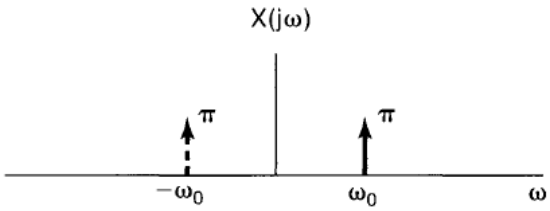
XMUT220 Signals and Systems

Tutorial 6c: Sampling and Reconstruction (Aliasing) - Solution

A. Aliasing in Signal Sampling

1. Aliasing influences the reconstruction of the original continuous-time signals from their discrete-time samples.



Given the frequency spectrum plot of a signal as shown above, for the following case studies of a sampling frequency at ω_s:

i. $\omega_o = \omega_s/6$

ii. $\omega_o = 2\omega_s/6$

iii. $\omega_o = 4\omega_s/6$

iv. $\omega_o = \frac{5\omega_s}{6}$

a. Sketch and describe the effect of aliasing in the frequency domain. [8 marks]

b. If the signal is a sinusoidal signal, sketch and describe the effect of aliasing in time domain. [16 marks]

Solution

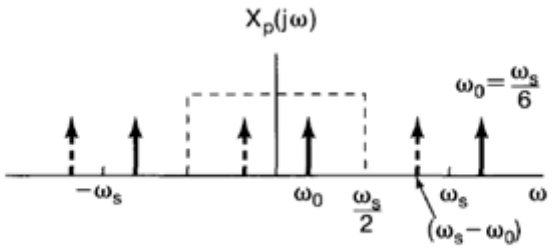
a. Sketch and describe the effect of aliasing in the frequency domain.

The case studies show the effect in the frequency domain of oversampling and undersampling:

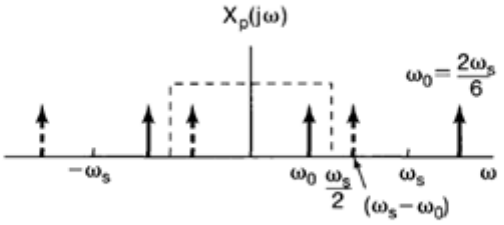
Figures (a), (b) spectrum of sampled signal with $\omega_s > 2\omega_o$; (c), (d) spectrum of sampled signal with $\omega_s < 2\omega_o$.

As we increase ω_o in moving from (a) through (d), the impulses drawn with solid lines move to the right, while the impulses drawn with dashed lines move to the left.

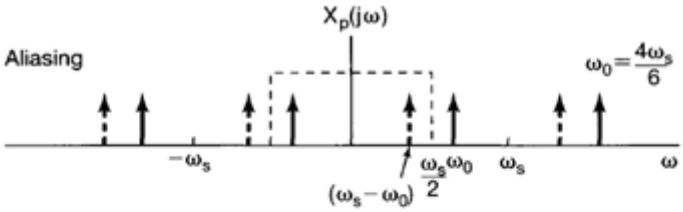
In figures (c) and (d), these impulses have moved sufficiently that there is a change in the ones falling within the passband of the ideal low-pass filter.



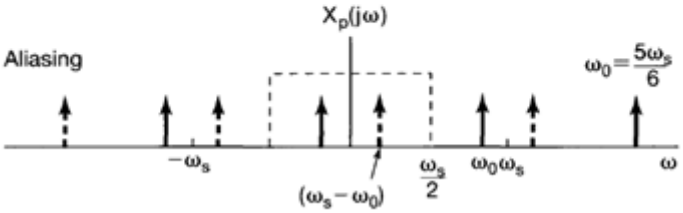
(a)



(b)



(c)

