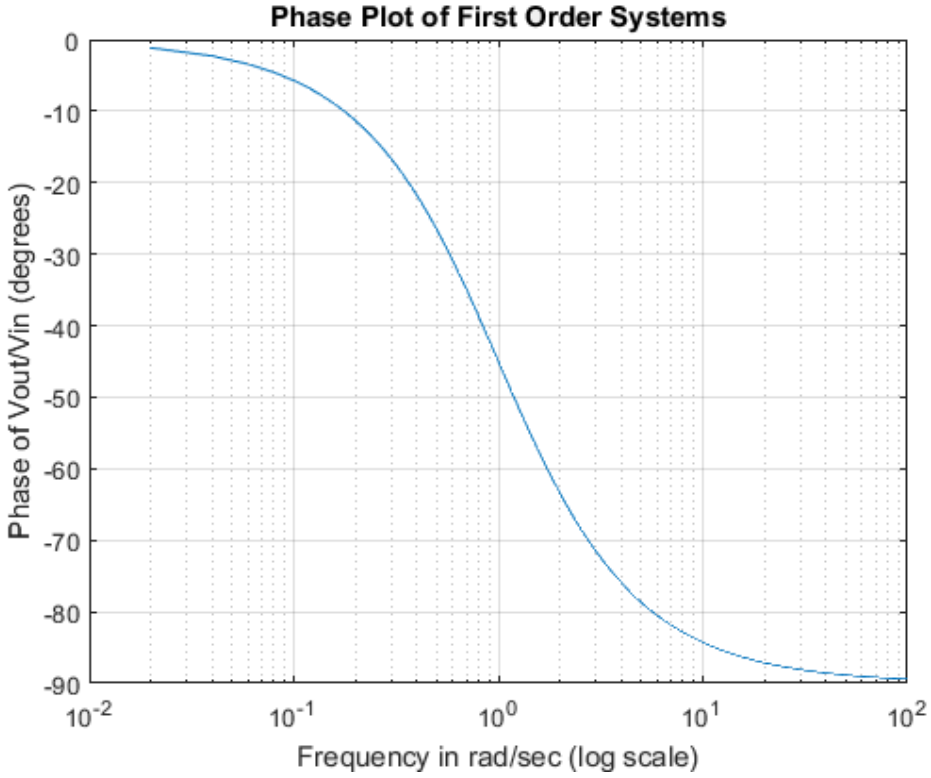
Determine points for the Bode plot of phase response of the given first order system.

$\boldsymbol{\omega}$ $\mathbf{(rad/s)}$	$\boldsymbol{\theta}$ (°)	$\boldsymbol{\theta}$ **(radian)**
0	0	0
0.1	-5.7	-π/14
0.5	-26.6	-π/6.77
1	-45	-π/4
5	-78.7	-π/2.29
10	-84.3	-π/2.14

i. Draw the Bode plot of phase response of the given first order system.



j. MATLAB simulation of Bode plot of phase response of the given first order system.



k. Describe the frequency response characteristics (phase) of the first-order system at the critical frequencies.

- As ω -> 0, the phase shift of the frequency response is

$$\theta = -\tan^{-1}(0) = 0°$$

- For ω = 1/RC, the phase shift of the frequency response is

$$\theta = -\tan^{-1}(1) = -45°$$

- As ω -> ∞ the phase shift of the frequency response is

$$\theta = -\tan^{-1}(\infty) = -90°$$

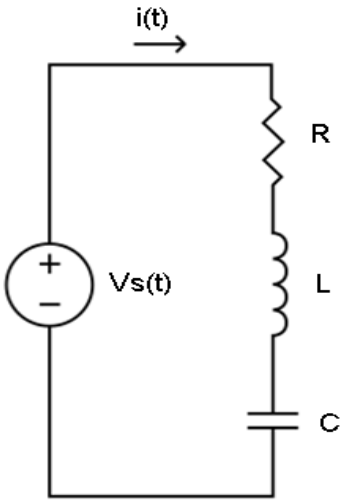
From the results of magnitude and phase plots, it is a first order system that has a low-pass filter characteristic (e.g. it allows low frequency signals to pass through).

C. Frequency Response Analysis of Second Order System

8. For the following systems, find the magnitude and phase equations of their frequency responses. Suppose the continuous-time second-order system represented as follows.

   Components that make up the circuit:

   - Voltage source ($v_s(t)$) across RLC.
   - Current ($i(t)$) flowing in the circuit
   - Passive electrical components in series (e.g. resistor (R), inductor (L) and capacitor (C)).



For the given second order RLC circuit, perform the following tasks:

a. Derive the equation for transient response of the system.

b. Derive equations for determining the magnitude and phase frequency response of the system.

c. Derive transfer function of the system when $R = L = C = 1$.

d. Determining points for drawing the Bode plots of magnitude (gain) response of the system.

e. Draw, simulate and describe at the critical frequencies the Bode plots of magnitude (gain) response of the system.

f. Determining points for drawing the Bode plots of phase response of the system.

g. Draw, simulate and describe at the critical frequencies the Bode plots of phase response of the system.

Solution

a. For the given RLC circuit, first of all, derive equations for the transient response of the system.

