ECEN310
Communications Engineering
Lecture 2 – Signals Review

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XMUT

Background Reading

- Reading: Proakis and Salehi *Fundamentals of Communication Systems* 2e, Pearson, Chapter 2.
- Important: Revision on ECEN220 Signals & Systems

Fourier Series of periodic functions

- Any *periodic deterministic* function, that is one where $g(t) = g(t \pm nT)$ (integer n, period T) can be expressed as a sum of sinusoids, that is

$$g(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_0 t}$$

  where f_0 is the *fundamental frequency*.

- One can show that the *Fourier Series coefficients* c_n are given by

$$c_n = \frac{1}{T} \int_{t_0}^{t_0+T} g(t) e^{j2\pi n f_0 t} dt$$

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- Each Fourier Series coefficient c_n represents a relative *weight* of the nth sinusoidal component (one at frequency nf_0) of $g(t)$
- Thus, a plot of the FS coefficients gives a graphical representation of the frequency (or *spectral*) content of $g(t)$
- The spectral lines are spaced $f_0 = 1/T$ apart

Fourier Transform

- For a *nonperiodic* deterministic signal $g(t)$, the frequency representation is computed by the Fourier Transform of $g(t)$:

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- $g(t)$ and $G(f)$ are said to form a *Fourier transform pair*: $g(t) \rightleftharpoons G(f)$

Fourier Transform

- shorthand notation:

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- The FT $G(f)$ is generally a complex function of frequency f, and thus can be expressed as

$$G(f) = |G(f)|e^{j\theta(f)}$$

where $|G(f)|$ and $\theta(f)$ are the *continuous amplitude spectrum* and *continuous phase spectrum* of $g(t)$, respectively

Example: Rectangular Pulse

- Consider a *rectangular pulse* defined by

$$\mathrm{rect}(t) \triangleq \begin{cases} 1 & |t| < 1/2 \\ 0 & |t| > 1/2 \end{cases}$$

- It is sometimes also labelled as $\Pi(t)$
- let $g(t) = A\,\mathrm{rect}(t/T)$. FT of $g(t)$ is given by

$$\begin{aligned} G(f) &= \int_{-T/2}^{T/2} (A) e^{-j2\pi f t} dt \\ &= AT \frac{\sin(\pi f T)}{\pi f T} = AT\,\mathrm{sinc}(\pi f T) \end{aligned}$$

- NOTE: in Lathi and ECEN 310 $\mathrm{sinc}\,\lambda \triangleq \frac{sin\lambda}{\lambda}$

FT Properties (1)

Some key properties of the Fourier Transform

- **Linearity (Superposition) Property:** let $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$. Then for all constants c_1 and c_2

$$c_1 g_1(t) + c_2 g_2(t) \rightleftharpoons c_1 G_1(f) + c_2 G_2(f)$$

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- **Time Scaling Property:** let $g(t) \rightleftharpoons G(f)$. Then

$$g(at) \rightleftharpoons \frac{1}{|a|} G(\frac{f}{a})$$

- *Proof: substitute* $\tau = at$ *into the definition for* $\mathcal{F}\{g(at)\}$

FT Properties (2): Duality

- **Duality Property:** let $g(t) \rightleftharpoons G(f)$. Then

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- *Homework: find and sketch the FT of* $g(t) = A\,\mathrm{sinc}(2Wt)$

Time Shifting

- **Time Shifting Property:** let $g(t) \rightleftharpoons G(f)$. Then the FT of $g(t)$ shifted in time by t_0 is

$$g(t - t_0) \rightleftharpoons G(f)e^{-j2\pi f t_0}$$

- *Proof:* using $\tau = t - t_0$, we have

$$\begin{aligned}\mathcal{F}\{g(t - t_0)\} &= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} g(\tau)e^{-j2\pi f \tau} d\tau \\ &= e^{-j2\pi f t_0} G(f)\end{aligned}$$

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- *magnitude* $G(f)$ will be unaffected by the time shift
- *phase* will be changed by a linear factor of $-2\pi f t_0$

Frequency Shifting (Modulation)

