XMUT220 Signals and Systems

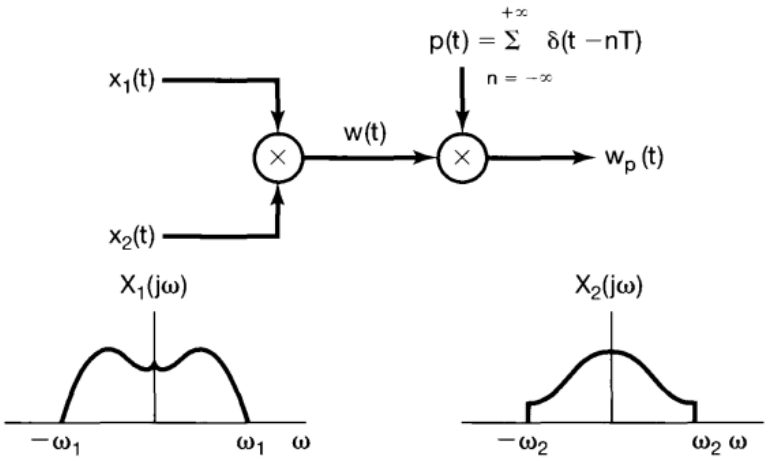
Tutorial 6b: Sampling and Reconstruction (Derivation of Equations) – Solution

A. Sampling of Signals

1. In the system shown in the figure given below, two functions of time, $x_1(t)$ and $x_2(t)$, are multiplied together, and the product $w(t)$ is sampled by a periodic impulse train. $x_1(t)$ is band limited to ω_1, and $x_2(t)$ is band limited to ω_2; that is,

$$X_1(j\omega) = 0, |\omega| \geq \omega_1$$

$$X_2(j\omega) = 0, |\omega| \geq \omega_2$$



Determine the maximum sampling interval T such that $w(t)$ is recoverable from $w_p(t)$ through the use of an ideal low-pass filter. [10 marks]

Solution

The signal ($w(t)$) is a multiple of two signals $x_1(t)$ and $x_2(t)$.

$$w(t) = x_1(t).x_2(t)$$

As a result, its Fourier transform is found from:

$$W(j\omega) = \frac{1}{2\pi}(X_1(j\omega) * X_2(j\omega))$$

Since $X_1(j\omega) = 0$ for $|\omega| \geq \omega_1$ and $X_2(j\omega) = 0$ for $|\omega| \geq \omega_2$, we may conclude that:

$$W(j\omega) = 0 \text{ for } |\omega| \geq \omega_1 + \omega_2$$

Consequently, the Nyquist rate for $w(t)$ is:

$$\omega_s = 2(\omega_1 + \omega_2)$$

Therefore, the maximum sampling period that allows $w(t)$ to be recovered is:

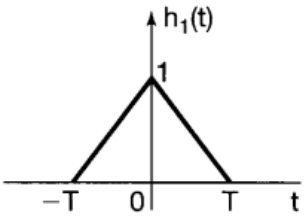
$$T = \frac{2\pi}{\omega_s} = \frac{\pi}{(\omega_1 + \omega_2)}$$

2. A signal $x(t)$ undergoes a zero-order hold operation with an effective sampling period T to produce a signal $x_0(t)$. Let $x_1(t)$ denote the result of a first-order hold operation on the samples of $x(t)$; i.e.,

$$x_1(t) = \sum_{n=-\infty}^{\infty} x(nT)h_1(t - nT)$$

Where: $h_1(t)$ is the function shown in the figure given below.

Specify the frequency response of a filter that produces $x_1(t)$ as its output when $x_0(t)$ is the input. [20 marks]



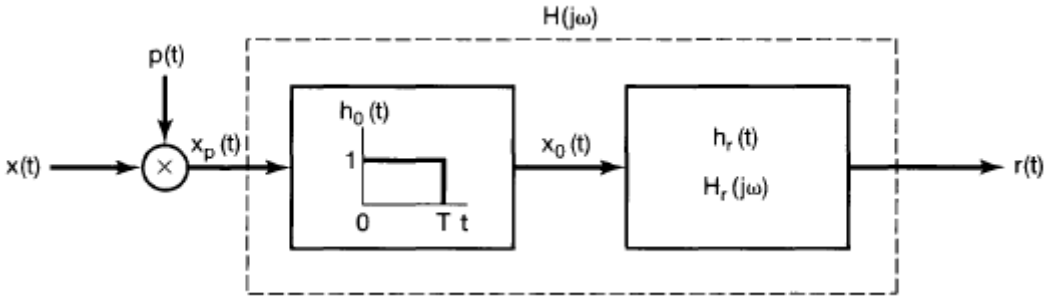
Solution

We note that the signal:

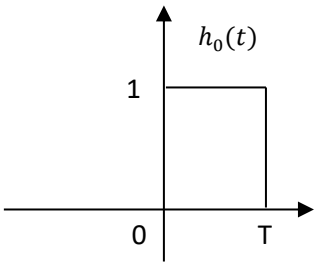
$$x_1(t) = h_1(t) * \left[\sum_{n=-\infty}^{\infty} x(nT).\delta(t - nT)\right]$$

The output of zero-order hold:

$$x_0(t) = h_0(t) * \left[\sum_{n=-\infty}^{\infty} x(nT).\delta(t - nT)\right]$$



Where $h_0(t)$ is as shown below.