Voltage across RLC is equal to voltage source as the total voltage across resistor, inductor and capacitor:

$$v_s(t) = v_{LCR}(t) = v_R(t) + v_L(t) + v_C(t)$$

Voltage across each of components:

Resistor voltage	Inductor voltage	Capacitor voltage
$v_R(t) = Ri(t)$	$v_L(t) = L\dfrac{di(t)}{dt}$	$v_C(t) = \dfrac{1}{C}\int_0^t i(\tau)d\tau$

Equating the voltages across resistor, inductor and capacitor:

$$v_{LCR}(t) = Ri(t) + L\left(\frac{di(t)}{dt}\right) + \frac{1}{C}\int_0^t i(\tau)d\tau + v(0)$$

Assume that the circuit is from scratch (e.g. no initial residual, so $v(0) = 0$) then:

$$v_{LCR}(t) = Ri(t) + L\left(\frac{di(t)}{dt}\right) + \frac{1}{C}\int_0^t i(\tau)d\tau$$

Applying transform to the equation above:

$$v_{LCR}(j\omega) = \left(R + j\omega L + \frac{1}{j\omega C}\right) I(j\omega)$$

Derive transfer function of the system (in terms of voltage across resistor). From the circuit:

$$V_{in}(j\omega) = v_{LCR}(j\omega) = \left(R + j\omega L + \frac{1}{j\omega C}\right) I(j\omega)$$

$$V_{out}(j\omega) = v_R(j\omega) = RI(j\omega)$$

Rearranging the equation of the circuit for V_{out}/V_{in}:

$$\frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \left(\frac{R}{j\omega L + \frac{1}{j\omega C} + R}\right)$$

As a result, the transfer function ($G(j\omega)$) for the circuit:

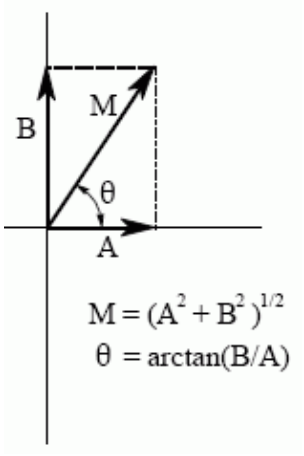
$$G(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)} = \frac{R}{j\omega L + \frac{1}{j\omega C} + R}$$

Rearrange the equation given above:

$$G(j\omega) = \frac{j\omega RC}{(j\omega)^2 LC + j\omega RC + 1}$$

$$= \frac{j\omega(R/L)}{-(\omega)^2 + j\omega(R/L) + 1/LC}$$

b. The magnitude and phase of frequency response of the system are:



The magnitude of frequency response:

$$|G(j\omega)| = \frac{\omega\left(\frac{R}{L}\right)}{\sqrt{\left[\frac{1}{LC} - \omega^2\right]^2 + \left[\omega\left(\frac{R}{L}\right)\right]^2}}$$

The phase of frequency response:

$$\angle G(j\omega) = \tan^{-1}\frac{\omega\left(\frac{R}{L}\right)}{0} - \tan^{-1}\frac{\left[\omega\left(\frac{R}{L}\right)\right]}{\left[\frac{1}{LC} - \omega^2\right]}$$

Thus, phase shift (θ) of the frequency response is:

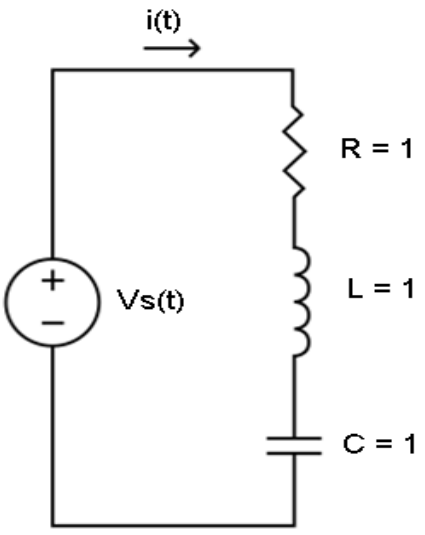
$$\theta = \frac{\pi}{2} - \tan^{-1}\left[\frac{\omega CR}{(1 - LC\omega^2)}\right]$$

c. Derive transfer function of the system when (for convenience) $R = L = C = 1$.

$$G(j\omega) = \frac{j\omega(R/L)}{-(\omega)^2 + j\omega(R/L) + 1/LC}$$

As a result, the transfer function of the circuit becomes:

$$G(j\omega) = \frac{j\omega}{-\omega^2 + j\omega + 1}$$



Determine equation for the magnitude of frequency response of the system.

$$|G(j\omega)| = \frac{\omega}{\sqrt{(1 - \omega^2)^2 + (\omega)^2}}$$

Determine the phase of frequency response of the system.

$$\theta = \tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left[\frac{\omega}{(1-\omega^2)}\right] = \frac{\pi}{2} - \tan^{-1}\left[\frac{\omega}{(1-\omega^2)}\right]$$

d. Knowing the magnitude (gain) of the frequency response of the system is calculated from:

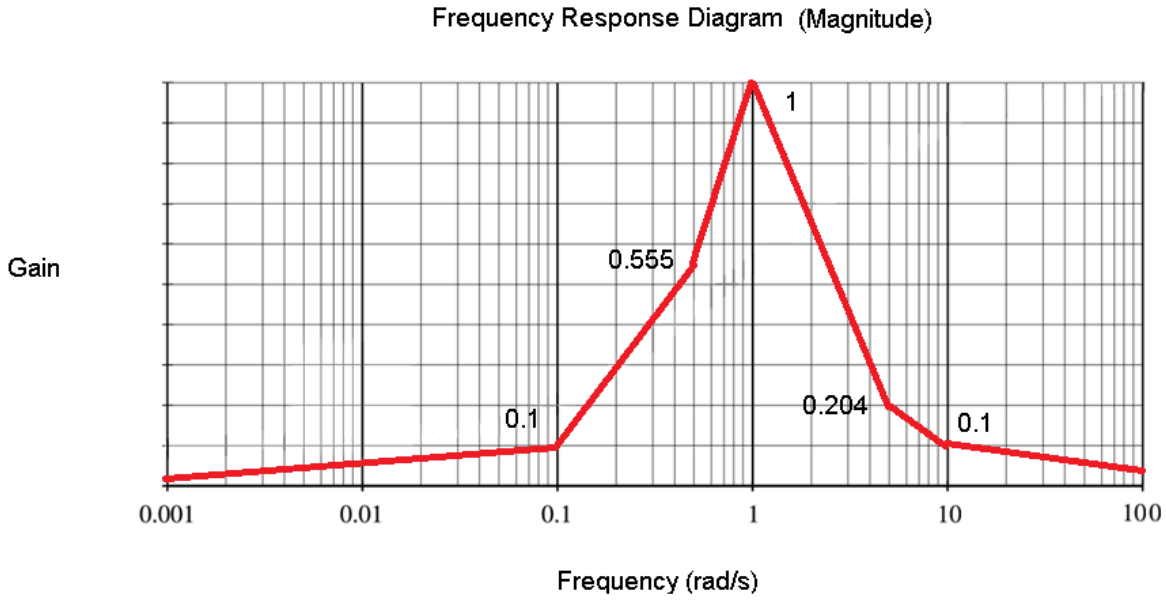
$$|G(j\omega)| = \frac{\omega}{\sqrt{(1-\omega^2)^2 + (\omega)^2}}$$

Determining points for drawing the Bode plots of magnitude (gain) response of the given second order system.

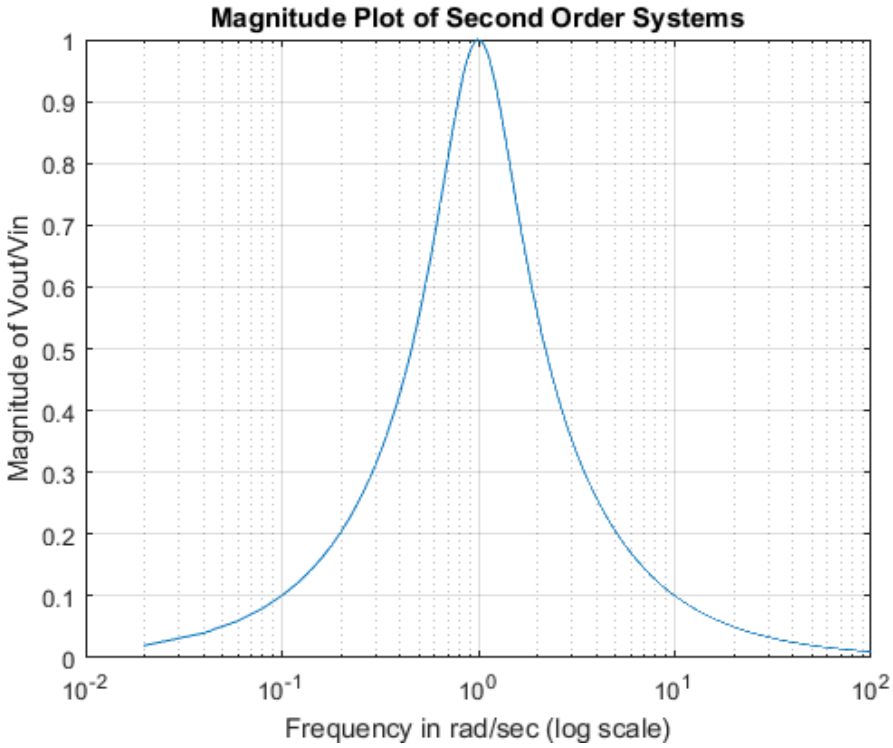
$\boldsymbol{\omega}$ $(\mathbf{rad/s})$	$\boldsymbol{\lvert G(j\omega)\rvert}$	$\mathbf{20\log\lvert G(j\omega)\rvert}$ $(\mathbf{dB})$
0	0	0
0.1	0.1	-20
0.5	0.555	-5.11
1	1	0
5	0.204	-13.8
10	0.1	-20

e. Draw, simulate and describe the Bode plots of magnitude (gain) response of the given second order system.

MATLAB simulation of the Bode plots of magnitude (gain) response of the given second order system.



MATLAB simulation of the Bode plots of magnitude (gain) response of the given second order system.



