XMUT220 Signals and Systems

Tutorial 7b: z-Transform (Applications) – Solution

A. Solving Discrete Systems and Differential Equations

1. Find the system function $H(z)$ for each causal discrete-time LTI system described below by means of a difference equation. Afterwards determine the impulse response of the system using partial fraction expansion.

   a. $y[n] = 0.9y[n-1] + x[n] - x[n-1]$ [6 marks]

   b. $y[n] = 1.7y[n-1] - 0.72y[n-2] + x[n] - 2x[n-1]$ [8 marks]

   c. $y[n] = 1.7y[n-1] - 0.72y[n-2] + x[n] + x[n-1] + x[n-2]$ [10 marks]

   d. $y[n] = y[n-1] - 0.11y[n-2] - 0.07y[n-3] + x[n-1]$ [10 marks]

Solution

a. Take the z-transform of both sides of the difference equation:

$$Y(z) = 0.9z^{-1}Y(z) + X(z) - z^{-1}X(z)$$

The system function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - 0.9z^{-1}} = \frac{z - 1}{z - 0.9}$$

Its partial fraction form is:

$$H(z) = \frac{10}{9} - \frac{1}{9}\left(\frac{z}{z - 0.9}\right)$$

The corresponding impulse response is:

$$h[n] = \frac{10}{9}\delta[n] - \frac{1}{9}(0.9)^n u[n]$$

b. Take the z-transform of both sides of the difference equation:

$$Y(z) = 1.7z^{-1}Y(z) - 0.72z^{-2}Y(z) + X(z) - 2z^{-1}X(z)$$

The system function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 2z^{-1}}{1 - 1.7z^{-1} + 0.72z^{-2}} = \frac{z(z - 2)}{(z - 0.9)(z - 0.8)}$$

Thus

$$\frac{H(z)}{z} = \frac{z - 2}{(z - 0.9)(z - 0.8)} = \frac{k_1}{z - 0.9} + \frac{k_2}{z - 0.8}$$

The residues are:

$$k_1 = \left.\frac{z - 2}{z - 0.8}\right|_{z = 0.9} = -11$$

$$k_2 = \left.\frac{z - 2}{z - 0.9}\right|_{z = 0.8} = 12$$

The partial fraction form of the system function is:

$$H(z) = -\frac{11z}{z - 0.9} + \frac{12z}{z - 0.8}$$

The impulse response is found as:

$$h[n] = -11(0.9)^n u[n] + 12(0.8)^n u[n]$$

c. Take the z-transform of both sides of the difference equation:

$$Y(z) = 1.7z^{-1}Y(z) - 0.72z^{-2}Y(z) + X(z) + z^{-1}X(z) + z^{-2}X(z)$$

The system function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1} + z^{-2}}{1 - 1.7z^{-1} + 0.72z^{-2}} = \frac{z^2 + z + 1}{(z - 0.9)(z - 0.8)}$$

Division of $H(z)$ by z yields:

$$\frac{H(z)}{z} = \frac{z^2 + z + 1}{z(z - 0.9)(z - 0.8)} = \frac{k_1}{z} + \frac{k_2}{z - 0.9} + \frac{k_3}{z - 0.8}$$

The residues are:

$$k_1 = \left.\frac{z^2 + z + 1}{(z - 0.9)(z - 0.8)}\right|_{z = 0} = 1.3889$$

$$k_2 = \left.\frac{z^2 + z + 1}{z(z - 0.8)}\right|_{z = 0.9} = 30.111$$

$$k_3 = \left.\frac{z^2 + z + 1}{z(z - 0.9)}\right|_{z = 0.8} = -30.5$$

Partial fraction form of the system function is:

$$H(z) = 1.3889 + \frac{30.111z}{z - 0.9} - \frac{30.5z}{z - 0.8}$$

The impulse response is found as:

$$h[n] = 1.3889\delta[n] + 30.1111(0.9)^n u[n] - 30.5(0.8)^n u[n]$$

d. Take the z-transform of both sides of the difference equation:

$$Y(z) = z^{-1}Y(z) - 0.11z^{-2}Y(z) - 0.07z^{-3}Y(z) + z^{-1}X(z)$$

The system function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} + 0.11z^{-2} + 0.07z^{-3}} = \frac{z^2}{z^3 - z^2 + 0.11z + 0.07}$$

In factored form the system function is:

$$H(z) = \frac{z^2}{(z - 0.7)(z - 0.5)(z + 0.2)}$$

Dividing $H(z)$ by z we get:

$$\frac{H(z)}{z} = \frac{z}{(z - 0.7)(z - 0.5)(z + 0.2)} = \frac{k_1}{z - 0.7} + \frac{k_2}{z - 0.5} + \frac{k_3}{z + 0.2}$$

The residues are:

$$k_1 = \left.\frac{z}{(z - 0.5)(z + 0.2)}\right|_{z = 0.7} = 2.7222$$

$$k_2 = \left.\frac{z}{(z - 0.7)(z + 0.2)}\right|_{z = 0.5} = -1.7857$$

$$k_3 = \left.\frac{z}{(z - 0.7)(z - 0.5)}\right|_{z = -0.2} = 0.0635$$

Partial fraction form of the system function is:

$$H(z) = \frac{2.7222z}{z - 0.7} - \frac{1.7857z}{z - 0.5} + \frac{0.0635z}{z + 0.2}$$

The impulse response is:

$$h[n] = 2.7222(0.7)^n u[n] - 1.7857(0.5)^n u[n] + 0.0635(-0.2)^n u[n]$$

2. Several causal discrete-time LTI systems are described below by means of their system functions. Find a difference equation for each system.

a. System 1 [8 marks]

$$H(z) = \frac{z + 1}{z^2 + 5z + 6}$$

b. System 2 [8 marks]

$$H(z) = \frac{(z - 1)^2}{\left(z + \frac{2}{3}\right)\left(z + \frac{1}{2}\right)\left(z - \frac{4}{5}\right)}$$

c. System 3 [6 marks]

$$H(z) = \frac{z^2 + 1}{z^3 + 1.2z^2 - 1.8}$$

Solution

a. Begin by writing $H(z)$ using negative powers of z:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-2}}{1 + 5z^{-1} + 6z^{-2}}$$

