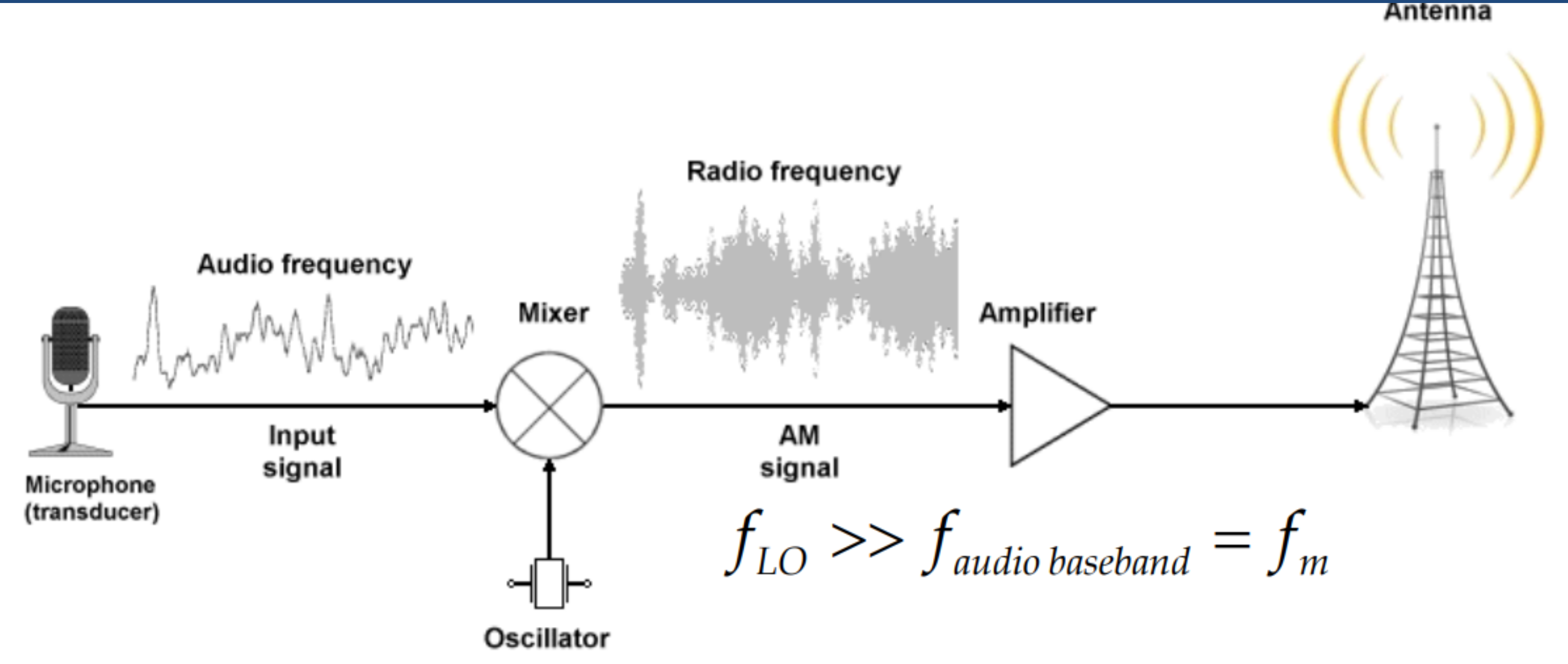
LECTURE 3:
AMPLITUDE MODULATION

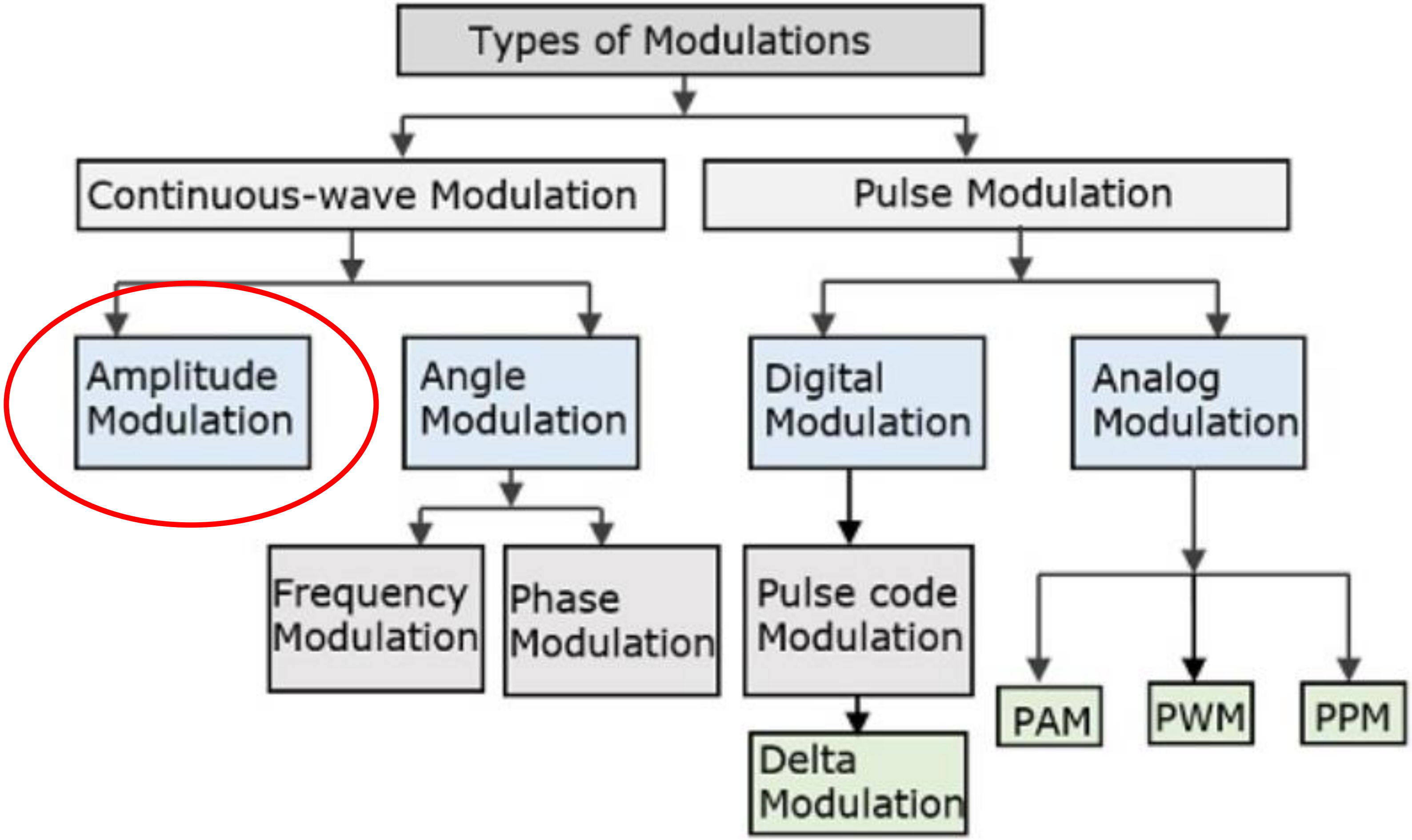


ECEN 310

Communication Engineering

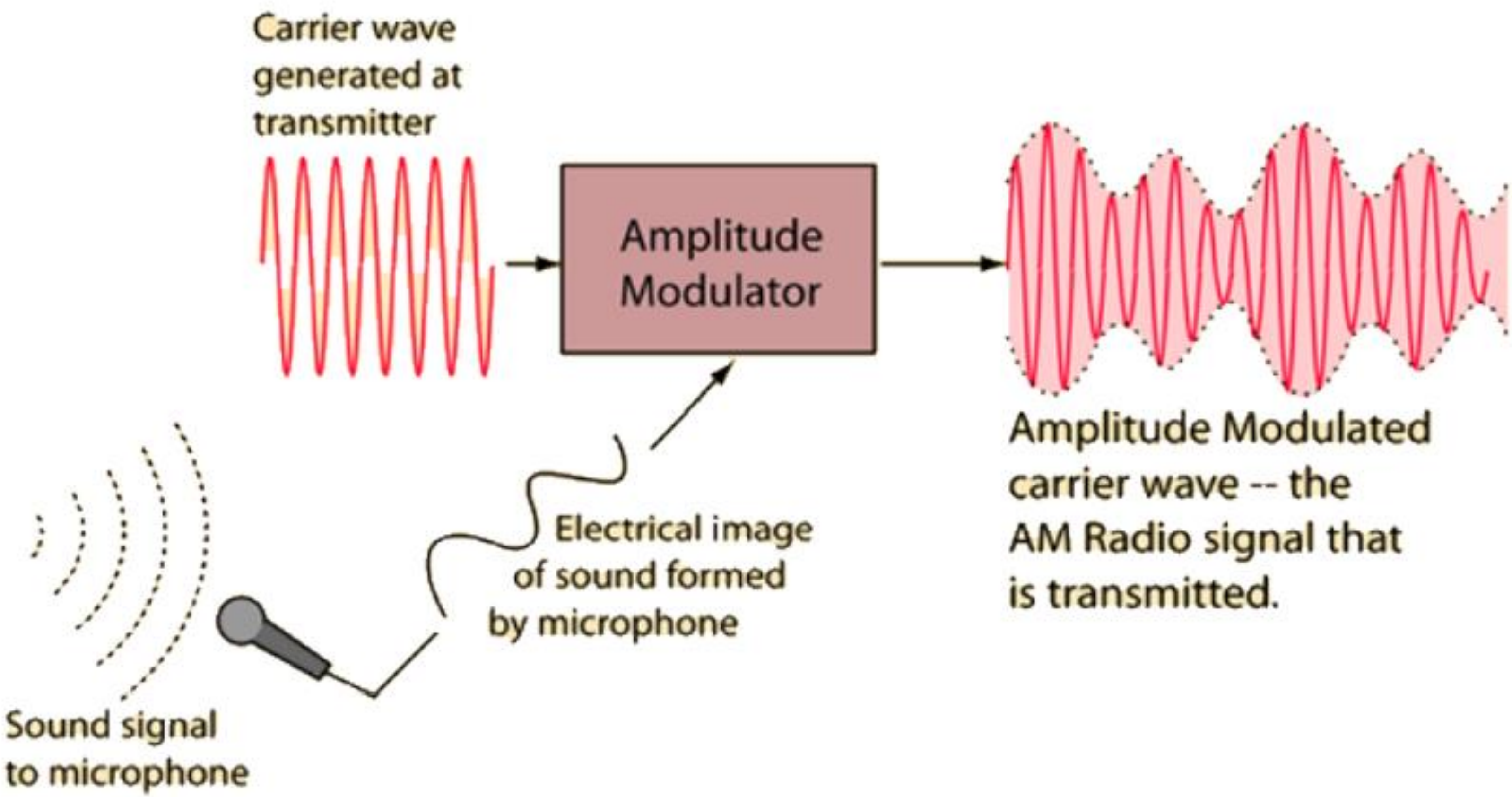
VICTORIA UNIVERSITY OF WELLINGTON TE HERENGA WAKA

Lecture Outline



This lecture

Read Proakis and Salehi, Chapter 3



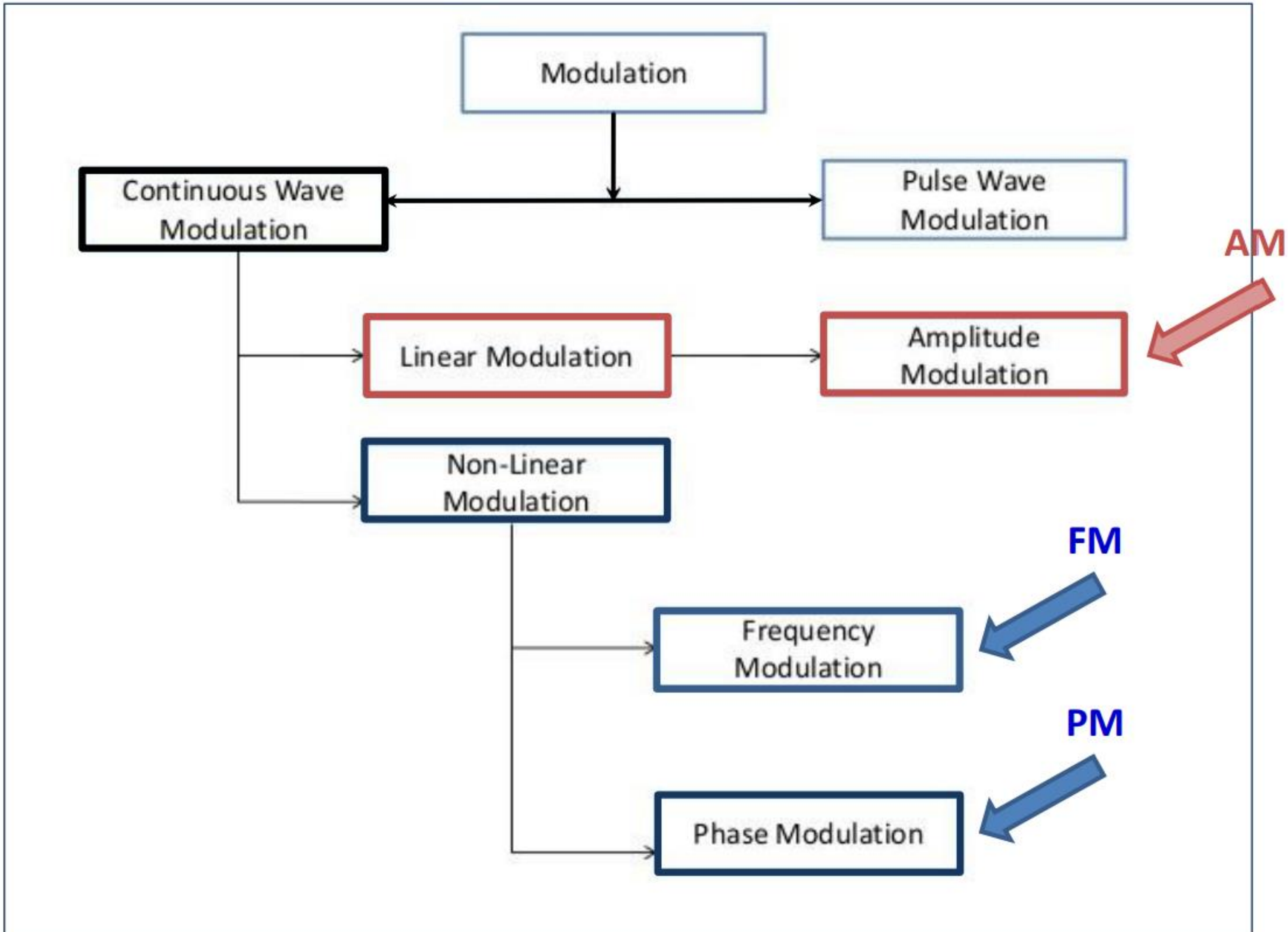
Baseband Communication

- baseband - frequency band of the original message signal from the source or input transducer
- **baseband transmission** refers to transmitting the signal directly, without any modification to the spectral content
  - most baseband signals (audio, video) contain significant low-frequency component - cannot be effectively transmitted over the wireless channel
  - baseband signals have overlapping band - results in severe interference if sharing a channel

Bandpass Communication

- *carrier modulation* shifts the frequency spectrum to a higher frequency band - **bandpass transmission**
  - modulating several signals to non-overlapping bands allows many users to share the wireless channel (*frequency division multiplexing*)
  - modulation onto higher frequencies is required for efficient power radiation using reasonably sized antennas

Modulation options



Modulation Overview

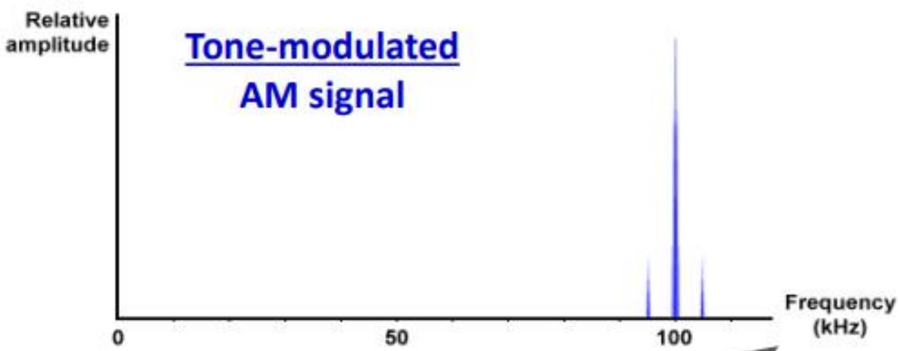
- Modulation is a process where some characteristic of a **carrier wave** is varied in accordance with an **information-bearing** signal.
- The carrier is needed to facilitate the transmission of the signal over a band-pass channel
- If the message signal is analogue (continuous in time and amplitude), we speak of *continuous-wave modulation* (continuous in time)

Continuous-Wave Modulation