- **Frequency Shifting Property:** let $g(t) \rightleftharpoons G(f)$. Then

$$e^{j2\pi f_c t} g(t) \rightleftharpoons G(f - f_c)$$

- Proof:

$$\begin{aligned}\mathcal{F}\left\{e^{j2\pi f_c t} g(t)\right\} &= \int_{-\infty}^{\infty} g(t) e^{-j2\pi t(f - f_c)} dt \\ &= \int_{-\infty}^{\infty} g(t) e^{-j2\pi t(f')} dt \\ &= G(f') = G(f - f_c)\end{aligned}$$

where $f' = f - f_c$

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- **multiplication by a factor $e^{j2\pi f_c t}$ is equivalent to shifting its Fourier transform $G(f)$ in the positive direction by f.**

Area under $g(t)$ and $G(f)$

- **Area Under $g(t)$:** let $g(t) \rightleftharpoons G(f)$. Then

$$\int_{-\infty}^{\infty} g(t)dt = G(0)$$

- That is the area under the function $g(t)$ is equal to its FT $G(f)$ at $f = 0$.
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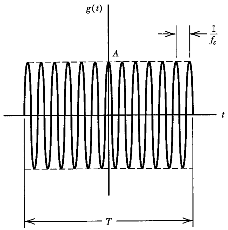
- using the fact that $\cos(2\pi f_c t) = \frac{1}{2}(e^{j2\pi f_c t} + e^{-j2\pi f_c t})$...
- and using the frequency shifting property...

$$\mathcal{F}\{y(t)\} = Y(f) = \frac{1}{2}\left(G(f - f_c) + G(f + f_c)\right)$$

Modulation example (a very important one for us)

- Example: Consider the rectangular pulse $g(t) = A\,\mathrm{rect}(t/T)$, multiplied by $\cos(2\pi f_c t)$, that is

$$y(t) = A\,\mathrm{rect}(t/T)\cos(2\pi f_c t)$$



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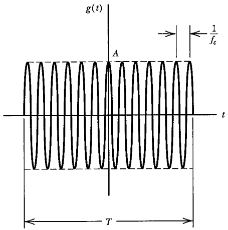

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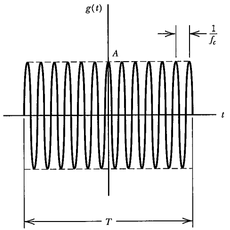

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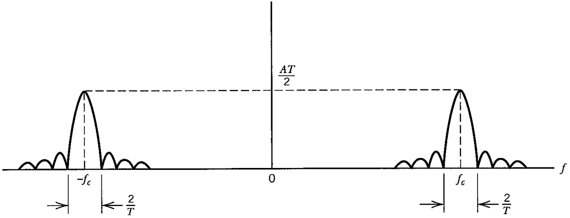
  - magnitude spectrum $|Y(f)|$



Dirac delta function (impulse)

- Now let's revisit the square pulse

$$g(t) = \begin{cases} 1/a & |t| < a/2 \\ 0 & |t| > a/2 \end{cases}$$

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- This limiting case is important and it is the *Dirac delta function*, denoted by $\delta(t)$.
- It has with infinite amplitude, infinitesimal duration and an integral of unity. It is formally defined as

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- informally, one can say

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

FT of a Delta Function

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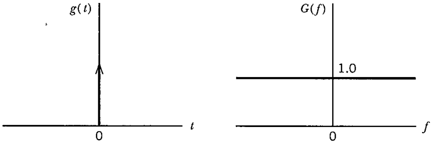

Delta Function: properties

- Recall the duality property of the FT: if $g(t) \rightleftharpoons G(f)$ then $G(t) \rightleftharpoons g(-f))$; Noting the symmetry of $\delta(t)$, we have

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- Note that letting $t_0 = 0$ and $f(t) = e^{-j2\pi f t}$, the sifting property can be easily used to derive the FT of $\delta(t)$...