The relationship between $X(z)$ and $Y(z)$ is:

$$Y(z)[1 + 5z^{-1} + 6z^{-2}] = X(z)[z^{-1} + z^{-2}]$$

Or, equivalently:

$$Y(z) + 5z^{-1}Y(z) + 6z^{-2}Y(z) = z^{-1}X(z) + z^{-2}X(z)$$

Taking inverse transforms of both sides leads to:

$$y[n] + 5y[n-1] + 6y[n-2] = x[n-1] + x[n-2]$$

b. Multiply out numerator and denominator factors, and write the result using negative powers of z to obtain:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 2z^{-1} + z^{-2}}{1 + \frac{11}{30}z^{-1} - \frac{3}{5}z^{-2} - \frac{4}{15}z^{-3}}$$

Using the same steps as in part (a):

$$Y(z)\left[1 + \frac{11}{30}z^{-1} - \frac{3}{5}z^{-2} - \frac{4}{15}z^{-3}\right] = X(z)[1 - 2z^{-1} + z^{-2}]$$

Thus:

$$Y(z) + \frac{11}{30}z^{-1}Y(z) - \frac{3}{5}z^{-2}Y(z) - \frac{4}{15}z^{-3}Y(z) = X(z) - 2z^{-1}X(z) + z^{-2}X(z)$$

And

$$y[n] + \frac{11}{30}y[n-1] - \frac{3}{5}y[n-2] - \frac{4}{15}y[n-3] = x[n] - 2x[n-1] + x[n-2]$$

c. The system function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-3}}{1 + 1.2z^{-1} - 1.8z^{-3}}$$

which leads to:

$$Y(z)[1 + 1.2z^{-1} - 1.8z^{-3}] = X(z)[z^{-1} + z^{-3}]$$

and to the corresponding difference equation:

$$y[n] + 1.2y[n-1] - 1.8\,y[n-3] = x[n-1] + x[n-3]$$

B. Time-Domain Response of Discrete Systems

3. A causal discrete-time LTI system is described by the difference equation

$$y[n] = -0.1y[n-1] + 0.56y[n-2] + x[n] - 2x[n-1]$$

a. Find the system function $H(z)$. [4 marks]

b. Determine the impulse response $h[n]$ of the system. [10 marks]

c. Using z-transform techniques, determine the unit-step response of the system. [8 marks]

d. Using z-transform techniques, find the response of the system to a time-reversed unit-step signal $x[n] = u[-n]$. [8 marks]

Solution

a. Take the z-transform of both sides of the difference equation:

$$Y(z) = -0.1z^{-1}Y(z) + 0.56z^{-2}Y(z) + X(z) - 2z^{-1}X(z)$$

The system function is:

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{1 - 2z^{-1}}{1 + 0.1z^{-1} - 0.56z^{-2}} = \frac{z(z-2)}{(z+0.8)(z-0.7)}$$

b. Division of $H(z)$ by z yields:

$$\frac{H(z)}{z} = \frac{z-2}{(z+0.8)(z-0.7)} = \frac{k_1}{z+0.8} + \frac{k_2}{z-0.7}$$

The residues are found as:

$$k_1 = \left.\frac{z-2}{z-0.7}\right|_{z=-0.8} = 1.8667$$

$$k_2 = \left.\frac{z-2}{z+0.8}\right|_{z=0.7} = -0.8667$$

Partial fraction form of the system function is:

$$H(z) = \frac{1.8667z}{z+0.8} - \frac{0.8667z}{z-0.7}, \qquad \text{ROC: } |z| > 0.8$$

The impulse response is found as:

$$h[n] = 1.8667(-0.8)^n u[n] - 0.8667(0.7)^n u[n]$$

c. If $x[n] = u[n]$, then:

$$X(z) = \frac{z}{z-1}, \qquad \text{ROC: } |z| > 1$$

The transform of the output signal is:

$$Y(z) = H(z)X(z)$$
$$= \frac{z^2(z-2)}{(z+0.8)(z-0.7)(z-1)}, \qquad \text{ROC: } |z| > 1$$

Partial fraction form of $Y(z)$ is:

$$Y(z) = \frac{0.8296z}{z+0.8} + \frac{2.0222z}{z-0.7} - \frac{1.8519z}{z-1}$$

This leads to the output signal:

$$y[n] = [0.8296(-0.8)^n + 2.0222(0.7)^n - 1.8519]u[n]$$

d. If $x[n] = u[-n]$, then:

$$X(z) = \frac{-1}{z-1}, \qquad \text{ROC: } z < 1$$

The transform of the output signal is:

$$Y(z) = H(z)X(z)$$
$$= \frac{-z(z-2)}{(z+0.8)(z-0.7)(z-1)}, \qquad \text{ROC: } 0.8 < z < 1$$

Partial fraction form of $Y(z)$ is:

$$Y(z) = \frac{1.0370z}{z+0.8} - \frac{2.8889z}{z-0.7} + \frac{1.8519z}{z-1}$$

This leads to the output signal:

$$y[n] = 1.0370(-0.8)^n u[n] - 2.8889(0.7)^n u[n] - 1.8519u[-n-1]$$

4. Repeat previous problem for an anti-causal discrete-time LTI system is described by the difference equation: [30 marks]

$$y[n] = -\frac{5}{6}y[n+1] - \frac{1}{6}y[n+2] + \frac{1}{6}x[n+1] + \frac{1}{6}x[n+2]$$

Solution

a. Take the z-transform of both sides of the difference equation:

$$Y(z) = -\frac{5}{6}zY(z) - \frac{1}{6}z^2Y(z) + \frac{1}{6}zX(z) + \frac{1}{6}z^2X(z)$$