By taking the Fourier transform of the above equations, we have:

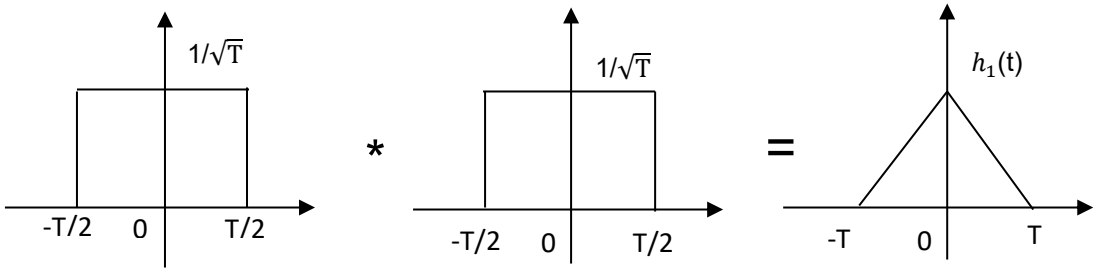
$$X_1(j\omega) = H_1(j\omega).X_p(j\omega)$$

$$X_0(j\omega) = H_0(j\omega).X_p(j\omega)$$

We need to determine a frequency response $H_d(j\omega)$ for a filter that produces $x_1(t)$ at its output when $x_0(t)$ is its input, therefore:

$$X_0(j\omega).H_d(j\omega) = X_1(j\omega)$$

The triangular function $h_1(t)$ may be obtained by convolving two rectangular pulses.



Therefore:

$$h_1(t) = \left[\frac{1}{\sqrt{T}}h_0(t + T/2)\right] * \left[\frac{1}{\sqrt{T}}h_0(t + T/2)\right]$$

Taking the Fourier transform of both sides of above equation:

$$H_1(j\omega) = \frac{1}{T}.e^{(j\omega T)}.H_0(j\omega).H_0(j\omega)$$

Therefore:

$$X_1(j\omega) = H_1(j\omega).X_p(j\omega)$$
$$= \frac{1}{T}e^{(j\omega T)}.H_0(j\omega).H_0(j\omega).X_p(j\omega)$$
$$= \frac{1}{T}e^{(j\omega T)}.H_0(j\omega).X_0(j\omega)$$

3. Let $x_c(t)$ be a continuous-time signal whose Fourier transform has the property that $X_c(j\omega) = 0$ for $|\omega| \geq 2{,}000\pi$. The continuous-time signal is part of a sampling system that produces a discrete sampled signal. From the sampling system, a discrete-time signal:

$$x_d[n] = x_c\left(n(0.5 \times 10^{-3})\right)$$

is obtained. For each of the following constraints on the Fourier transform $X_d(e^{j\omega})$ of $x_d[n]$, determine the corresponding constraint on $X_c(j\omega)$:

a. $X_d(e^{j\omega})$ is real. [2 marks]

b. The maximum value of $X_d(e^{j\omega})$ over all ω is 1. [2 marks]

c. $X_d(e^{j\omega}) = 0$ for $3\pi/4 \leq |\omega| \leq \pi$. [6 marks]

d. $X_d(e^{j\omega}) = X_d(e^{j(\omega - 1/T)})$. [2 marks]

Solution

We know discrete-time Fourier transform of $x_d[n]$ as a result of convolution of continuous – time signal with sampling impulse is:

$$X_d(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_C\left[\frac{j(\omega - 2\pi k)}{T}\right]$$

a. Since $X_d(e^{j\omega})$ is formed by shifting and summing of replicas of $X_c(j\omega)$, we may assume that $X_d(e^{j\omega})$ is real then $X_c(j\omega)$ must be real.

b. $X_d(e^{j\omega})$ consists of replicas of $X_c(j\omega)$ which are scaled by $1/T$. Therefore if $X_d(j\omega)$ has a maximum of 1, then $X_c(j\omega)$ should have a maximum of $T = 0.5 \times 10^{-3}$.

c. Region $3/4\ \pi \leq |\omega| \leq \pi$ in the discrete time domain in the continuous time domain corresponds to the following region in the continuous-time domain:

$$\frac{3\pi}{4T} \leq |\omega| \leq \frac{\pi}{T}$$

Therefore, if

$$X_d(e^{j\omega}) = 0, \quad \text{for } 3\pi/4 \leq |\omega| \leq \pi$$

Then

$$X_c(j\omega) = 0, \quad \text{for } 1500\pi \leq |\omega| \leq 2000\pi$$

But since we already know:

$$X_c(j\omega) = 0, \quad \text{for } |\omega| \geq 2000\pi$$

We have

$$X_c(j\omega) = 0, \quad \text{for } |\omega| \geq 1500\pi$$

d. In this case, since π in the discrete-time frequency domain corresponds to 2000π in the continuous-time-frequency domain, this condition translates to:

$$X_d\left(e^{(j\omega}\right) = X_d\left(e^{(j(\omega-\pi)}\right) \quad \leftrightarrow \quad X_c(j\omega) = X_c j(\omega - 2000\pi)$$