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt = e^{-j2\pi f(0)} = 1$$

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- ... and similarly, the invers FT of $\delta(f)$

$$\mathcal{F}^{-1}\{\delta(f)\} = \int_{-\infty}^{\infty} \delta(f) e^{j2\pi f t} df = e^{-j2\pi(0)f} = 1$$

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- First, we have that

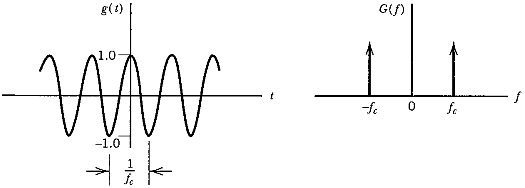
$$\begin{aligned}\mathcal{F}\{e^{j2\pi f_c t}\} &= \int_{-\infty}^{\infty} e^{j2\pi f_c t} e^{-j2\pi f t} dt \\ &= \int_{-\infty}^{\infty} e^{-j2\pi (f-f_c) t} dt = \delta(f - f_c)\end{aligned}$$

where we used a substitution of $f' = f - f_c$, and treated the integral as a FT of 1.

Fourier Transform of a Sinusoid

- it is now trivial to show that

$$\cos(2\pi f_c t) \rightleftharpoons \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$$



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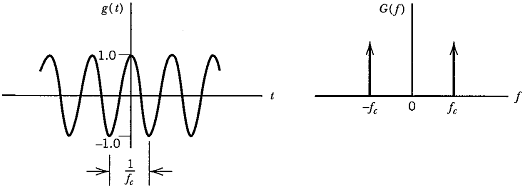

- Applying similar approach to $\sin(2\pi f_c t)$, one can show that
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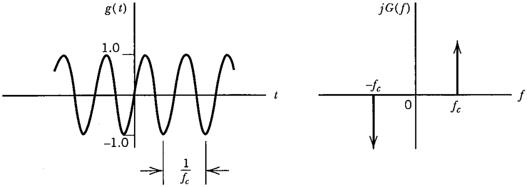

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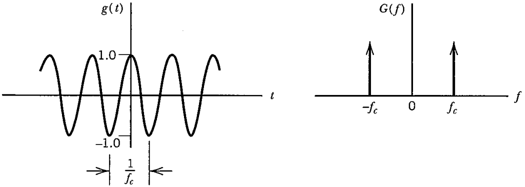

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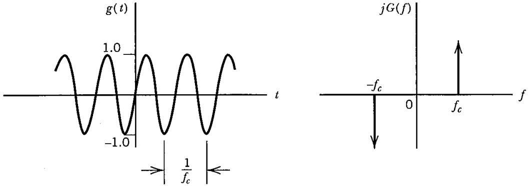
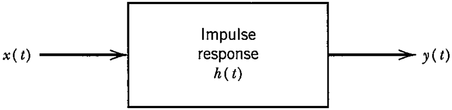


LTI Systems

- Given an arbitrary input $x(t)$ to a linear, time invariant system with an impulse response $h(t)$, determine the output $y(t)$



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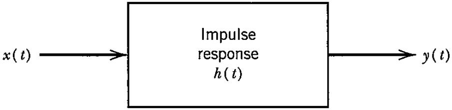

- convolution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

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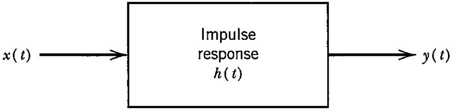

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- If $x(t) \rightleftharpoons X(f)$, $h(t) \rightleftharpoons H(f)$ and $y(t) \rightleftharpoons Y(f)$, then

$$Y(f) = X(f)H(f)$$

Convolution: sanity check

- recall the sifting property of $\delta(t)$, that is

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t) dt = f(t_0)$$

- let the input $x(t) = \delta(t)$, substitute into the convolution integral

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \\ &= \int_{-\infty}^{\infty} \delta(\tau) h(t-\tau) d\tau \\ &= h(t) \end{aligned}$$

as expected, from the definition of the impulse response, ie the output due to a delta input!