The system function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z(z+1)}{(z+2)(z+3)}, \qquad \text{ROC: } |z| < 2$$

b. Division of $H(z)$ by z yields:

$$\frac{H(z)}{z} = \frac{z+1}{(z+2)(z+3)} = \frac{k_1}{z+2} + \frac{k_2}{z+3}$$

The residues are found as:

$$k_1 = \left.\frac{z+1}{z+3}\right|_{z=-2} = -1$$

$$k_2 = \left.\frac{z+1}{z+2}\right|_{z=-3} = 2$$

Partial fraction form of the system function is:

$$H(z) = \frac{-z}{z+2} + \frac{2z}{z+3}, \qquad \text{ROC: } z < 2$$

The impulse response is found as:

$$h[n] = (-2)^n u[-n-1] - 2(-3)^n u[-n-1]$$

c. If $x[n] = u[n]$ then:

$$X(z) = \frac{z}{z-1}, \qquad \text{ROC: } z > 1$$

The transform of the output signal is:

$$Y(z) = H(z)X(z)$$

$$= \frac{z^2(z+1)}{(z-1)(z+2)(z+3)}, \qquad \text{ROC: } 1 < |z| < 2$$

Partial fraction form of $Y(z)$ is:

$$Y(z) = \frac{1}{6}\left(\frac{z}{z-1}\right) - \frac{2}{3}\left(\frac{z}{z+2}\right) + \frac{3}{2}\left(\frac{z}{z+3}\right)$$

This leads to the output signal:

$$y[n] = \frac{1}{6}u[n] + \frac{2}{3}(-2)^n u[-n-1] - \frac{3}{2}(-3)^n u[-n-1]$$

d. If $x[n] = u[-n]$ then:

$$X(z) = \frac{-1}{z-1}, \qquad \text{ROC: } z < 1$$

The transform of the output signal is:

$$Y(z) = H(z)X(z)$$
$$= \frac{-z(z+1)}{(z-1)(z+2)(z+3)}, \quad \text{ROC: } |z| < 1$$

Partial fraction form of $Y(z)$ is:

$$Y(z) = -\frac{1}{6}\left(\frac{z}{z-1}\right) - \frac{1}{3}\left(\frac{z}{z+2}\right) + \frac{1}{2}\left(\frac{z}{z+3}\right)$$

This leads to the output signal:

$$y[n] = \frac{1}{6}u[-n-1] + \frac{1}{3}(-2)^n u[-n-1] - \frac{1}{2}(-3)^n u[-n-1]$$

5. A causal discrete-time LTI system is characterised by the system function:

$$H(z) = \frac{0.04z}{z-0.96}$$

a. Using z-transform techniques, determine the response of the system to the causal sinusoidal signal: [6 marks]

$$x[n] = \sin(0.01n)\, u[n]$$

b. Determine the response of the system to the non-causal sinusoidal signal: [4 marks]

$$x[n] = \sin(0.01n)$$

This is the steady-state response of the system.

c. Compare the responses in parts (a) and (b). Approximately how many samples does it take for the response to the causal sinusoidal signal in part (a) to be almost equal to the steady-state response found in part (b)? [4 marks]

Solution

a. The transform of the causal sinusoidal signal is:

$$X(z) = \frac{\sin(0.01)\, z}{z^2 - 2\cos(0.01)\, z + 1}$$
$$= \frac{\sin(0.01)\, z}{(z - e^{j0.01})(z - e^{-j0.01})}$$

The transform of the output signal is:

$$Y(z) = H(z)X(z)$$
$$= \frac{0.04\sin(0.01)\, z^2}{(z-0.96)(z - e^{j0.01})(z - e^{-j0.01})}$$

$$= \frac{0.2264z}{z-0.96} + \frac{0.4856\left(e^{-j1.8061}\right)z}{z-e^{j0.01}} + \frac{0.4856\left(e^{j1.8061}\right)z}{z-e^{-j0.01}}$$

As a result, the output signal is:

$$y[n] = 0.2264(0.96)^n u[n] + 0.4856e^{-j1.8061}e^{j0.01n} + 0.4856e^{j1.8061}e^{-j0.01n}$$
$$= 0.2264(0.96)^n u[n] + 0.9713\cos(0.01n - 1.8061)\,u[n]$$

b. The system function evaluated at ω_0 = 0.01 rad/s is:

$$H(0.01) = H(z)\big|_{z=e^{j0.01}} = 0.9445 - j0.2264 = 0.9713e^{-j0.2353}$$

The steady-state response of the system is:

$$y_{ss}[n] = 0.9713\sin(0.01n - 0.2353)$$
$$= 0.9713\cos(0.01n - 0.2353 - \pi/2)$$
$$= 0.9713\cos(0.01n - 1.8061)$$

c. The difference between the two responses is the transient term:

$$y_t[n] = 0.2264(0.96)^n u[n]$$

It can be shown that $y_t[140] \approx 0.0007$ which is less than 0.1 percent of the amplitude of the steady-state response.

6. The impulse response of a discrete-time LTI system is:

$$h[n] = (0.8)^n\cos(0.2\pi n)\,u[n]$$

a. Determine the system function $H(z)$ and indicate its ROC. [4 marks]

b. Draw the pole-zero plot for the system function and determine its stability. [6 marks]

c. Write a difference equation for this system. [2 marks]

d. Determine the unit-step response of the system using z-transform techniques. [6 marks]

Solution

a. Starting with the z-transform of the impulse response of $\cos(0.2\pi n)\,u[n]$:

$$Z\{\cos(0.2\pi n)\,u[n]\} = \frac{z[z-\cos(0.2\pi)]}{z^2 - 2\cos(0.2\pi)\,z + 1}$$
$$= \frac{z(z-0.8090)}{z^2 - 1.6180z + 1}$$

Using the multiplication by an exponential property of the z-transform,

$$(Z_0)^n x[n] \leftrightarrow X(z/z_0), \qquad \text{ROC} = |z_0|R_x$$

the system function is obtained as:

$$Z\{(0.8)^n \cos(0.2\pi n)\, u[n]\} = \frac{\left(\frac{z}{0.8}\right)\left[\left(\frac{z}{0.8}\right) - 0.8090\right]}{\left(\frac{z}{0.8}\right)^2 - 1.6180\left(\frac{z}{0.8}\right) + 1}$$

$$= \frac{z(z - 0.5178)}{z^2 - 1.2944z + 0.64}$$

Factorising the system function,

$$H(z) = \frac{z(z - 0.5178)}{(z - 0.6472 + j0.4702)(z - 0.6472 - j0.4702)}$$

As a result, the ROC of the $|z| > 0.6472 + j0.4702$.

