XMUT220 Signals and Systems

Tutorial 6a: Sampling and Reconstruction (Introduction) – Solution

A. Introduction to Sampling and Reconstruction

1. Find the Nyquist rate for each of the following signals:

   a. $x(t) = 25\cos(500\pi t)$ [2 marks]

   b. $x(t) = 3\cos(50\pi t) + 10\sin(300\pi t) - \cos(100\pi t)$ [2 marks]

   c. $x(t) = \text{sinc}(200\pi t)$ [2 marks]

   d. $x(t) = \text{sinc}^2(200\pi t)$ [2 marks]

   e. $x(t) = \text{sinc}(200\pi t) + \text{sinc}^2(200\pi t)$ [2 marks]

   f. $x(t) = [\sin(1000\pi t)/\pi t] * y(t)$ [2 marks]

   with: $X(j\omega) = \begin{cases} 1, & |\omega| < 2000\pi \\ 0, & \text{otherwise} \end{cases}$

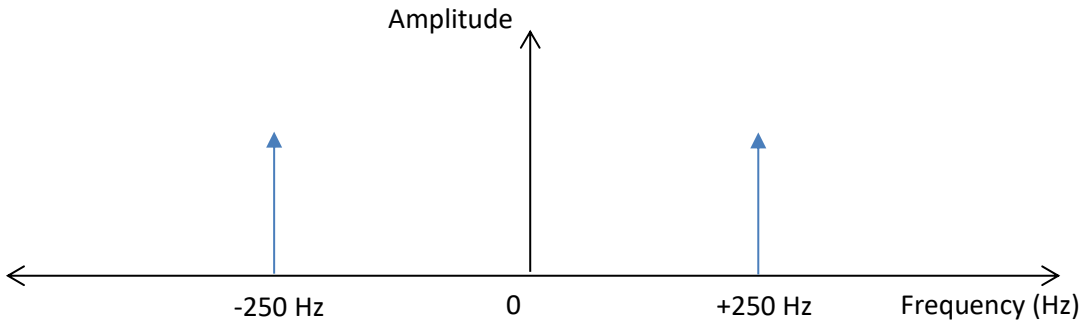
   g. $x(t) = 2\cos(5000\pi t)\cos(500k\pi t)$ [2 marks]

Solution

a. The frequency spectrum of the signal $x(t) = 25\cos(500\pi t)$ shows:

$$X(f) = 12.5\,[\delta(f - 250) + \delta(f + 250)]$$

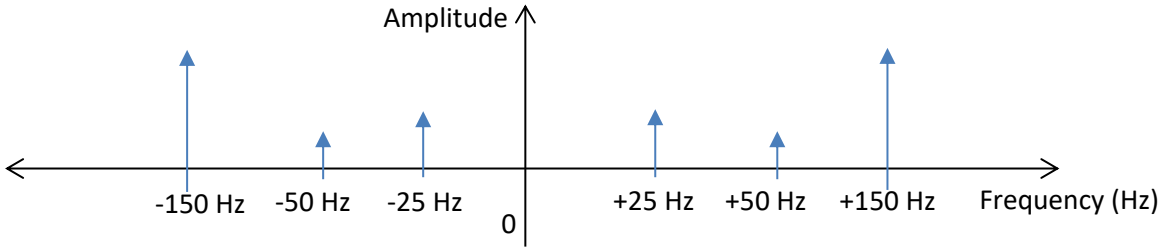
The highest frequency (and the only frequency) present in this signal is f = 250 Hz. As a result, the Nyquist rate is 500 Hz.



b. The frequency spectrum of the signal $\mathrm{x(t)} = 3\cos(50\pi t) + 10\sin(300\pi t) - \cos(100\pi t)$ shows:

$$X(f) = 1.5\,[\delta(f-25) + \delta(f+25)] + 5\,[\delta(f-150) + \delta(f+150)] - 0.5\,[\delta(f-50) + \delta(f+50)]$$

The frequency of the $3\cos(50\pi t)$ component is 25 Hz, for the $10\sin(300\pi t)$component, it is 150 Hz, and for the $-\cos(100\pi t)$, it is 50 Hz.

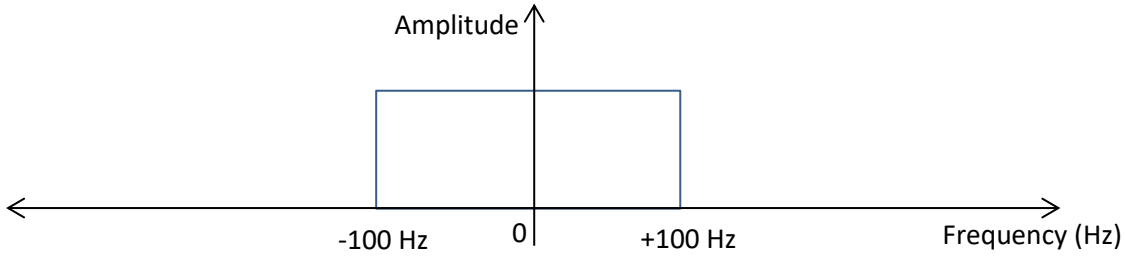


From these three signal components, the component that has the highest frequency is the $10\sin(300\pi t)$. The frequency of this component is 150 Hz and as a result, the Nyquist rate of the signal is 300 Hz.

c. The given $x(t) = \mathrm{sinc}(200\pi t)$ function can be treated as:

$$x(t) = \frac{\sin(200\pi t)}{\pi t}$$

The frequency spectra of a sinc() function is a rectangle like shape in the frequency spectrum.