Convolution: graphical explanation

- for an intuitive understanding of convolution $y(t) = x(t) * (t)h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$, consider a visual explanation
  1. express $x(t)$ and $h(t)$ in terms of a dummy variable τ
  2. flip $h(\tau)$ around the τ-axis: $h(\tau) \rightarrow h(-\tau)$
  3. add a time offset t and 'slide' $h(-\tau + t)$ along the τ-axis
  4. for each t, compute the integral of the product $x(\tau)h(-\tau + t)$
  5. this results in $y(t)$ for all t

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  2. flip $h(\tau)$ around the τ-axis: $h(\tau) \to h(-\tau)$
  3. add a time offset t and 'slide' $h(-\tau + t)$ along the τ-axis
  4. for each t, compute the integral of the product $x(\tau)h(-\tau + t)$
  5. this results in $y(t)$ for all t
- note that convolution is commutative, that is

$$x_1(t) * x_2(t) = x_2(t) * x_1(t)$$

Convolution: graphical explanation

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- this means that the above procedure can also be done by interchanging the roles of $x(t)$ and $h(t)$ - that is flip and slide $x(t)$ rather than $h(t)$
- classic example: rectangular input $x(t)$ to a system with a rectangular impulse response $h(t)$

Frequency Response

- Claim: if $x(t) \rightleftharpoons X(f)$, $h(t) \rightleftharpoons H(f)$ and $y(t) \rightleftharpoons Y(f)$, and

$$y(t) = x(t) * h(t)$$

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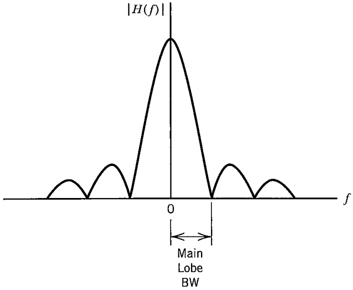
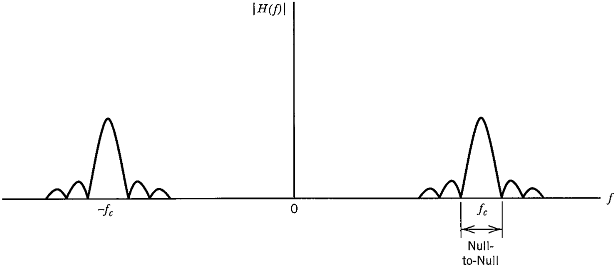
- for a system with impulse response $h(t)$, $H(f)$ is called the *transfer function* of the system
- Proof: see ECEN 320 notes!

Bandwidth

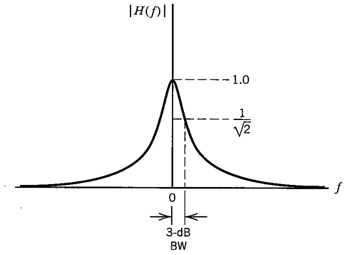
- *Bandwidth* of a signal measures the extent of *significant spectral content* of the signal for positive frequencies.
- For strictly bandlimited signals, such as a sinc pulse, the bandwidth is clearly defined
- Many signals are not strictly band limited. Definition of *significant spectral content* can vary
    - main lobe / null-to-null bandwidth (for symmetric signals with a main lobe bounded by a null)
    - 3-dB bandwidth
    - rms bandwidth

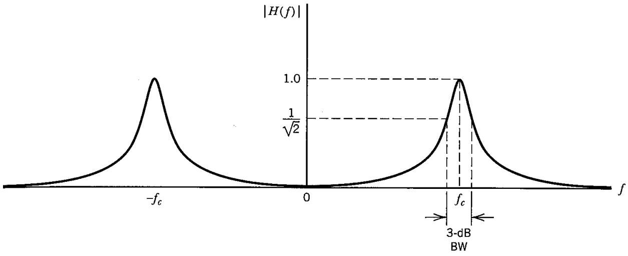
Null-to-null bandwidth





3-dB bandwidth





rms bandwidth

- root mean square (rms) bandwidth - square root

rms bandwidth

- root mean square (rms) bandwidth - square root of the second moment

rms bandwidth

- root mean square (rms) bandwidth - square root of the second moment of a squared amplitude spectrum

rms bandwidth

- root mean square (rms) bandwidth - square root of the second moment of a squared amplitude spectrum , normalised

$$W_{\mathsf{rms}} = \left(\frac{\int_{-\infty}^{\infty} f^2 |G(f)|^2 df}{\int_{-\infty}^{\infty} |G(f)|^2 df} \right)^{1/2}$$

Homework

- Week 2: Amplitude Modulation (AM)
- Please read Proakis & Salehi Chapter 3