b. With the z-transform of the system function is:

$$H(z) = \frac{z(z - 0.5178)}{(z - 0.6472 + j0.4702)(z - 0.6472 - j0.4702)}$$

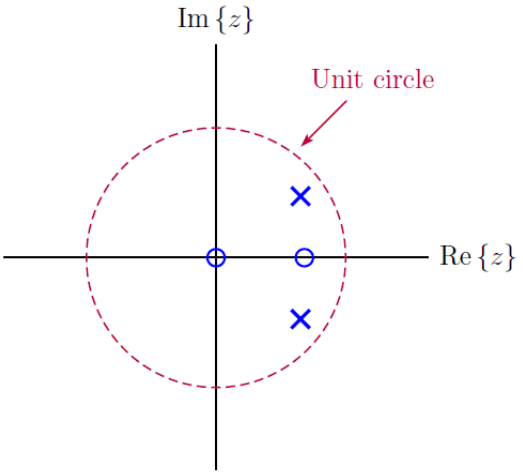
The zeros of the system function are at:

$$z_1 = 0, \qquad z_2 = 0.5178$$

and its poles are at:

$$p_{1,2} = 0.6472 \pm 0.4702$$

The pole and zero diagram of the system is shown in the figure below.



Notice that the poles are inside the unit circle. Since $h[n]$ is causal, it is also stable.

c. Taking the inverse transform of the system function,

$$\frac{Y[z]}{X[z]} = \frac{z(z-0.5178)}{z^2 - 1.2944z + 0.64} = \frac{1 - 0.5178z^{-1}}{1 - 1.2944z^{-1} + 0.64z^{-2}}$$

the difference equation for the system is:

$$y[n] - 1.2944y[n-1] + 0.64y[n-2] = x[n] - 0.5178x[n-1]$$

d. The z-transform of the output signal is:

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \left(\frac{z(z-0.5178)}{z^2 - 1.2944z + 0.64}\right)\left(\frac{z}{z-1}\right) \\ &= \frac{z^2(z-0.5178)}{z^3 - 2.2944z^2 + 1.9344z - 0.64} \end{aligned}$$

Taking the partial fraction expansion, the z-transform of the output signal is:

$$Y(z) = \frac{1.0209z}{z-1} + \frac{(-0.0104 - j0.6804)z}{z - 0.6472 - j0.4702} + \frac{(-0.0104 + j0.6804)z}{z - 0.6472 + j0.4702}$$

Applying Euler's identity, the z-transform of the output signal is now:

$$Y(z) = \frac{1.0209z}{z-1} + \frac{0.6804e^{-j1.5861}z}{z - 0.8e^{j0.6283}} + \frac{0.6804e^{j1.5861}z}{z - 0.8e^{-j0.6283}}$$

Taking the inverse z-transform from the table, the output signal is:

$$\begin{aligned} y[n] &= 1.0209u[n] + 0.6804e^{-j1.5861}\left(0.8e^{j0.6283}\right)^n u[n] \\ &\quad + 0.6804e^{j1.5861}\left(0.8e^{j0.6283}\right)^n u[n] \\ &= 1.0209u[n] + 1.3608(0.8)^n \cos(0.6283n - 1.5861) \end{aligned}$$

7. A discrete-time LTI system has the system function:

$$H(z) = \frac{z^2 + 3z}{z^2 - 1.4z + 0.85}$$

Find the steady-state response of the system to each of the following signals:

a. $x[n] = (0.8)^n e^{j0.4\pi n}$ [8 marks]

b. $x[n] = (0.9)^n \cos(0.3\pi n)$ [8 marks]

Solution

a. The input signal can be expressed as $x[n] = z_0^n$ with:

$$z_0 = 0.8e^{j0.4\pi}$$

The steady-state response of the system for a complex exponential signal is:

$$y[n] = H(z_0)x[n] = H(z_0)z_0^n$$

The system function can be evaluated at $z = z_0$ to yield:

$$H(z_0) = H\left(0.8e^{j0.4\pi}\right) = \frac{\left(0.8e^{j0.4\pi}\right)^2 + 3\left(0.8e^{j0.4\pi}\right)}{(0.8ej^{0.4\pi})^2 - 1.4(0.8e^{j0.4\pi}) + 0.85} = 3.8717e^{j3.0777}$$

Thus

$$Y(z) = H(z)X(z) = 3.8717e^{j(3.0777)}(0.8)^n e^{j0.4\pi n}$$

Therefore:

$$y[n] = 3.8717(0.8)^n e^{j(0.4\pi n + 3.0777)}$$

b. Let $z_1 = (0.9)^n e^{0.3\pi}$. The input signal can be expressed as:

$$x[n] = 0.5z_1^n + 0.5(z_1^*)^n$$

The system function can be evaluated at $z = z_1$ and at $z = z_1^*$ to yield:

$$H(z_1) = H(0.9e^{j0.3\pi}) = \frac{\left(0.9e^{j0.3\pi}\right)^2 + 3\left(0.9e^{j0.3\pi}\right)}{(0.9e^{j0.3\pi})^2 - 1.4(0.9e^{j0.3\pi}) + 0.85} = 11.3347e^{-j3.0515}$$

and

$$H(z_1^*) = H(0.9e^{-j0.3\pi}) = H^*(0.9e^{j0.3\pi}) = 11.3347e^{j3.0515}$$

In the last step, we recognized that $H(z)$ has only real coefficients and therefore $H(z_1^*) = H^*(z_1)$.

$$Y(z) = H(z)X(z) = 11.3347e^{-j3.0515}\left[0.5\left(0.9e^{j0.3\pi}\right)^n\right] + 11.3347e^{j3.0515}\left[0.5\left(0.9e^{-j0.3\pi}\right)^n\right]$$

The output signal is:

$$y[n] = (0.5)11.3347e^{-j3.0515}\left(0.9e^{j0.3\pi}\right)^n + (0.5)11.3347e^{j3.0515}\left(0.9e^{-j0.3\pi}\right)^n$$
$$= 11.3347(0.9)^n \cos(0.3\pi n - 3.0515)$$

C. Frequency-Domain Response of Discrete Systems

8. Sketch the pole-zero diagram of the frequency response for the system with its transfer function is given as: [8 marks]