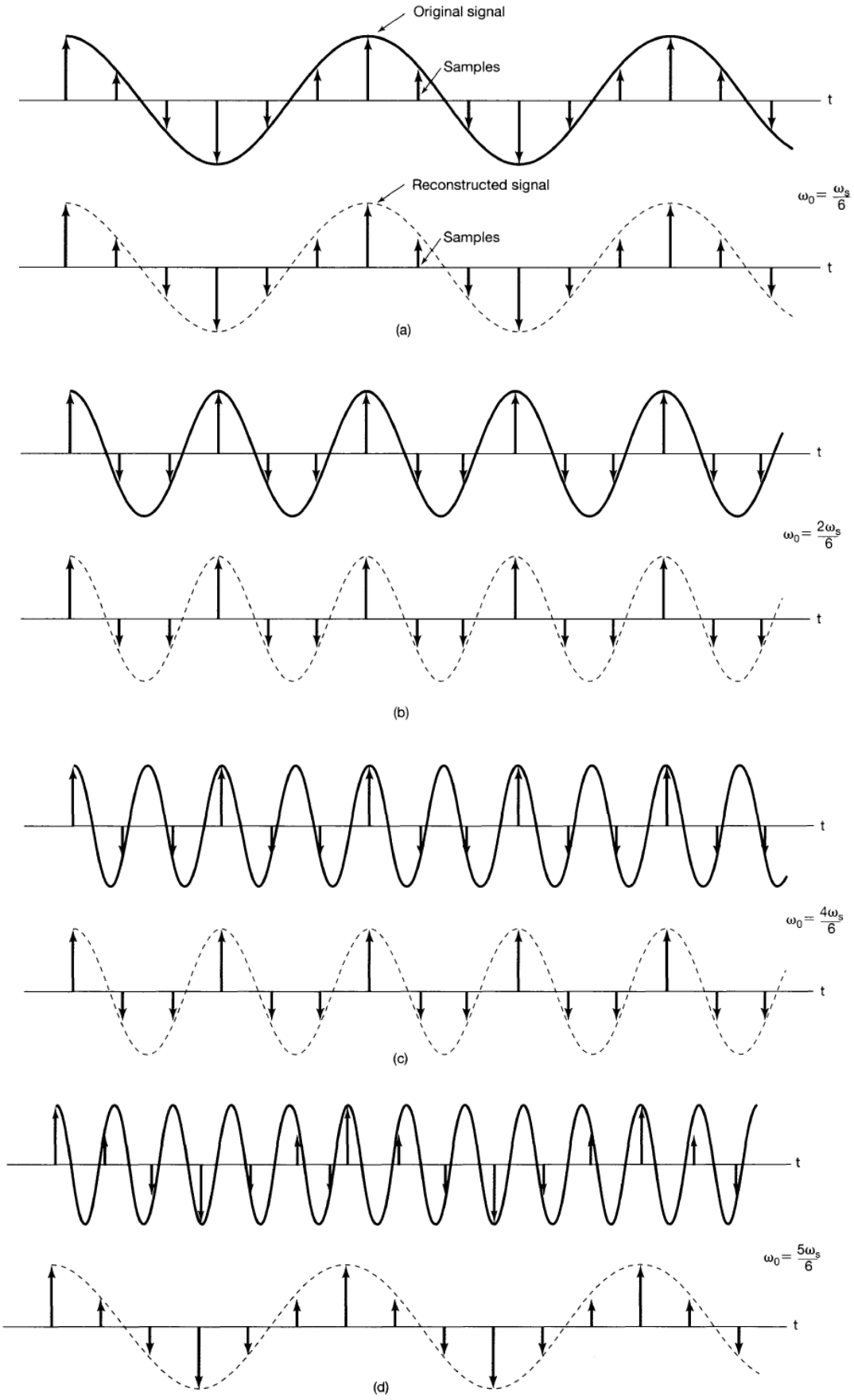


(d)

b. Sketch and description of the effect of aliasing if the signal is a sinusoidal signal.

For each of four values of ω_o, the original sinusoidal signal (solid curve), its samples, and the reconstructed signal (dashed curve) are illustrated: (a) $\omega_o = \omega_s/6$; (b) $\omega_o = 2\omega_s/6$; (c) $\omega_o = 4\omega_s/6$; (d) $\omega_o = 5\omega_s/6$.

In figures (a) and (b) no aliasing occurs, whereas in (c) and (d) there is aliasing.


Original signal
Samples
t
Reconstructed signal
$\omega_0 = \frac{\omega_s}{6}$
Samples
t
(a)
t
$\omega_0 = \frac{2\omega_s}{6}$
t
(b)
t
$\omega_0 = \frac{4\omega_s}{6}$
t
(c)
t
$\omega_0 = \frac{5\omega_s}{6}$
t
(d)

2. You are given a task to do signal sampling of a sinusoidal signal of frequency f_0 as follows with a sampling frequency $F = 8$ Hz.

$$x(t) = \sin(2\pi f_0 t)$$

Where: $t_0 = 1/F = 1/8$ is the time interval between two consecutive samples.

a. Considering the sampling effects in time domain as well as in frequency domain, determine and sketch original signals, reconstructed signals and spectrum of reconstructed signals for three different signal frequencies f_0 = 3, 5 and 9 Hz. [20 marks]

b. Comment further on the marginal case where the signal frequency $f_0 = F/2$ is precisely the Nyquist frequency. [4 marks]

Solution

a. The proof of the sampling theorem can usually be illustrated in the frequency domain and also be illustrated in the time domain. Consider sampling a sinusoidal signal of frequency f_0 with a sampling frequency $F = 8$ Hz.

$$x(t) = \sin(2\pi f_0 t) = \frac{1}{2j}\left[e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right]$$

The resulting samples are represented as:

$$x|m| = x(t)|_{t=mt_0} = x(mt_0) = x(m/F) = x(m/8) \qquad m = 0, \pm 1, \pm 2, \ldots$$

Where: $t_0 = 1/F = 1/8$ is the time interval between two consecutive samples.

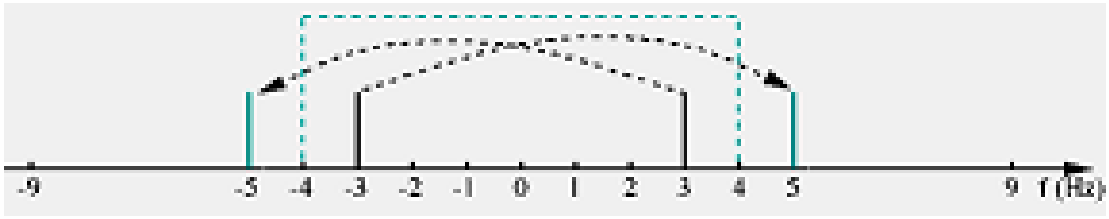
Considering the sampling effects in time domain as well as in frequency domain, sketch original signals, reconstructed signal and spectrum of reconstructed signal for three different signal frequencies f_0= 3, 5 and 9 Hz.

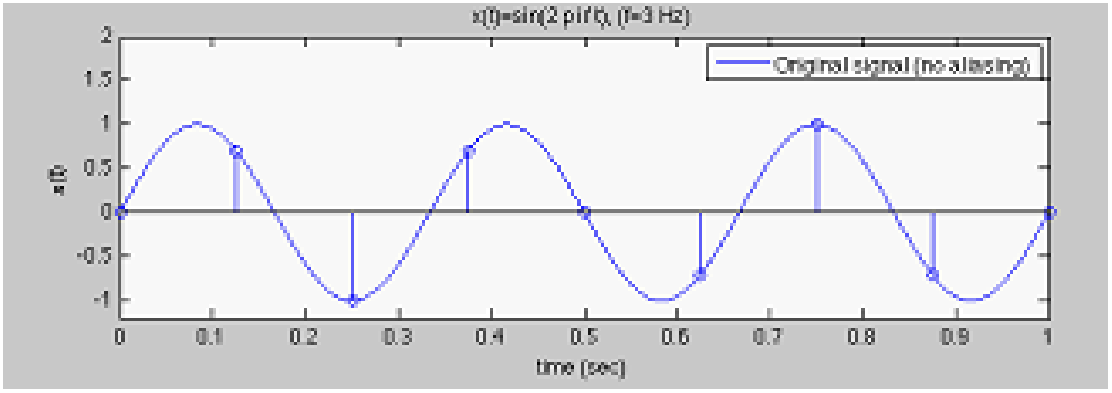
- For $f_0 = 3$ Hz (in this case $f_0 < F/2 = 4$ Hz):

$$x|m| = \sin(2\pi 3t)\,|_{t=m/8} = \sin(2m\pi 3/8)$$

$$= \frac{1}{2j}\left[e^{j2m\pi 3/8} - e^{-j2m\pi 3/8}\right]$$

The two frequency components $f_0 = \pm 3$ are inside the range $-4 < f < 4$. As the spectrum is periodic, they also appear at $f = \pm 3 + 8k$ for any integer k. But these replicas are all outside the range $-4 < f < 4$, causing no aliasing.