Describe the frequency domain characteristics (magnitude) of this system at the critical frequencies:

- The gain of this system is -> 0 at low and high frequencies.
- The frequency at ω = 1 is named cut-off frequency (F_c).
- At the cut off frequency, its phase angle is 0.

f. Knowing the phase shift of the frequency response of the system is calculated from:

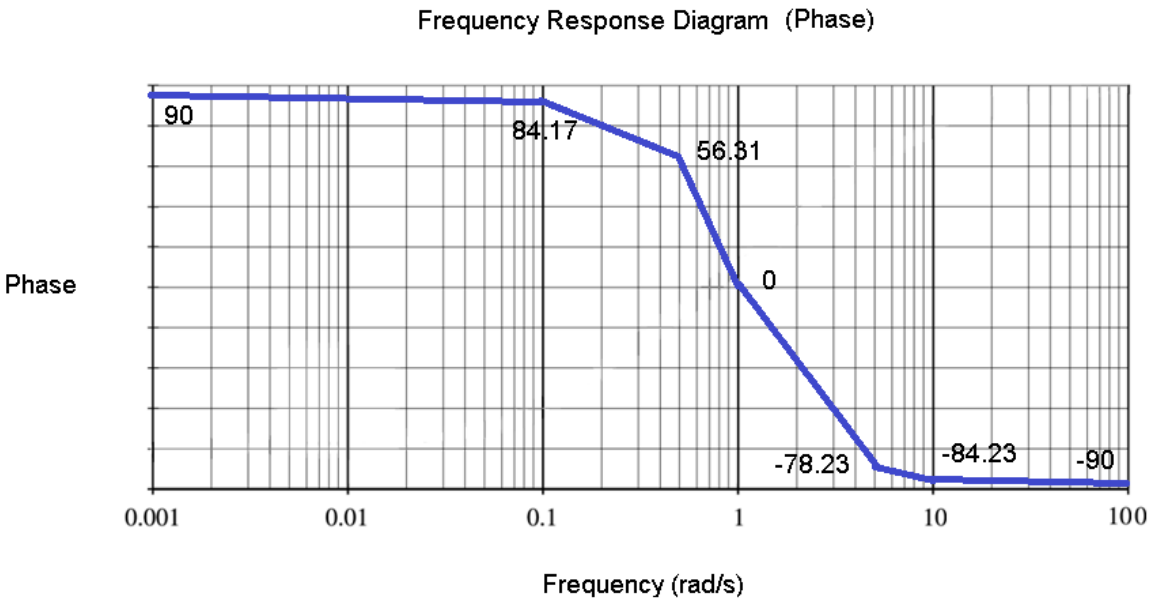
$$\theta = \frac{\pi}{2} - \tan^{-1}\left[\frac{\omega}{(1-\omega^2)}\right]$$

Determine the points for drawing Bode plots of phase response of the given second order system (note: $\tan -\theta = (\pi - \tan\theta)$).

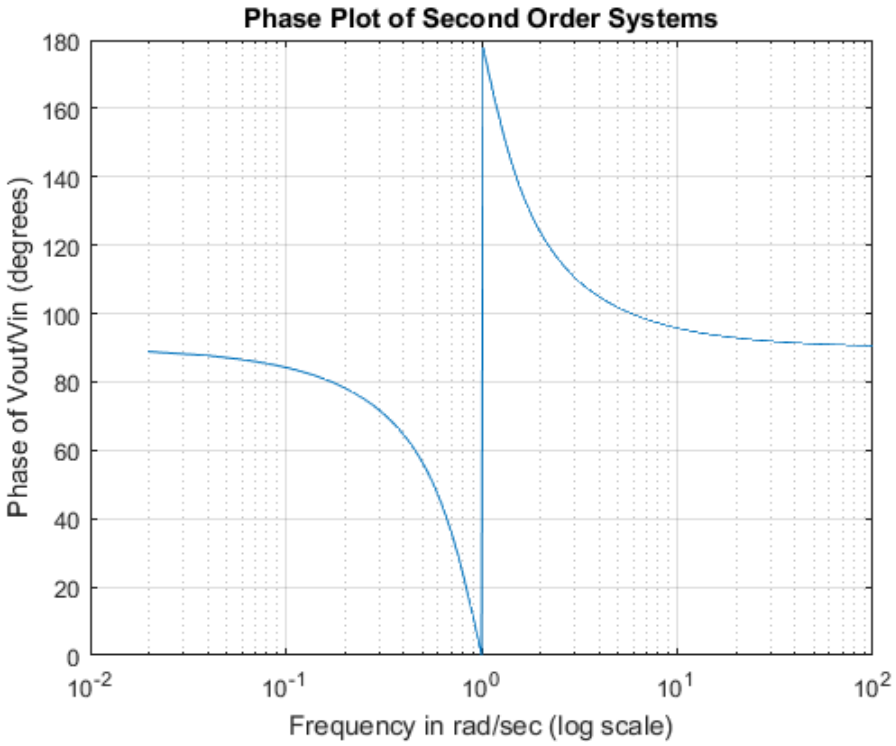
$\boldsymbol{\omega}$ $\mathbf{(rad/s)}$	$\boldsymbol{\theta}$ (°)	$\boldsymbol{\theta}$ **(radian)**
0	90	$\pi/2$
0.1	84.17	$\pi/2.14$
0.5	56.31	$\pi/3.2$
1	0	0
5	-78.23	$-\pi/2.3$
10	-84.23	$-\pi/2.13$

g. Draw, simulate and describe the Bode plots of magnitude (phase) response of the given second order system.

Draw Bode plots of the phase response of the given second order system.



MATLAB simulation of the Bode plots of phase response of the given second order system.



Note: the result of the phase plot simulation shows an asymptote of infinity phase shift at frequency of 10^0 rad/s, whereas we did not observe it in the sketch.

Describe the frequency domain characteristics (phase) of this system at the critical frequencies:

- At low frequency, it has approximately phase angle of $\pi/2$ as ω -> 0.
- At high frequency, its phase angle is about $-\pi/2$ as ω -> ∞.
- At the cut off frequency, its frequency at $\omega = 1$ is named cut-off frequency (F_c), the phase angle is 0.