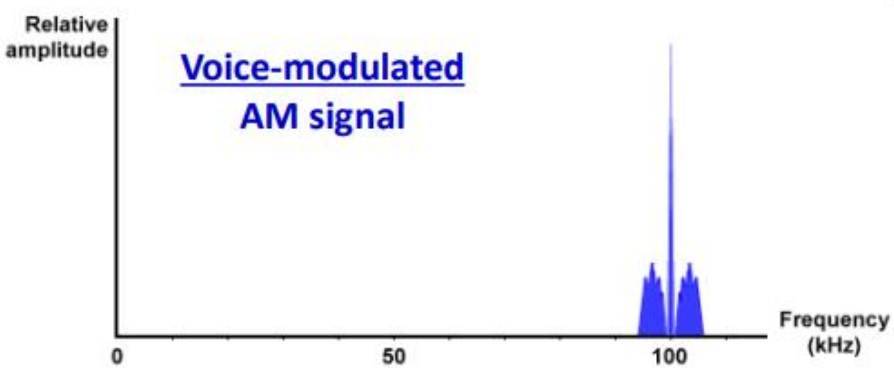
- There are two types of continuous-wave modulation
  - amplitude modulation - the amplitude of the carrier is varied according to the message signal (AM, SSB-SC, VSB)
  - angle modulation - and phase or frequency of the carrier is varied according to the message signal (FM, PM)
- we will discuss different variants of amplitude and angle modulations, focusing on
  - transmitter and receiver **complexity**
  - required **power**
  - required **bandwidth**
- today: Amplitude Modulation...

AM in pictures

Frequency Domain



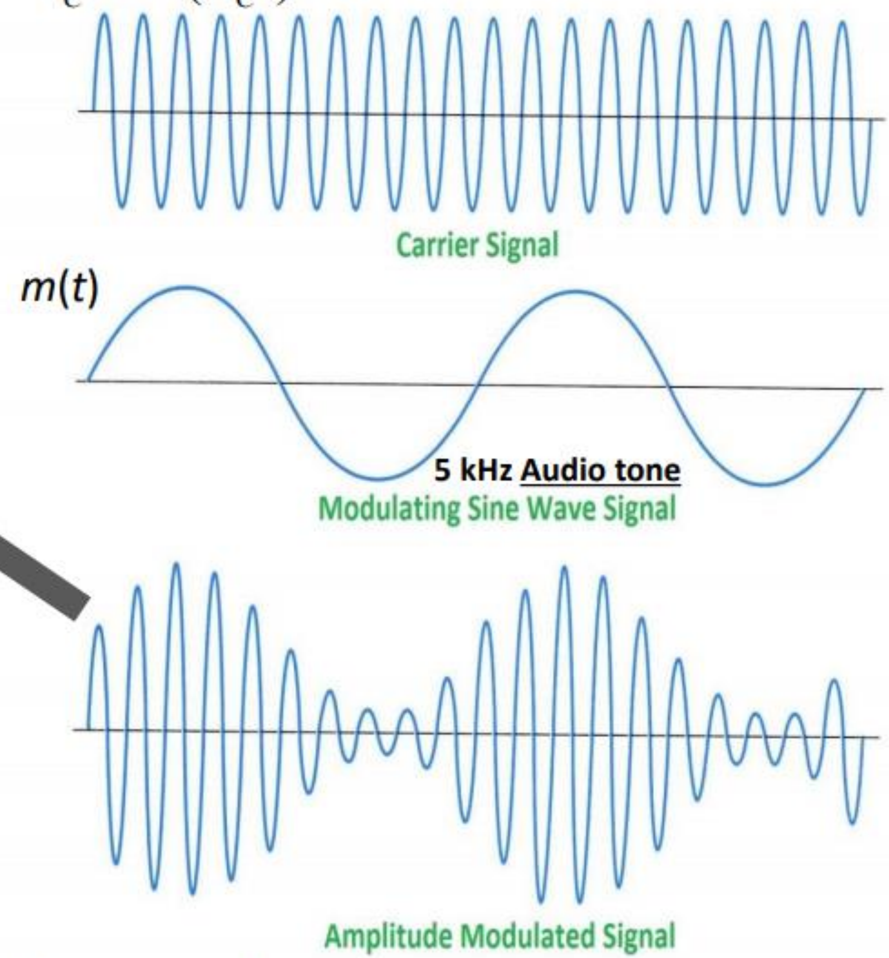
100 kHz carrier modulated by a 5kHz audio tone



100 kHz carrier modulated by an audio signal (frequencies up to 6 kHz)

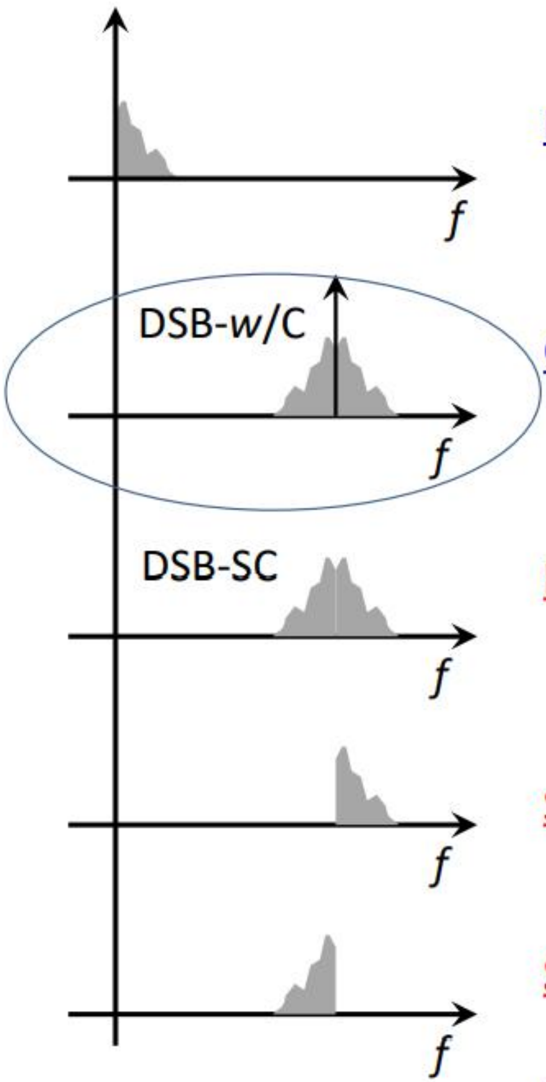
Time Domain

$A_C \cdot \cos(\omega_C t)$



$[A_C + m(t)]\cos(\omega_C t)$

4 variants of AM



Baseband spectrum (message spectrum $M(f)$)

Conventional AM (**Double-SideBand With Carrier**)

Special cases of AM:

Double-Sideband-Suppressed Carrier (DSB-SC)

Single-Sideband /Upper Sideband (SSB/USB)

Single-Sideband /Lower Sideband (SSB/LSB)

Also Vestigial Sideband and Amplitude Companded SSB

Amplitude Modulation

- We will study 4 variants of amplitude modulation
  - (conventional) amplitude modulation (AM)
  - double sideband-suppressed carrier (DSB-SC)
  - single sideband (SSB)
  - vestigial sideband (VSB)
- roughly speaking

type	complexity	power requirement	spectral efficiency
AM	low	high	low
DSB-SC	moderate	moderate	low
SSB	high	low	high
VSB	high	low	high

Note: a real valued signal has a symmetric frequency spectrum - thus only one half of it is required to convey all its information