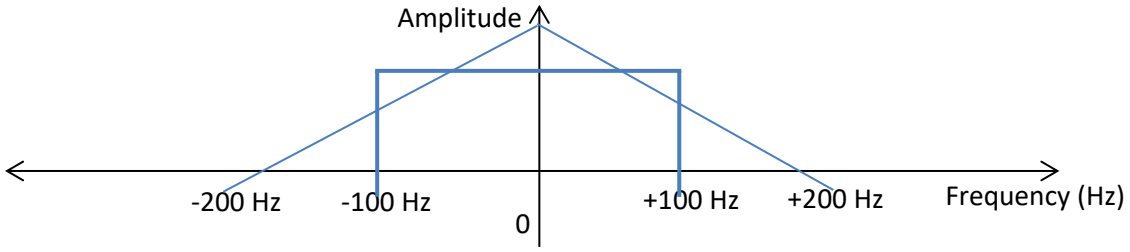


As a result, the highest frequency of the signal is 100 Hz and its Nyquist rate is 200 Hz.

d. The signal $\mathrm{x(t)} = \mathrm{sinc}^2(200\pi t)$ can be treated as an (auto) convolution of two sinc() function signals:

$$x(t) = \mathrm{sinc}\,(200\pi t) * \mathrm{sinc}(200\pi t)$$

The original sinc() signal has a square like spectra shape in the frequency spectrum and the resulting convoluted signal (rectangle * rectangle) has a triangle like spectra shape.



The highest frequency of the convoluted signals is twice the frequency of its original signals i.e. 2 x 100 Hz = 200 Hz. As a result, its Nyquist rate is 400 Hz.

e. The highest frequency of the signal $\mathrm{x(t)} = \mathrm{sinc}(200\pi\mathrm{t}) + \mathrm{sinc}^2(200\pi\mathrm{t})$ is from its $\mathrm{sinc}^2(200\pi t)$ component. The frequency of this component is 200 (e.g. its highest frequency is 2 x 100 Hz), so its Nyquist rate is 400 Hz.

f. For the $x(t) = [\sin(1000\pi t)/\pi t] * y(t)$ signal:

$$X(j\omega) = \begin{cases} 1, & |\omega| < 2000\pi \\ 0, & \text{otherwise} \end{cases}$$

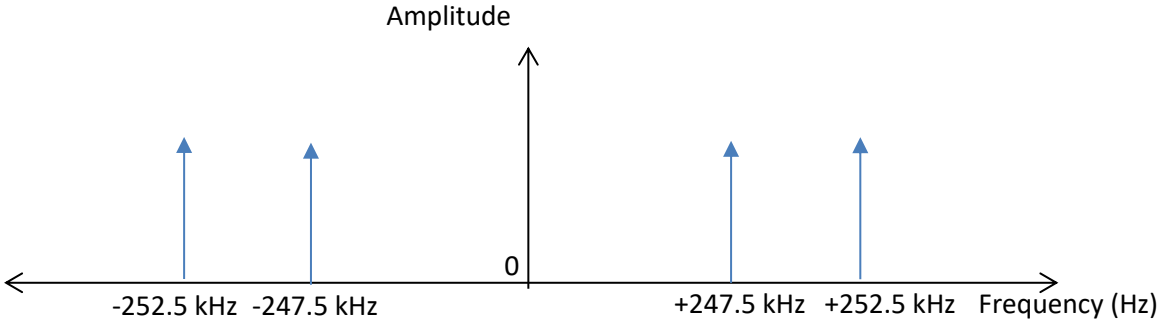
As a result, the highest sampling frequency of the signal is 1000 Hz from the $y(t)$ part of the signal and hence, its Nyquist rate is at least 2000 Hz.

g. Knowing that $2\cos\theta_1\cos\theta_2 = \cos(\theta_1+\theta_2) + \cos(\theta_1-\theta_2)$, as a result:

$$2\cos(5000\pi t)\cos(500\mathrm{k}\pi t) = \cos(500\mathrm{k}\pi t + 5000\pi\mathrm{t}) + \cos(500\mathrm{k}\pi t - 5000\pi\mathrm{t})$$

The frequency spectrum of the signal $x(t) = 2\cos(5000\pi t)\cos(500\mathrm{k}\pi t)$ is:

$$X(f) = \frac{2}{2}[\delta(250{,}000 + 2{,}500) + \delta(250{,}000 - 2{,}500)]$$



The highest frequency in the $x(t)$ signal is f_m = 252.5 kHz. Therefore, the Nyquist rate is 505 kHz.

2. A continuous-time signal ($x(t)$) can be represented in its samples ($x_p(t)$) and can be reconstructed back to its original signal when sampling frequency, f_s is greater than or equal to the twice the highest frequency component of the signal.