- For $f_0 = 5$ Hz (in this case $f_0 > F/2 = 4$ Hz):

  $x|m| \quad = \sin(2\pi 5t)\,|_{t=m/8} = \sin(2m\pi\, 5/8)$

  $= \frac{1}{2j}\left[e^{j2m\pi 5/8} - e^{-j2m\pi 5/8}\right]$

  At $F = 1/t_0 = 8$ samples/second, the 5 Hz frequency component is aliased to 3 Hz frequency component.

  $x|m| \quad = \frac{1}{2j}\left[e^{j2m\pi(8-3)/8} - e^{-j2m\pi(8-3)/8}\right]$

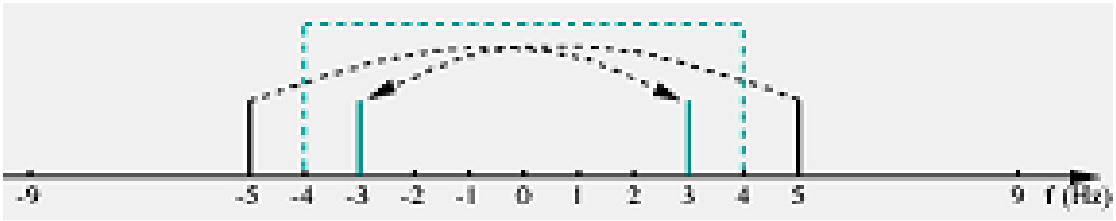
  $= \frac{1}{2j}\left[e^{-j2m\pi 3/8} - e^{j2m\pi 3/8}\right]$

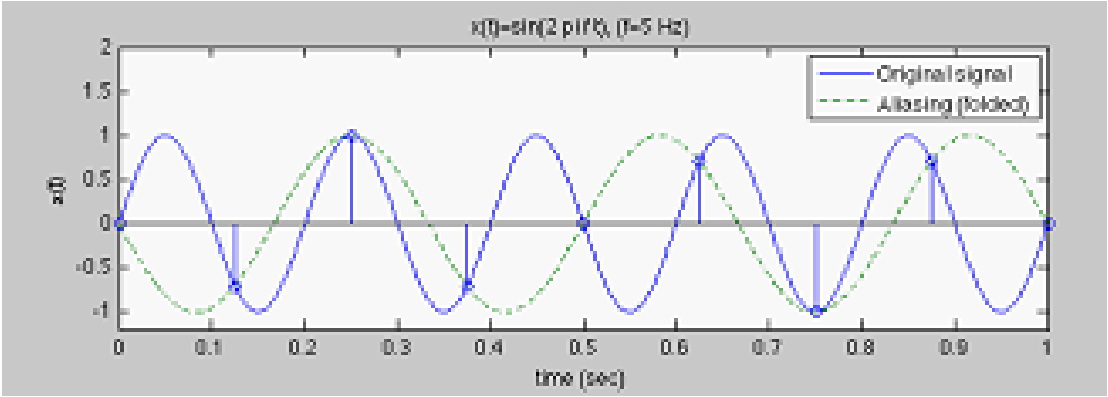
  $= -\sin(2m\pi\, 3/8)$

  This signal equation is identical to the discrete version of a signal $-\sin(2\pi 3t)$ with frequency $f = 3$ Hz, i.e., aliasing occurred.

  Moreover, the aliased signal also has a phase shift of 180°. This effect is also called folding.

  In frequency domain, the two real frequency components at $f = \pm 5$ are outside the range $-4 < f < 4$, but as $X(f + 8) = X(f)$ is periodic, the two replicas of the actual frequency components appear inside the range at $f = 5 - 8 = -3$ Hz and $f = -5 + 8 = 3$ Hz, respectively, causing aliasing.





- For $f_0 = 9$ Hz (in this case $f_0 > F/2 = 4$ Hz):

$$x|m| = \sin(2\pi 9t)\,|_{t=m/8} = \sin(2m\pi\, 9/8)$$

$$= \frac{1}{2j}\left[e^{j2m\pi 9/8} - e^{-j2m\pi 9/8}\right]$$

At $F = 1/t_0 = 9$ samples/second, the 9 Hz frequency component is aliased to 1 Hz frequency component.

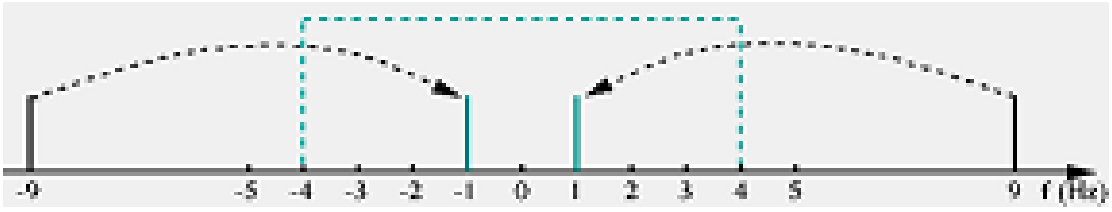
$$x|m| = \frac{1}{2j}\left[e^{j2m\pi(8+1)/8} - e^{-j2m\pi(8+1)/8}\right]$$

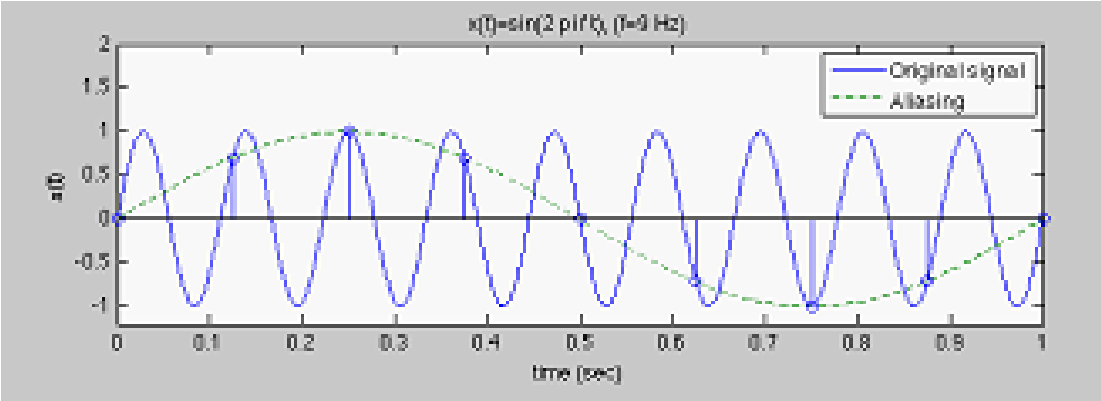
$$= \frac{1}{2j}\left[e^{-j2m\pi/8} - e^{j2m\pi/8}\right]$$

$$= -\sin(2m\pi/8)$$

The equation as above is identical to the discrete version of a different signal $\sin(2\pi t)$ with frequency $f = 1$ Hz, i.e., aliasing occurred as $f = 9 > F/2 = 4$.