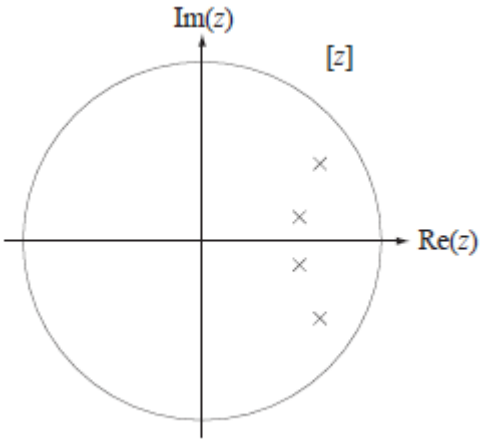
$$H(z) = \frac{0.0686}{(z^2 - 1.087z + 0.3132)(z^2 - 1.315z + 0.6182)}$$

Solution

This transfer function of the system can be factored into:

$$H(z) = \frac{0.0686}{(z-0.5435+j0.1333)(z-0.5435-j0.1333)(z-0.6575+j0.4312)(z-0.6575-j0.4312)}$$

The pole-zero diagram of the system is illustrated in the figure below.



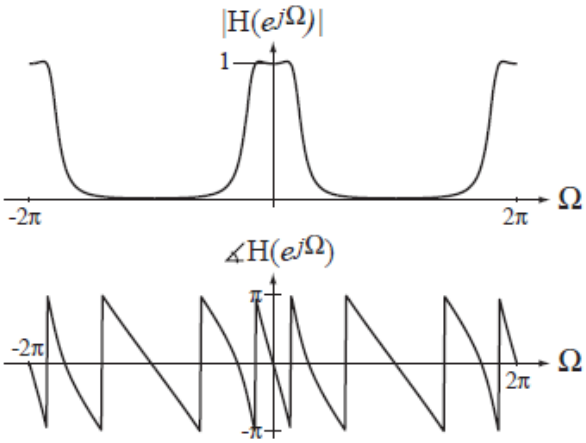
Notice the complex pairs of poles at $0.5435 \pm j0.1333$ and $0.6575 \pm j0.4312$. All poles of the system are located inside the unit circle in the z-plane.

The magnitude and phase plots of the frequency response of the system are illustrated in the figure below. As shown in the magnitude plot, the gain of the system is maximum at the poles of the system.

$$|H(e^{j\Omega})| = \frac{1}{|e^{j\Omega}-p_1||e^{j\Omega}-p_2||e^{j\Omega}-p_3||e^{j\Omega}-p_4|}$$

As for the phase plot, the phase of the system is maximum when the pole's vector at the minimum and is minimum when the pole's vector at the maximum.

$$\measuredangle H(e^{j\Omega}) = -\measuredangle(e^{j\Omega}-p_1) - \measuredangle(e^{j\Omega}-p_2) - \measuredangle(e^{j\Omega}-p_3) - \measuredangle(e^{j\Omega}-p_4)$$



9. A system with transfer function given in the equation below is excited by a unit sequence.

$$H(z) = \frac{z}{(z-0.3)(z+0.8)} \qquad |z| > 1$$

Find the overall response of the system using the following method:

a. Z-transform. [10 marks]

b. Discrete-time Fourier transform. Comment on the difference between two methods. [32 marks]

Solution

a. The z-transform of the response is:

$$Y(z) = H(z)X(z) = \left[\frac{z}{(z-0.3)(z+0.8)}\right]\left(\frac{z}{z-1}\right) \qquad |z| > 1$$

Expanding in partial fractions, the equation above becomes:

$$Y(z) = \frac{z^2}{(z-0.3)(z+0.8)(z-1)} = \frac{0.1169}{z-0.3} + \frac{0.3232}{z+0.8} + \frac{0.7937}{z-1} \qquad |z| > 1$$

Therefore, the overall response is:

$$y[n] = [-0.1169(0.3)^{n-1} + 0.3232(-0.8)^{n-1} + 0.7937]u[n-1]$$

b. This problem can also be analysed using the discrete-time Fourier transform, but the notation is significantly clumsier, mainly because the discrete-time Fourier transform of a unit sequence is:

$$\frac{1}{1-e^{-j\Omega}} + \pi\delta_{2\pi}(\Omega)$$

Then, the transfer function equation of the system becomes:

$$H(e^{j\Omega}) = \frac{e^{j\Omega}}{(e^{j\Omega}-0.3)(e^{j\Omega}+0.8)}$$

The discrete-time Fourier transform of the system response is:

$$Y(e^{j\Omega}) = H(e^{j\Omega})X(e^{j\Omega}) = \left[\frac{e^{j\Omega}}{(e^{j\Omega}-0.3)(e^{j\Omega}+0.8)}\right]\left[\frac{1}{1-e^{j\Omega}} + \pi\delta_{2\pi}(\Omega)\right]$$

$$= \frac{e^{j\Omega}}{(e^{j\Omega}-0.3)(e^{j\Omega}+0.8)(1-e^{j\Omega})} + \pi\left[\frac{e^{j\Omega}}{(e^{j\Omega}-0.3)(e^{j\Omega}+0.8)}\right]\delta_{2\pi}(\Omega)$$

Expanding in partial fractions, the equation above becomes:

$$Y(e^{j\Omega}) = \frac{-0.1169}{e^{j\Omega}-0.3} + \frac{0.3232}{e^{j\Omega}+0.8} + \frac{0.7937}{e^{j\Omega}-1} + \left(\frac{\pi}{(1-0.3)(1+0.8)}\right)\delta_{2\pi}(\Omega)$$

Using the equivalence property of the impulse and the periodicity of both $\delta_{2\pi}(\Omega)$ and $e^{j\Omega}$, the equation above becomes.

$$Y(e^{j\Omega}) = \frac{-0.1169e^{-j}}{1-0.3e^{-j\Omega}} + \frac{0.3232e^{-j\Omega}}{1+0.8e^{-j\Omega}} + \frac{0.7937e^{-j\Omega}}{1-e^{-j\Omega}} + 2.4933\delta_{2\pi}(\Omega)$$

Then, manipulating this expression into a form for which the inverse discrete-time Fourier transform is direct:

$$Y(e^{j\Omega}) = \frac{0.2269e^{-j\Omega}}{1-0.3e^{-j\Omega}} + \frac{0.3232e^{-j\Omega}}{1+0.8e^{-j\Omega}} + 0.7937\left[\frac{e^{-j\Omega}}{1-e^{-j\Omega}} + \delta_{2\pi}(\Omega)\right] - 0.7937\pi\delta_{2\pi}(\Omega) + 2.4933\delta_{2\pi}(\Omega)$$