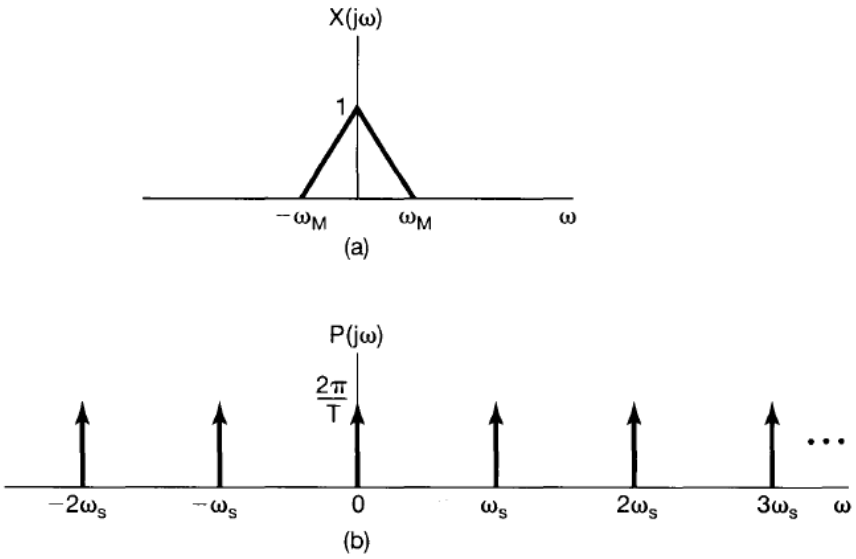
   a. Describe the effect of sampling in the frequency domain. Illustrate the effect in terms of spectrum plots of original signal and sampling function and show also the difference between spectrum plots of sampled signal with $\omega_s > 2\omega_M$ and spectrum of sampled signal with $\omega_s < 2\omega_M$. [8 marks]

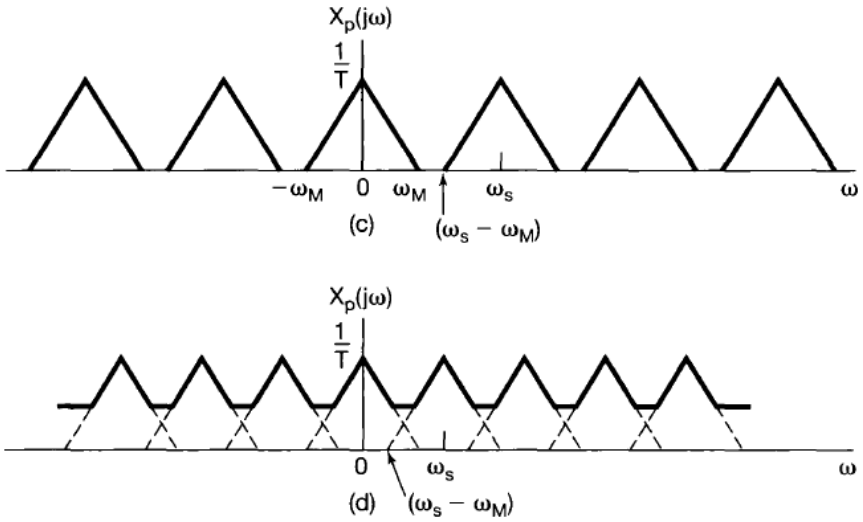
   b. Describe the effect of reconstructions of a continuous-time signal from its samples using an ideal low-pass filter: Illustrate your answer with a block diagram of system for sampling and reconstruction, representative spectrum for $x(t)$, corresponding spectrum for $x_p(t)$, ideal low-pass filter to recover $X(j\omega)$ from $X_p(j\omega)$, and spectrum of reconstructed signal, $X_r(t)$.

   [8 marks]

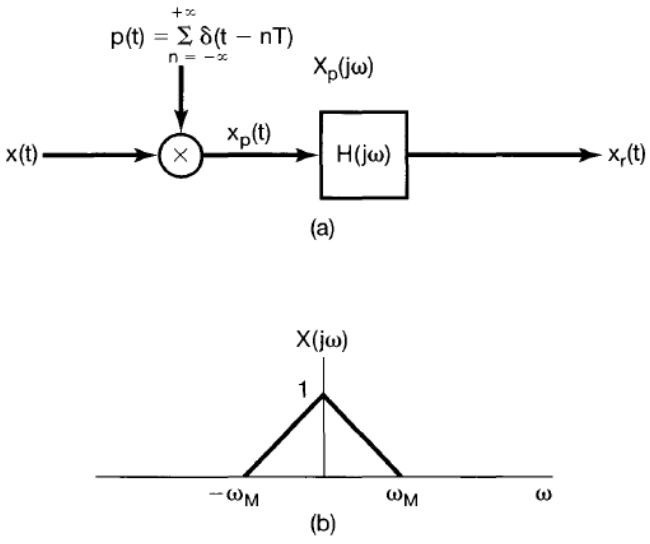
Solution

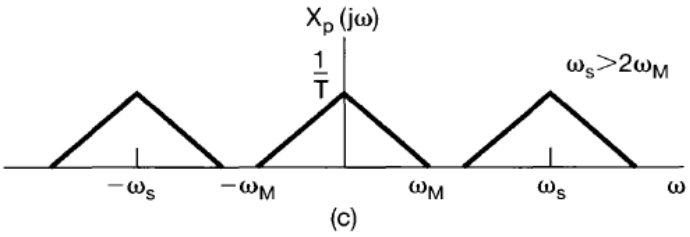
a. Effect in the frequency domain of sampling in the time domain: (a) spectrum of original signal; (b) spectrum of sampling function; (c) spectrum of sampled signal with $\omega_s > 2\omega_M$; (d) spectrum of sampled signal with $\omega_s < 2\omega_M$.



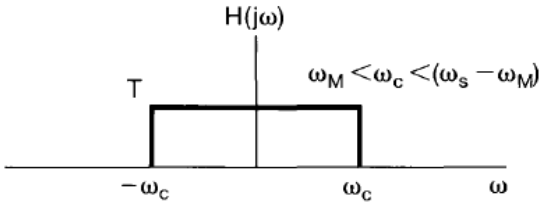


b. Exact recovery of a continuous-time signal from its samples using an ideal low-pass filter: (a) system for sampling and reconstruction; (b) representative spectrum for $x(t)$; (c) corresponding spectrum for $x_p(t)$; (d) ideal low-pass filter to recover $X(j\omega)$ from $X_p(j\omega)$; (e) spectrum of $X_r(t)$.