In frequency domain, the two actual frequency components $f = \pm 9$ Hz are outside the range $-4 < f < 4$, but their replicas appear inside the range at $f = 9 - 8 = 1$ Hz and $f = -9 + 8 = -1$ Hz, respectively.





b. Finally, we consider the marginal case where the signal frequency $f_0 = F/2$ is precisely the Nyquist frequency. Assume again a sinusoidal signal $x(t) = \sin(2\pi f_0 t + \varphi)$is sampled at frequency $F = 2f_0$ to produce discrete samples:

$$x|m = x(m/F) = \sin(2m\pi f_0/F + \varphi) = \sin(m\pi + \varphi)$$

The value of $x|m|$ depends on the phase φ. For example, If $\varphi = 0$, $x|m| = 0$ for all m, aliasing occurred.

But, if $\varphi = \pi/2$, $x|m| = 1$ for even m and $x|m| = -1$ for odd m, i.e., the samples still represent $x(t)$ correctly. If φ takes any other value, $x|m|$ still takes alternating signs depending on whether m is even or odd, with its absolute value between 0 and 1.

3. Given a signal composed of two frequency components $f_1 = 7$ Hz and $f_2 = 2$ Hz:

$$x(t) = 2\cos(2\pi 7t) + \cos(2\pi 2t)$$

a. Assume the sampling frequency is $F = 1/t_0 = 10$ samples/second, and the sampling period is $t = 0.1$ seconds. Determine and sketch the discrete signal after sampling. [10 marks]

b. If the first frequency component is increased to $f_1 = 8$ Hz, while the sampling frequency is still $F = 1/t_0 = 10$. Determine and sketch the discrete signal after sampling. [10 marks]

c. Consider this signal given below at the same sampling rate of 10 Hz. Determine the discrete signal after sampling. [10 marks]

$$x(t) = \cos(2\pi 8t) - \cos(2\pi 2t)$$

Solution

For the given signal:

a. Consider a signal composed of two frequency components $f_1 = 7$ Hz and $f_2 = 2$ Hz:

$$x(t) = 2\cos(2\pi(7)t) + \cos(2\pi(2)t)$$

Assume the sampling frequency is $F = 1/t_0 = 10$ samples/second, and the sampling period is $t = 0.1$ seconds, then the discrete signal after sampling is:

$$x[n] \quad = x(nt_0)$$

$$= x(n/10)$$

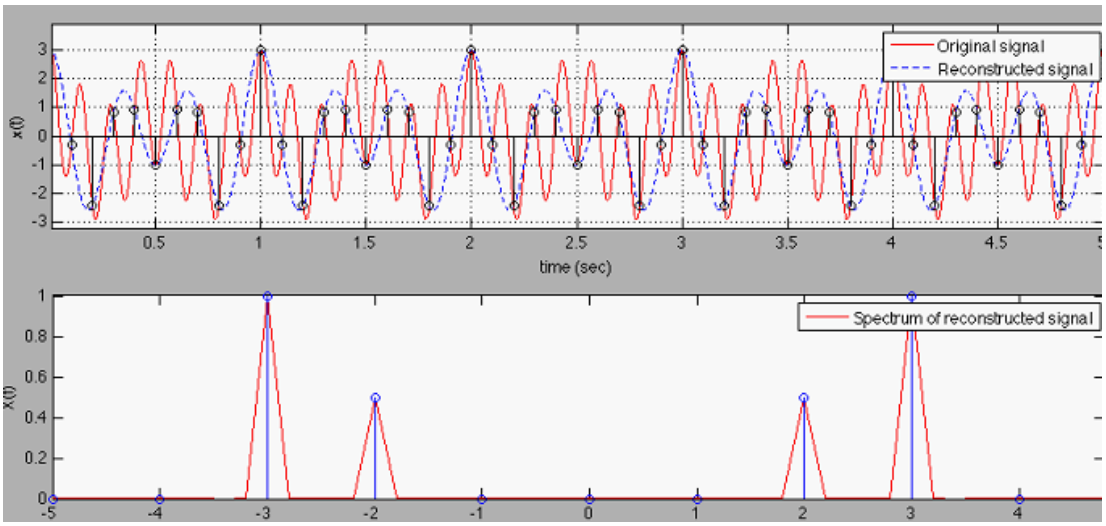
$$= 2\cos(2\pi 7n/10) + \cos(2\pi 2n/10)$$

$$= e^{j2\pi 7n/10} + e^{-j2\pi 7n/10} + \frac{1}{2}e^{j2\pi 2n/10} + \frac{1}{2}e^{-j2\pi 2n/10}$$

At $F = 1/t_0 = 10$ samples/second, the 7 Hz frequency component is aliased to 3 Hz frequency component.