AM theory

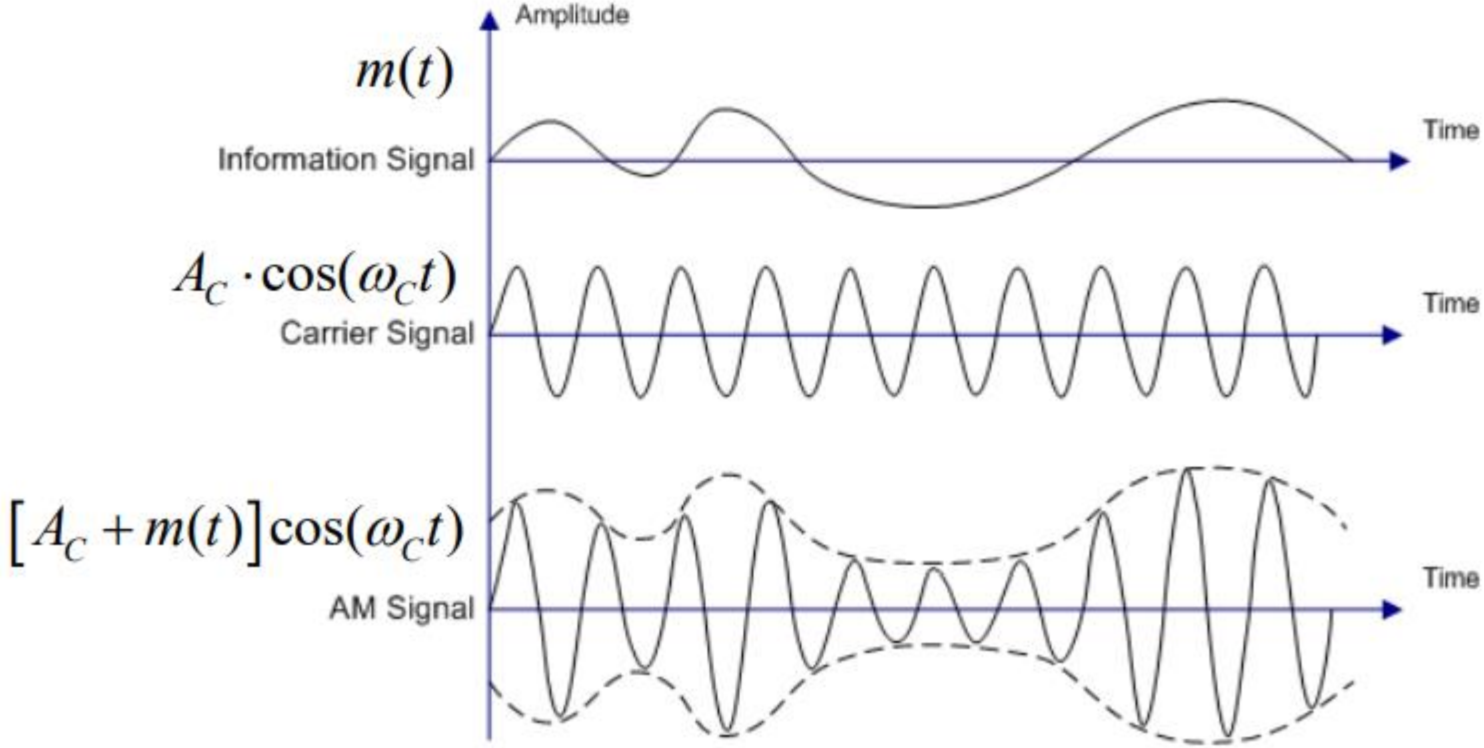
- Now, in light of the requirement to generate a coherent carrier at the receiver, let us consider a slightly modified technique - (conventional AM)
- In Amplitude Modulation (AM), the amplitude of the carrier $c(t)$ is varied about a mean (A) linearly with $m(t)$, that is

$$\phi_{\text{AM}}(t) = [A + m(t)]\cos(2\pi f_c t) = \underbrace{A\cos(2\pi f_c t)}_{\textit{carrier}} + \underbrace{m(t)\cos(2\pi f_c t)}_{\text{modulated message}}$$

- In other words, we transmit an unmodulated carrier in addition to the modulated message.

$$\Phi_{\text{AM}}(f) = \frac{A}{2}[\delta(f + f_c) + \delta(f - f_c)] + \frac{1}{2}[M(f + f_c) + M(f - f_c)]$$

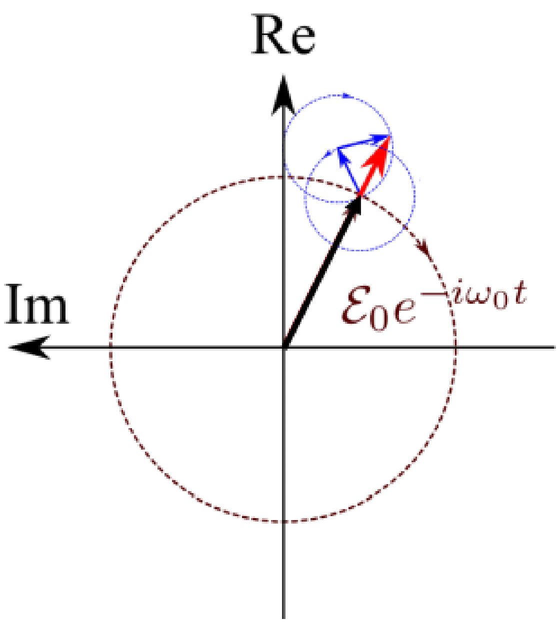
Time domain AM signal

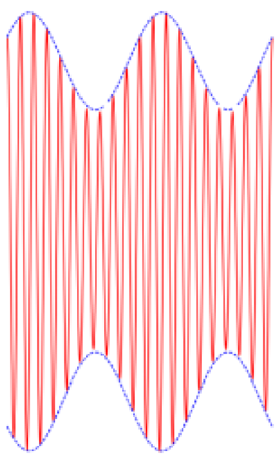


Phasor view of AM signal

Example showing tone modulation

a) Amplitude modulation phasor diagram





Black vector $\Rightarrow$ Carrier signal

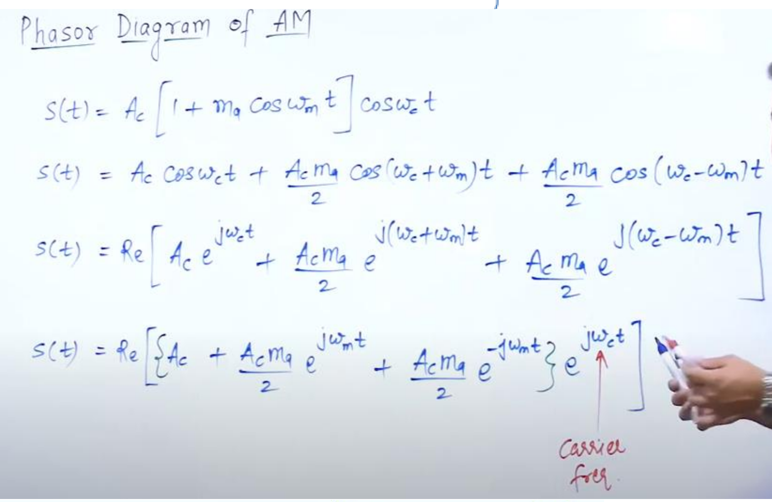
Red vector $\Rightarrow$ AM modulated signal

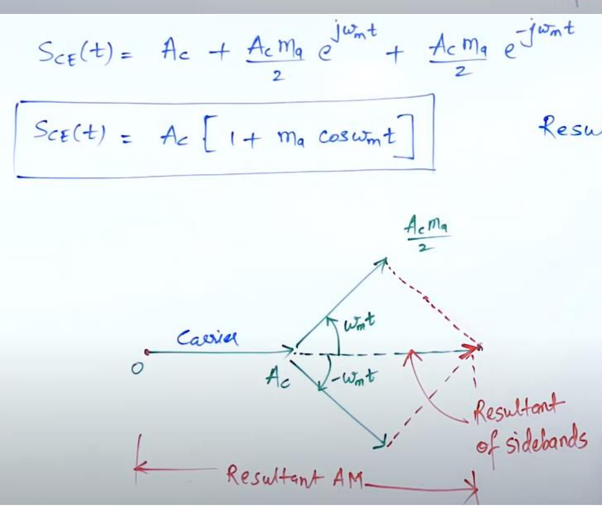
Modulated oscillation is a sum of these three vectors an is given by the red vector. In the case of amplitude modulation (AM), the modulated oscillation vector is always in phase with the carrier field while its length oscillates with the modulation frequency. The time dependence of its projection onto the real axis gives the signal strength as drawn to the right of the corresponding phasor diagram.

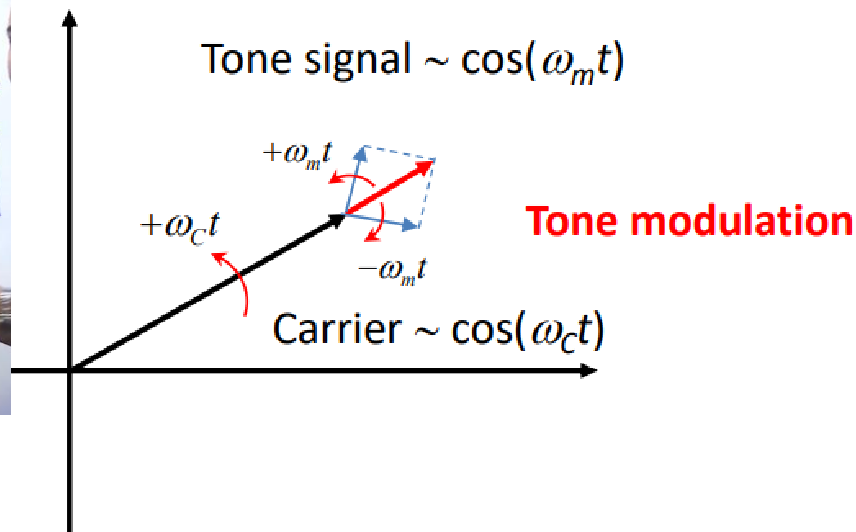
Phasor view of AM signal

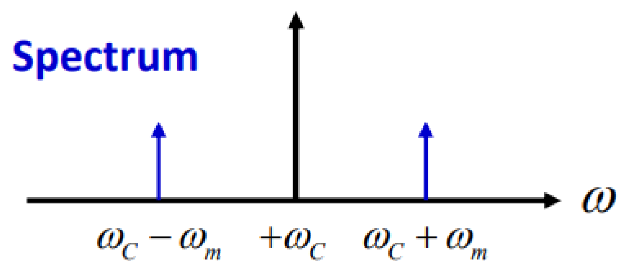
Complex Exponential Format

$$\phi_{AM}(t) = \mathrm{Re}\left[e^{j\omega_C t}\left(1 + \frac{e^{j\omega_m t}}{2} + \frac{e^{-j\omega_m t}}{2}\right)\right]$$