With $0.7937\pi\delta_{2\pi}(\Omega) + 2.4933\delta_{2\pi}(\Omega) = 0$, thus:

$$Y(e^{j\Omega}) = \frac{0.2269e^{-j\Omega}}{1-0.3e^{-j\Omega}} + \frac{0.3232e^{-j\Omega}}{1+0.8e^{-j\Omega}} + 0.7937\left[\frac{e^{-j\Omega}}{1-e^{-j\Omega}} + \delta_{2\pi}(\Omega)\right]$$

Taking the inverse discrete-time Fourier transform of the equation above, the overall response:

$$y[n] = [-0.1169(0.3)^{n-1} + 0.3232(-0.8)^{n-1} + 0.7937]u[n-1]$$

The result is the same, but the effort and the probability of error are considerably greater.

10. A discrete-time LTI system has the system function:

$$H(z) = \frac{z-0.4}{z(z^2-1.4z+0.85)}$$

a. Draw a pole-zero diagram to show poles and zeros of the system function as well as the unit circle of the z-plane. [6 marks]

b. Using the graphical method discussed in the time and frequency characteristics topic, determine the magnitude and the phase of the system function at the angular frequency $\Omega = \pi/6$. [24 marks]

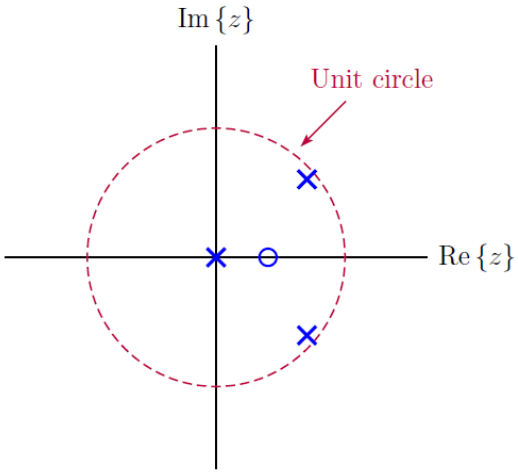
c. Repeat part (b) at $\Omega = \pi/4$. [24 marks]

Solution

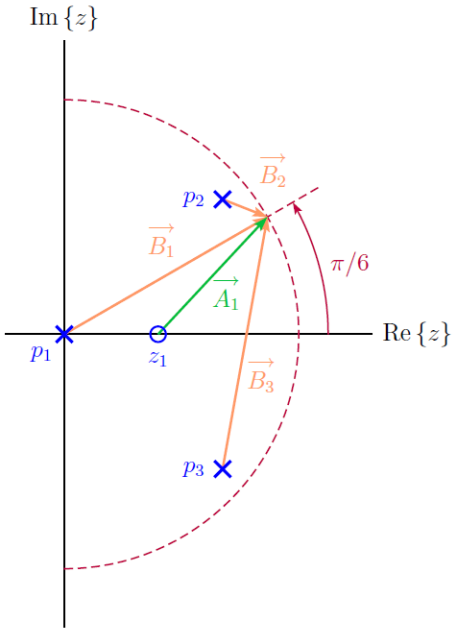
a. The zero and poles of the system function are at:

$$z_1 = 0.4, \qquad p_1 = 0, \qquad p_{2,3} = 0.7 \pm 0.6$$

The pole and zero diagram of the system is shown in the figure below.



b. The vectors from the poles and zero to the point of interest are shown in the figure below.



The magnitude and angle (phase) of the vectors are:

- $|A_1|$ = 0.6835 and $\measuredangle A_1$ = 0.8206 rad.
- $|B_1|$ = 1 and $\measuredangle B_1$ = 0.5236 rad.
- $|B_2|$ = 0.1938 and $\measuredangle B_2$ = −0.5421 rad.
- $|B_3|$ = 1.1125 and $\measuredangle B_3$ = 1.4210 rad.

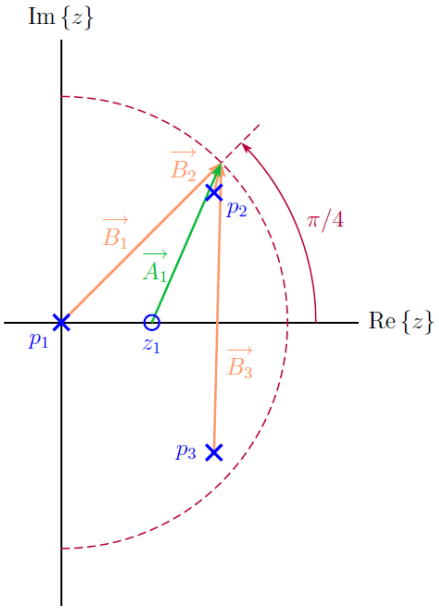
The magnitude and phase of the system function are:

$$H\left(\frac{\pi}{6}\right) = \frac{|A_1|}{|B_1||B_2||B_3|} = \frac{0.6835}{(1)(0.1938)(1.1125)} = 3.1701$$

And

$$\measuredangle H\left(\frac{\pi}{6}\right) = \measuredangle A_1 - \measuredangle B_1 - \measuredangle B_2 - \measuredangle B_3 = 0.8206 - 0.5236 + 0.5421 - 1.4210 = -0.5819 \text{ rad}$$

c. The vectors from the poles and zero to the point of interest are shown in the figure below.



The magnitude and phase of the system function are:

- $|A_1|$ = 0.7709 and $\measuredangle A_1$ = 1.1611 rad.
- $|B_1|$ = 1 and $\measuredangle B_1$ = 0.7854 rad.
- $|B_2|$ = 0.1073 and $\measuredangle B_2$ = 1.5045 rad.
- $|B_3|$ = 1.3071 and $\measuredangle B_3$ = 1.5654 rad.

