XMUT220 Signals and Systems

Tutorial 7a: z-Transform (Introduction) - Solution

A. z-Transform

1. Find the z-transform of the following signals:

   a. $x[n] = \delta[n+5]$ [4 marks]

   b. $x[n] = u[n] - u[n-4]$ [4 marks]

   c. $x[n] = (-1)^n u[n]$ [4 marks]

   d. $x[n] = 2(0.8)^n u[n]$ [4 marks]

   e. $x[n] = (0.5)^{n+1} u[n+3]$ [4 marks]

   f. $x[n] = (0.25)^{-n} u[n-2]$ [4 marks]

   g. $x[n] = (2)^n u[-n] + (0.25)^n u[n-1]$ [6 marks]

   h. $x[n] = 1, 2, 3, 4$ [4 marks]

Answers

a. $x[n] = \delta[n+5]$.

Applying z-transform to the sequence given above:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta[n+5]\, z^{-n}$$

Applying the shifting property of the z-transform, the discrete unit impulse $\delta[n+5]$ is non-zero only when the argument $n+5=0$, which occurs at $n=-5$. At this point, $\delta[0]=1$, thus:

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n+5]\, z^{-n}$$

$$= 1 \cdot z^{-(-5)}$$

$$= z^5$$

With the Region of Convergence (ROC) is all z, except possibly $z = 0$ and $z = \infty$ because it is a finite duration signal.

b. $x[n] = u[n] - u[n-4]$.

Applying the z-transform to the sequence given above:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Using linearity property of z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} \{u[n] - u[n-4]\}\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u[n]\, z^{-n} - \sum_{n=-\infty}^{\infty} u[n-4]\, z^{-n}$$

$$= \mathbb{Z}\{u[n]\} - \mathbb{Z}\{u[n-4]\}$$

From the z-transform table, the z-transform of a unit step $u[n]$ is:

$$\mathbb{Z}\{u[n]\} = \frac{z}{z-1} \qquad \text{for } |z| > 1$$

Applying time-shifting property of z-transform, $x[n-k] \rightarrow z^{-k}X(z)$, thus:

$$\mathbb{Z}\{u[n-4]\} = z^{-4}\left(\frac{z}{z-1}\right) = \frac{z^{-3}}{z-1} \qquad \text{for } |z| > 1$$

Substituting the individual transforms back into the expression for $X(z)$ and simplifying the expression, thus:

$$X(z) = \frac{z}{z-1} - \frac{z^{-3}}{z-1} = \frac{z - z^{-3}}{z-1}$$

Multiplying the numerator and denominator by z^3 to get rid negative powers:

$$X(z) = \left(\frac{z^3}{z^3}\right)\left(\frac{z - z^{-3}}{z-1}\right) = \frac{z^4 - 1}{z^3(z-1)}$$

Factorising the numerator $z^4 - 1 = (z^2 - 1)(z^2 + 1) = (z-1)(z+1)(z^2+1)$, thus:

$$X(z) = \frac{(z-1)(z+1)(z^2+1)}{z^3(z-1)} = \frac{z^3 + z^2 + z + 1}{z^3}$$

Expressing the equation above as a polynomial in z^{-1}, the z-transform of the sequence given above is:

$$X(z) = \frac{z^3}{z^3} + \frac{z^2}{z^3} + \frac{z}{z^3} + \frac{1}{z^3} = 1 + z^{-1} + z^{-2} + z^{-3}$$

The resulting ROC of the z-transform is modified due to the finite duration of $x[n]$, i.e. it is non-zero only for n = 0, 1, 2, 3. The ROC becomes the entire z-plane i.e. $-\infty < |z| < \infty$, except for $z = 0$ (due to negative powers of z).

c. $x[n] = (-1)^n u[n]$.