B. Reconstruction of Signals

4. A discrete-time signal $x_d[n]$ has a Fourier transform $X_d\left(e^{j\omega}\right)$ with the property that $X_d\left(e^{j\omega}\right)$ = 0 for $3\pi/4 \leq |\omega| \leq \pi$. The signal is converted into a continuous-time signal below where: T = 10^{-3}:

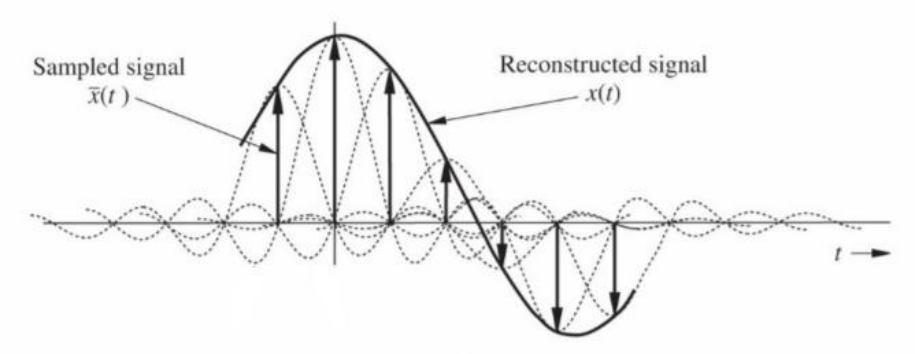
$$x_c(t) = T \sum_{n=-\infty}^{\infty} x_d[n] \left(\frac{\sin \frac{\pi}{T}(t - nT)}{\pi(t - nT)} \right)$$

a. Describe the reconstruction method used. [2 marks]

b. Determine the values of ω for which the Fourier transform $X_c(j\omega)$ of $x_c(t)$ is guaranteed to be zero. [2 marks]

Solution

a. The reconstruction method used for the given signal is the sinc function reconstruction method. The above given reconstruction formula represents a linear convolution between the sequence x_d and scaled and shifted samples of the sinc function.

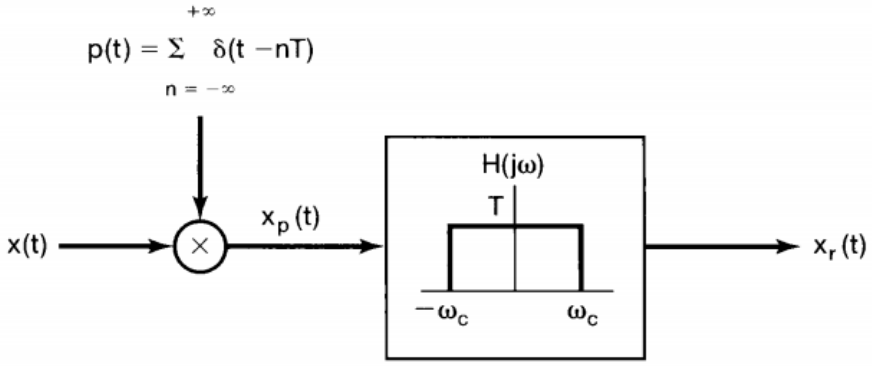


b. We know that the discrete and continuous-time frequencies Ω and ω are related by $\Omega = \omega T$. Therefore, in this case for $\Omega = 3\pi/4$, we find the corresponding value of ω to be:

$$\omega = \left(\frac{3\pi}{4}\right)\left(\frac{1}{T}\right) = \frac{3000\pi}{4} = 750\pi$$

5. In figure given below $p(t)$ is a sampler, followed by an ideal low-pass filter $H(j\omega)$, for reconstruction of $x(t)$ from its samples $x_p(t)$. From the sampling theorem, we know that if $\omega_s = 2\pi/T$ is greater than twice the highest frequency present in $x(t)$ and $\omega_c = \omega_s/2$, then the reconstructed signal $x_r(t)$ will exactly equal $x(t)$. If this condition on the bandwidth of $x(t)$ is violated, then $x_r(t)$ will not equal $x(t)$. We seek to show in this problem that if $\omega_c = \omega_s/2$, then for any choice of T, $x_r(t)$ and $x(t)$ will always be equal at the sampling instants; that is,

$$x_r(kT) = x(kT), k = 0, \pm 1, \pm 2, \ldots$$



To obtain this result, consider the following equation which expresses $x_r(t)$ in terms of the samples of $x(t)$:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)T\frac{\omega_c}{\pi}\frac{\sin[\omega_c(t-nT)]}{\omega_c(t-nT)}$$

With $\omega_c = \omega_s/2$, this becomes:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\sin\left[\frac{\pi}{T}(t-nT)\right]}{\frac{\pi}{T}(t-nT)}$$

By considering the values of α for which $[\sin(\alpha)]/\alpha = 0$, show from the equation given above that, without any restrictions on $x(t)$, $x_r(kT) = x(kT)$ for any integer value of k. [10 marks]