The magnitude and the angle (phase) of the system function are:

$$H\left(\frac{\pi}{4}\right) = \frac{|A_1|}{|B_1||B_2||B_3|} = \frac{0.7709}{(1)(0.1073)(1.3071)} = 5.4944$$

And

$$\measuredangle H\left(\frac{\pi}{4}\right) = \measuredangle|A_1| - \measuredangle|B_1| - \measuredangle|B_2| - \measuredangle|B_3|$$

$$= \angle 1.1611 - \angle 0.7854 - \angle 1.5045 - \angle 1.5654 = -2.6942 \text{ rad}$$

D. Analysis of Discrete Systems

11. For each function $H(z)$ given below, determine if it could be the system function of a causal and stable system. In each case provide the reasoning behind your answer.

    a. System 1 [6 marks]

$$H(z) = \frac{(z-2)(z+2)}{\left(z+\frac{1}{2}\right)}$$

b. System 2 [6 marks]

$$H(z) = \frac{(z+1)^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

c. System 3 [6 marks]

$$H(z) = \frac{z(z+1)}{\left(z+\frac{1}{2}\right)\left(z+\frac{3}{2}\right)}$$

Solution

a. The system has zeros at $z = \pm 2$ and a pole at $z = -1/2$. The system function has a second-order numerator and a first-order denominator and therefore does not converge at $|z| \to \infty$. The ROC may be one of the following:

- $|z| < 1/2$: Neither stable nor causal.
- $1/2 < |z| < \infty$: Stable, but not causal.

In the first case the ROC includes neither the unit circle nor infinity. In the second case it includes the unit circle but excludes infinity. Consequently, this cannot be the system function of a system that is both stable and causal.

b. The system has two zeros at $z = -1$ and poles at $z = -1/3$ and $z = 1/2$. The ROC may be one of the following:

- $|z| < 1/3$: Neither stable nor causal.
- $1/3 < |z| < 1/2$: Neither stable nor causal.
- $|z| > 1/2$: Stable and causal.

This system function could correspond to a system that is both stable and causal.

c. The system has zeros at $z = 0$ and $z = -1$. Its poles are at $z = -1/2$ and $z = -3/2$. The ROC may be one of the following:

- $|z| < 1/2$: Neither stable nor causal.
- $1/2 < |z| < 3/2$: Stable, but not causal.
- $|z| > 3/2$: Causal, but not stable.

This cannot be the system function of a system that is both stable and causal.

12. Input and output signal pairs are listed below for several discrete time LTI systems. For each case, determine the system function $H(z)$ along with its ROC. Also indicate if the system considered is stable and/or causal.

a. $x[n] = (1/2)^n u[n]$ and $y[n] = 3(1/2)^n u[n] + 2(3/4)^n u[n]$ [6 marks]

b. $x[n] = (1/2)^n u[n]$ and $y[n] = (1/2)^n + 3u[n+2] + (1/2)^n + 2u[n+1]$ [10 marks]

c. $x[n] = u[n]$ and $y[n] = (n-1)u[n-1]$ [6 marks]

d. $x[n] = 1.25\delta[n] - 0.25(0.8)^n u[n]$ and $y[n] = (0.8)^n u[n]$ [6 marks]

Solution

a. The z-transform of the system function is:

$$X(z) = \frac{z}{z-\frac{1}{2}}, \qquad \text{ROC: } |z| > \frac{1}{2}$$

And

$$Y(z) = \frac{3z}{z-\frac{1}{2}} + \frac{2z}{z-\frac{3}{4}} = \frac{5z\left(z-\frac{13}{20}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{3}{4}\right)}, \qquad \text{ROC: } |z| > \frac{3}{4}$$

The transfer function equation of the system is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{5\left(z-\frac{13}{20}\right)}{z-\frac{3}{4}}, \qquad \text{ROC: } |z| > \frac{3}{4}$$

As a result, the system is both causal and stable.

b. The z-transform of the system function is:

$$X(z) = \frac{z}{z-\frac{1}{2}}, \qquad \text{ROC: } |z| > \frac{1}{2}$$

The output signal may be written as:

$$y[n] = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n+2} u[n+2] + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{n+1} u[n+1]$$

This leads to the transform:

$$Y(z) = \left(\frac{1}{2}\right) z^2 \left(\frac{z}{z-\frac{1}{2}}\right) + \left(\frac{1}{2}\right) z \left(\frac{z}{z-\frac{1}{2}}\right) = \frac{\left(\frac{1}{2}\right) z^2 (z+1)}{z-\frac{1}{2}}, \qquad \text{ROC: } \frac{1}{2} < |z| < \infty$$

The transfer function equation of the system is:

$$H(z) = \frac{Y(z)}{X(z)} = \left(\frac{1}{2}\right) z(z\ + 1), \qquad \text{ROC: } |z| < \infty$$

Thus, the impulse response is:

$$h[n] = \{0.5, 0.5\}$$

As a result, the system is stable but not causal.

c. The z-transform of the system function is:

$$X(z) = \frac{z}{z-1}, \qquad \text{ROC: } |z| > 1$$

And

$$Y(z) = z^{-1}\left|\frac{z}{(z-1)^2}\right| = \frac{1}{(z-1)^2}, \qquad \text{ROC: } |z| > 1$$

The transfer function equation of the system is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z(z-1)}, \qquad \text{ROC: } |z| > 1$$

As a result, the system is causal but not stable.

d. The z-transform of the system function is:

$$X(z) = 1.25 - \frac{0.25z}{z-0.8} = \frac{z-1}{z-0.8}, \qquad \text{ROC: } |z| > 0.8$$

And

$$Y(z) = \frac{z}{z-0.8}, \qquad \text{ROC: } |z| > 0.8$$

The transfer function equation of the system is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z-1}, \qquad \text{ROC: } |z| > 1$$

As a result, the system is causal but not stable.