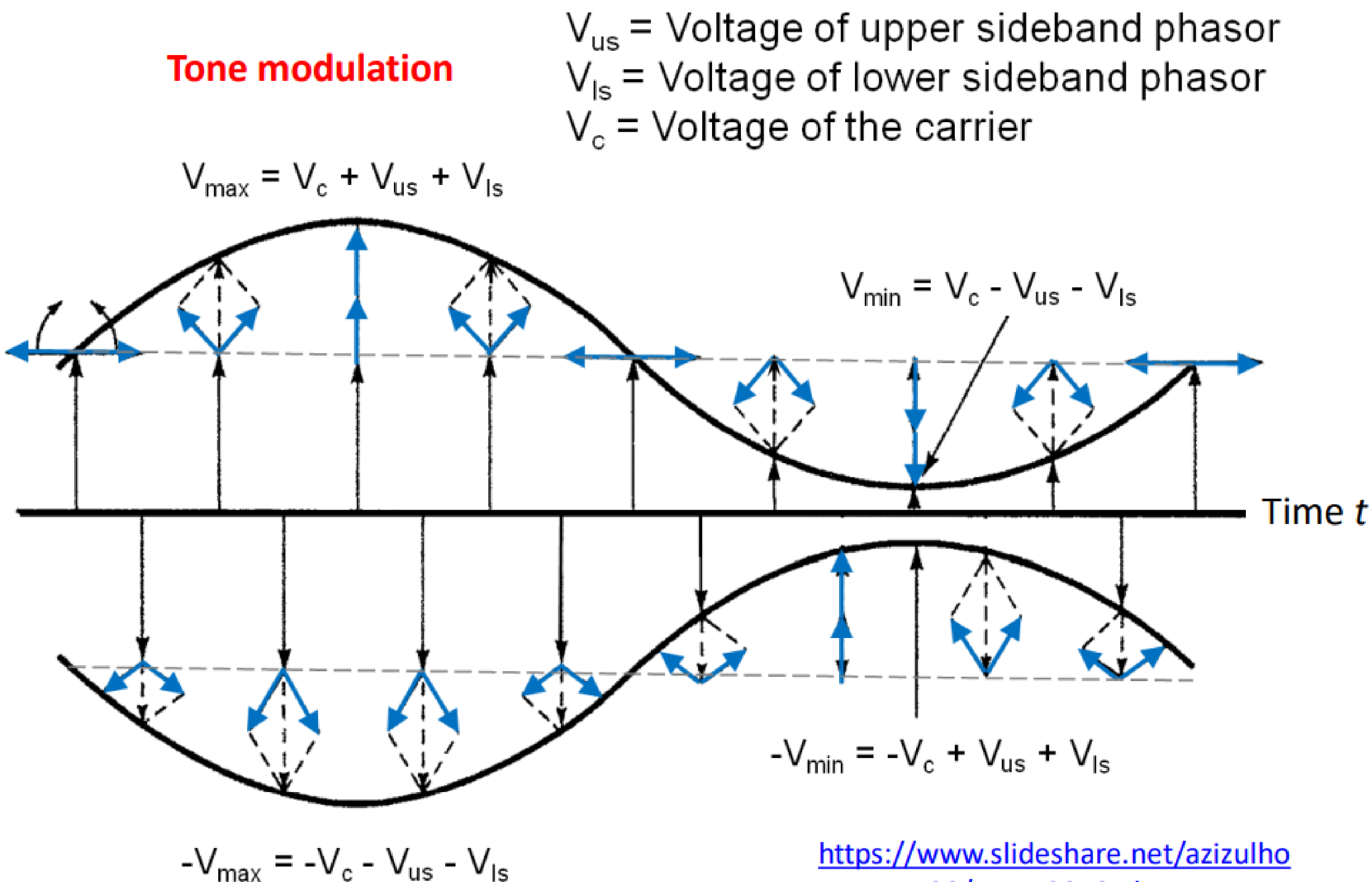








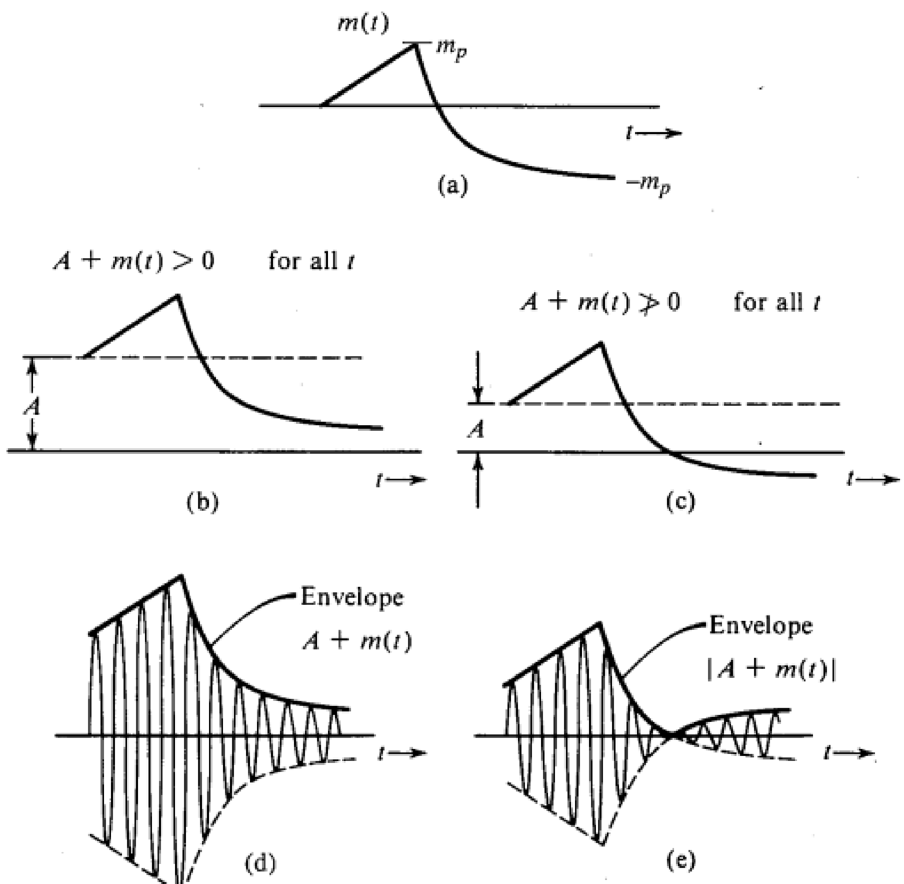
Phasor view of AM Signal



https://www.slideshare.net/azizulhoque539/eeng-3810-chapter-4

AM theory

- AM is identical to DSB-SC *with* $A + m(t)$ *as the modulating signal*
- The carrier component oscillates between the *envelope* $|A + m(t)|$ and its negative image $-|A + m(t)|$



AM theory

- The envelope is an accurate representation of the message, provided
  1. the carrier frequency f_c is much greater than the highest frequency component of $m(t)$, which is the *message bandwidth* W:
  $$f_c >> B$$
  2. the message amplitude satisfies
  $$A + m(t) \geq 1, \text{ for all } t$$
- Condition 1 relates to overlap of the frequency spectrum components
- Condition 2 ensures that the message can be recovered from the envelope.

AM theory

- If $m(t) \geq 0 \; \forall t$, then we can let $A = 0$ (no additional carrier component needed). Here, the signal is actually DSB-SC, with an envelope of $m(t)$. Envelope detection is possible.
- Suppose $m(t)$ has zero offset, and $\pm m_p$ denotes the maximum and minimum values of $m(t)$, respectively.
  - That is $m(t) \geq -m_p$ and condition for envelope detection is

$$A \geq -m_p$$

  - Define *modulation index* μ

$$\mu = \frac{m_p}{A}$$

  - Thus, envelope detection is distortionless if $0 \leq \mu \leq 1$
  - The condition $\mu > 1$ is referred to as overmodulation, and envelope detection is not possible.

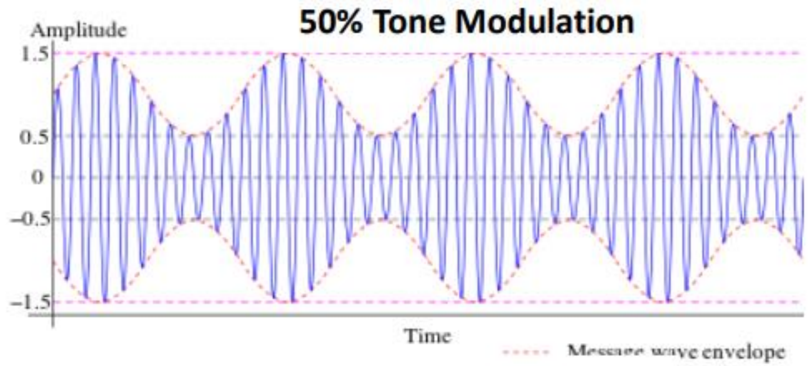
AM theory

- Suppose $m(t)$ has *nonzero* offset, with m_{max} and m_{min} denotes the maximum and minimum values of $m(t)$, respectively. Here $m_{\text{min}} \neq -m_{\text{max}}$
- One can show that envelope detection is distortionless if $0 \leq \mu \leq 1$ but with the modified modulation index

$$\mu = \frac{m_{\text{max}} - m_{\text{min}}}{2A + m_{\text{max}} + m_{\text{min}}}$$

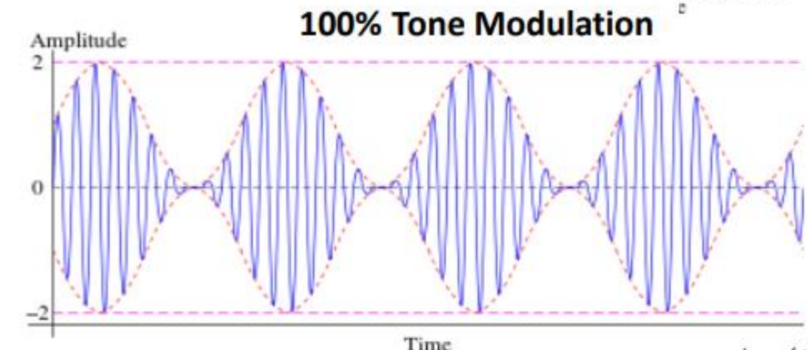
AM Modulation Index

50%



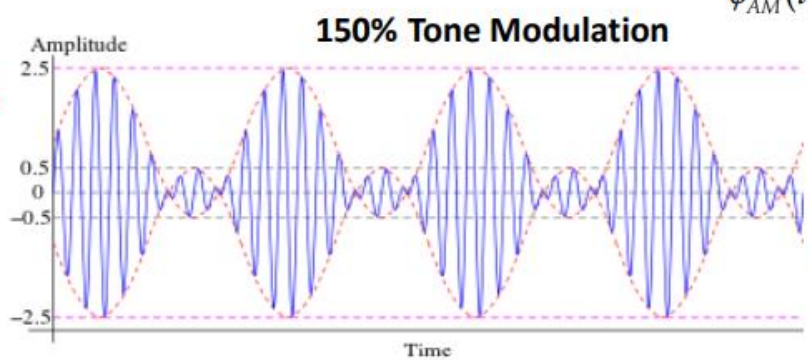
$$\mu = \frac{m_p}{A_C}$$

100%



$$\phi_{AM}(t) = A_C\left[1 + \mu \cdot \cos(\omega_m t)\right] \cdot \cos(\omega_C t)$$

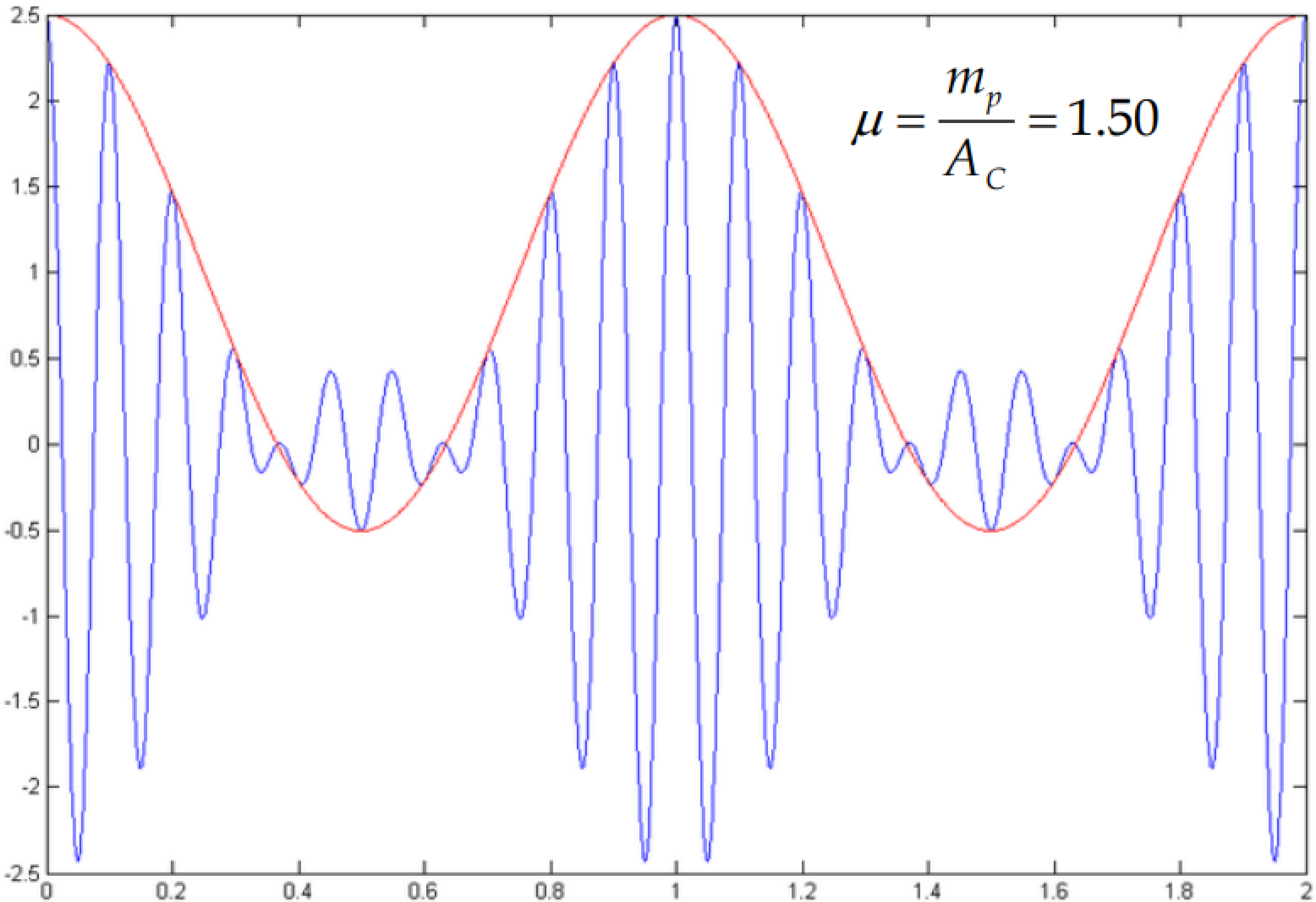
150%



Overmodulation
or
Envelope Distortion

https://en.wikipedia.org/wiki/Amplitude_modulation

Envelope detection is not for overmodulation



AM: transmitted power

- The advantage of envelope detection comes at a price of reduced power efficiency.