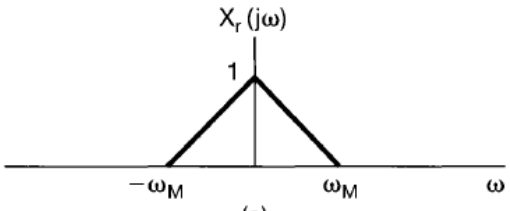




(c)



(d)



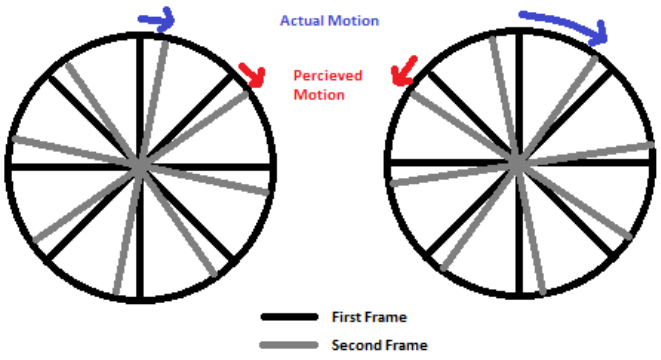
(e)

3. Aliasing refers to the incorrect measurement of a signal's frequency due to an inadequate digital sampling rate.

   a. Describe what happens to a signal which is not adequately sampled using enough data points? [2 marks]

   b. What is required to avoid aliasing? [2 marks]

   c. With regard to aliasing, what is the “wagon wheel effect”? [2 marks]



Solution

a. If a signal is not sampled using enough data points, its true frequency will be underestimated. This falsely estimated signal will be indistinguishable from (i.e., be an "alias" to) another signal having that true lower frequency.

b. To avoid aliasing, the digital sampling rate must be at least twice the highest frequency contained within that signal. This is known as the Nyquist sampling theorem.

c. The "wagon wheel effect" is a typical example of aliasing. In this optical illusion, spokes on a wheel appear to rotate at different rates or even backwards depending on the digital frame rate of the video.

4. Where f_H is highest frequency of the sampled signal and B is the bandwidth of the sampled signal, the minimum sampling rate is given as:

$$f_{s\,(min)} > \frac{2f_H}{\lfloor f_H/B \rfloor}$$

Let a signal have no nonzero spectral components outside the following range of frequencies, what is the minimum sampling rate (f_s) that avoids aliasing?

a. $0 < |f| < 600\ \text{kHz}$ [2 marks]

b. $15\ \text{kHz} < |f| < 45\ \text{kHz}$ [2 marks]

Solution

a. For the given low pass filter with range of its operating frequencies between: $0 < |f| < 600\ \text{kHz}$, the minimum sampling frequency of the signal is:

$$f_{s(min)} > \frac{2f_H}{\lfloor f_H/B \rfloor} = \frac{2 \times 600\ \text{kHz}}{\lfloor 600\ \text{kHz}/\ 600\ \text{kHz} \rfloor} = 1200\ \text{kHz}$$

b. For the given band pass filter with its operating frequency range between: $15\ \text{kHz} < |f| < 45\ \text{kHz}$, the minimum sampling frequency of the signal is:

$$f_{s(min)} > \frac{2f_H}{\lfloor f_H/B \rfloor} = \frac{2 \times 45\ \text{kHz}}{\lfloor 45\ \text{kHz}/30\ \text{kHz} \rfloor} = 60\ \text{kHz}$$

Note that signal with bandwidth limitation has less minimum frequency requirement than the one without.

B. Applications of Sampling and Reconstruction

5. Suppose a signal that is to be acquired by a data acquisition system is known to have an amplitude spectrum that is flat out to 100 kHz and drops suddenly there to zero. The fastest rate at which our data acquisition system can sample the signal is 60 kHz.

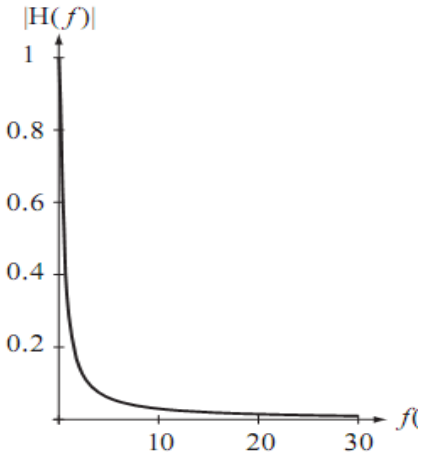
   a. Design an anti-aliasing filter that will reduce the signal's amplitude spectrum at 30 kHz to less than 1% of its value at very low frequencies so that aliasing will be minimised. [4 marks]

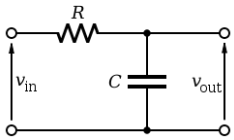
   b. Determine the CR time constant of the filter. [10 marks]

Answer

a. As the filter is required to attenuate certain higher frequency components of the signals as a fraction of its lower frequency counterparts, then we need to consider a low-pass filter.

The frequency response of this filter is similar to a unity-gain low-pass filter, and the filter is realised as a resistor-capacitor (RC) circuit as shown below.





$$H(f) = \frac{1}{1 + j2\pi fCR}$$

b. As a design consideration, the corner frequency (-3 dB frequency) of this filter is specified at 300 Hz, which is 100 times lower than the Nyquist frequency of 30 kHz.