Applying z-transform to the sequence given above:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} (-1)^n\, z^{-n}$$

This can be written as:

$$X(z) = \sum_{n=0}^{\infty} \left(\frac{-1}{z}\right)^n$$

The above expression is a geometric series $\sum_{n=0}^{\infty} r^n$ with $r = -1/z$. This series converges to $1/(1-r)$ if $|r| < 1$.

$$X(z) = \frac{1}{1-\left(\frac{-1}{z}\right)} = \frac{1}{1+\frac{1}{z}}$$

Simplifying the expression for $X(z)$:

$$X(z) = \frac{1}{\frac{(z+1)}{z}} = \frac{z}{z+1}$$

The region of convergence (ROC) is determined by the condition $|r| < 1$, so:

$$\left|\frac{-1}{z}\right| < 1$$

Thus, the ROC: $|z| > 1$

d. $x[n] = 2(0.8)^n u[n]$.

Applying z-transform to the sequence given above, the unit step function $u[n]$ makes the signal non-zero only for $n \geq 0$. The summation becomes:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} 2(0.8)^n\, z^{-n}$$

Factoring out the constant and rearranging the terms to form a standard geometric series with common ratio, $r = 0.8z^{-1}$:

$$X(z) = 2\sum_{n=0}^{\infty}(0.8z^{-1})^n$$

Using the geometric series sum formula $\sum_{n=0}^{\infty} r^n = 1/(1-r)$ for $|r| < 1$, we get:

$$X(z) = 2\left(\frac{1}{1-0.8z^{-1}}\right) = \frac{2}{1-0.8z^{-1}}$$

The region of convergence (ROC) is defined by the condition $|0.8z^{-1}| < 1$, which simplifies to $|z^{-1}| < 1/0.8$ or $|z| > 0.8$.

Or, alternatively

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= 2\sum_{n=0}^{\infty}(0.8)^n\, z^{-n}$$

$$= \frac{2}{1-0.8z^{-1}}$$

With the ROC: $|z| > 0.8$

e. $x[n] = (0.5)^{n+1}u[n+3]$.

Applying the z-transform to the sequence above, given $x[n] = (0.5)^{n+1}u[n+3]$, the unit step function $u[n+3]$ is 1 for $n \geq -3$ and 0 otherwise. The summation limits therefore change from $n = -3$ to ∞.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=-3}^{\infty}(0.5)^{n+1}\, z^{-n}$$

Using the standard geometric series formula, reindex the sum by letting $k = n+3$, which means $n = k-3$. When $n = -3$ and $k = 0$.

$$X(z) = \sum_{k=0}^{\infty}(0.5)^{(k-3)+1}\, z^{-(k-3)}$$

$$= \sum_{k=0}^{\infty}(0.5)^{k-2}\, z^{-k+3}$$

Factoring out the constant terms with respect to k:

$$X(z) = (0.5)^{-2} z^3 \sum_{k=0}^{\infty} (0.5)^k \, z^{-k}$$

$$= 4z^3 \sum_{k=0}^{\infty} \left(\frac{0.5}{z}\right)^k$$

The infinite geometric series $\sum_{k=0}^{\infty} r^k$ converges to $1/(1-r)$ if $|r| < 1$. Here, $r = 0.5/z$. The condition for convergence is $|0.5/z| < 1$, which simplifies to the Region of Convergence (ROC) $|z| > 0.5$. Applying the formula:

$$X(z) = 4z^3 \left(\frac{1}{1 - 0.5z^{-1}}\right)$$

By multiplying top and bottom of the equation in the bracket by z, simplify the expression algebraically:

$$X(z) = 4z^3 \left(\frac{z}{z - 0.5}\right)$$

$$= \frac{4z^4}{z - 0.5}$$

The Region of Convergence (ROC), $|z| > 0.5$.

Or, alternatively

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \, z^{-n}$$

$$= \sum_{n=-3}^{\infty} (0.5)^{n+1} \, z^{-n}$$

$$= \sum_{n=0}^{\infty} (0.5)^{n-2} \, z^{-n+3}$$

$$= (0.5)^{-2} z^3 \sum_{n=0}^{\infty} (0.5)^n \, z^{-n}$$

$$= \frac{4z^3}{1 - (0.5)z^{-1}}$$

$$= \frac{4z^4}{z - 0.5}$$

With the ROC: $|z| > 0.5$.

f. $x[n] = (0.25)^{-n} u[n-2]$.

