XMUT220 Signals and Systems

Tutorial 7b: z-Transform (Applications)

A. Solving Discrete Systems and Differential Equations

1. Find the system function $H(z)$ for each causal discrete-time LTI system described below by means of a difference equation. Afterwards determine the impulse response of the system using partial fraction expansion.

   a. $y[n] = 0.9y[n-1] + x[n] - x[n-1]$ [6 marks]

   b. $y[n] = 1.7y[n-1] - 0.72y[n-2] + x[n] - 2x[n-1]$ [8 marks]

   c. $y[n] = 1.7y[n-1] - 0.72y[n-2] + x[n] + x[n-1] + x[n-2]$ [10 marks]

   d. $y[n] = y[n-1] - 0.11y[n-2] - 0.07y[n-3] + x[n-1]$ [10 marks]

2. Several causal discrete-time LTI systems are described below by means of their system functions. Find a difference equation for each system.

   a. System 1 [8 marks]

$$H(z) = \frac{z+1}{z^2+5z+6}$$

   b. System 2 [8 marks]

$$H(z) = \frac{(z-1)^2}{\left(z+\frac{2}{3}\right)\left(z+\frac{1}{2}\right)\left(z-\frac{4}{5}\right)}$$

   c. System 3 [8 marks]

$$H(z) = \frac{z^2+1}{z^3+1.2z^2-1.8}$$

B. Time-Domain Response of Discrete Systems

3. A causal discrete-time LTI system is described by the difference equation:

$$y[n] = -0.1y[n-1] + 0.56y[n-2] + x[n] - 2x[n-1]$$

a. Find the system function $H(z)$. [4 marks]

b. Determine the impulse response $h[n]$ of the system. [10 marks]

c. Using z-transform techniques, determine the unit-step response of the system. [8 marks]

d. Using z-transform techniques, find the response of the system to a time-reversed unit-step signal $x[n] = u[-n]$. [8 marks]

4. Repeat previous problem for an anti-causal discrete-time LTI system is described by the difference equation: [30 marks]

$$y[n] = -\frac{5}{6}y[n+1] - \frac{1}{6}y[n+2] + \frac{1}{6}x[n+1] + \frac{1}{6}x[n+2]$$

5. A causal discrete-time LTI system is characterised by the system function:

$$H(z) = \frac{0.04z}{z - 0.96}$$

a. Using z-transform techniques, determine the response of the system to the causal sinusoidal signal: [6 marks]

$$x[n] = \sin(0.01n)\, u[n]$$

b. Determine the response of the system to the non-causal sinusoidal signal: [4 marks]

$$x[n] = \sin(0.01n)$$

This is the steady-state response of the system.

c. Compare the responses in parts (a) and (b). Approximately how many samples does it take for the response to the causal sinusoidal signal in part (a) to be almost equal to the steady-state response found in part (b)? [4 marks]

6. The impulse response of a discrete-time LTI system is:

$$h[n] = (0.8)^n \cos(0.2\pi n)\, u[n]$$

a. Determine the system function $H(z)$ and indicate its ROC. [4 marks]

b. Draw the pole-zero plot for the system function and determine its stability. [6 marks]

c. Write a difference equation for this system. [2 marks]

d. Determine the unit-step response of the system using z-transform techniques. [6 marks]

7. A discrete time LTI system has the system function:

$$H(z) = \frac{z^2 + 3z}{z^2 - 1.4z + 0.85}$$

Find the steady-state response of the system to each of the following signals:

a. $x[n] = (0.8)^n e^{j0.4\pi n}$ [8 marks]

b. $x[n] = (0.9)^n \cos(0.3\pi n)$ [8 marks]

C. Frequency-Domain Response of Discrete Systems

8. Sketch the pole-zero diagram of the frequency response for the system with its transfer function is given as: [8 marks]

$$H(z) = \frac{0.0686}{(z^2 - 1.087z + 0.3132)(z^2 - 1.315z + 0.6182)}$$

9. A system with transfer function given in the equation below is excited by a unit sequence.

$$H(z) = \frac{z}{(z - 0.3)(z + 0.8)} \qquad |z| > 1$$

Find the total response of the system using the following method:

a. Z-transform. [10 marks]

b. Discrete-time Fourier transform. Comment on the difference between two methods. [32 marks]

10. A discrete-time LTI system has the system function:

$$H(z) = \frac{z - 0.4}{z(z^2 - 1.4z + 0.85)}$$

a. Draw a pole-zero diagram to show poles and zeros of the system function as well as the unit circle of the z-plane. [6 marks]

b. Using the graphical method discussed in the time and frequency characteristics topic, determine the magnitude and the phase of the system function at the angular frequency $\Omega = \pi/6$. [24 marks]

c. Repeat part (b) at $\Omega = \pi/4$. [24 marks]