$$\phi_{\mathsf{AM}}(t) = \underbrace{A\cos(\omega_c t)}_{\textit{carrier}} + \underbrace{m(t)\cos(\omega_c t)}_{\text{sidebands}}$$

  - Carrier power is the mean square value of $A\cos\omega_c t$, that is $P_c = \frac{A^2}{2}$
  - Similarly, sideband power is $P_s = \frac{1}{2}\overline{m^2(t)}$, where $\bar{\cdot}$ denotes mean square value, ie

$$\overline{m^2(t)} = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} m^2(t)\ dt$$

- Power efficiency η is thus

$$\eta = \frac{\text{useful power}}{\text{total power}} = \frac{P_s}{P_c + P_s} = \frac{\overline{m^2(t)}}{A^2 + \overline{m^2(t)}}$$

AM: transmitted power

- Consider the special case of tone modulation, that is

$$m(t) = \mu A \cos \omega_m t$$

  and

$$\overline{m^2(t)} = \frac{(\mu A)^2}{2}$$

- power efficiency is

$$\eta = \frac{\mu^2}{2 + \mu^2}$$

- Recall the condition $0 \leq \mu \leq 1$; The efficiency increases monotonically with μ, with

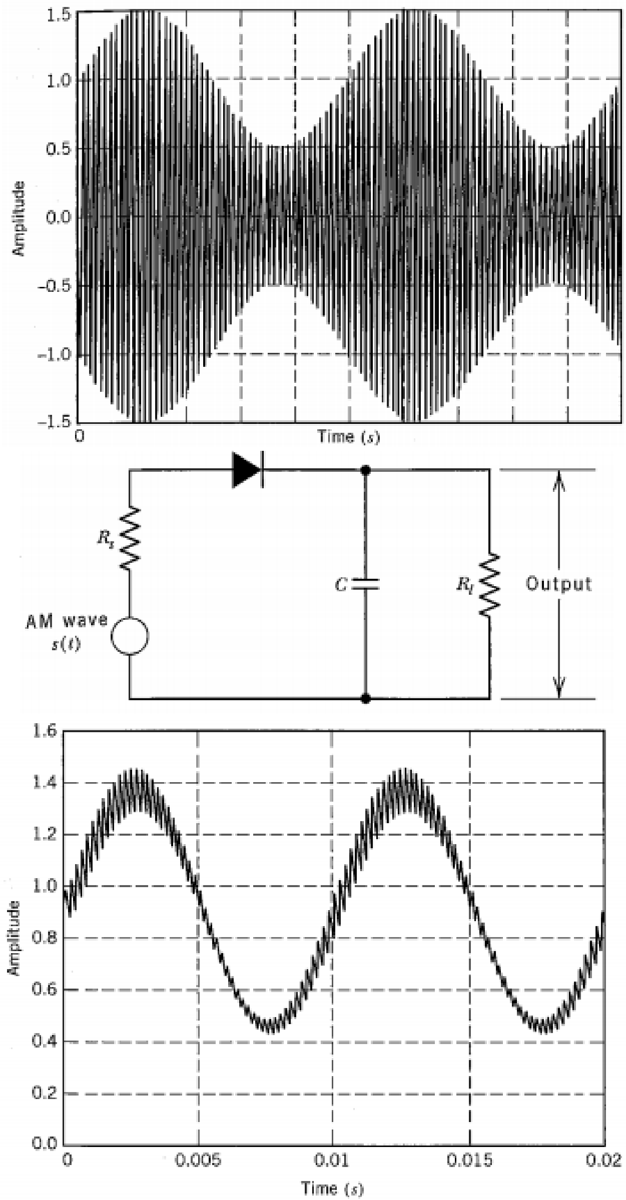
$$\eta_{\text{max}} = 33\%$$

AM power efficiency

Modulation index = μ	$\eta = \frac{\mu^2}{2+\mu^2}$
0.25	0.0303 or 3.03 %
0.5	0.111 or 11.1 %
1.0	0.333 or 33.3 %

Conclusion: AM power efficiency is low (this is undesirable).

AM: envelope detection

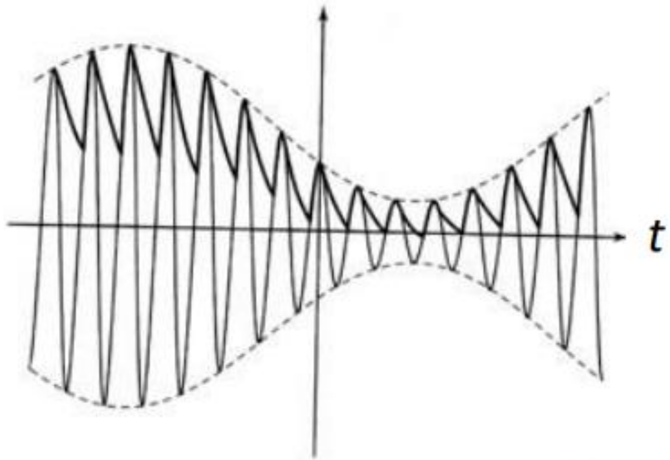


- One of the simplest methods of AM demodulation: envelope detection
- found in almost all AM radios
- diode, RC filter:
  1. on the positive half-cycle of the AM signal diode is forward-biased
  2. capacitor charges up to the peak of the signal $s(t)$
  3. when $s(t)$ falls below the peak, capacitor discharges slowly via resistor
  4. discharge continues until next positive half-cycle
  5. when $s(t)$ reaches capacitor's voltage, the diode is once again forward-biased
  6. process continues

we will study the process in the lab

Envelope detection

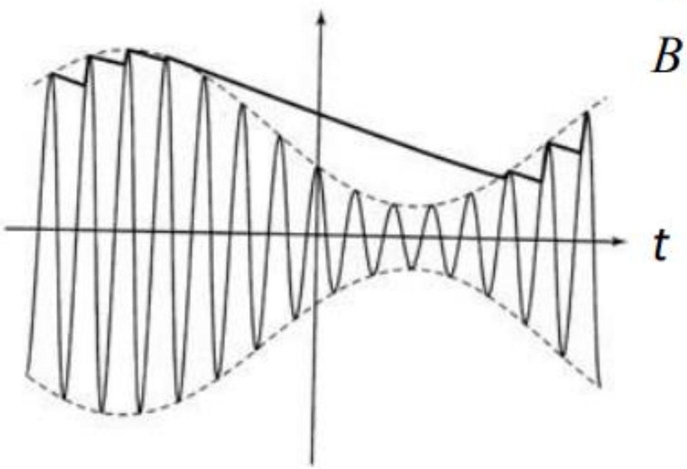
How the envelope is captured



Time constant $\tau = RC$ too short.

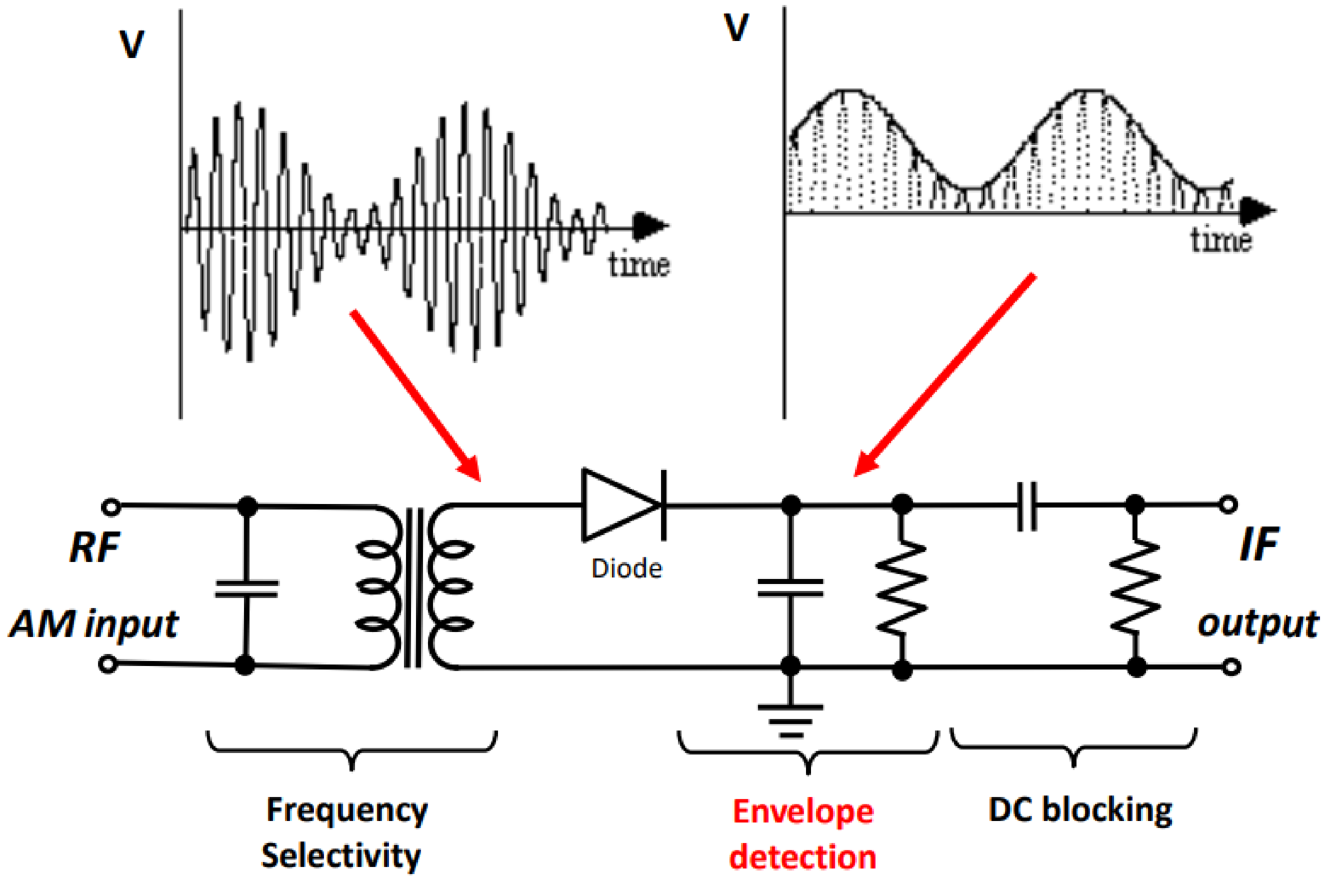
Design criteria is $2\pi B < \frac{1}{RC} << 2\pi f_c$

B Hz is bandwidth of message signal.



Time constant $\tau = RC$ too long.

Practical demodulation of AM signal



The user has a choice of changing the filter to meet their needs.

AM: summary

1. Conventional AM is simple to generate and to demodulate.
2. Conventional AM is wasteful of transmitted power: the unmodulated carrier $c(t)$ carries no information, yet it is transmitted.
3. Conventional AM is wasteful of channel bandwidth: for real valued signal, whose spectra are symmetric, transmission of both sidebands takes up twice as much bandwidth as required.