For the given signal, the term $(0.25)^{-n}$ can be rewritten as:

$$(0.25)^n = \left(\frac{1}{4}\right)^{-n} = (4^{-1})^{-n} = 4^n$$

So, the sequence becomes $x[n] = 4^n u[n-2]$. Substituting $x[n]$ and using the property of the unit step function $u[n-2]$ (which is 1 for $n \geq 2$ and 0 otherwise), the summation limits change from $n = 2$ to ∞.

$$X(z) = \sum_{n=2}^{\infty} 4^n z^{-n} = \sum_{n=2}^{\infty} \left(\frac{4}{z}\right)^n$$

This is a geometric series of the form $\sum_{n=k}^{\infty} r^n$, which converges to $r^k/(1-r)$ if $|r| < 1$. Here, $r = 4/z$ and $k = 2$, thus:

$$X(z) = \frac{\left(\frac{4}{z}\right)^2}{1 - \left(\frac{4}{z}\right)}$$

Simplify the expression for $X(z)$:

$$X(z) = \frac{\left(\frac{16}{z^2}\right)}{\frac{(z-4)}{z}} = \frac{16}{z(z-4)}$$

The condition for convergence is $r < 1$, which means $|4/z| < 1$ or $|z| > 4$.

Or, alternatively

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \\ &= \sum_{n=2}^{\infty} (0.25)^{-n}\, z^{-n} \\ &= \sum_{n=0}^{\infty} (0.25)^{-n-2}\, z^{-n-2} \\ &= (0.25)^{-2} z^{-2} \sum_{n=0}^{\infty} (0.25)^n\, z^{-n} \\ &= 16\left[\frac{z^{-2}}{1 - 0.25z^{-1}}\right] \\ &= \frac{16}{z(z-4)} \end{aligned}$$

With the ROC: $|z| > 4$.

g. $x[n] = (2)^n u[-n] + (0.25)^n u[n-1]$.

There are two components of the z-transform of the sequence given above i.e. $x_1[n] = (2)^n u[-n]$ and $x_2[n] = (0.25)^n u[n-1]$.

For the z-transform of the first component $x_1[n] = (2)^n u[-n]$, its z-transform is:

$$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]\, z^{-n} = \sum_{n=-\infty}^{0} 2^n\, z^{-n} = \sum_{n=-\infty}^{0} (2z^{-1})^n$$

Letting $m = -n$, the summation becomes:

$$X_1(z) = \sum_{m=0}^{\infty} (2z^{-1})^{-m} = \sum_{m=0}^{\infty} (2^{-1}z)^m = \sum_{m=0}^{\infty} \left(\frac{z}{2}\right)^m$$

This is a geometric series that converges if $|z/2| < 1$, which means $|z| < 2$. The sum is:

$$X_1(z) = \frac{1}{1-\left(\frac{z}{2}\right)} = \frac{2}{2-z} = -\left(\frac{2}{z-2}\right)$$

With the Region of Convergence (ROC) for $X_1(z)$ is $|z| < 2$.

For the z-transform of the second component $x_2[n] = (0.25)^n u[n-1]$, the $u[n-1]$ term means the summation starts from $n = 1$.

$$X_2(z) = \sum_{n=1}^{\infty} (0.25)^n\, z^{-n} = \sum_{n=1}^{\infty} (0.25z^{-1})^n$$

We can express this starting from $n = 0$ by substratcting the $n = 0$ term:

$$X_2(z) = \sum_{n=1}^{\infty} (0.25)^n\, z^{-n} = \sum_{n=1}^{\infty} (0.25z^{-1})^n - (0.25z^{-1})^0$$

The infinite sum is a standard result for $a^n u[n]$ (where $a = 0.25$): $1/(1-az^{-1})$ for $|z| > |a|$.

$$X_2(z) = \frac{1}{1-0.25z^{-1}} - 1 = \frac{0.25z^{-1}}{1-0.25z^{-1}} = \frac{0.25}{z-0.25}$$

With the ROC for $X_2(z)$ is $|z| > 0.25$.