D. Analysis of Discrete Systems

11. For each function $H(z)$ given below, determine if it could be the system function of a causal and stable system. In each case provide the reasoning behind your answer.

    a. System 1 [6 marks]

$$H(z) = \frac{(z-2)(z+2)}{\left(z+\frac{1}{2}\right)}$$

    b. System 2 [6 marks]

$$H(z) = \frac{(z+1)^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

    c. System 3 [6 marks]

$$H(z) = \frac{z(z+1)}{\left(z+\frac{1}{2}\right)\left(z+\frac{3}{2}\right)}$$

12. Input and output signal pairs are listed below for several discrete time LTI systems. For each case, determine the system function $H(z)$ along with its ROC. Also indicate if the system considered is stable and/or causal.

    a. $x[n] = (1/2)^n u[n], y[n] = 3(1/2)^n u[n] + 2(3/4)^n u[n]$ [6 marks]

    b. $x[n] = (1/2)^n u[n], y[n] = (1/2)^n + 3u[n+2] + (1/2)^n + 2u[n+1]$ [10 marks]

    c. $x[n] = u[n], y[n] = (n-1)u[n-1]$ [6 marks]

    d. $x[n] = 1.25\delta[n] - 0.25(0.8)^n u[n], y[n] = (0.8)^n u[n]$ [6 marks]

13. Several causal discrete time LTI systems are described below by their difference equations. For each system, determine the system function $H(z)$. Draw a pole-zero diagram for the system and indicate the ROC for the system function. Determine if the system is stable.

    a. $y[n] = 1.5y[n-1] - 0.54y[n-2] + x[n] + 3x[n-1]$ [12 marks]

    b. $y[n] = -0.64y[n-2] + 2x[n]$ [12 marks]

    c. $y[n] = 0.25y[n-1] - 0.125y[n-2] - 0.5y[n-3] + x[n]$ [12 marks]

    d. $y[n] = 0.25y[n-1] - 0.5y[n-2] - 0.75y[n-3] + x[n] + x[n-1]$ [12 marks]

Appendix 1 - z-Transform

No	Sequence	Transform	ROC
1	$\delta[n]$	1	All z
2	$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\|>1$
3	$-u[n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\|<1$
4	$\delta[n-m]$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5	$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\|>\|a\|$
6	$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\|<\|a\|$
7	$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|>\|a\|$
8	$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|<\|a\|$
9	$\begin{cases} a^n, & 0\le n\le N-1 \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\|>0$
10	$\sin(\omega_0 n)u[n]$	$\frac{\sin\omega_0\, z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\|>1$
11	$\cos(\omega_0 n)u[n]$	$\frac{1-\cos\omega_0\, z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\|>1$
12	$[a^n \sin\omega_0 n]u[n]$	$\frac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\|>\|a\|$
13	$[a^n \cos\omega_0 n]u[n]$	$\frac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\|z\|>\|a\|$

Appendix 2 - z-transform Properties

No	Property	Sequence	Transform	ROC
1	Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2.
2	Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the region.
3	Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X\left(e^{-j\omega_0}z\right)$	R
		$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$z_0 R$
		$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e. the set of points $\{\|a\|z\}$for z in R.
4	Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R(i.e. R^{-1} is equal to the set of point z^{-1}, where z is in R).
5	Time expansion	$x_{(k)}[n]\begin{cases}x[r], n = rk \\ 0, n \neq rk\end{cases}$ for some integer r	$X\left(z^k\right)$	$R^{1/k}$(i.e. the set of point $z^{1/k}$, where z is in R).
6	Conjugation	$x^*[n]$	$X^*(z^*)$	R
7	Convolution	$x_1[n] * x_2[n]$	$X_1(z) \cdot X_2(z)$	At least the intersection of R_1 and R_2.
8	First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$.
9	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\left(\frac{1}{1 - z^{-1}}\right)X(z)$	At least the intersection of R and $\|z\| > 1$.
10	Differentiation in the z-domain	$nx[n]$	$-z\left[\frac{dX(z)}{dz}\right]$	R
11	Initial value theorem	If $x[n] = 0$ for $x < 0$, then $x[0] = \lim_{z\to\infty} X(z)$		

Note:

- For a sequence $x[n]$, its z-transform is $X(z)$ and its ROC is R.
- For a sequence $x_1[n]$, its z-transform is $X_1(z)$ and its ROC is R_1.
- For a sequence $x_2[n]$, its z-transform is $X_2(z)$ and its ROC is R_2.