From the results of magnitude and phase plots, it is a second order system that has a band-pass filter characteristic (e.g. it allows the signals with the frequencies inside the band to pass through).

D. Sketch and Simulation of Frequency Response

9. Sketch and simulate in MATLAB the frequency response (magnitude and phase) plots of the following systems. Also describe the characteristics of the frequency response of these systems:

   a. System (i):

$$G(j\omega) = \frac{5j\omega}{(0.1j\omega + 5)}$$

   b. System (ii):

$$G(j\omega) = \frac{(0.5j\omega + 3)}{(j\omega)^3 + 12(j\omega)^2 + 20j\omega}$$

Solution

a. The frequency response plots of the System (i) are:

$$G(j\omega) = \frac{5j\omega}{(0.1j\omega + 5)}$$

Normalise the transfer function equation

$$G(j\omega) = \frac{(5)}{(0.1)}\left[\frac{j\omega}{\left(j\omega + \frac{5}{0.1}\right)}\right] = 50\left[\frac{j\omega}{(j\omega + 50)}\right]$$

- Numerators: $j\omega = 0$
- Denominators: $j\omega = 50$

Calculate the points for plotting the frequency response of the System (i):

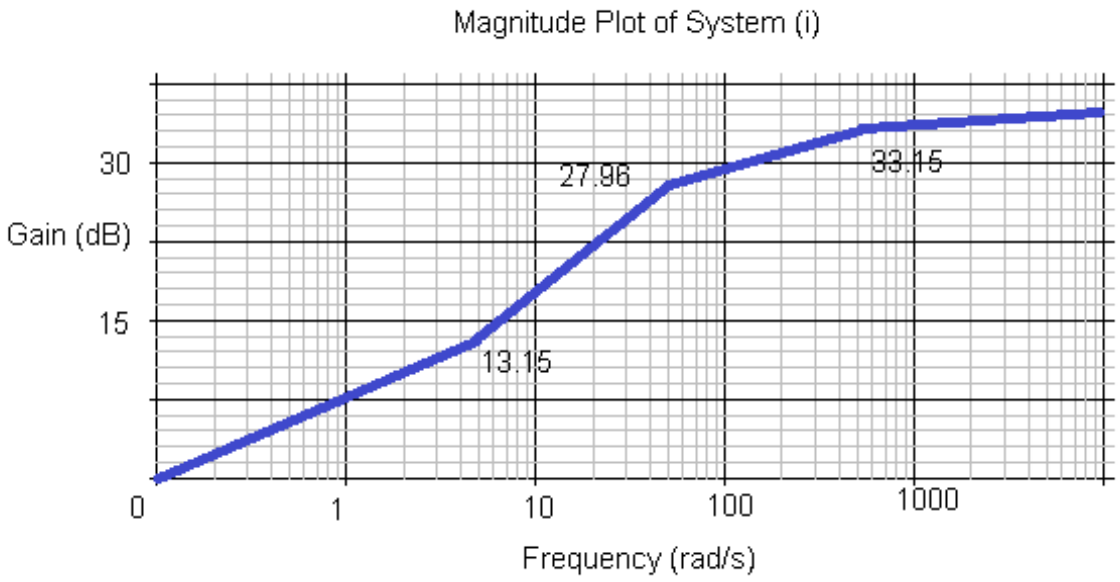
- For plotting the magnitude plot:

When frequency = 0 rad/s, calculate the DC gain:

$$|G(0)| = 20\log\left\{50\left[\frac{0}{(0+50)}\right]\right\} = 0$$

When frequency = 50 rad/s, the breakpoint in magnitude plot.

Sketch of magnitude response plot of System (i):



- For plotting the phase plot:

  When frequency = 0 rad/s, calculate the phase at DC:

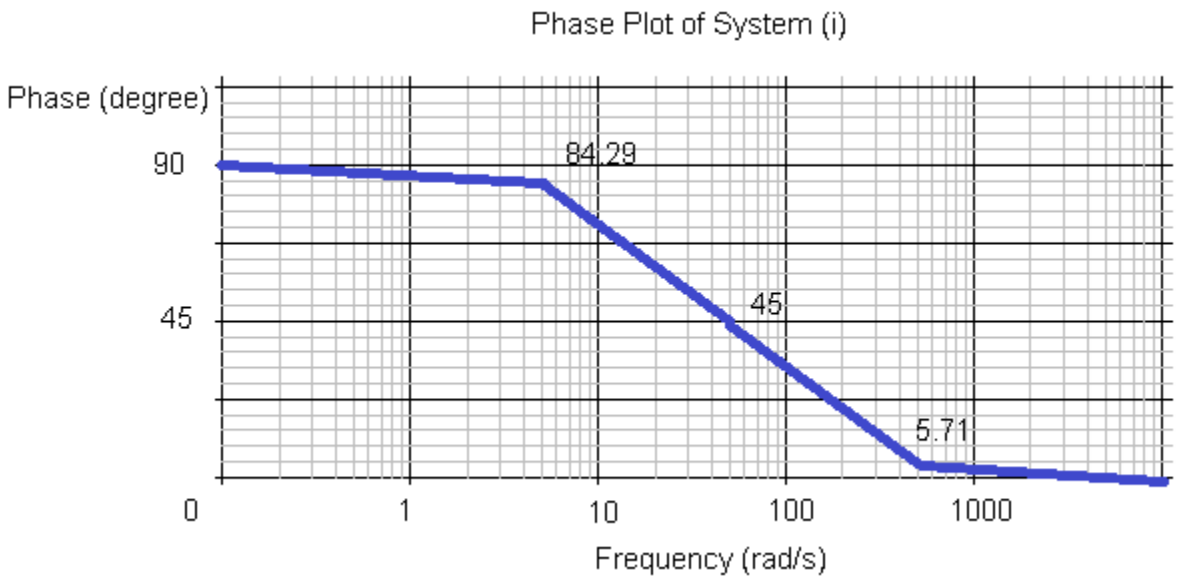
$$\angle G(0) = \frac{\tan^{-1}\left(\frac{0^+}{0}\right)}{\tan^{-1}\left(\frac{0^+}{50}\right)} = 90° - 0° = 90°$$