Solution

Here, $x_r(kT)$ can be written as:

$$x_r(kT) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\sin[\pi(k-n)]}{\pi(k-n)}$$

Note that when $n \neq k$

$$\frac{\sin[\pi(k-n)]}{\pi(k-n)} = 0$$

And when $n = k$

$$\frac{\sin[\pi(k-n)]}{\pi(k-n)} = 1$$

As

$$\text{sinc } 0 = \lim_{x\to 0}\frac{\sin(ax)}{ax} = 1 \quad \text{for all real and } a \neq 0$$

Therefore:

$$x_r(kT) = x(kT)$$

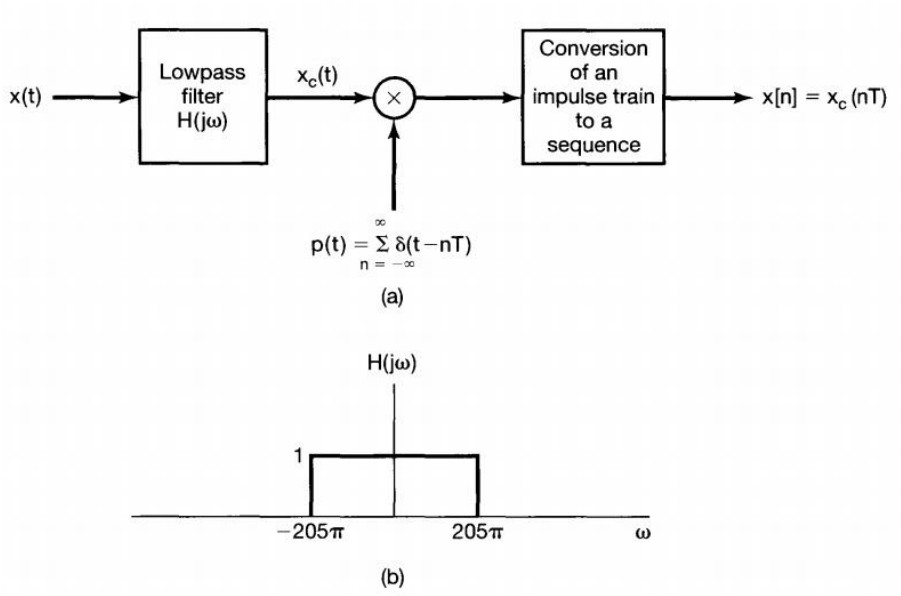
6. Figure (a) given below shows a system that converts a continuous-time signal to a discrete-time signal. The input $x(t)$ is periodic with a period of 0.1 second. The Fourier series coefficients of $x(t)$ are:

$$a_k = \left(\frac{1}{2}\right)^{|k|}, -\infty < k < +\infty$$

The low-pass filter $H(j\omega)$ has the frequency response shown in figure (b) below. The sampling period T = 5 x 10^{-3} second.

a. Show that $x[n]$ is a periodic sequence and determine its period. [18 marks]

b. Determine the Fourier series coefficients of $x[n]$. [4 marks]



Solution

a. Fundamental frequency of $x(t)$ is 20π rad/s. Fourier transform of $x(t)$ is given by:

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_K \, \delta(\omega - 20\pi k)$$

Fourier transform $X_C(j\omega)$ of signal $x_C(t)$ is as sketched in the figures below. Note that the spectrum of sampling signal is:

$$P(j\omega) = \frac{2\pi}{5 \times 10^{-3}} \sum_{k=-\infty}^{\infty} \frac{\delta(\omega - 2\pi k)}{5 \times 10^{-3}}$$