By using the linearity property of the z-transform, the overall transform $X(z)$ is the sum of $X_1(z)$ and $X_2(z)$.

$$X(z) = X_1(z) + X_2(z) = -\left(\frac{2}{z-2}\right) + \frac{0.25}{z-0.25}$$

The overall ROC is the intersection of the individual ROCs: $|z| < 2$ and $|z| > 0.25$, which is the annular region $0.25 < |z| < 2$.

Or, alternatively

$$X(z) = \sum_{n=-\infty}^{\infty} x_1[n]\, z^{-n} + \sum_{n=-\infty}^{\infty} x_2[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{0} (2)^n z^{-n} + \sum_{n=1}^{\infty} (0.25)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (2)^{-n} z^n + \sum_{n=0}^{\infty} (0.25)^{n+1} z^{-n-1}$$

$$= \sum_{n=0}^{\infty} (1/2)^n z^n + 0.25z^{-1} \sum_{n=0}^{\infty} (0.25)^n z^{-n}$$

$$= \frac{1}{1-(1/2)z^1} + \frac{0.25z^{-1}}{1-0.25z^{-1}}$$

$$= \frac{-2z^{-1}}{1-2z^{-1}} + \frac{0.25z^{-1}}{1-0.25z^{-1}}$$

$$= \frac{-2}{z-2} + \frac{0.25}{z-0.25}$$

With the ROC: $0.25 < |z| < 2$

h. $x[n] = 1, 2, 3, 4.$

The one-sided z-transform of a discrete sequence $x[n]$ is defined by the formula:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

For a finite sequence starting at $n = 0$, the summation limit becomes finite.

The given sequence is $x[0] = 1$, $x[1] = 2$, $x[2] = 3$, and $x[3] = 4$, with $x[n] = 0$ for $n \geq 4$. Substituting these values into the formula:

$$X(z) = x[0]z^{-0} + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \sum_{n=4}^{\infty} 0\, z^{-n}$$

Thus

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

With the ROC: $|z| > 0$

B. Inverse z-Transform

2. Find the inverse z-transforms of the following signals:

a. $X(z) = \frac{2}{(z^{-1}-2)(z^{-1}+1)}$ [10 marks]

b. $X(z) = \frac{(z^{-1}+0.8)}{(z^{-1}-0.5)(z^{-1}+0.2)}$ [10 marks]

c. $X(z) = \frac{(z^{-1}-1)(z^{-1}+0.8)}{(z^{-1}-0.5)(z^{-1}+0.2)}$ [12 marks]

d. $X(z) = \frac{(z^{-2}-1)(z^{-1}+0.8)}{(z^{-1}-0.5)^2(z^{-1}+0.2)}$ [14 marks]

Solution

a. For the given signal:

$$X(z) = \frac{2}{(z-2)(z+1)}$$

The equation becomes partial fraction as follows:

$$\frac{X(z)}{z} = \frac{c_1}{z-2} + \frac{c_2}{z+1}$$

Equating both sides:

$$\frac{2}{(z-2)(z+1)} = \frac{c_1}{z-2} + \frac{c_2}{z+1}$$

Multiple both sides by $(z-2)(z+1)$ to solve partial fraction:

$c_1 = \frac{X(z)}{z}(z-2)$ When z = 2,

$$c_1 = \frac{2}{(2+1)} = 2/3$$

$c_2 = \frac{X(z)}{z}(z+1)$ When z = -1,

$$c_2 = \frac{2}{(-1-2)} = -2/3$$

As a result, the equation becomes:

$$X(z) = \frac{2}{3(z-3)} - \frac{2}{3(z+1)}$$

Taking inverse z-transform to obtain signal in time domain:

$$x[n] = \frac{2}{3[(3)^n - (-1)^n]u[n]}$$

b. For the given signal

$$X(z) = \frac{z+0.8}{(z-0.5)(z+0.2)}$$

The equation becomes partial fraction as follows:

$$X(z) = \frac{c_1}{z-0.5} + \frac{c_2}{z+0.2}$$

Equating both sides:

$$\frac{(z+0.8)}{(z-0.5)(z+0.2)} = \frac{c_1}{z-0.5} + \frac{c_2}{z+0.2}$$

Multiple both sides by $(z-0.5)(z+0.2)$ to solve partial fraction:

$c_1 = \frac{X(z)}{z}(z-0.5)$ When z = 0.5,

$$c_1 = \frac{(0.5+0.8)}{(0.5+0.2)} = 1.857$$

$c_2 = \frac{X(z)}{z}(z+0.2)$ When z = -0.2,

$$c_2 = \frac{(-0.2+0.8)}{(-0.2-0.5)} = -0.8571$$

As a result, the equation becomes:

$$X(z) = \frac{1.857}{z-0.5} - \frac{0.8571}{z+0.2}$$

Taking inverse z-transform to obtain signal in time domain:

$$x[n] = [1.857(0.5)^n - 0.8571(-0.2)^n]u[n]$$

c. For the given signal:

$$X(z) = \frac{(z-1)(z+0.8)}{(z-0.5)(z+0.2)}$$

The equation becomes partial fraction as follows:

$$\frac{X(z)}{z} = \frac{c_1}{z} + \frac{c_2}{z-0.5} + \frac{c_3}{z+0.2}$$

Equating both sides:

$$\frac{(z-1)(z+0.8)}{(z-0.5)(z+0.2)} = \frac{c_1}{z} + \frac{c_2}{z-0.5} + \frac{c_3}{z+0.2}$$

Multiple both sides by z$(z-0.5)(z+0.2)$ to solve partial fraction:

$c_1 = \frac{X(z)}{z}z$ When $z = 0$,

$$c_1 = \frac{(0-1)(0+0.8)}{(0-0.5)(0+0.2)} = 8$$

$c_2 = \frac{X(z)}{z}(z-0.5)$ When z = 0.5,

$$c_2 = \frac{(0.5-1)(0.5+0.8)}{0.5(0.5+0.2)} = -1.857$$

$c_3 = \frac{X(z)}{z}(z+0.2)$ When z = -0.2,

$$c_3 = \frac{(-0.2-1)(-0.2+0.8)}{-0.2(-0.2-0.5)} = -5.143$$

As a result, the equation becomes:

$$X(z) = \frac{8}{z} - \frac{1.857z}{z-0.5} - \frac{5.143z}{z+0.2}$$

Taking inverse z-transform to obtain signal in time domain:

$$x[n] = 8\delta[n] - 1.857(0.5)^n u[n] - 5.143(-0.2)^n u[n]$$

d. For the given signal:

$$X(z) = \frac{(z^2-1)(z+0.8)}{(z-0.5)^2(z+0.2)}$$

The equation becomes partial fraction as follows:

$$\frac{X(z)}{z} = \frac{c_1}{z} + \frac{c_2}{z+0.2} + \frac{c_3}{z-0.5} + \frac{c_4}{(z-0.5)^2}$$

Equating both sides:

$$\frac{(z^2-1)(z+0.8)}{(z-0.5)^2(z+0.2)} = \frac{c_1}{z} + \frac{c_2}{z+0.2} + \frac{c_3}{z-0.5} + \frac{c_4}{(z-0.5)^2}$$

Multiple both sides by $z(z-0.5)^2(z+0.2)$ to solve partial fraction:

$c_1 = \frac{X(z)}{z}z$ When $z = 0$,

$$c_1 = \frac{((0)^2-1)(0+0.8)}{(0-0.5)^2(0+0.2)} = -16$$

$c_2 = \frac{X(z)}{z}(z+0.2)$ When z = -0.2,

$$c_2 = \frac{((-0.2)^2-1)(-0.2+0.8)}{-0.2(-0.2-0.5)^2} = 5.88$$

$c_4 = \frac{X(z)}{z}(z-0.5)^2$ When z = 0.5,

$$c_4 = \frac{((0.5)^2-1)(0.5+0.8)}{0.5(0.5+0.2)} = -2.79$$

Finding c_3:

$$(z^2-1)(z+0.8) = (z-0.5)^2(z+0.2)(-16) + z(z-0.5)^2(5.88) + z(z-0.5)(z+0.2)c_3 + z(z+0.2)(-2.79))$$

