XMUT220 Signals and Systems

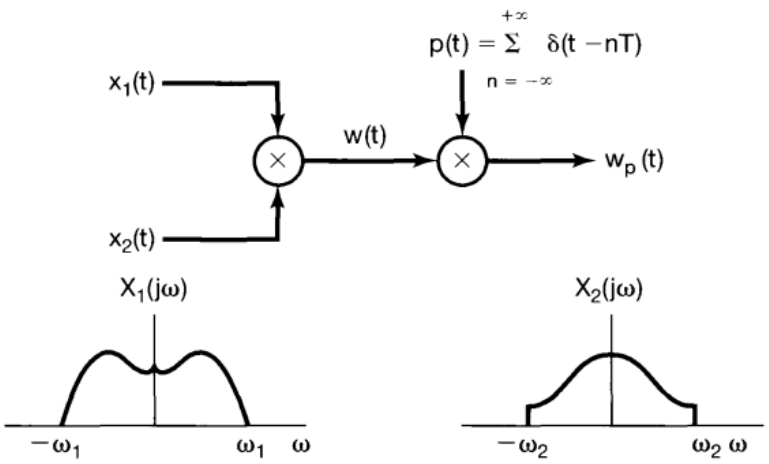
Tutorial 6b: Sampling and Reconstruction (Derivation of Equations)

A. Sampling of Signals

1. In the system shown in the figure given below, two functions of time, $x_1(t)$ and $x_2(t)$, are multiplied together, and the signals product $w(t)$ is sampled by a periodic impulse train. $x_1(t)$ is band limited to ω_1, and $x_2(t)$ is band limited to ω_2; that is,

$$X_1(j\omega) = 0, |\omega| \geq \omega_1$$

$$X_2(j\omega) = 0, |\omega| \geq \omega_2$$



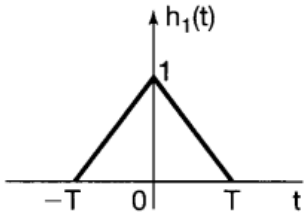
Determine the maximum sampling interval T such that $w(t)$ is recoverable from $w_p(t)$ through the use of an ideal low-pass filter. [10 marks]

2. A signal $x(t)$ undergoes a zero-order hold operation with an effective sampling period T to produce a signal $x_0(t)$. Let $x_1(t)$ denote the result of a first-order hold operation on the samples of $x(t)$; i.e.,

$$x_1(t) = \sum_{n=-\infty}^{\infty} x(nT)h_1(t - nT)$$

Where: $h_1(t)$ is the function shown in the figure given below.

Specify the frequency response of a filter that produces $x_1(t)$ as its output when $x_0(t)$ is the input. [20 marks]



3. Let $x_c(t)$ be a continuous-time signal whose Fourier transform has the property that $X_c(j\omega)$ = 0 for $|\omega| \geq 2{,}000\pi$. The continuous-time signal is part of a sampling system that produces a discrete sampled signal. From the sampling system, a discrete-time signal:

$$x_d[n] = x_c\left(n(0.5 \times 10^{-3})\right)$$

is obtained. For each of the following constraints on the Fourier transform $X_d(e^{j\omega})$ of $x_d[n]$, determine the corresponding constraint on $X_c(j\omega)$:

a. $X_d(e^{j\omega})$ is real. [2 marks]

b. The maximum value of $X_d(e^{j\omega})$ over all ω is 1. [2 marks]

c. $X_d(e^{j\omega})$ = 0 for $3\pi/4 \leq |\omega| \leq \pi$. [6 marks]

d. $X_d(e^{j\omega}) = X_d(e^{j(\omega - 1/T)})$. [2 marks]

B. Reconstruction of Signals

4. A discrete-time signal $x_d[n]$ has a Fourier transform $X_d(e^{j\omega})$ with the property that $X_d(e^{j\omega})$ = 0 for $3\pi/4 \leq |\omega| \leq \pi$. The signal is converted into a continuous-time signal where: T = 10^{-3}:

$$x_c(t) = T \sum_{n=-\infty}^{\infty} x_d[n] \left(\frac{\sin \frac{\pi}{T}(t - nT)}{\pi(t - nT)} \right)$$

a. Describe the reconstruction method used. [2 marks]