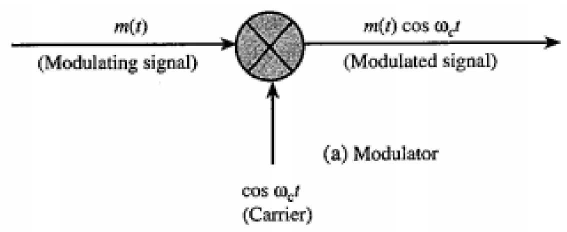
Double Sideband-Suppressed Carrier (DSB-SC) modulation

- (The name will become clear a bit later on...)
- in general, in amplitude modulation the amplitude of a information bearing carrier is a linear function of the message
- Consider the sinusoidal carrier $c(t)$, with carrier frequency f_c, that is

$$c(t) = \cos(2\pi f_c t)$$

- *Modulating* the carrier is an analogue message signal $m(t)$, that is

$$\phi_{\text{DSB}}(t) = c(t)m(t) = m(t)\cos(2\pi f_c t)$$



- Since the modulation is a simple multiplication of the message by a carrier, the DSB-SC modulator is referred to as a *product modulator*

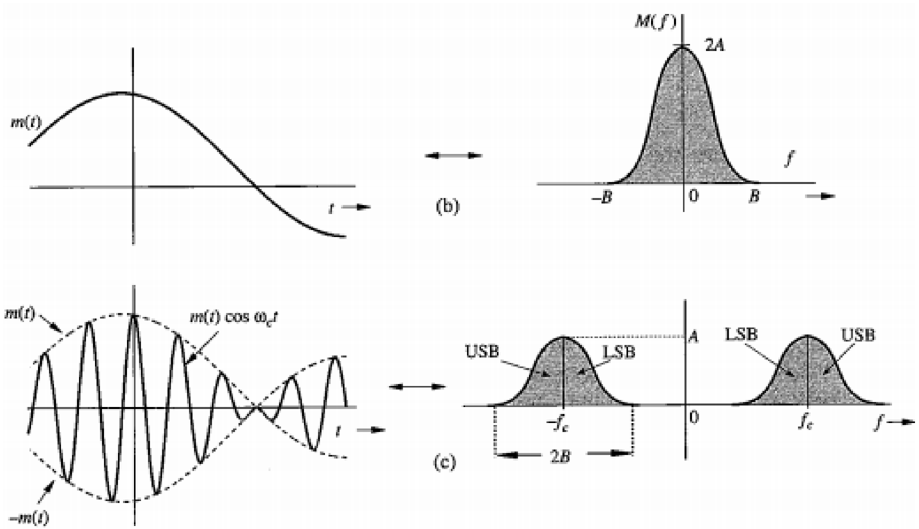
DSB-SC spectrum

- Assume $m(t) \rightleftharpoons M(f)$, with bandwidth B, and $f_c > B$
- The spectrum of $s(t)$ is a translation to the left and right by f_c

$$\Phi_{\mathsf{DSB}}(f) = \frac{1}{2}[M(f - f_c) + M(f + f_c)]$$

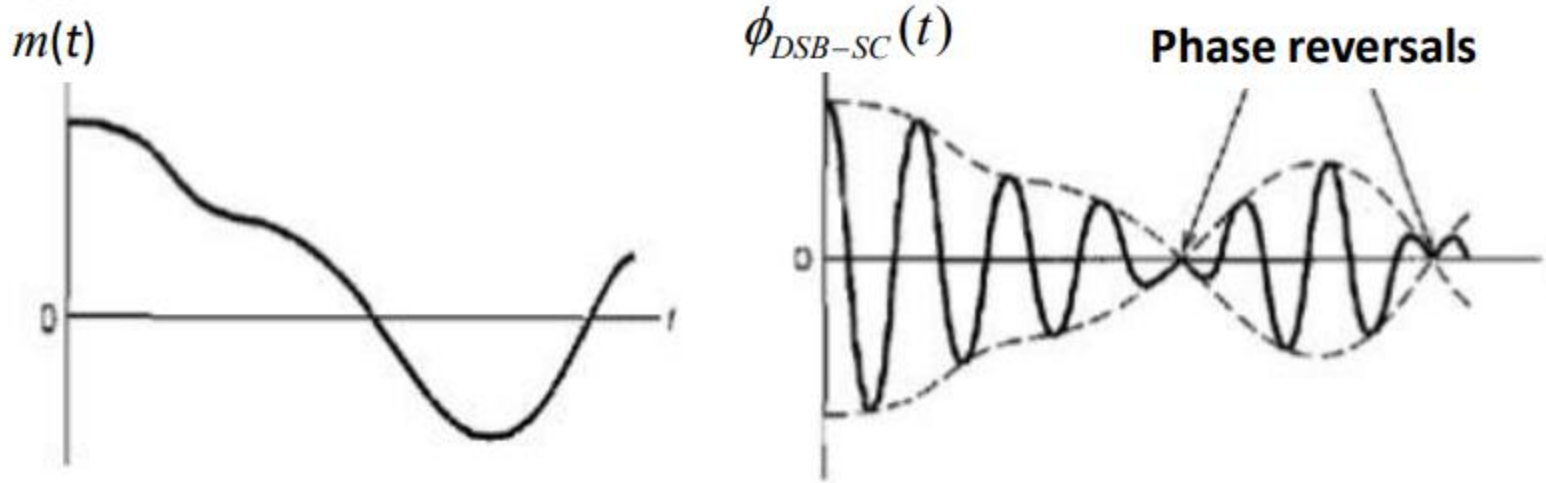
where we have used

- $\cos(2\pi f_c t) = \frac{1}{2}[e^{-j2\pi f_c t} + e^{-j2\pi f_c t}]$
- $m(t)e^{j2\pi f_c t} \rightleftharpoons M(f - f_c)$



Can't use envelope detection!

- This modulated wave undergoes a phase reversal whenever the message signal m(t) crosses zero, as illustrated in figure below



(a) Baseband signal (b) DSB-SC modulated wave

The penalty for using DSB-SC is a complex detection scheme is needed.

We can't use simple envelope detection.

DSB-SC spectrum

- Note:
  1. The negative frequencies become 'visible' after modulation
  2. For positive frequencies the spectrum above/below f_c is referred to as the *upper/lower sideband*
  3. for positive frequencies, the span of frequencies present, that is $f_c - B < f < f_c + B$ is the *transmission bandwidth*, given by

$$B_T = 2B$$

- Recall condition that $f_c > B$: if $f_c < B$, the sidebands will overlap resulting in distortion
- e.g. AM: $B = 5$ kHz, carrier frequency $f_c = 550 - 1600$ kHz for $f_c/B >> 1$
- Unless the message contains an impulse at DC, the modulated signal does note contain a discrete component of the carrier frequency f_c - hence the name **double sideband suppressed carrier** modulation

DSB-SC: coherent detection

- Consider the demodulation at the receiver
- Modulation translates the frequency spectrum, now need to re-translate to original position.
- Note the trig identity

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

- If we multiply $\phi_{\text{DSB}}(t)$ by $\cos(2\pi f_c t)$, the constant term $(\frac{1}{2})$ allows us to recover (a scaled copy of) the original message $m(t)$

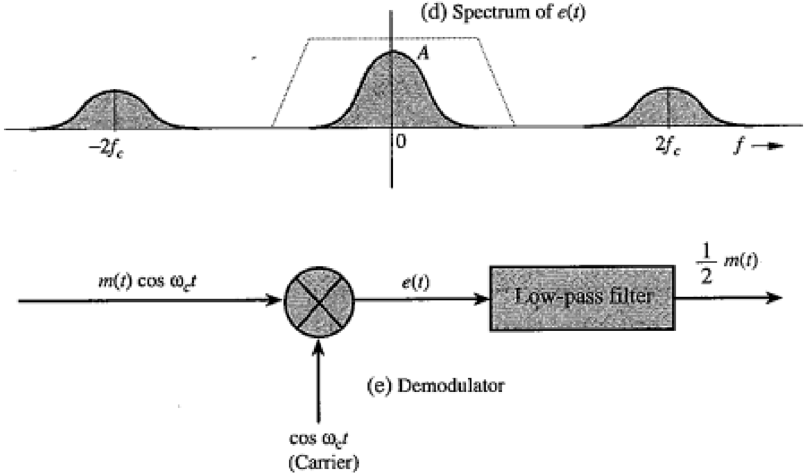
$$\begin{aligned} e(t) &= \phi_{\text{DSB}}\cos(\omega_c t) = m(t)\cos^2(\omega_c t) \\ &= \frac{1}{2}m(t) + \frac{1}{2}m(t)\cos(2\omega_c t) \end{aligned}$$

DSB-SC: coherent detection

- The spectrum is now

$$E(f) = \frac{1}{2}M(f) + \frac{1}{4}\left[M(f+2f_c) + M(f-2f_c)\right]$$

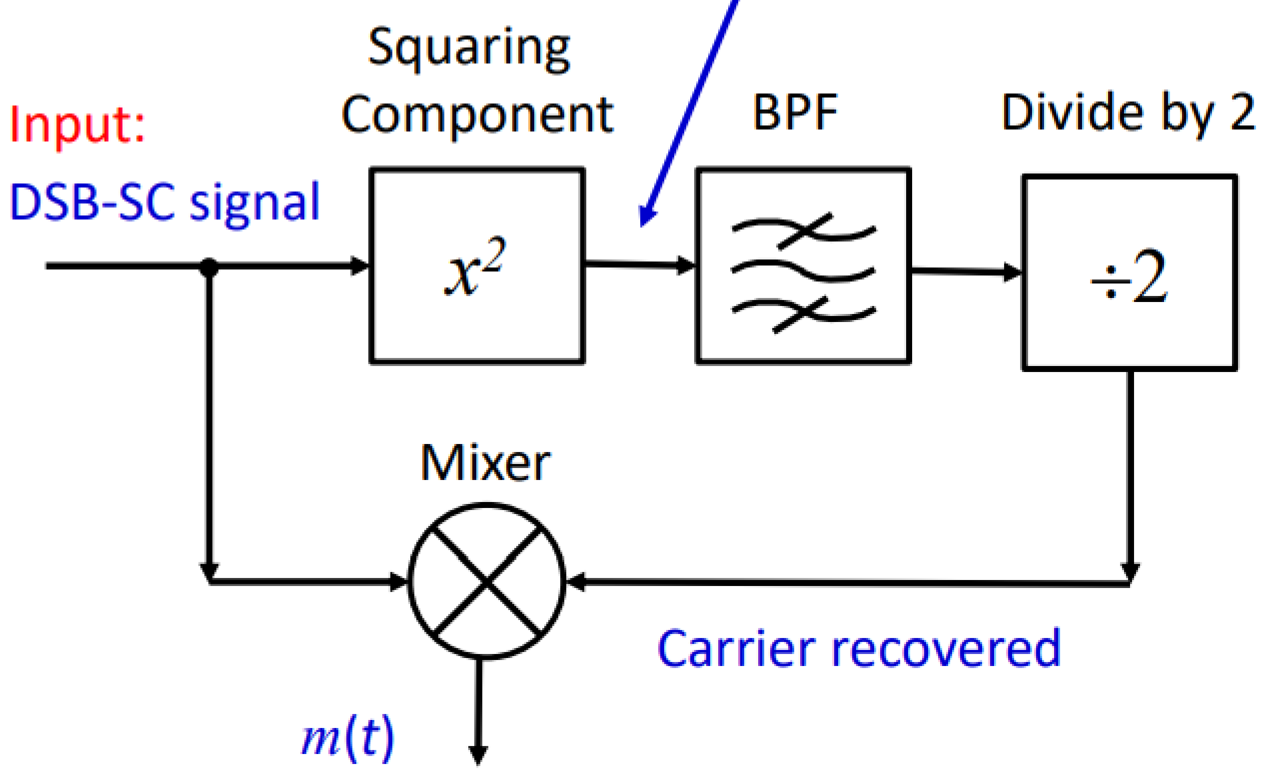
- We can remove the double frequency component by low-pass filtering



- This assumes, that the receiver can generate $\cos(2\pi f_c t + \phi)$, with $\phi = 0$ (or, exactly in phase with the carrier at the transmitter) - this is why we refer to this detection technique as **coherent** or **synchronous** detection.

DSB-SC synchronous demodulation

$$\cos^2\omega t = \frac{1}{2}[1+\cos(2\omega t)]$$



DSB-SC: summary

1. DSB-SC simple to generate and to demodulate.
2. DSB-SC is wasteful of channel bandwidth: for real valued signal, whose spectra are symmetric, transmission of both sidebands takes up twice as much bandwidth as required.
3. It requires a coherent demodulator, that is a receiver with a carrier synchronised to that at the transmitter

... and so...