13. Several causal discrete time LTI systems are described below by their difference equations. For each system, determine the system function $H(z)$. Draw a pole-zero diagram for the system and indicate the ROC for the system function. Determine if the system is stable.

a. $y[n] = 1.5y[n-1] - 0.54y[n-2] + x[n] + 3x[n-1]$ [12 marks]

b. $y[n] = -0.64y[n-2] + 2x[n]$ [12 marks]

c. $y[n] = 0.25y[n-1] - 0.125y[n-2] - 0.5y[n-3] + x[n]$ [12 marks]

d. $y[n] = 0.25y[n-1] - 0.5y[n-2] - 0.75y[n-3] + x[n] + x[n-1]$ [12 marks]

Solution

a. Taking the z-transform of the difference equation leads to:

$$Y(z) = 1.5z^{-1}\,Y(z) - 0.54z^{-2}\,Y(z) + X(z) + 3z^{-1}X(z)$$

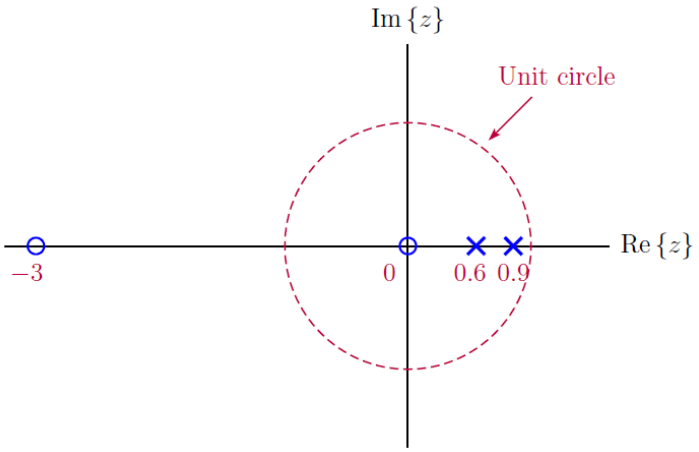
From which the system function can be obtained as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 3z^{-1}}{1 - 1.5z^{-1} + 0.54z^{-2}}$$

The factored form of the system function is:

$$H(z) = \frac{z(z+3)}{(z-0.6)(z-0.9)}$$

The pole-zero diagram of the system is shown in the figure below.



Since the system is specified to be causal, the ROC of the system function is:

$$|z| > 0.9$$

As the ROC includes the unit circle, therefore the system is stable.

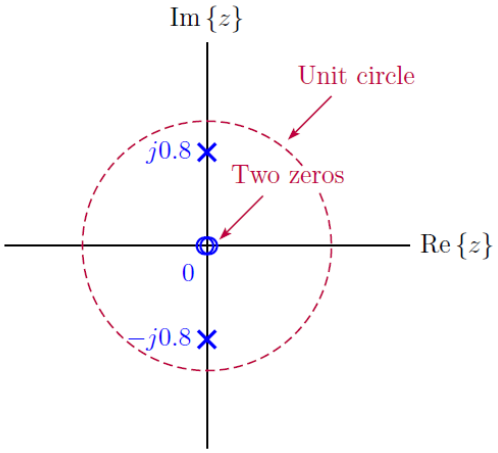
b. Taking the z-transform of the difference equation leads to:

$$Y(z) = -0.64z^{-2}Y\,(z) + 2X(z)$$

From which the system function can be obtained as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1 + 0.64z^{-2}} = \frac{2z^2}{(z - j0.8)(z + j\,0.8)}$$

The pole-zero diagram of the system is shown in the figure below.



Since the system is specified to be causal, the ROC of the system function is:

$$|z| > 0.8$$

As the ROC includes the unit circle, therefore the system is stable.

c. Taking the z-transform of the difference equation leads to:

$$Y(z) = 0.25z^{-1}Y(z) - 0.125z^{-2}Y(z) - 0.5z^{-3}Y(z) + X(z)$$

From which the system function can be obtained as:

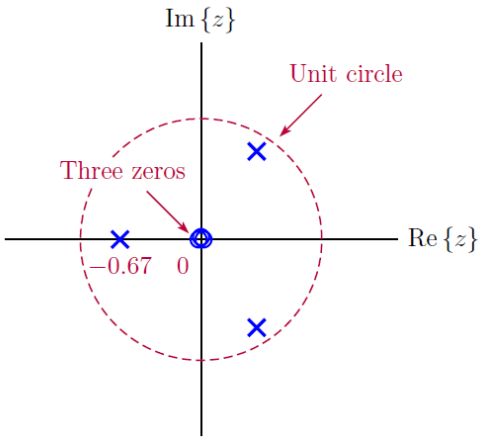
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 0.25z^{-1} + 0.125z^{-2} + 0.5z^{-3}}$$

Or, using non-negative powers of z:

$$H(z) = \frac{z^3}{z^3 - 0.25z^2 + 0.125z + 0.5}$$

There are three zeros at $z = 0$. Poles are at $z = -0.6718$ and $z = 0.4609 \pm 0.7293$.

The pole-zero diagram of the system is shown in the figure below.



Since the system is specified to be causal, the ROC of the system function is:

$$|z| > 0.8627$$

As the ROC includes the unit circle, therefore the system is stable.

d. Taking the z-transform of the difference equation leads to:

$$Y(z) = 0.25z^{-1}Y(z) - 0.5z^{-2}Y(z) - 0.75z^{-3}Y(z) + X(z) + z^{-1}X(z)$$

From which the system function can be obtained as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - 0.25z^{-1} + 0.5z^{-2} + 0.75z^{-3}}$$

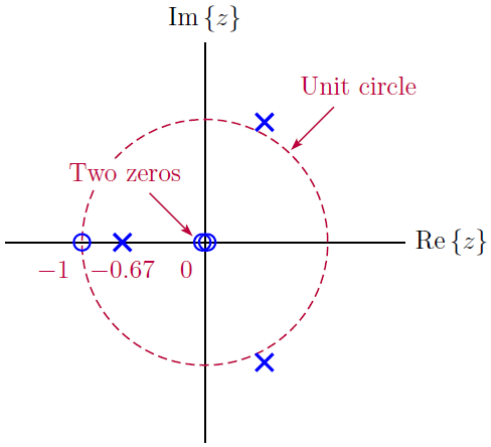
Or, using non-negative powers of z:

$$H(z) = \frac{z^2(z+1)}{z^3 - 0.25z^2 + 0.5z + 0.75}$$

The factored form of the system function is:

$$H(z) = \frac{z^2(z+1)}{(z + 0.6709)(z - 0.4605 - j0.9518)(z - 0.4605 + j0.9518)}$$

The pole-zero diagram of the system is shown in the figure below.



Since the system is specified to be causal, the ROC of the system function is:

$$|z| > 1.0573$$

As the ROC does not include the unit circle, therefore the system is not stable.