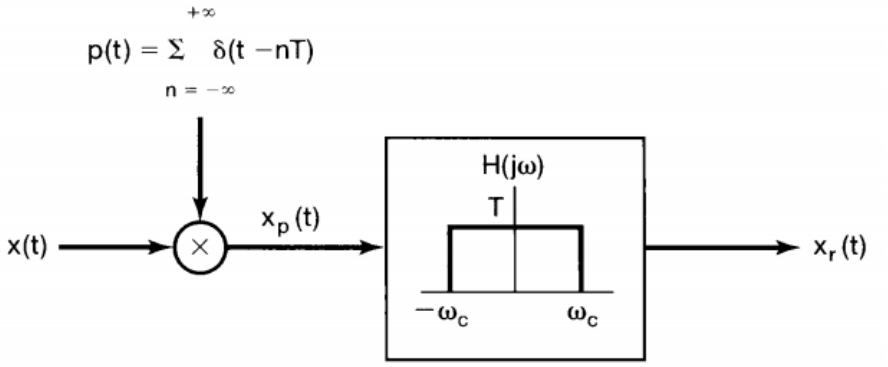
b. Determine the values of ω for which the Fourier transform $X_c(j\omega)$ of $x_c(t)$ is guaranteed to be zero. [2 marks]

5. In figure given below $p(t)$ is a sampler, followed by an ideal low-pass filter $H(j\omega)$, for reconstruction of $x(t)$ from its samples $x_p(t)$.

From the sampling theorem, we know that if $\omega_s = 2\pi/T$ is greater than twice the highest frequency present in $x(t)$ and $\omega_c = \omega_s/2$, then the reconstructed signal $x_r(t)$ will exactly equal $x(t)$.

If this condition on the bandwidth of $x(t)$ is violated, then $x_r(t)$ will not equal $x(t)$. We seek to show in this problem that if $\omega_c = \omega_s/2$, then for any choice of T, $x_r(t)$ and $x(t)$ will always be equal at the sampling instants; that is:

$$x_r(kT) = x(kT), k = 0, \pm 1, \pm 2, \ldots$$



To obtain this result, consider the following equation which expresses $x_r(t)$ in terms of the samples of $x(t)$:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)T\frac{\omega_c}{\pi}\frac{\sin[\omega_c(t-nT)]}{\omega_c(t-nT)}$$

With: $\omega_c = \omega_s/2$, this becomes:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\sin\left[\frac{\pi}{T}(t-nT)\right]}{\frac{\pi}{T}(t-nT)}$$

By considering the values of α for which $[\sin(\alpha)]/\alpha = 0$, show from the equation given above that, without any restrictions on $x(t)$, $x_r(kT) = x(kT)$ for any integer value of k. [10 marks]