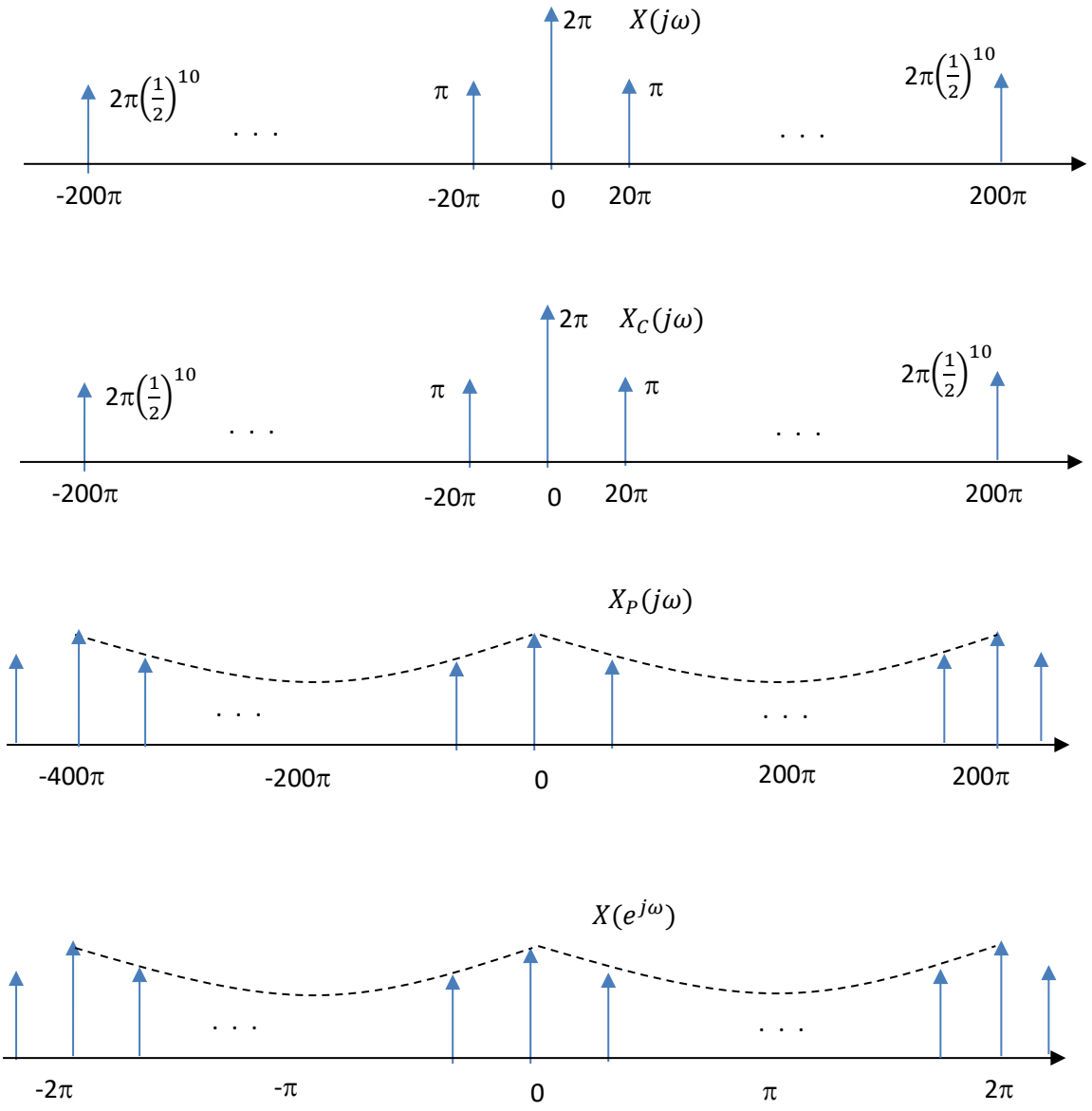
And

$$X_P(j\omega) = \frac{1}{2\pi}[X_C(j\omega) * P(j\omega)]$$

Therefore $X_p(j\omega)$ is as shown in the figure below. Note that impulses from adjacent replicas of $X_C(j\omega)$ add up at 200π. Now, the Fourier transform $X(e^{j\omega})$ of the sequence $x[n]$ is given by:

$$X(e^{j\Omega}) = X_P(j\omega)|_{\omega=\Omega T}$$

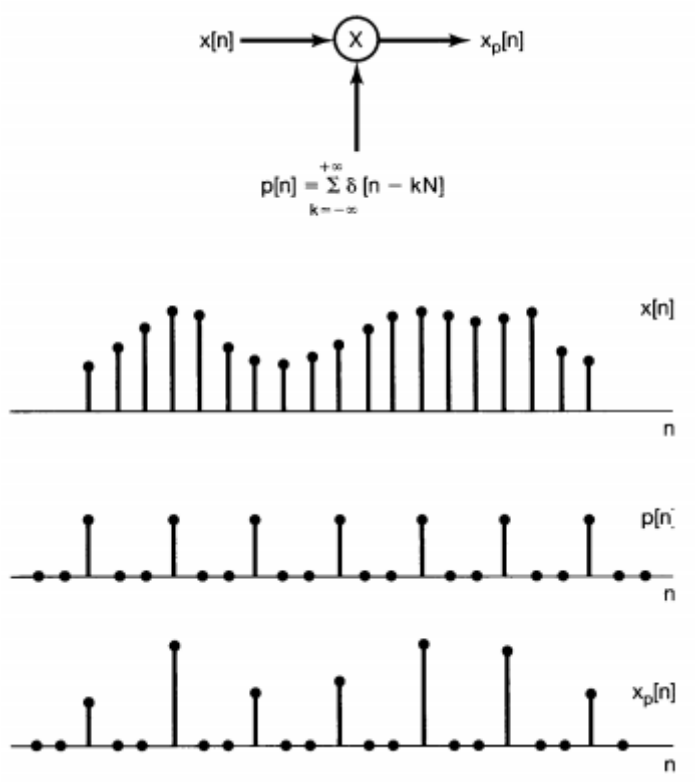
This is shown in the figure below. Since impulse in $X(e^{j\omega})$ are located at multiples of a 0.1π, the signal $x[n]$ is periodic. The fundamental period is $2\pi/(0.1\pi) = 20$.



b. The Fourier series coefficient of $x[n]$ is:

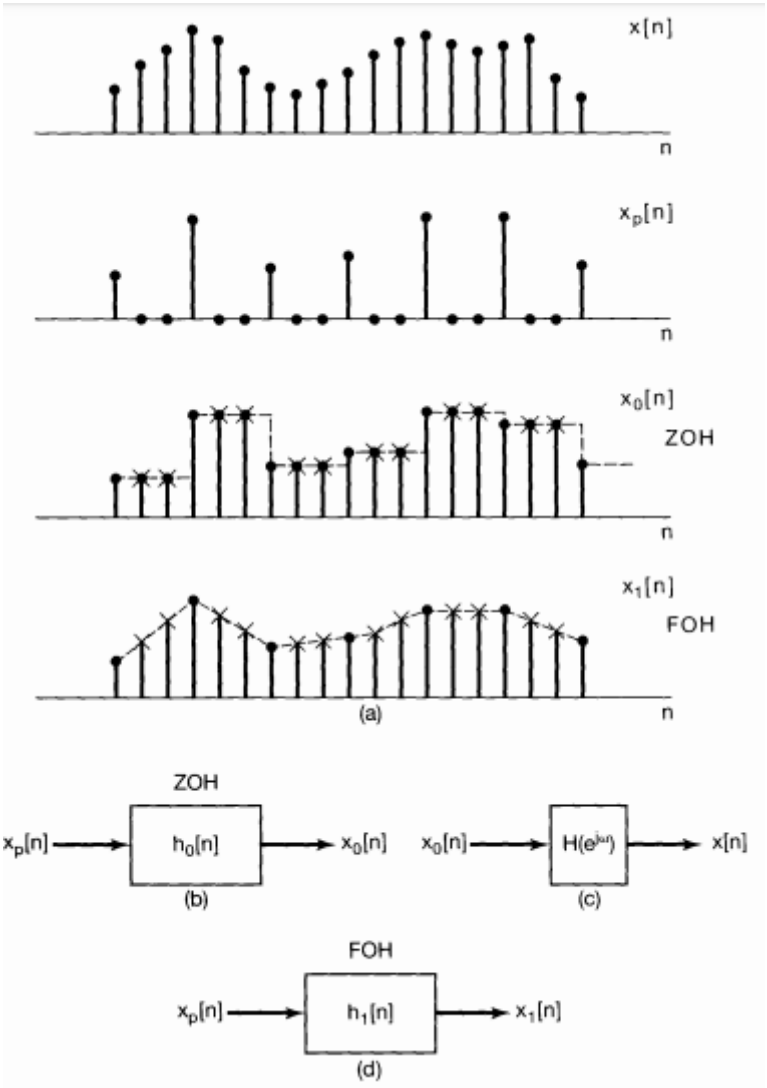
$$a_k \begin{cases} \frac{2\pi}{T}\left(\frac{1}{2}\right)^k & k = 0, \pm 1, \pm 2, \ldots, \pm 9 \\ \frac{4\pi}{T}\left(\frac{1}{2}\right)^{10} & k = 10 \end{cases}$$

7. In this problem, we consider the discrete-time counterparts of the zero-order hold and first-order hold. Let $x[n]$ be a sequence to which discrete-time sampling, as illustrated in figure below:



Suppose the conditions of the discrete-time sampling theorem are satisfied; that is, $\omega_s > 2\omega_M$, where ω_s is the sampling frequency and $X(e^{j\omega}) = 0$, $\omega_M < |\omega| \leq \pi$. The original signal $x[n]$ is then exactly recoverable from $X_p[n]$ by ideal low-pass filtering corresponds to band-limited interpolation.

The zero-order hold represents an approximate interpolation whereby every sample value is repeated (or held) $N - 1$ successive times, as illustrated in the figure below for the case of $N = 3$. The first-order hold represents a linear interpolation between samples, as illustrated in the same figure.



a. The zero-order hold can be represented as an interpolation in the form of equation below:

$$x_r[n] = \sum_{k=-\infty}^{+\infty} x[kN]h_r[n - kN]$$

or, equivalently, the system in part (b) of the figure given above. Determine and sketch $h_0[n]$ for the general case of a sampling period N. [4 marks]

b. Sequence $x[n]$ can be exactly recovered from the zero-order-hold sequence $x_0[n]$ using an appropriate LTI filter $H(e^{j\omega})$, as indicated in part (c) of the figure given above. Determine $H(e^{j\omega})$. [8 marks]

c. The first-order-hold (linear interpolation) can be represented as an interpolation in the form of equation given in part (a) of the question or, equivalently, the system in part (d) of the figure given above. Determine $h_1[n]$ for the general case of a sampling period N. [2 marks]

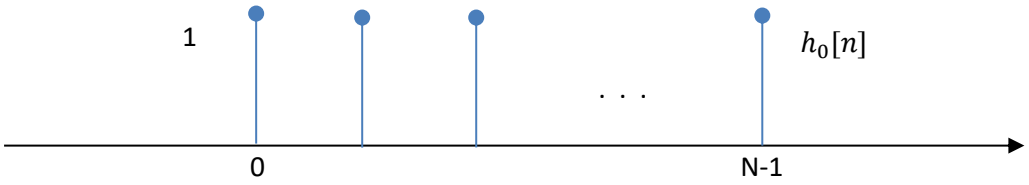
d. Sequence $x[n]$ can be exactly recovered from the first-order-hold sequence $x_1[n]$ using an appropriate LTI filter with frequency response $H(e^{j\omega})$. Determine $H(e^{j\omega})$. [4 marks]

Solution

a. We have $h_0[n]$ for the general case of a sampling period N for the given discrete-time sequence:

$$h_0[n] = u[n] - u[n-N]$$

This is as shown in the following figure:



b. We require that $H(e^{j\omega}).H_0(e^{j\omega}) = N$ for $|\omega| < \omega_s/2$ and zero otherwise. Here $\omega_s = 2\pi/N$, so $\omega_s/2 = \pi/N$.