$$x[n] = e^{j2\pi 3n/10} + e^{-j2\pi 3n/10} + \frac{1}{2}e^{j2\pi 2n/10} + \frac{1}{2}e^{-j2\pi 2n/10}$$

$$= 2\cos(2\pi 3n/10) + \cos(2\pi 2n/10)$$



b. Now if the first frequency is increased to $f_1 = 8$ Hz, while the sampling frequency is still $F = 1/t_0 = 10$. As a result, we have:

$$x(t) = 2\cos(2\pi(8)t) + \cos(2\pi(2)t)$$

The discrete signal after sampling is:

$$x[n] = x(nt_0)$$

$$= x(n/10)$$

$$= 2\cos(2\pi 8n/10) + \cos(2\pi 2n/10)$$

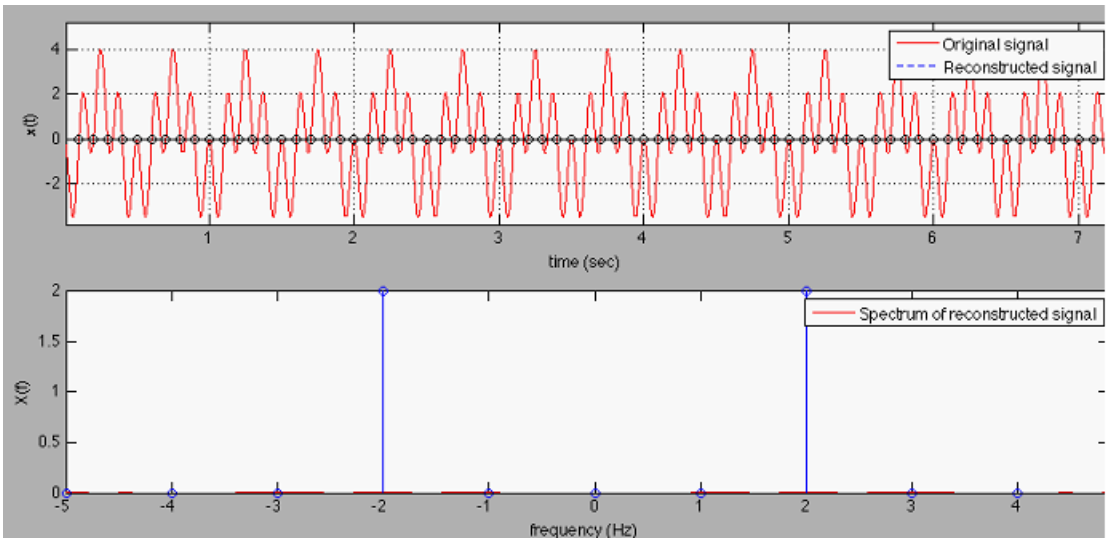
$$= e^{j2\pi 8n/10} + e^{-j2\pi 8n/10} + \frac{1}{2}e^{j2\pi 2n/10} + \frac{1}{2}e^{-j2\pi 2n/10}$$

At $F = 1/t_0 = 10$ samples/second, the 8 Hz frequency component is aliased to 2 Hz frequency component.

$$x[n] = e^{j2\pi 2n/10} + e^{-j2\pi 2n/10} + \frac{1}{2}e^{j2\pi 2n/10} + \frac{1}{2}e^{-j2\pi 2n/10}$$

$$= \frac{3}{2}e^{j2\pi 2n/10} + \frac{3}{2}e^{-j2\pi 2n/10}$$

$$= 3\cos(2\pi 2n/10)$$



c. Finally, consider this signal at the same sampling rate at 10 Hz:

$$x(t) = \cos(2\pi(8)t) - \cos(2\pi(2)t)$$

The discrete signal $x[n]$ after sampling is:

$$\begin{aligned} x[n] &= x(nt_0) \\ &= x(n/10) \\ &= \cos(2\pi 8n/10) - \cos(2\pi 2n/10) \\ &= \tfrac{1}{2}e^{j2\pi 8n/10} + \tfrac{1}{2}e^{-j2\pi 8n/10} - \tfrac{1}{2}e^{j2\pi 2n/10} - \tfrac{1}{2}e^{-j2\pi 2n/10} \end{aligned}$$

At $F = 1/t_0 = 10$ samples/second, the 8 Hz frequency component is aliased to 2 Hz frequency component.

$$\begin{aligned} x[n] &= \tfrac{1}{2}e^{j2\pi 2n/10} + \tfrac{1}{2}e^{-j2\pi 2n/10} - \tfrac{1}{2}e^{j2\pi 2n/10} - \tfrac{1}{2}e^{-j2\pi 2n/10} \\ &= 0 \end{aligned}$$

As a result, the two frequency components in the spectrum cancel each other.

Correspondingly, in time domain, the signal happens to be sampled at the zero-crossing, i.e., all samples are zero.

4. Providing a continuous signal as shown below:

$$x(t) = 26\cos(2\pi(2)t + \pi/2) + 14\cos(2\pi(5)t + \pi/2)$$

a. If the sampling rate is $F = 12$, determine and sketch the original signal, reconstructed signal and spectrum of reconstructed signals. [8 marks]

b. If the sampling rate is $F = 7$, determine and sketch the original signal, reconstructed signal and spectrum of reconstructed signals. [8 marks]

Solution

Consider the given continuous signal equation:

$$x(t) = 26\cos(2\pi(2)t + \pi/2) + 14\cos(2\pi(5)t + \pi/2)$$

The spectrum of this signal has four non-zero terms:

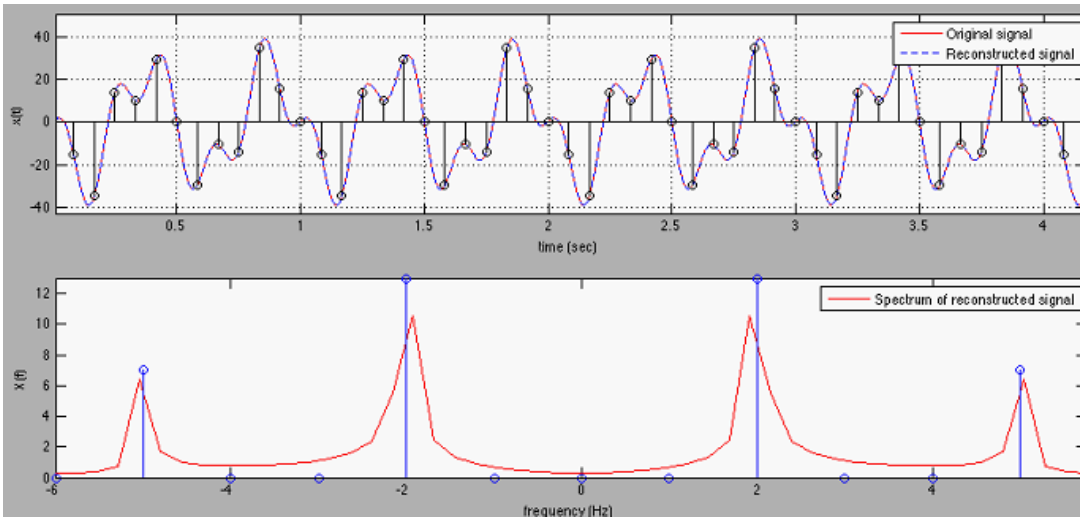
$$c_2 = 13e^{j\pi/2} \quad c_{-2} = 13e^{-j\pi/2}$$

$$c_5 = 7e^{-j\pi/2} \quad c_{-5} = 7e^{j\pi/2}$$

a. If the signal is sampled at a rate $F = 12$, the non-zero DFT coefficients are the same as the Fourier series coefficients:

$$X[2] = c_2 = 13e^{j\pi/2} \quad X[-2] = c_{-2} = 13e^{-j\pi/2}$$

$$X[5] = c_5 = 7e^{-j\pi/2} \quad X[-5] = c_{-5} = 7e^{j\pi/2}$$



b. However, if the sampling rate is $F = 7$, the signal frequencies 5 and -5 Hz aliased (folded) respectively to $5 - 7 = -2$ and $-5 + 7 = 2$ Hz, and superposed with the other frequency components.