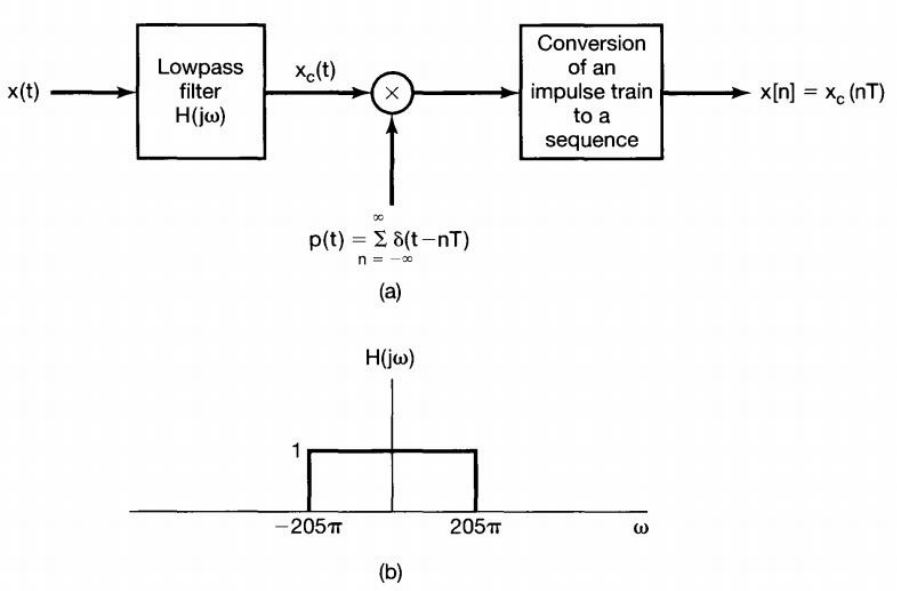
6. Figure (a) given below shows a system that converts a continuous-time signal to a discrete-time signal. The input $x(t)$ is periodic with a period of 0.1 second. The Fourier series coefficients of $x(t)$ are:

$$a_k = \left(\frac{1}{2}\right)^{|k|}, -\infty < k < +\infty$$

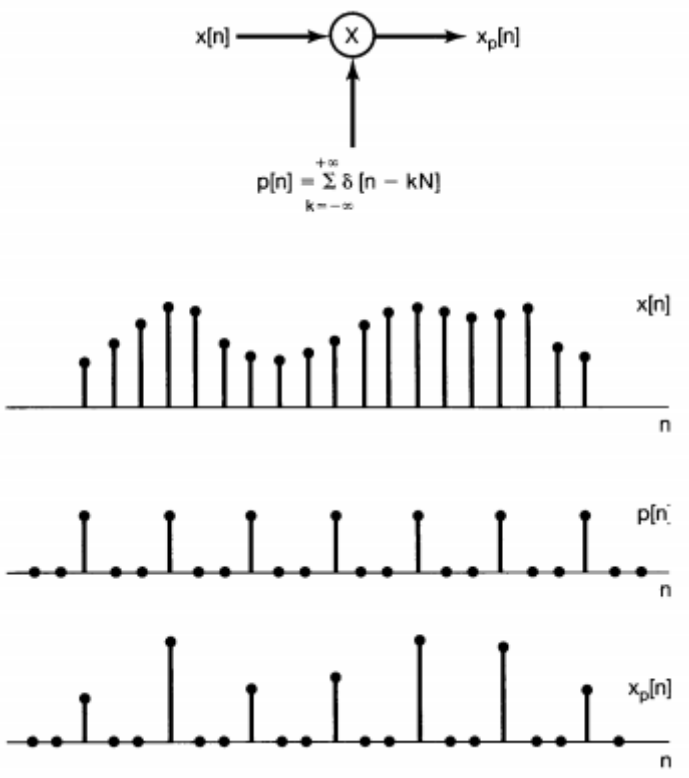
The low-pass filter $H(j\omega)$ has the frequency response shown in figure (b) below. The sampling period T = 5 x 10^{-3} second.

a. Show that $x[n]$ is a periodic sequence and determine its period. [18 marks]

b. Determine the Fourier series coefficients of $x[n]$. [4 marks]