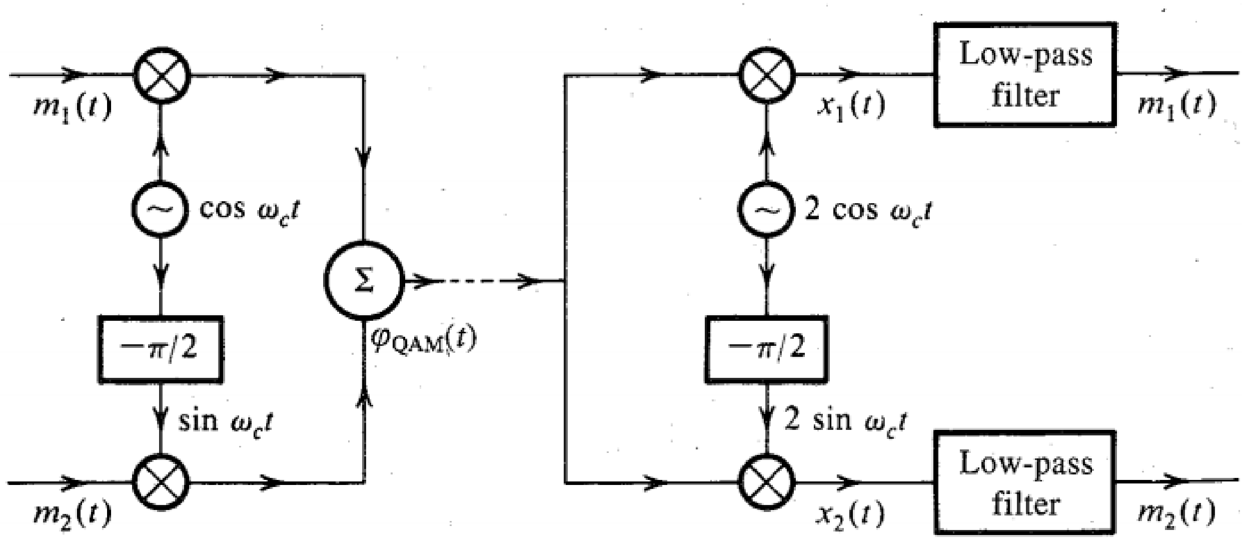
Question:

- Consider coherent demodulation of a DSB-SC signal, however where the local oscillator at the receiver produces a carrier with a phase offset ϕ, that is $\tilde{c}(t) = \cos(\omega_c t + \phi)$.
  - Find the expression for the recovered message as a function of ϕ.
  - What happens if ϕ is time varying, that is $\phi = \phi(t)$?
  - What happens if $\phi = \pi/2$?

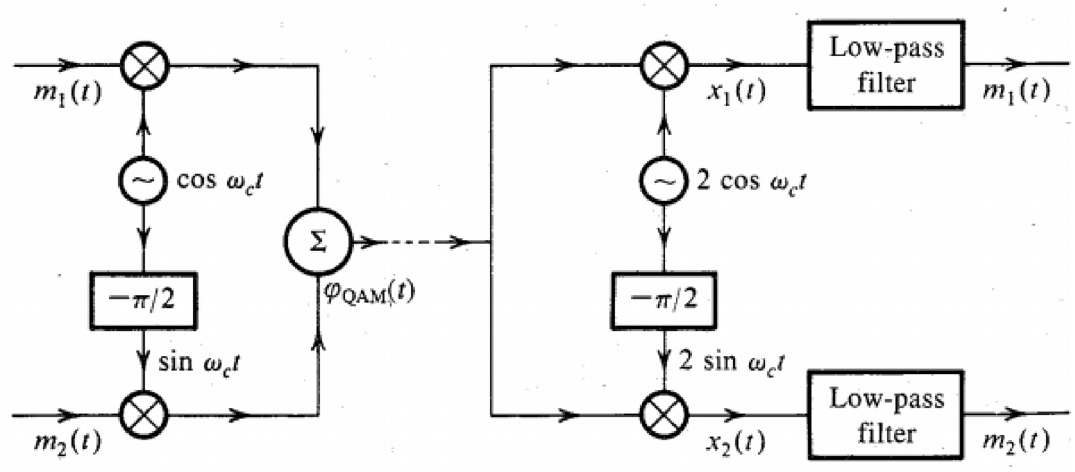
Quadrature Amplitude Modulation (QAM)

- We can take advantage of orthogonality of $\cos$ and $\sin$ to transmit twice the data over the same bandwidth as DSB-SC or AM.
- This scheme is referred to as Quadrature Amplitude Modulation (QAM) or Quadrature Modulation
- We transmit two message signals $m_1(t)$ and $m_2(t)$ using two carriers $\pi/2$ out of phase (in *phase quadrature*)

$$\phi_{\text{QAM}}(t) = m_1(t)\cos(\omega_c t) + m_2(t)\sin(\omega_c t)$$



Quadrature Amplitude Modulation



- at the receiver the composite signal is coherently demodulated by two carriers, again, in phase quadrature
- The two multiplier outputs are given by

$$x_1(t) = 2\phi_{\mathsf{QAM}}(t)\cos\omega_c t = m_1(t) + m_1(t)\cos 2\omega_c t + m_2(t)\sin 2\omega_c t$$
$$x_2(t) = 2\phi_{\mathsf{QAM}}(t)\sin\omega_c t = m_2(t) - m_2(t)\cos 2\omega_c t + m_1(t)\sin 2\omega_c t$$

- Following lowpass filtering, we have recovered both signals have been recovered.
- The upper ($\cos$) channel and the lower ($\sin$) channel are referred to as the *in-phase* (I) and *quadrature* (Q) channels, respectively

Quadrature Amplitude Modulation

- The drawback of QAM is that it requires perfect synchronisation - a carrier phase error will result in interference between the I- and Q- branches
- The two multiplier outputs are given by

$$\begin{aligned} x_1(t) &= 2[m_1(t)\cos\omega_c t + m_2(t)\sin\omega_c t]\cos(\omega_c t + \theta) \\ &= m_1(t)\cos\theta + m_2(t)\sin\theta + m_1(t)\cos(2\omega_c t + \theta) + m_2(t)\sin(2\omega_c t + \theta) \end{aligned}$$

- which after low-pass filtering leaves

$$\hat{m}_1(t) = m_1(t)\cos\theta + m_2(t)\sin\theta$$

- This is referred to as *co-channel interference*

QAM – quadrature carrier multiplexing

Quadrature-carrier multiplexing allows for transmitting two message signals on the same carrier frequency.

(1) Two quadrature carriers are multiplexed together,

(2) Signal $m_I(t)$ modulates the carrier $\cos(\omega_C t)$, and Signal $m_Q(t)$ modulates the carrier $\sin(\omega_C t)$.

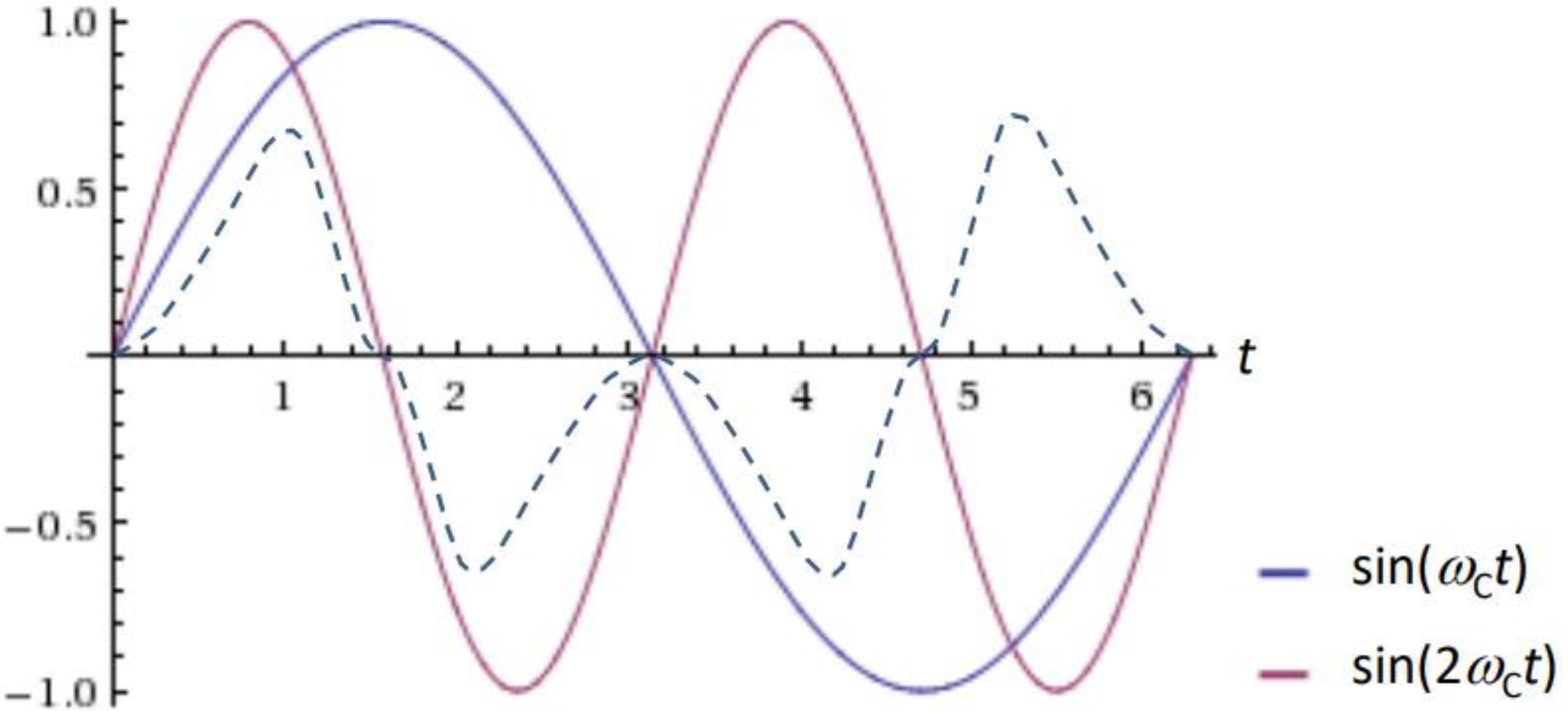
(3) The two modulated signals are added together & transmitted over the channel as

$$\boxed{\varphi_{QAM}(t) = m_I(t)\cdot\cos(\omega_C t) + m_Q(t)\cdot\sin(\omega_C t)}$$

Features on QAM

1. QAM transmits two DSB-SC signals in the bandwidth of one DSB-SC signal.
2. Interference between the two modulated signals of the same frequency is prevented by using two carriers in phase quadrature. This is because they are **orthogonal** to each other.
3. The In-phase (I-phase) channel modulates the cos($\omega_C t$) signal and the Quadrature-phase (Q-phase) channel modulates the sin ($\omega_C t$) signal.
4. The carriers used in the transmitter and receiver are synchronous with each other. In fact, they must be almost exactly in quadrature with each other, otherwise they experience cochannel interference.
5. Low-pass filters are used to extract the baseband signals $m_I(t)$ and $m_Q(t)$ in the receiver.

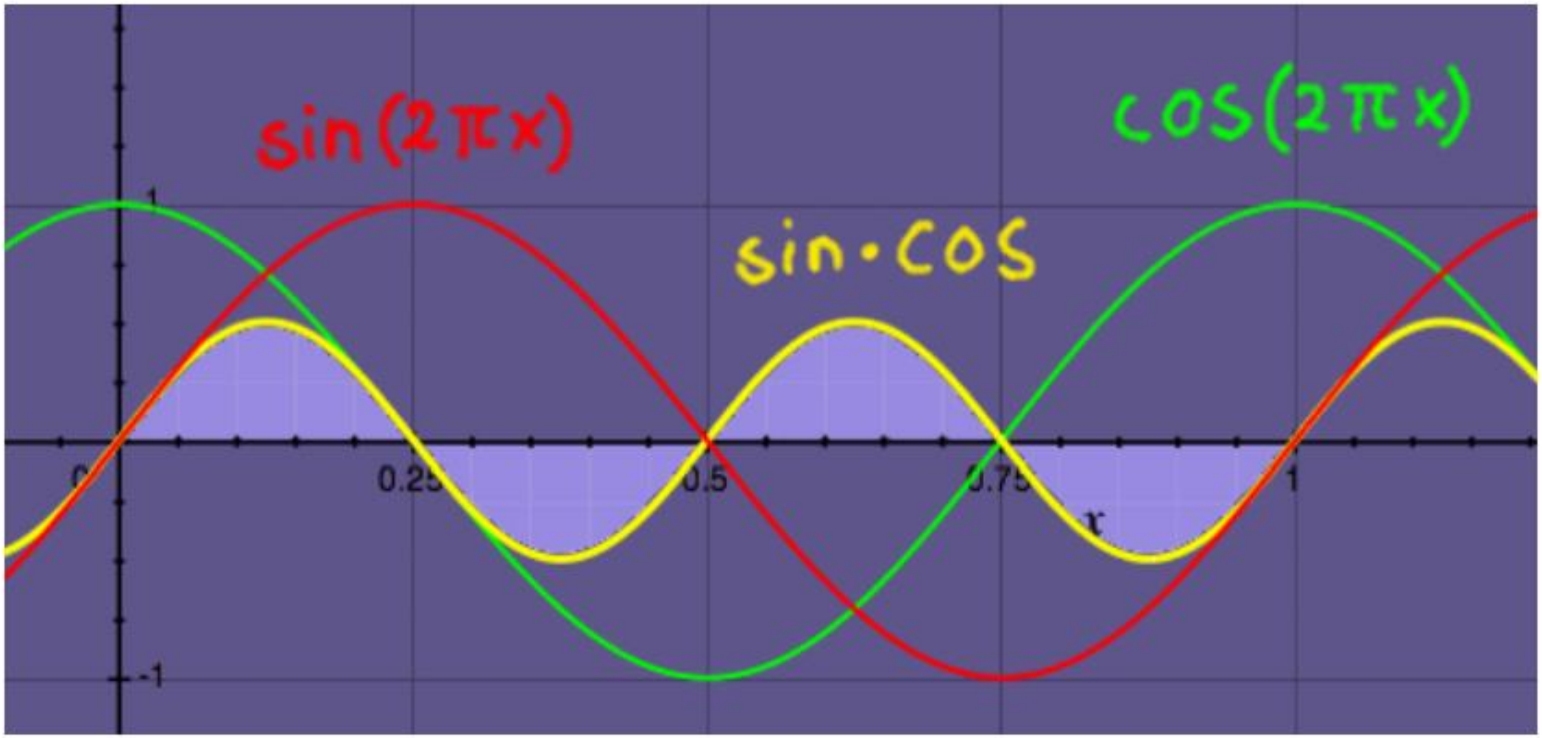
Orthogonality



Signals are harmonically related.