  When frequency = 5 rad/s, a decade before (0.1 time) of the breakpoint.

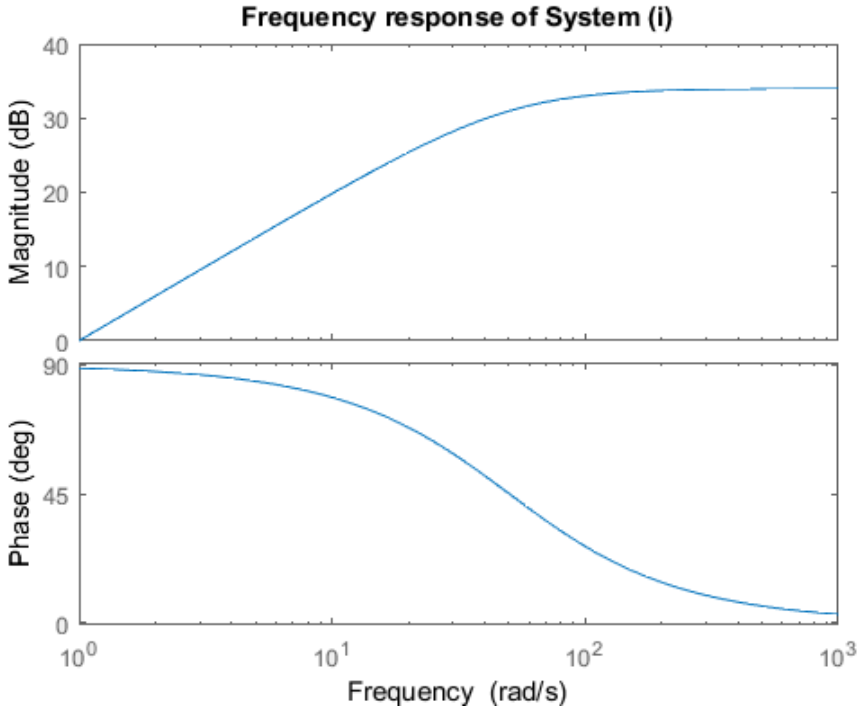
  When frequency = 50 rad/s, the break point.

  When frequency = 500 rad/s, a decade after (10 times) the break point.

Sketch of phase response plot of System (i):



Result of frequency response simulation of System (i) in MATLAB



System (i) appears to have a frequency response of:

Frequency (rad/s)	Gain (dB)	Phase (degree)

0	0	90
5	13.15	84.29
50	27.96	45
500	33.15	5.71

Magnitude plot:

- When $\omega = 0$ rad/s, gain of the system is 0 dB.
- When $\omega = 5$ rad/s, gain of the system is 13.15 dB.
- When $\omega = 50$ rad/s, gain of the system is 27.96 dB.
- When $\omega = 500$ rad/s, gain of the system is 33.14 dB.

Phase plot:

- When $\omega = 0$ rad/s, phase shift of the system is 90°.
- When $\omega = 5$ rad/s, phase shift of the system is 84.29°.
- When $\omega = 50$ rad/s, phase shift of the system is 45°.
- When $\omega = 500$ rad/s, phase shift of the system is 5.71°.

The System (i) resemblances a system with a zero gain and phase lead at low frequency and a system with a system with a constant gain at high frequency.

It has a break frequency at 50 rad/s where the gain of the system is 27.96 dB and its phase shift is 45°.

b. The frequency response plots of the System (ii) are:

The transfer function of System (ii) is:

$$G(j\omega) = \frac{(0.5j\omega + 3)}{(j\omega)^3 + 12(j\omega)^2 + 20j\omega}$$

Thus,

$$G(j\omega) = \frac{(0.5j\omega + 3)}{j\omega(j\omega + 2)(j\omega + 10)}$$

Normalise the transfer function equation

$$G(j\omega) = (0.5)\left[\frac{(j\omega + 6)}{j\omega(j\omega + 2)(j\omega + 10)}\right]$$

- Numerators: $j\omega = 6$
- Denominators: $j\omega = 0$, $j\omega = 2$, and $j\omega = 10$

Points for plotting the frequency response of the system:

- For sketching the magnitude plot of the frequency response:

  When frequency = 0 rad/s, calculate the DC gain:

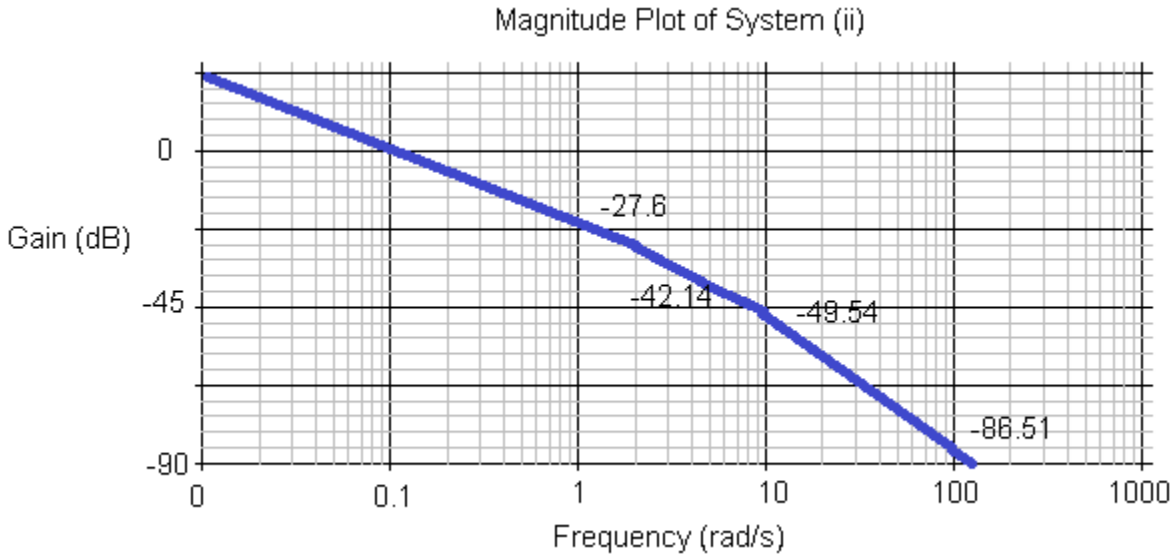
$$|G(0)| = 20\log\left\{(0.5)\left[\frac{(0 + 6)}{(0)(0 + 2)(0 + 10)}\right]\right\} = \infty$$

  When frequency = 2 rad/s, the break point 1.

  When frequency = 6 rad/s, the break point 2.

  When frequency = 10 rad/s, the break point 3.

Sketch of magnitude response plot of System (ii):



- For sketching the phase plot of the frequency response:

  When frequency = 0 rad/s, calculate the phase at DC:

$$\angle G(0) = \frac{\tan^{-1}\left(\frac{0^+}{6}\right)}{\tan^{-1}\left(\frac{0^+}{0}\right) + \tan^{-1}\left(\frac{0^+}{2}\right) + \tan^{-1}\left(\frac{0^+}{10}\right)}$$

$$= 0° - 90° - 0° - 0° = -90°$$

When frequency = 0.2 rad/s, a decade (0.1 time) before of break point 1.

When frequency = 0.6 rad/s, a decade (0.1 time) before break point 2.

When frequency = 1 rad/s, a decade (0.1 time) before break point 3.

When frequency = 2 rad/s, the break point 1.

When frequency = 6 rad/s, the break point 2.

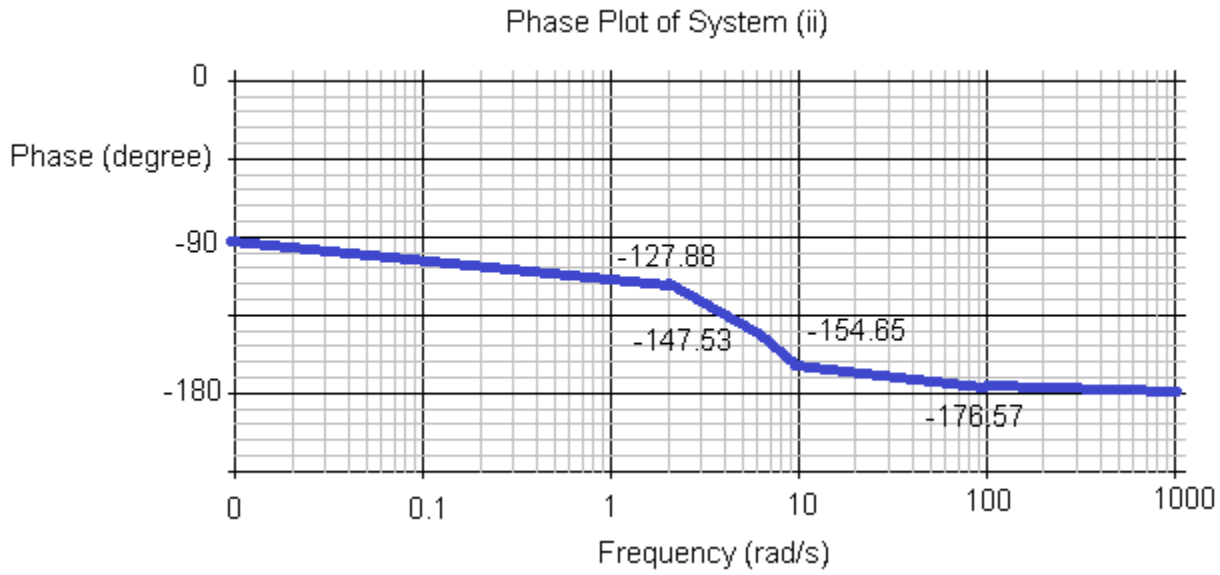
When frequency = 10 rad/s, the break point 3.