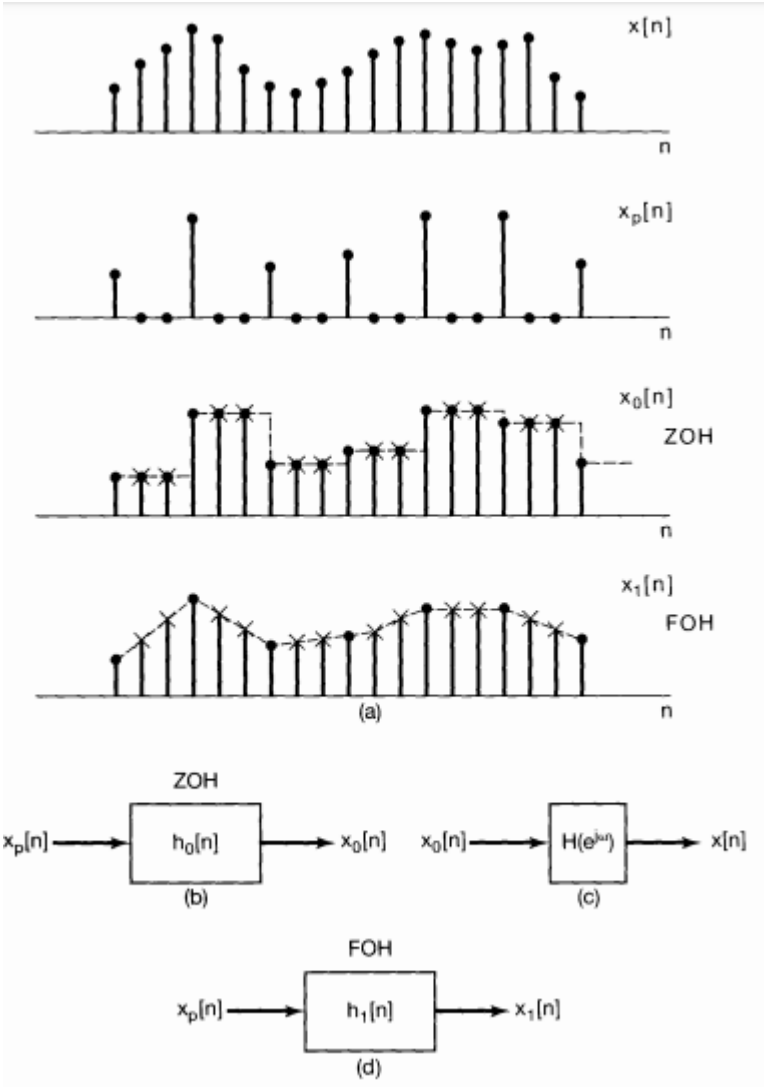


7. In this problem, we consider the discrete-time counterparts of the zero-order hold and first-order hold. Let $x[n]$ be a sequence to which discrete-time sampling, as illustrated in figure below.



Suppose the conditions of the discrete-time sampling theorem are satisfied; that is, $\omega_s > 2\omega_M$, where ω_s is the sampling frequency and $X(e^{j\omega}) = 0$, $\omega_M < |\omega| \leq \pi$. The original signal $x[n]$ is then exactly recoverable from $X_p[n]$ by ideal low-pass filtering corresponds to band-limited interpolation.

The zero-order hold represents an approximate interpolation whereby every sample value is repeated (or held) $N-1$ successive times, as illustrated in the figure below for the case of $N=3$. The first-order hold represents a linear interpolation between samples, as illustrated in the same figure.



a. The zero-order hold can be represented as an interpolation in the form of equation below:

$$x_r[n] = \sum_{k=-\infty}^{+\infty} x[kN]h_r[n-kN]$$

or, equivalently, the system in part (b) of the figure given above. Determine and sketch $h_0[n]$ for the general case of a sampling period N. [4 marks]

b. Sequence $x[n]$ can be exactly recovered from the zero-order-hold sequence $x_0[n]$ using an appropriate LTI filter $H(e^{j\omega})$, as indicated in part (c) of the figure given above. Determine $H(e^{j\omega})$. [8 marks]

c. The first-order hold (linear interpolation) can be represented as an interpolation in the form of equation given in part (a) of the question or, equivalently, the system in part (d) of the figure given above. Determine $h_1[n]$ for the general case of a sampling period N.