The magnitude of the frequency response is given as follows:

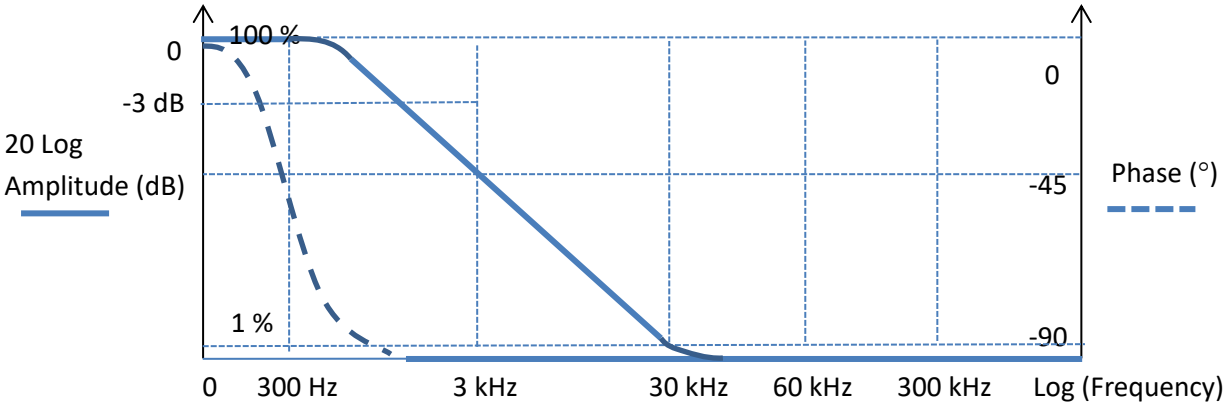
$$|H(f)| = \frac{1}{\sqrt{(2\pi fCR)^2 + (1)^2}}$$

The phase of the frequency response is given as follows:

$$\angle H(f) = -\tan^{-1}\left(\frac{2\pi fCR}{1}\right)$$

From the equation given above, its value at very low frequencies approaches one (100%).

The following diagram shows the frequency domain characteristics of this filter.



By setting the CR time constant so that at 30 kHz, the squared magnitude of $H(f)$ is $(0.01)^2$. Entering values to the equation and rearranging the equation to find CR time constant.

$$|H(30{,}000)|^2 = \frac{1}{(2\pi \times 30{,}000 \times CR)^2 + 1} = (0.01)^2$$

As a result, CR time constant of the low pass filter is 0.5305 ms.

If we have to choose values for resistor and capacitor, select capacitor value of C = 1 μF, hence resistor value of R = 530.5 Ω, so nearest E12 standard resistor value is 520 Ω.

6. Describe the following signal reconstruction methods

   a. Zero-order hold. [2 marks]

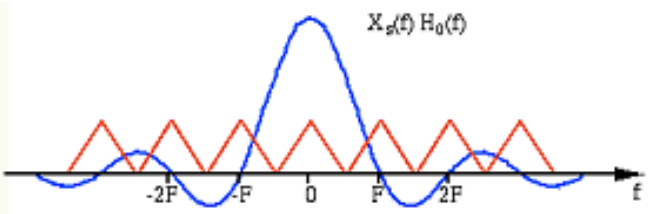
   b. First order hold. [2 marks]

   c. Sinc function. [2 marks]

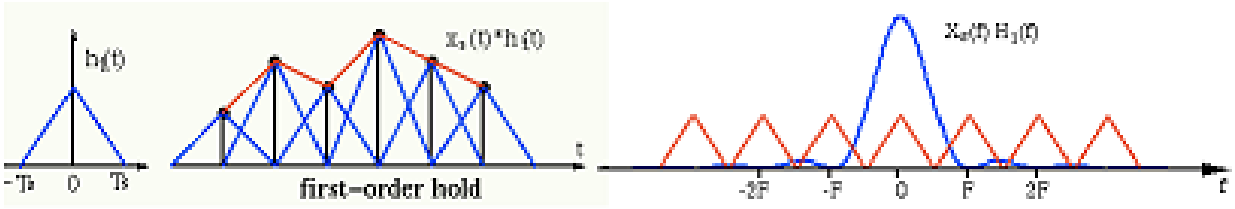
Solution

a. In a zero-order reconstruction method, it converts a discrete-time signal to a continuous-time signal by holding each sample value for one sample interval.





b. For a first-order hold method, the signal is reconstructed as a piecewise linear approximation to the original signal that was sampled.

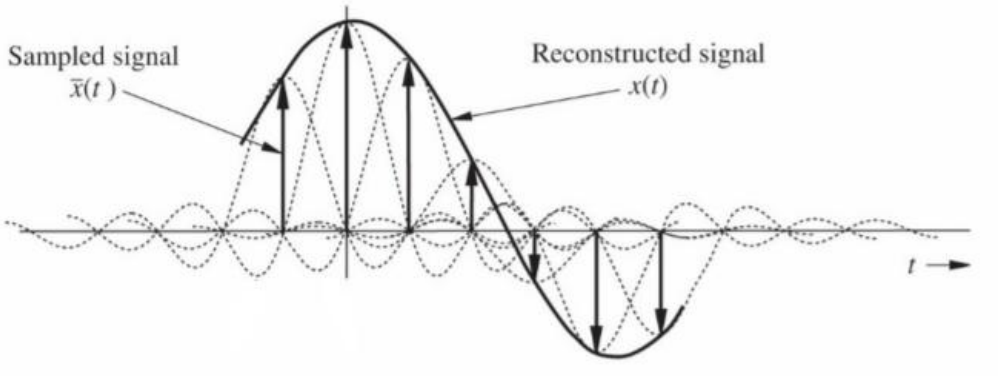


c. The sinc interpolation formula is defined as:

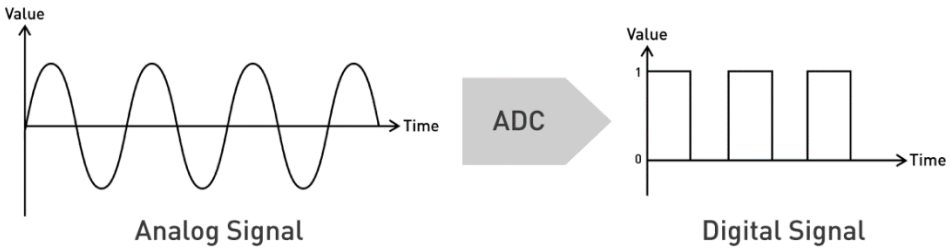
$$x(t) = \sum_{n=-\infty}^{+\infty} x_n \,\mathrm{sinc}\left[\frac{\pi}{T}(t - nT)\right]$$

Where: T is the sampling period used to determine x_n from the original signal, and x_t is the reconstructed signal. The function $\mathrm{sinc}(x)$ is defined by $\mathrm{sinc}(x) = \sin(x)/x$ for $\neq 0$, with $\mathrm{sinc}(0) = 1$.

The above formula represents a linear convolution between the sequence x_n and scaled and shifted samples of the sinc function.



7. Continuous-time signal sampling to its discrete-time samples can be performed by an analogue-to-digital conversion system.



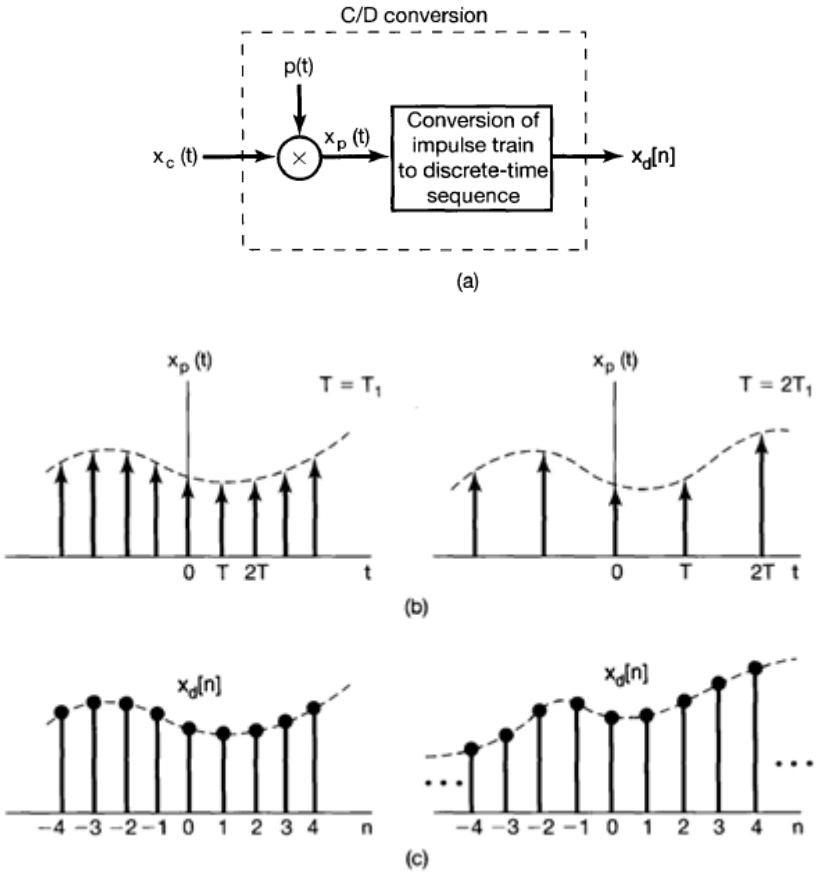
a. Describe sampling with a periodic impulse train followed by conversion to a discrete-time sequence for two different sampling rates (e.g. $T_1 = T$ and $T_2 = 2T$).

Include the following in your description: (a) overall system; (b) discrete-time sampled signals, $x_p(t)$ and its original continuous-time signal, $x_c(t)$; and (c) the output sequence, $x_d(n)$. [6 marks]