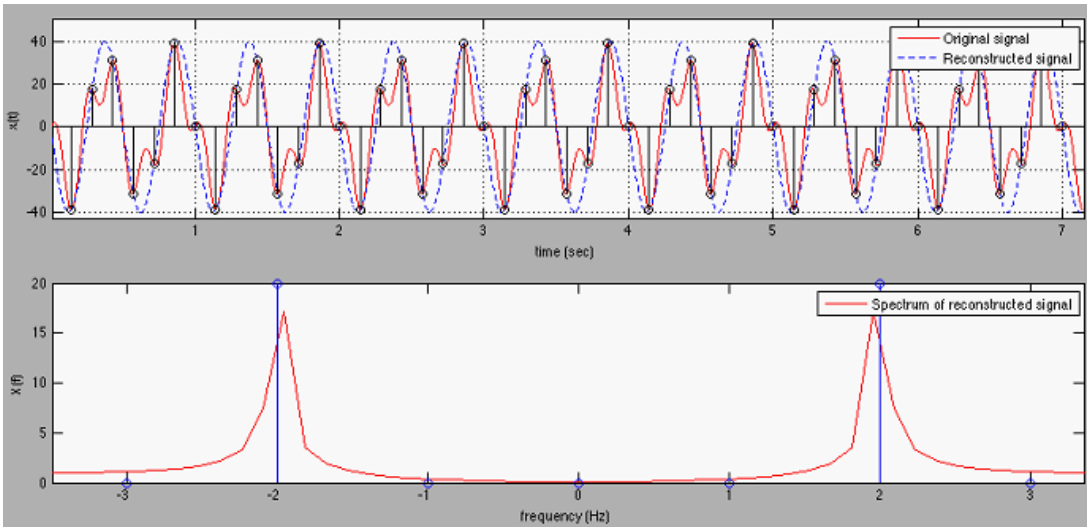
The resulting spectrum has two non-zero coefficients:

$$X[2] = 13e^{j\pi/2} + 7e^{j\pi/2} = 20e^{j\pi/2}$$

$$X[-2] = 13e^{-j\pi/2} + 7e^{-j\pi/2} = 20e^{-j\pi/2}$$

The reconstructed signal becomes:

$$y(t) = 40\cos(2\pi 2t + \pi/2) = -40\sin(2\pi 2t)$$

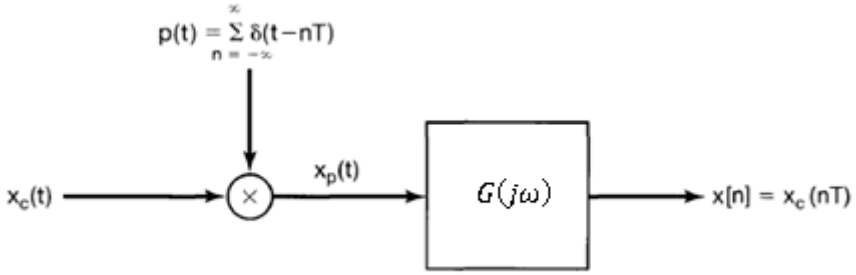


B. Aliasing in Applications of Sampling and Reconstructions

5. Consider a band-limited signal $x_c(t)$ that is sampled at a rate higher than the Nyquist rate.

   The samples, spaced T seconds apart, are then converted to a sequence $x[n]$, *i*s indicated in the figure given below.

   Given in the figure $G(j\omega)$ is a function in the system that performs conversion of the impulse trains to the sequence of discrete signals.



   Determine the relation between the energy E_d of the sequence, the energy E_c of the original signal, and the sampling interval T. How aliasing of the signal during the sampling process is prevented from happening? [14 marks]

   The energy of a sequence $x[n]$ is defined as:

$$E_d = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

   The energy in a continuous-time function $X_c(t)$ is defined as:

$$E_c = \int_{-\infty}^{+\infty} |x_c(t)|^2 dt$$

Solution

For the given bandwidth-limited signal and its sampling system, knowing that $X_c = 0$ for $|\omega| \geq \omega_0$, we get:

$$E_c = \int_{-\infty}^{\infty} |x_c(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} |X_c(j\omega)|^2 \, d\omega \qquad (Eq.1)$$

Parseval's theorem states:

$$E_d = \int_{-\infty}^{\infty} |x[n]|^2 \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|X(e^{j\Omega})\right|^2 \, d\Omega$$

Since $X\left(e^{j\Omega}\right) = (1/T)X_c(j\Omega/T)$ for $-\pi \leq \Omega \leq \pi$, the total power of the discrete sequence is:

$$E_d = \frac{1}{2\pi T^2} \int_{-\pi}^{\pi} |X_c(j\Omega/T)|^2 \, d\Omega$$

Replacing Ω/T by ω, we get:

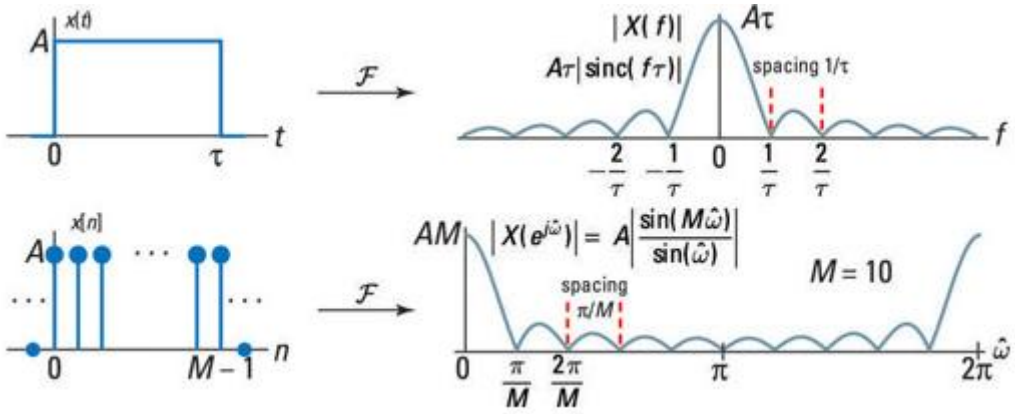
$$E_d = \frac{1}{2\pi T} \int_{-\pi/T}^{\pi/T} |X_c(j\omega)|^2 \, d\omega \qquad (Eq.2)$$

Aliasing can be prevented from happening if $2\pi/T \geq 2\omega_0$, so $\pi/T = \omega_0$.

Equating equation (1) with (2), as a result, the above equation becomes:

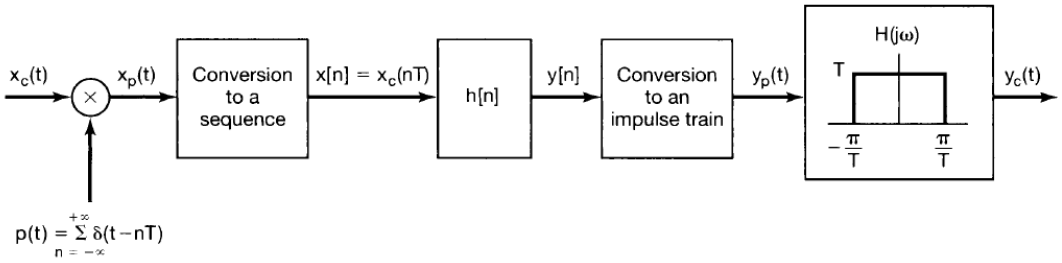
$$E_d = \frac{1}{2\pi T} \int_{-\omega_0}^{\omega_0} |X_c(j\omega)|^2 \, d\omega = \frac{E_c}{T}$$

The following diagram illustrates the relationships between these energies of continuous time signal with its discrete sequence.



6. In the system shown in the figure below, the input $x_c(t)$ is band limited with $X_c(j\omega) = 0, |\omega| > 2\pi \times 10^4$. The digital filter $h[n]$ is described by the input-output relation:

$$y[n] = T\sum_{k=-\infty}^{n} x[k]$$



a. What is the maximum value of T allowed if aliasing is to be avoided in the transformation from $x_c(t)$ to $x_p(t)$. [2 marks]

b. With the discrete-time LTI system specified through equation given above, determine its impulse response $h[n]$. [4 marks]

c. Determine whether there is any value of T for which:

$$\lim_{n\to\infty} y[n] = \lim_{t\to\infty} \int_{-\infty}^{t} x_c(\tau)d\tau$$

If so, determine the maximum value. If not, explain and specify how T would be chosen so that the equality in equation given above is best approximated. [16 marks]

Solution

For the given discrete-time LTI system

a. Knowing the Nyquist rate for $x_c(t) = 4\pi \times 10^4$, time constant for sampling $x(t)$ is:

$$T_{max} = \frac{2\pi}{4\pi \times 10^4} = 5 \times 10^{-3} \text{ second}$$

b. We have the following equation:

$$y[n] = T\sum_{k=-\infty}^{\infty} x[k] = T\sum_{k=-\infty}^{\infty} x[k]u[n-k] = T\{x[n] * u[n]\}$$

Therefore:

$$h[n] = T.u[n]$$

c. We have the following equation:

$$\lim_{n\to\infty} y[n] = \lim_{n\to\infty} T\sum_{k=-\infty}^{\infty} x[k] = TX(e^{j0})$$

Also

$$\lim_{t\to\infty} x_c[t] = X_c(j0)$$

Therefore, the given equation of:

$$\lim_{n\to\infty} y[n] = \lim_{t\to\infty} \int_{-\infty}^{t} x_c(\tau)d\tau$$

The above equation requires:

$$TX\left(e^{j\omega}\right) = X_c(j0)$$

Now

$$X\left(e^{j0}\right) = X_p(j\omega/T)$$

And

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j(\omega - 2\pi k/T)\right)$$

To avoid aliasing at $\omega = 0$ in $X_p(\omega)$, we require that:

$$2\pi/T > 2\pi \times 10^4$$

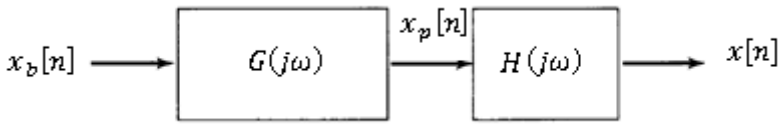
This implies that:

$$T < 10^{-4}$$

With this condition, the Fourier transform of the discrete signal becomes:

$$X\left(e^{j0}\right) = \frac{1}{T}X_c(j0)$$

7. Suppose the impulse response of an ideal discrete-time low-pass filter with cut-off frequency $\pi/2$ is interpolated to obtain an up sampling by a factor of 2.

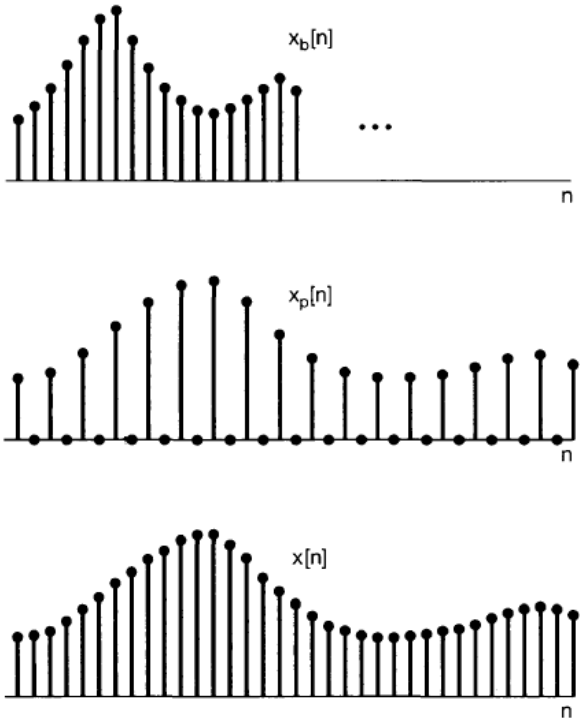


In accordance with the figure given above, $G(j\omega)$ is a function that performs conversion of decimated sequence to sampled sequence and $H(j\omega)$ is an ideal low-pass filter ($H(e^{j\omega})$).

