XMUT220 Signals and Systems

Tutorial 7a: z-Transform (Introduction)

A. Z-Transform

1. Find the z-transform of the following signals:

   a. $x[n] = \delta[n+5]$ [4 marks]

   b. $x[n] = u[n] - u[n-4]$ [4 marks]

   c. $x[n] = (-1)^n u[n]$ [4 marks]

   d. $x[n] = 2(0.8)^n u[n]$ [4 marks]

   e. $x[n] = (0.5)^{n+1} u[n+3]$ [4 marks]

   f. $x[n] = (0.25)^{-n} u[n-2]$ [4 marks]

   g. $x[n] = (2)^n u[-n] + (0.25)^n u[n-1]$ [6 marks]

   h. $x[n] = 1, 2, 3, 4$ [4 marks]

B. Inverse Z-Transform

2. Find the inverse z-transforms of the following signals:

   a. $X(z) = \frac{2}{(z^{-1}-2)(z^{-1}+1)}$ [10 marks]

   b. $X(z) = \frac{(z^{-1}+0.8)}{(z^{-1}-0.5)(z^{-1}+0.2)}$ [10 marks]

   c. $X(z) = \frac{(z^{-1}-1)(z^{-1}+0.8)}{(z^{-1}-0.5)(z^{-1}+0.2)}$ [12 marks]

   d. $X(z) = \frac{(z^{-2}-1)(z^{-1}+0.8)}{(z^{-1}-0.5)^2(z^{-1}+0.2)}$ [14 marks]

C. Properties of Z-Transform

3. Describe and prove the linearity property of z-transform. For any two complex numbers: α and β we have: [8 marks]

$$Z[\alpha x[n] + \beta y[n]] = \alpha X(z) + \beta Y(z) \quad \text{for} \quad r < |z| < R$$

4. Describe and prove the time-reversal property of Z-transform: [6 marks]

$$Z\big[x[-n]\big] = X(z^{-1}) \quad \text{for} \quad \frac{1}{R_x} < |z| < \frac{1}{r_x}$$

5. By applying the linear property of Z-transform, determine the Z-transform of the sequence: [6 marks]

$$x[n] = \cos(n\omega)\, u[n] \qquad \omega \in \mathbb{C}$$

6. By applying the time reversal and differentiation properties of Z-transform, determine the Z-transform of the sequence: [6 marks]

$$x[n] = n3^n u[-n]$$

7. With help of conjugation property of Z-transform, if:

$$Y(z) = Z\big[(1+j)^n u[n]\big] \qquad \text{or} \qquad Y(z) = \frac{z}{z-(1+j)}, |z| > \sqrt{2}$$

Determine the Z-transform of: [4 marks]

$$x[n] = (1-j)^n u[n]$$

8. By applying the Parseval's theorem property of Z-transform, determine the Z-transform of the following: [12 marks]

$$\sum_{n=-\infty}^{+\infty} x[n]y[n]$$

Where: $x[n] = \left(\frac{1}{2}\right)^n u[n]$ and $y[n] = \left(\frac{1}{3}\right)^n u[n]$

9. By applying the linearity and time reversal properties of Z-transform, determine the z-transform of the sequence: [5 marks]

$$x[n] = \left(-\frac{1}{2}\right)^{|n|}$$

Appendix 1 - z-Transform

No	Sequence	Transform	ROC
1	$\delta[n]$	1	All z
2	$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\|>1$
3	$-u[n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\|<1$
4	$\delta[n-m]$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\|>\|a\|$
6	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\|<\|a\|$
7	$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|>\|a\|$
8	$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|<\|a\|$
9	$\begin{cases} a^n, & 0\le n\le N-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\|>0$
10	$\sin(\omega_0 n)u[n]$	$\frac{\sin\omega_0 z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\|>1$
11	$\cos(\omega_0 n)u[n]$	$\frac{1-\cos\omega_0 z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\|>1$
12	$[a^n \sin\omega_0 n]u[n]$	$\frac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\|>\|a\|$
13	$[a^n \cos\omega_0 n]u[n]$	$\frac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\|>\|a\|$

Appendix 2 - z-transform Properties

No	Property	Sequence	Transform	ROC
1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2.
2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the region.
3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X\left(e^{-j\omega_0}z\right)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e. the set of points $\{\|a\|z\}$for z in R.
4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R(i.e. R^{-1} is equal to the set of point z^{-1}, where z is in R).
5	Time expansion	$x_{(k)}[n]\begin{cases}x[r], n = rk \\ 0, n \neq rk\end{cases}$ for some integer r	$X\left(z^k\right)$	$R^{1/k}$(i.e. the set of point $z^{1/k}$, where z is in R).
6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
7	Convolution	$x_1[n] * x_2[n]$	$X_1(z) \cdot X_2(z)$	At least the intersection of R_1 and R_2.
8	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$.
9	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\left(\frac{1}{1-z^{-1}}\right)X(z)$	At least the intersection of R and $\|z\| > 1$.
10	Differentiation in the z-domain	$nx[n]$	$-z\left[\frac{dX(z)}{dz}\right]$	R
11	Initial value theorem	If $x[n] = 0$ for $x < 0$, then $x[0] = \lim_{z\to\infty} X(z)$		

Note:

- For a sequence $x[n]$, its z-transform is $X(z)$ and its ROC is R.
- For a sequence $x_1[n]$, its z-transform is $X_1(z)$ and its ROC is R_1.
- For a sequence $x_2[n]$, its z-transform is $X_2(z)$ and its ROC is R_2.