[2 marks]

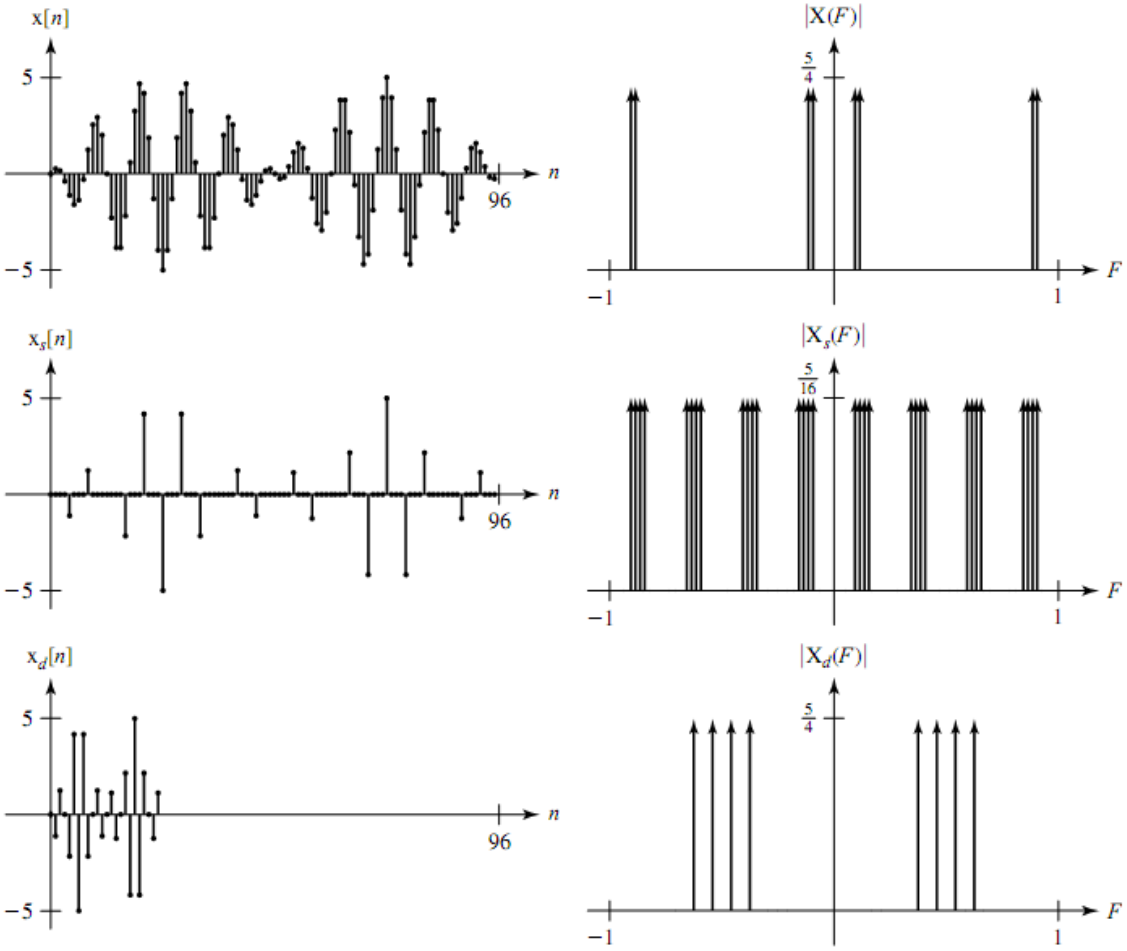
d. Sequence $x[n]$ can be exactly recovered from the first-order hold sequence $x_1[n]$ using an appropriate LTI filter with frequency response $H(e^{j\omega})$. Determine $H(e^{j\omega})$. [4 marks]

C. Decimation and Extrapolation

8. For the given sampling and reconstruction system, the given signal is undergone decimation process first, before the resulting decimated signal experience interpolation process.

a. The following figures outline original, sampled and down sampled (decimated) discrete-time signals and their DTFTs. Describe the significance of the time and frequency domain plots.

[12 marks]



b. The figures shown below show original, up sampled (extrapolated) and discrete-time low pass filtered discrete-time signals. Describe the significance of the given time domain plots.

[6 marks]