When frequency = 20 rad/s, a decade (10 times) after break point 1.

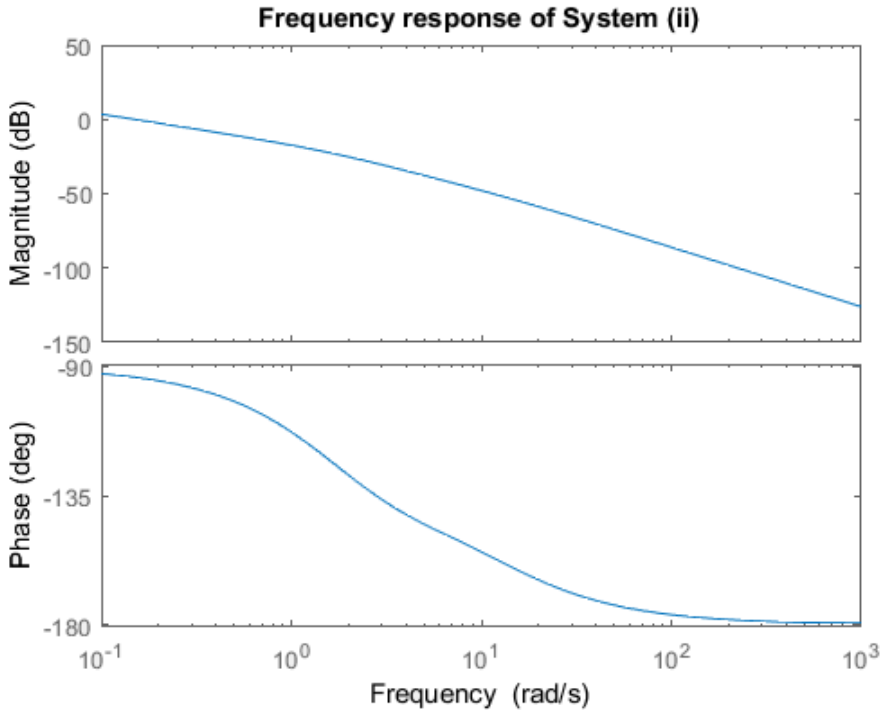
When frequency = 60 rad/s, a decade (10 times) after break point 2.

When frequency = 100 rad/s, a decade (10 times) after break point 3.

Sketch of phase response plot of System (ii):



Result of frequency response simulation of System (ii) in MATLAB.



System (ii) appears to have a frequency response of:

Frequency (rad/s)	Gain (dB)	Phase (degree)
0	∞	-90
2	-27.6	-127.88
6	-42.14	-147.53
10	-49.54	-154.65
100	-86.51	-176.57

Magnitude plot:

- When $\omega = 0$ rad/s, gain of the system is ∞ dB
- When $\omega = 2$ rad/s, gain of the system is -27.6 dB
- When $\omega = 6$ rad/s, gain of the system is -42.14
- When $\omega = 10$ rad/s, gain of the system is -49.54 dB
- When $\omega = 100$ rad/s, gain of the system is -88.51 dB

Phase plot:

- When $\omega = 0$ rad/s, phase shift of the system is -90°.
- When $\omega = 2$ rad/s, phase shift of the system is -127.88°.
- When $\omega = 6$ rad/s, phase shift of the system is -147.53°.
- When $\omega = 10$ rad/s, phase shift of the system is -154.65°.
- When $\omega = 100$ rad/s, phase shift of the system is -176.57°.

The System (ii) has a gain of approaching ∞ and a phase shift of -90° at low frequency and it has a slope of -40 dB/dec and a phase shift approaching -180° at high frequency.

It has break frequencies at 0, 2, 6 and 10 rad/s.

MATLAB Files:

First Order System:

Frequency Response (Magnitude)

```
% 1stordermag.m
% Magnitude characteristics of Bode plot of a first order systems
w=0:0.02:10;
RC=1;
magGjw=1./sqrt(1+1./(w.*RC).^2);
semilogx(w,magGjw);
xlabel('Frequency in rad/sec (log scale)');
ylabel('Magnitude of Vout/Vin');
title('Magnitude Characteristics of First Order Systems');
grid
```

Frequency Response (Phase)

```
% 1storderphase.m
% Phase characteristics of Bode plot of a first order systems
w=0:0.02:10;
RC=1;
phaseGjw=-atan(w.*RC).*180./pi;
semilogx(w,phaseGjw);
xlabel('Frequency in rad/sec (log scale)');
ylabel('Phase of Vout/Vin (degrees)');
title('Phase Characteristics of First Order Systems');
grid
```

Second Order System:

Frequency Response (Magnitude)

```
% 2ndordermag.m
% Magnitude plot of Bode plot of a second order systems
```

```
w=0:0.02:100;
magGjw=w./sqrt((1-w.^2).^2+w.^2);
semilogx(w,magGjw);
xlabel('Frequency in rad/sec (log scale)');
ylabel('Magnitude of Vout/Vin');
title('Magnitude Plot of Second Order System');
grid
```

Frequency Response (Phase)

```
% 2ndorderphase.m
% Phase plot of a second order systems
w=0:0.02:100;
phaseGjw=90-atan(w./(1-w.^2))*180./pi;
semilogx(w,phaseGjw);
xlabel('Frequency in rad/sec (log scale)');
ylabel('Phase of Vout/Vin (degrees)');
title('Phase Plot of Second Order System');
grid
```