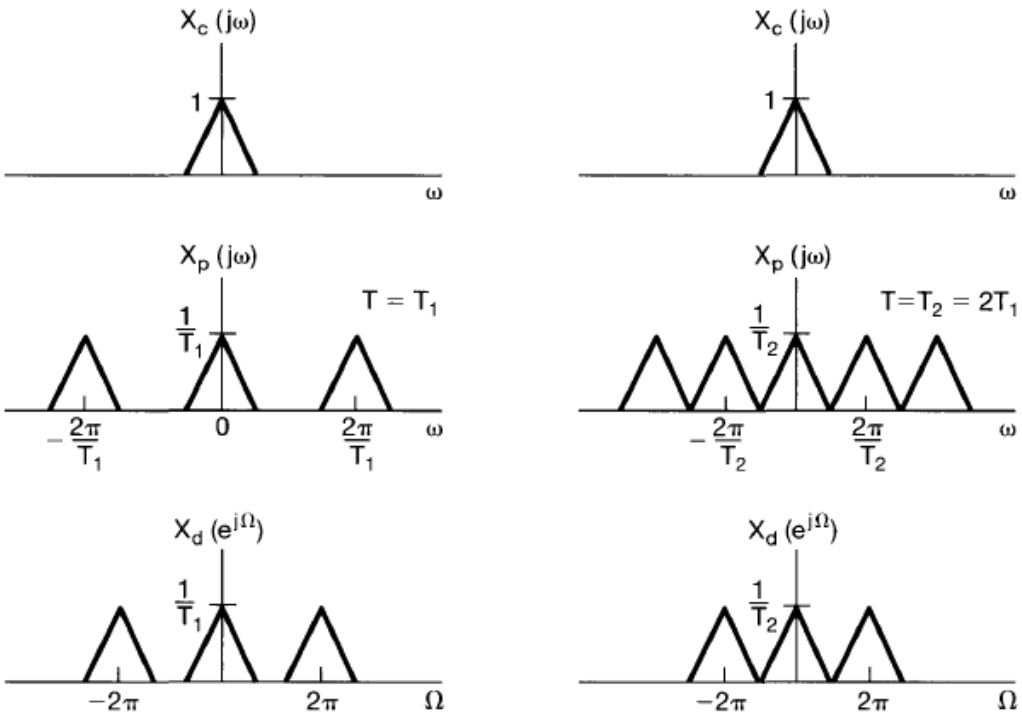
b. Describe the relationship between $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$ for the given different sampling rates. [6 marks

Solution

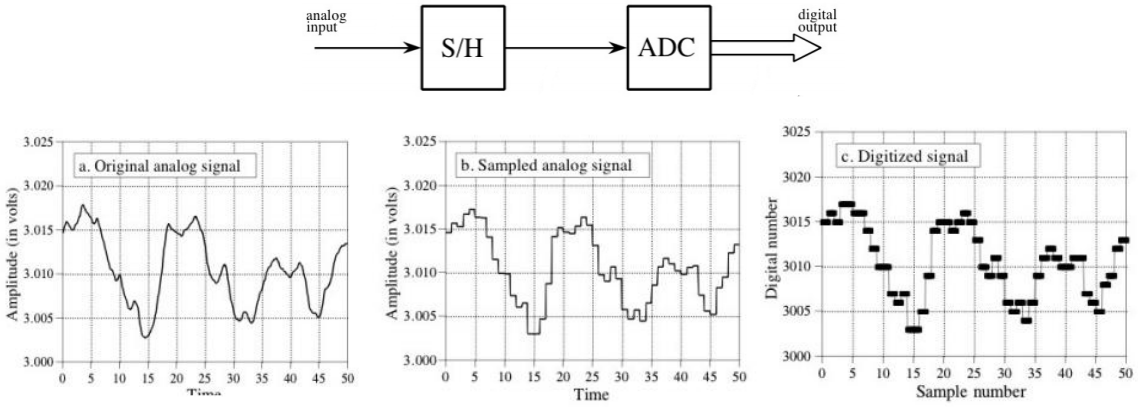
a. Sampling with a periodic impulse train followed by conversion to a discrete-time sequence: (a) overall system; (b) $x_p(t)$ for two sampling rates. The dashed envelope represents $x_c(t)$; (c) the output sequence for the two different sampling rates.



b. Relationship between $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$ for two different sampling rates.



8. You are given the following analogue-to-digital (ADC) system. The given system employs sample-and-hold (S/H) process and quantization process inside the ADC system.



a. Referring to the diagrams, describe the two stages of the conversion process: sample-and-hold and quantisation. Describe the effect of quantisation. [2 marks]

b. What happen to the sampled signal after these sample and hold and quantisation process? [2 marks]

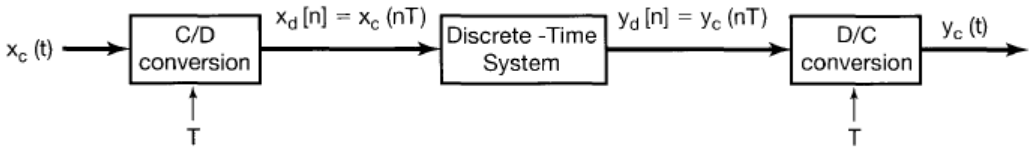
Solution

a. In the sample-and-hold process, the output only changes at periodic instants of time. The independent variable now takes values in a discrete set. Whereas in the quantisation process, each flat region in the sampled signal is “rounded-off” to the nearest member of a

set of discrete values (e.g., nearest integer). Main effect of quantisation is that it introduces error in the sampled signal.

b. After the sample-and-hold and quantisation processes, the sampled signal is encoded into bits. These discrete-time pulses will be used for further processing e.g. digital processing, computation, statistical analysis, etc.

9. Digital signal processing (DSP) is fundamental to digital technology, especially for digital telecommunication and wireless communications. DSP is applicable to both streaming data and static (stored) data.



a. Describe the main reason why digital signal processing is preferred compared with analogue signal processing. [2 marks]

b. Describe the components that make up a typical DSP system. [6 marks]