When z = -0.5, and knowing $c_1 = -16, c_2 = 5.88, c_4 = -2.79$ as a result $c_3 = 11.12$

$$X(z) = \frac{-16}{z} + \frac{5.88}{z+0.2} + \frac{11.12}{z-0.5} - \frac{-2.79}{(z-0.5)^2}$$

Taking reversed z-transform to obtain the signal in time domain:

$$x[n] = -16\delta[n] + 5.88(-0.2)^n u[n] + 11.12(0.5)^n u[n] - 2.79n(0.5)^n u[n]$$

C. Properties of Z-Transform

3. Describe and prove the linearity property of z-transform. For any two complex numbers: α and β we have: [8 marks]

$$Z[\alpha x[n] + \beta y[n]] = \alpha X(z) + \beta Y(z) \quad \text{for} \quad r < |z| < R$$

Solution

Linearity property of z-transform is when two or more individual discrete signals are multiplied by constants, their respective z-transforms will also be multiplied by the same constants.

Let

$$w[n] = \alpha x[n] + \beta y[n]$$

Then

$$W(z) = \sum_{n=-\infty}^{\infty} [\alpha x[n] + \beta y[n]] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \alpha x[n]\, z^{-n} + \sum_{n=-\infty}^{\infty} \beta y[n]\, z^{-n}$$

$$= \alpha \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} + \beta \sum_{n=-\infty}^{\infty} y[n]\, z^{-n}$$

$$= \alpha X(z) + \beta Y(z)$$

ROC of $W(z) = \alpha X(z) + \beta Y(z)$ contains all z such that:

$$z \in \mathrm{ROC}(X(z)) \cap \mathrm{ROC}(Y(z))$$

Then

$$(r_x < |z| < R_x) < |z| < (r_y < |z| < R_y)$$

As a result

$$\max(r_x, r_y) < |z| < \min(R_x, R_y)$$

If the linear combination cancelled, some poles the region of convergence may be larger.

For example, the sequences $c^n u[n]$ and $c^n u[n-1]$ both have a region of convergence defined by $|z| > |c|$, but the sequence corresponding to the difference $c^n u[n] - c^n u[n-1] = \delta[n]$ has the entire z-plane as its ROC.

4. Describe and prove the time-reversal property of z-transform: [6 marks]

$$Z[x[-n]] = X(z^{-1}) \quad \text{for} \quad \frac{1}{R_x} < |z| < \frac{1}{r_x}$$

Solution

Time-reversal property is inverse in the time domain affects the z- domain by z^{-1}

Applying z-transform to the given equation:

$$Z[x[-n]] = \sum_{n=-\infty}^{\infty} x[-n]\, z^{-n}$$

Let $m = -n$, then the above equation becomes:

$$\sum_{n=-\infty}^{\infty} x[m]\,(z)^m = \sum_{m=-\infty}^{\infty} x[m](z^{-1})^{-m} = X(z^{-1})$$

Then, the region of convergence is:

$$\text{ROC} \quad r_x < |z| < R_x \quad \text{or} \quad \frac{1}{R_x} < |z| < \frac{1}{r_x}$$

5. By applying the linear property of z-transform, determine the z-transform of the sequence: [6 marks]

$$x[n] = \cos(n\omega)\, u[n] \qquad \omega \in \mathbb{C}$$

Solution

By using Euler's identity:

$$\cos(n\omega) = \frac{e^{n\omega j} + e^{-n\omega j}}{2}$$

The $X(z)$ will be:

$$X(z) = Z[\cos(n\omega)u[n]] = Z\left[\left(\frac{e^{n\omega j} + e^{-n\omega j}}{2}\right)u[n]\right]$$

By using the linear property of z-transform, we get:

$$X(z) = \frac{1}{2}\{Z[e^{n\omega j}u[n]] + Z[e^{-n\omega j}u[n]]\}$$

6. By applying the time-reversal and differentiation properties of z-transform, determine the z-transform of the sequence: [6 marks]

$$x[n] = n3^n u[-n]$$

Solution

With $\alpha = 1/3 \quad \left(|\alpha| = \left|\frac{1}{3}\right| = \frac{1}{3}\right)$, we get:

$$Z\left[\left(\frac{1}{3}\right)^n u[n]\right] = \frac{z}{z - \frac{1}{3}} \qquad \text{for} \quad |z| > \frac{1}{3}$$