Note that the illustrated sampling processes given below are associated with sequences and spectra for up sampling by a factor of 2.

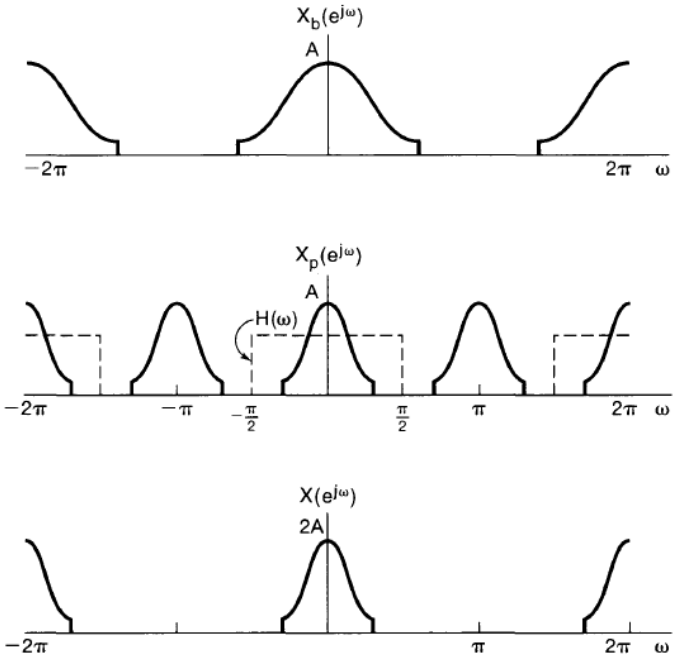
What is the frequency spectrum corresponding to this up sampled impulse response?

For up sampling and down sampling processes, how do you preserve the sampling of the signal without aliasing? [8 marks]



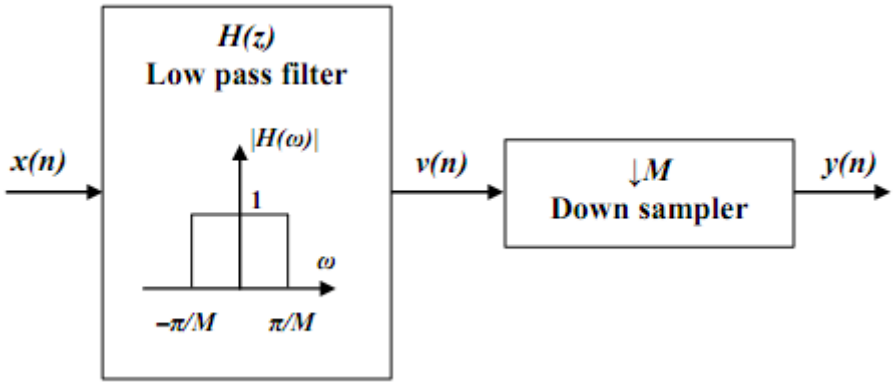
Solution

For the stated ideal discrete time low-pass filter, referring to the given frequency spectrum graph in the figure below, it is clear that interpolation by a factor of 2 results in frequency spectrum compressed by a factor of 2. Interpolation also results in a magnitude scaling by a factor of 2.



As a result, interpolated impulse response will correspond to an ideal low-pass filter with $f_c = 2\pi f/2 = \pi f$ and a gain of 2.

Although up sampling process will not result the signal sampling into aliasing problem, down sampling by N causes aliasing of any frequencies in the original signal above $|\omega| > \pi/N$, typically, the input signal may need to be first low pass-filtered to prevent this aliasing.



8. A real-valued discrete-time signal $x[n]$ has a Fourier transform $X(e^{j\omega})$ that is zero for $3\pi/14 \leq |\omega| \leq \pi$.

   By considering aliasing free sampling process, the nonzero portion of the Fourier transform of one period of $X(e^{j\omega})$ can be made to occupy the region $|\omega| < \pi$ by first performing upsampling by a factor of L and then performing down sampling by a factor of M.

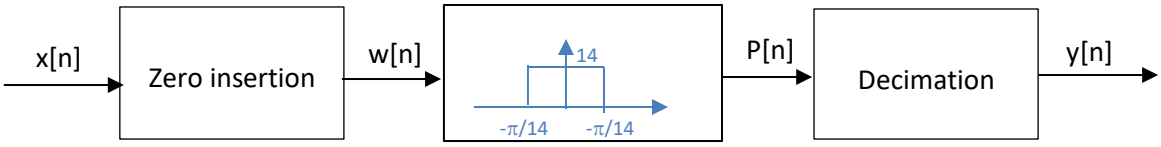
   Specify the values of L and M. How do you know that the sampling process is aliasing free?

   [8 marks]

Solution

For the given discrete-time signal, to make $X(e^{j\omega})$ to occupy entire region from $-\pi$ to π, signal $x[n]$ must be down-sampled by a factor of 14/3.

Since it is not possible to directly down-sample by a non-integer factor, first up-sample the signal by 3. Then, after up-sampling, reduce sampling rate by $14/3 \times 3 = 14$.



$$w[n] = \begin{cases} \frac{x[n]}{3} & n = 0, \pm 3, \pm 6 \\ 0, & \text{otherwise} \end{cases} \qquad y[n] = P[14n]$$

For the zero-insertion method that we implement to realise an up-sampling process we transform the sequence by a factor of L = 3.

$$w[n] = \begin{cases} \frac{x[n]}{3}, & n = 0, \pm 3, \pm 6 \\ 0, & \text{otherwise} \end{cases}$$

For the decimation method that we implement to realise a down-sampling process, we transform the sequence by a factor of M = 14.

$$y[n] = P[14n]$$

The sampling process is aliasing free since Fourier transform of one period of $X(e^{j\omega})$ is $|\omega| < \pi$ e.g. for this condition to happen, sampling frequency is more than the highest frequency component of the signal to be sampled.