Following Tustin approximation for conversion from continuous- to discrete-time systems:

$$e^{j\omega T_s} \approx \frac{1 + j\omega T_s/2}{1 - j\omega T_s/2}$$

The frequency spectrum of discrete zero-order hold function is:

$$H_0(e^{j\omega}) = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

Therefore, the zero-order hold function of the discrete sequence is:

$$H(e^{j\omega}) = \begin{cases} N\left[\dfrac{1 - e^{-j\omega}}{1 - e^{-j\omega N}}\right], & |\omega| < \pi/N \\ 0, & \pi/N \le |\omega| \le \pi \end{cases}$$

c. Evaluating on both sides of the ranges, we have the first-order hold function of discrete time sequence.

$$h_1[n] = \frac{1}{N}[h_0[n] * h_0[-n]]$$

d. We have $H(e^{j\omega}) = N/H_1(e^{j\omega})$ for $|\omega| < \pi/N$ and zero otherwise. But, from part (c):

$$H_1(e^{j\omega}) = \left(\frac{1}{N^2}\right)[H_0(e^{j\omega})]^2$$

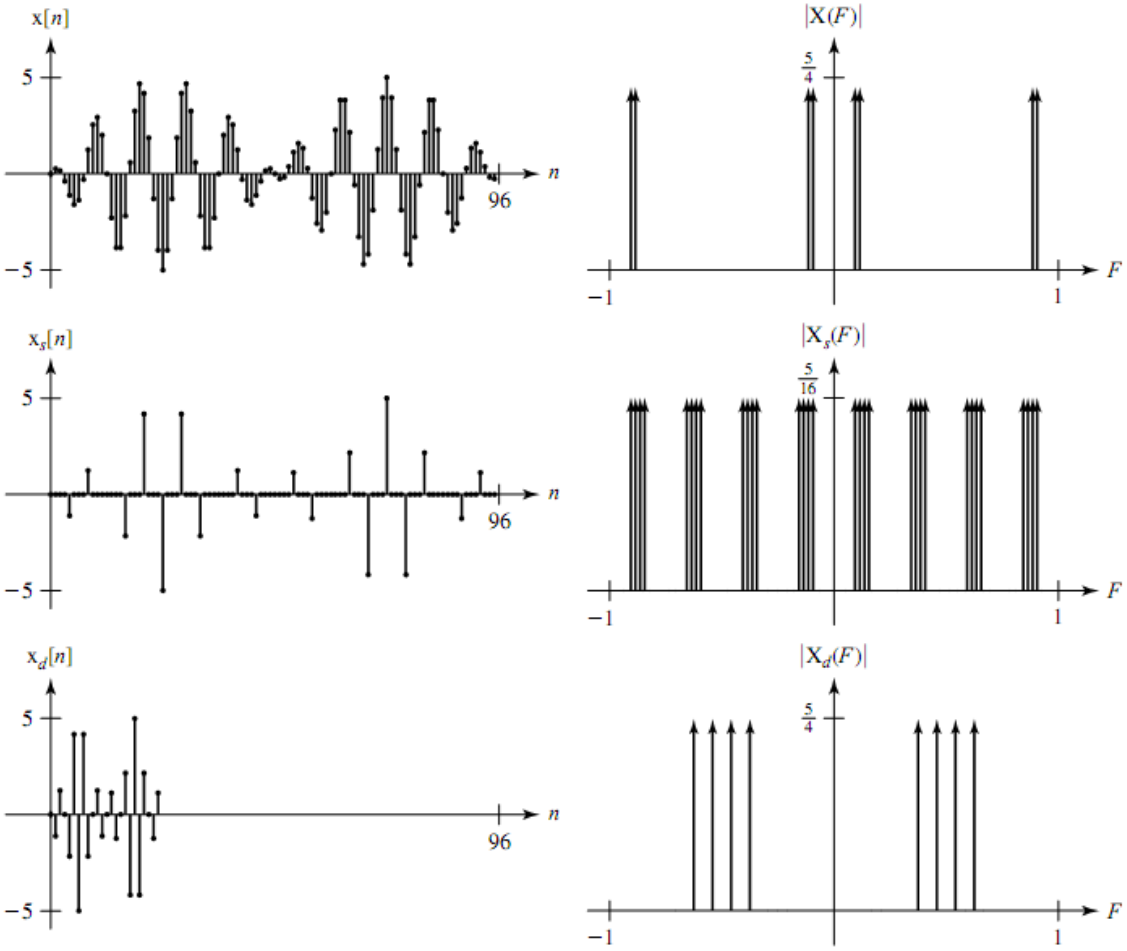
Therefore, the first-order hold function of the discrete sequence is:

$$H(e^{j\omega}) = \begin{cases} N^2 \left[\dfrac{1-e^{-j\omega}}{1-e^{-j\omega N}}\right]^2, & |\omega| < \pi/N \\ 0, & \pi/N \le |\omega| \le \pi \end{cases}$$