Then, by time reversal property, we obtain:

$$Z[3^n u[-n]] = \frac{z^{-1}}{z^{-1} - \frac{1}{3}} = \frac{3}{3 - z} \qquad \text{for} \quad |z| < 3$$

Finally, using differentiation (multiplication by n) property, it follows that:

$$Z[n3^n u[-n]] = -z\left[\frac{d}{dz}\left(\frac{3}{3 - z}\right)\right] = \frac{-3z}{(3 - z)^2} \qquad \text{for} \quad |z| < 3$$

7. With help of conjugation property of z-transform, if:

$$Y(z) = Z\left[(1 + j)^n u[n]\right] \qquad \text{or} \qquad Y(z) = \frac{z}{z - (1 + j)}, |z| > \sqrt{2}$$

Determine the z-transform of: [4 marks]

$$x[n] = (1 - j)^n u[n]$$

Solution

Note that $x[n] = [(1 + j)^n u[n]]^*$.

Using conjugation property:

$$Z[x[n]] = Y^*(z^*) = \left[\frac{z^*}{z^* - (1 + j)}\right]^* \qquad \text{for} \quad |z| > \sqrt{2}$$

As a result:

$$x[n] = \left(\frac{-1}{2}\right)^{|n|} \qquad \text{for} \quad |z| > \sqrt{2}$$

8. By applying the Parseval's theorem property of z-transform, determine the z-transform of the following: [12 marks]

$$\sum_{n=-\infty}^{+\infty} x[n]y[n]$$

Where: $x[n] = \left(\frac{1}{2}\right)^n u[n]$ and $y[n] = \left(\frac{1}{3}\right)^n u[n]$

Solution

From the Parseval's theorem, since $y[n]$ is real, we use the expression:

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi j}\oint X_1(v)\, X_2^*\left(\frac{1}{v^*}\right)\frac{dv}{v}$$

We obtain:

$$\sum_{n=-\infty}^{\infty} x[n]y[n] = \frac{1}{2\pi j}\oint_c X(z)\, Y\left[\frac{1}{z}\right]\frac{dz}{z}$$

Thus,

$$X(z) = \frac{z}{z-\frac{1}{2}} \qquad \text{for } |z| > \frac{1}{2}$$

As a result,

$$Y\left(\frac{1}{z}\right) = \frac{3}{3-z} = -\frac{3}{z-3} \qquad \text{for } |z| < 3$$

So, the ROC of the $X(z)$ and $Y\left(\frac{1}{z}\right)$ is $1/2 < |z| < 3$

$$\sum_{n=-\infty}^{\infty} x[n]\, y[n] = \frac{1}{2\pi j}\oint_c \left(\frac{z}{z-\frac{1}{2}}\right)\left(\frac{-3}{z-3}\right)dz = \frac{1}{2\pi j}\oint_c \frac{-3}{\left(z-\frac{1}{2}\right)(z-3)}dz$$

Where c is any circle with radius r, where $1/2 < r < 3$

$$\sum_{n=-\infty}^{\infty} x[n]y[n] = \text{Res}\left[-\frac{3}{\left(z-\frac{1}{2}\right)(z-3)}, \frac{1}{2}\right] = \frac{6}{5}$$

9. By applying the linearity and time reversal properties of z-transform, determine the z-transform of the sequence: [6 marks]

$$x[n] = \left(-\frac{1}{2}\right)^{|n|}$$

Solution

We can write $x[n]$ as:

$$x[n] = \left(-\frac{1}{2}\right)^{|n|} u[n] + \left(-\frac{1}{2}\right)^{|n|} u[-n] - \delta[n]$$

Using the linearity and time reversal properties, we get:

$$X[z] = \frac{z}{z+\frac{1}{2}} + \frac{z^{-1}}{z^{-1}+\frac{1}{2}} - 1 \quad \text{for} \quad |-1/2| < |z| < 1/|-1/2|$$

Then

$$X[z] = \frac{3z}{(2z+1)(z+2)} \quad \text{for} \quad 1/2 < |z| < 2$$