Solution

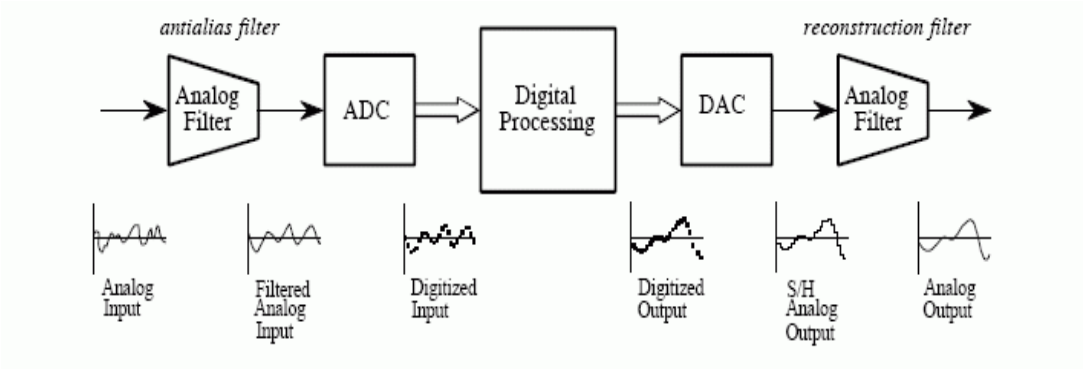
a. Analogue signal processing is complex in terms of its analysis and design and the outcome depends on accuracy and tolerance of the analogue devices used for constructing the processing circuits. Furthermore, application of digital computation to signal processing has many advantages over analogue processing in many applications, e.g. complex algorithms, error detection and correction, data compression, etc.

b. For a typical digital signal processing system, you might expect to see the following components of the system:

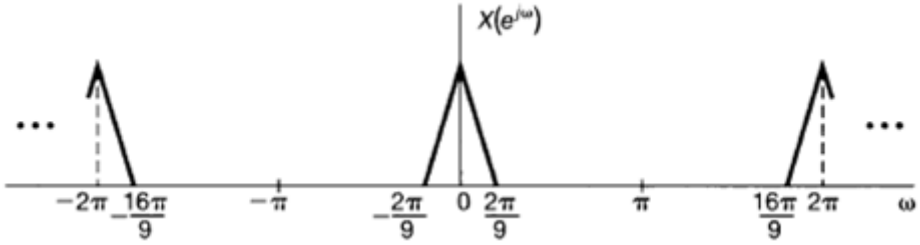
- Analogue Filter (Input) – Remove unwanted signals and noises.
- Digital Processing – Main signal processing of the system e.g. wave shaping, algorithm based processing, signal modulation, etc.
- Analogue Filter (Output) – Increase the quality and performance of the processed signal for the targeted system.

More specifically, the digital processing unit of the DSP system is usually constructed as follow:

- Hardware unit e.g. wave shaping, digital filtering, advanced signal conversions.
- Software unit e.g. algorithm of digital signal processing for manipulating the characteristics and properties of the signals.



10. Consider the sequence $x[n]$ whose Fourier transform $X(e^{j\omega})$ is illustrated in the figure below. The lowest rate at which impulse-train sampling may be used on this sequence without incurring aliasing is $2\pi/9$. This corresponds to sampling every 4th value of $x[n]$.



a. Decimate the sampling by a factor of 4. Describe the issue that you see from the reconstruction result. [4 marks]

b. Referring to the results in part (a), can this reconstruction process be improved further? Hint: application of up sampling and down sampling. [6 marks]

Solution

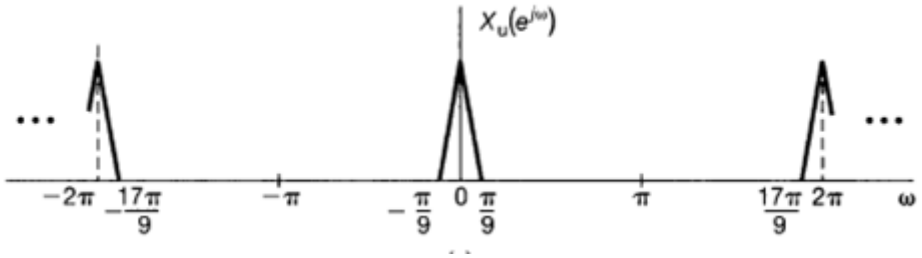
a. If the result of such sampling is decimated by a factor of 4, we obtain a sequence $x_b[n]$ whose spectrum is shown in the figure below. Clearly, there is still no aliasing of the original spectrum. However, this spectrum is zero for $8\pi/9 < |\omega| < \pi$, which suggests there is room for further down sampling.



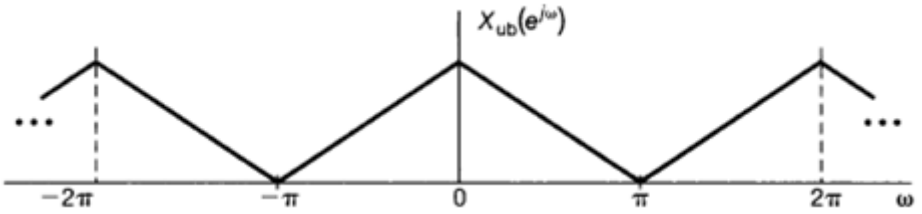
b. Specifically, by examining the frequency spectrum of the signal to be reconstructed, we see that if we could scale frequency by a factor of 9/2, the resulting spectrum would have nonzero values over the entire frequency interval from -π to π.

However, since 9/2 is not an integer, we can't achieve this purely by down sampling. Rather we must first up sample $x[n]$ by a factor of 2 and then down sample by a factor of 9. In

particular, the spectrum of the signal $x_u[n]$ obtained when $x[n]$ is up sampled by a factor of 2, is displayed in the figure given below.



When $x_u[n]$ is then down sampled by a factor of 9, the spectrum of the resulting sequence $x_{ub}[n]$ is as shown in the figure given below.



This combined result effectively corresponds to down sampling $x[n]$ by a noninteger amount, 9/2. Assuming that $x[n]$ represents unaliased samples of a continuous-time signal $x_c(t)$, our interpolated and decimated sequence represents the maximum possible (aliasing-free) down sampling of $x_c(t)$.