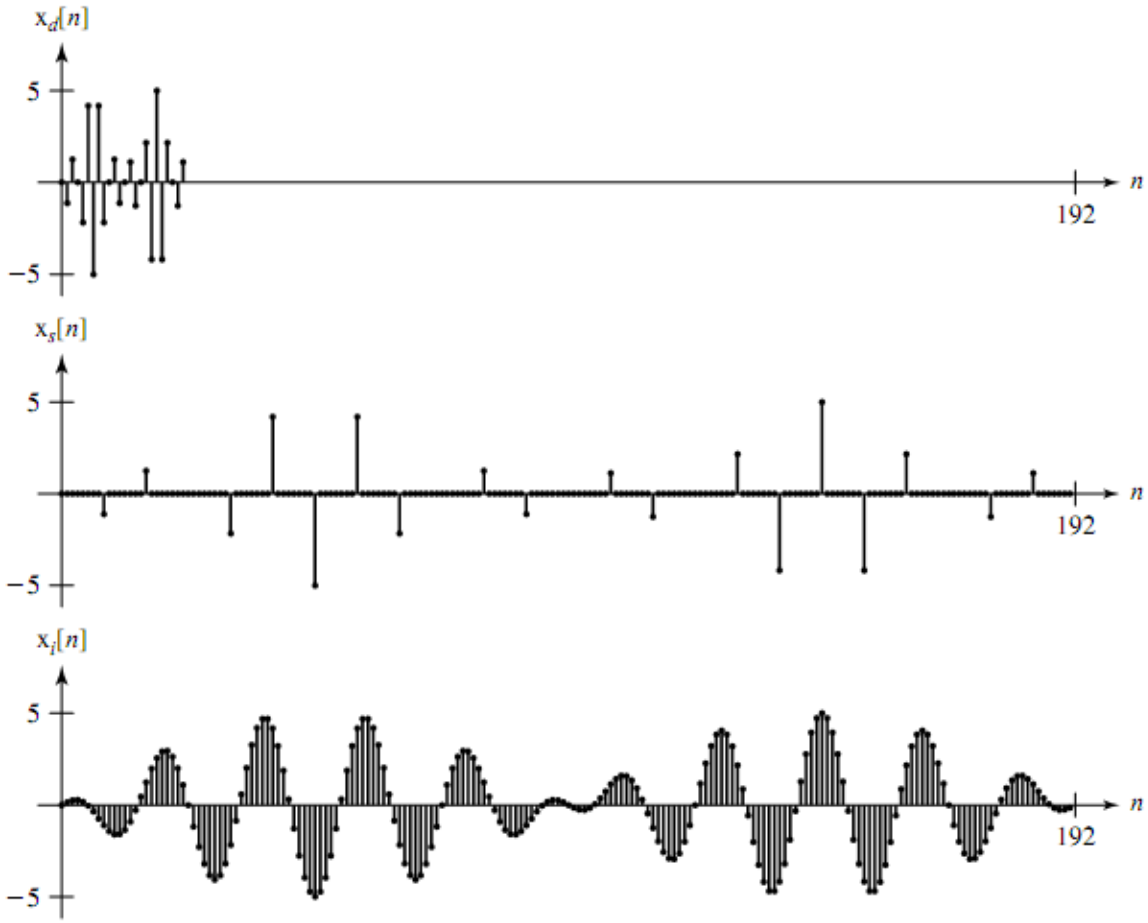
C. Decimation and Extrapolation

8. For the given sampling and reconstruction system, the given signal is undergone decimation process first, before the resulting decimated signal experience interpolation process.

   a. The following figures outline original, sampled and down sampled (decimated) discrete-time signals and their DTFTs. Describe the significance of the given time and frequency domain plots. [12 marks]



   b. The figures shown below show original, up sampled (extrapolated) and discrete-time low pass filtered discrete-time signals. Describe the significance of the given time domain plots. [6 marks]



Solution

a. Looking into the time-domain plots e.g. transients of the decimation process:

- Original sequence signal, $x[n]$ is sampled at the normal sampling rate.
- Sampling sequence $x_s[n]$ for decimation by a factor $M = 5$.
- Sequence $x_d[n]$ that has been decimated is 5 times faster than original sequence signal.

Looking into the frequency-domain plots e.g. frequency spectra of the decimation process:

- Original sequence signal, $X[f]$ is sampled at the normal sampling rate.
- Sampling sequence $X_s[f]$ for decimation by a factor $M = 5$.
- Sequence $X_d[f]$ that has been decimated has 1/5 times of the frequency spectrum ofthe original sequence signal.

b. Looking into the time-domain plots e.g. transients of the interpolation process:

- The sequence signal, $x[n]$ that has been decimated is shown at the normal sampling rate after decimation.
- Sampling sequence $x_s[n]$ for interpolation by a factor $L = 5$.
- Decimated sequence $x_d[n]$ that has been interpolated is 5 times slower than the original sequence signal before interpolation.