$$\int_0^T \sin(\omega_C t)\cdot\sin(2\omega_C t)dt = 0 \quad \text{and} \quad \int_0^T \cos(\omega_C t)\cdot\cos(2\omega_C t)dt = 0$$

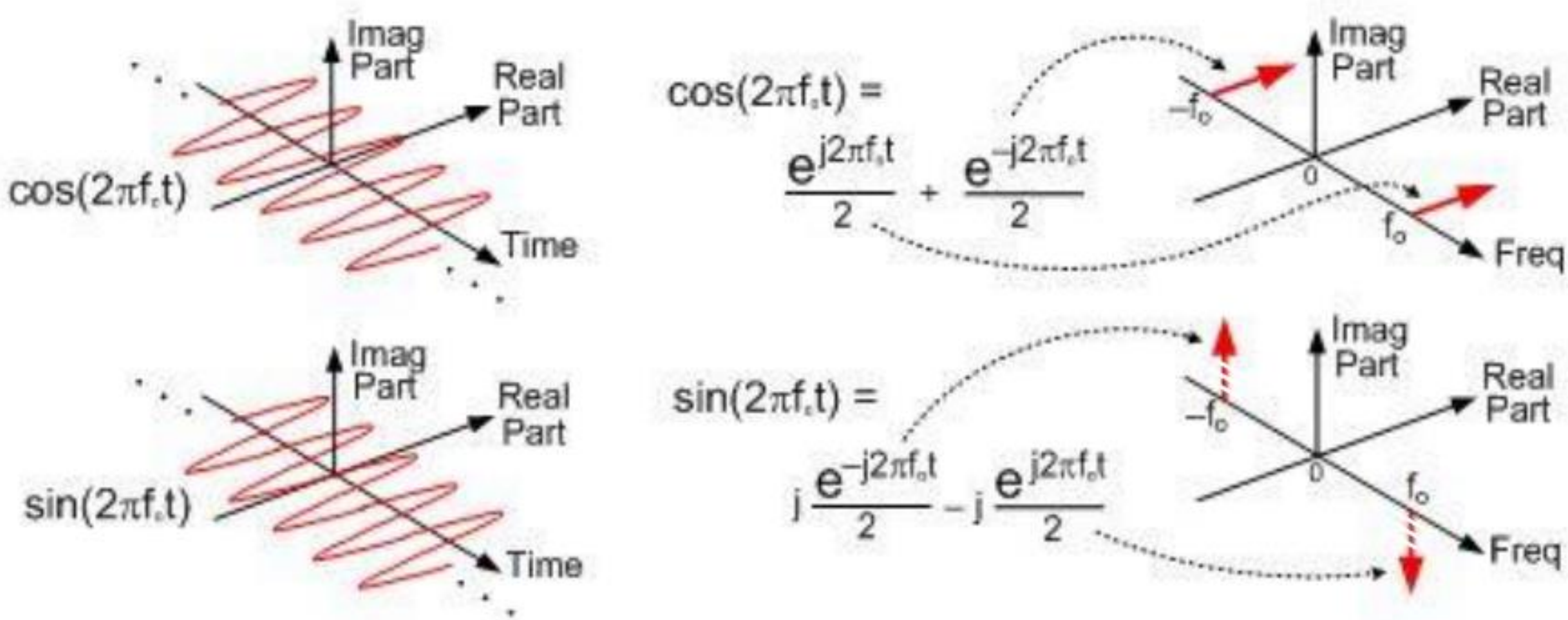
Orthogonality



Area of product over one period is equal to zero.
Therefore, the signals are orthogonal.

$$\int_0^T \sin(\omega_C t)\cdot\cos(\omega_C t)dt = 0$$

Spectral lines for Sin and Cos



Question:

The local carrier in a DSB-SC receiver and a QAM receiver is $2\cos(\omega_c t + \alpha)$, while the signal carrier at the input of each receiver is $\cos(\omega_c t)$. That means the signal carrier and local carrier in the receivers are phase shifted relative to each other. Derive expressions for the demodulated output signals for both receivers. Compare your results for DSB-SC and QAM.

For the DSB-SC receiver (NOT QAM):

Phase error $= \alpha$

$$x(t) = 2\cos(\omega_c t + \alpha) \cdot \phi_{DSB-SC}(t) = 2m(t) \cdot \cos(\omega_c t) \cdot \cos(\omega_c t + \alpha)$$

$$x(t) = m(t) \cdot \cos(\alpha) + m(t) \cdot \cos(\omega_c t + \alpha)$$

Therefore, low-pass filering gives $y(t) = m(t) \cdot \cos(\alpha)$

For the QAM receiver:

$$z_I(t) = 2\cos(\omega_C t + \alpha)\cdot\phi_{QAM}(t) = 2\cos(\omega_C t + \alpha)\left[m_I(t)\cdot\cos(\omega_C t) + m_Q(t)\cdot\sin(\omega_C t)\right]$$

$$z_I(t) = 2m_I(t)\cdot\cos(\omega_C t)\cdot\cos(\omega_C t + \alpha) + 2m_Q(t)\cdot\sin(\omega_C t)\cdot\cos(\omega_C t + \alpha)$$

$$z_I(t) = m_I(t)\cdot\cos(\alpha) + m_1(t)\cdot\cos(2\omega_C t + \alpha) - m_Q(t)\cdot\sin(\alpha) + m_Q(t)\cdot\sin(2\omega_C t + \alpha)$$

and

$$z_Q(t) = 2\sin(\omega_C t + \alpha)\cdot\phi_{QAM}(t) = 2\sin(\omega_C t + \alpha)\left[m_I(t)\cdot\cos(\omega_C t) + m_Q(t)\cdot\sin(\omega_C t)\right]$$

$$z_Q(t) = 2m_I(t)\cdot\cos(\omega_C t)\cdot\sin(\omega_C t + \alpha) + 2m_Q(t)\cdot\sin(\omega_C t)\cdot\sin(\omega_C t + \alpha)$$

$$z_Q(t) = m_I(t)\cdot\sin(\alpha) + m_1(t)\cdot\sin(2\omega_C t + \alpha) + m_Q(t)\cdot\cos(\alpha) - m_Q(t)\cdot\cos(2\omega_C t + \alpha)$$

The low-pass filters suppress the terms centered at $2\omega_C$

Therefore,

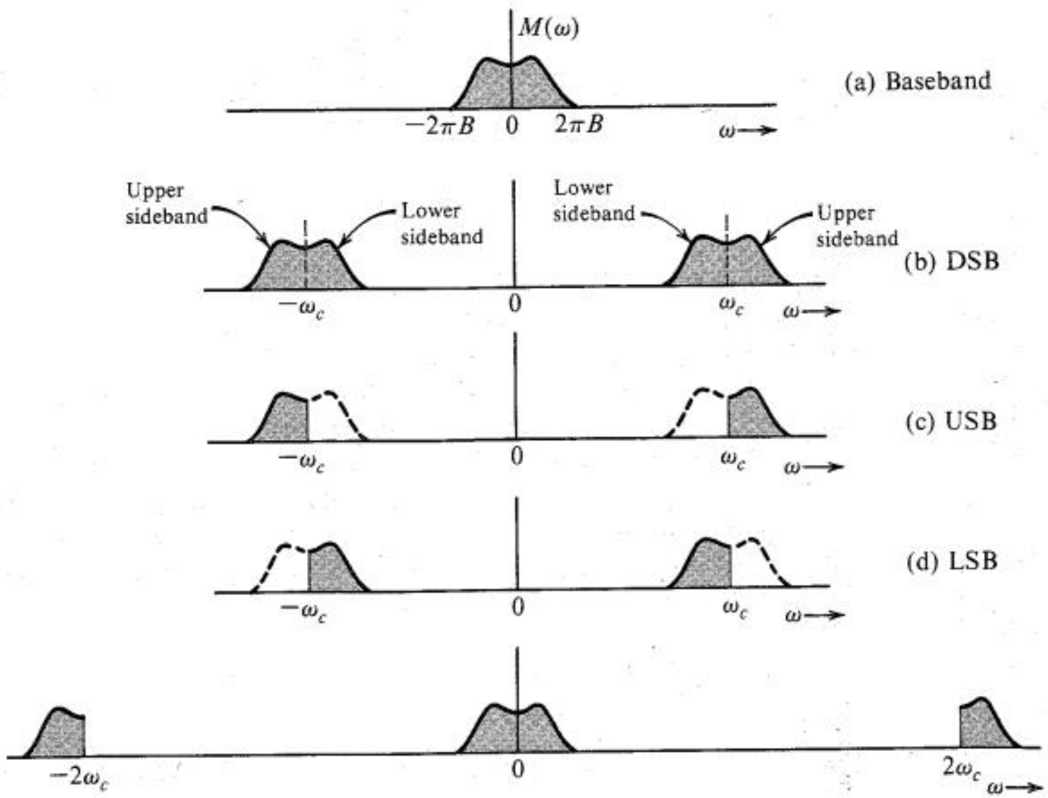
$$\left.\begin{aligned} y_I(t) &= m_I(t)\cdot\cos(\alpha) - m_Q(t)\cdot\sin(\alpha) \\ y_Q(t) &= m_Q(t)\cdot\cos(\alpha) + m_I(t)\cdot\sin(\alpha) \end{aligned}\right\}\ \text{Co-channel interference}$$

Single-Sideband (SSB) Modulation

- Thus far, conventional AM offered low complexity, but at a price of power and bandwidth inefficiency
- DSB-SC: suppresses the carrier to reduce the power requirement (decoding complexity increased)
- Single Sideband (SSB) Modulation: remove either lower or upper sideband to reduce the bandwidth requirement, resulting in upper or lower sideband (USB/LSB) signal
- Recall for real valued signals, whose spectrum is symmetric, all the information is contained in either sideband
- There are a number of ways to generate SSB signals
    - filter out the lower/upper sideband
    - apply a *Hilbert Transform*

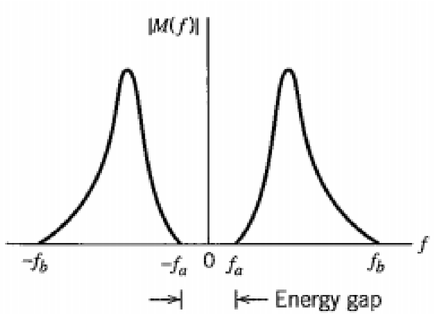
Single-Sideband (SSB) Demodulation

- SSB signals can be coherently demodulated, just as was done in DSB-SC

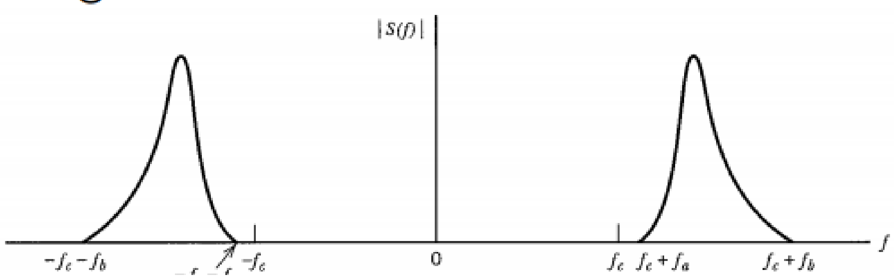


SSB: generation using a band-pass filter

- Starting with a DSB-SC signal, in theory we can remove one of the sidebands by passing it through a band-pass filter $H(f)$ - one that passes either the lower or upper sideband only.
- In theory this is simple, however practical construction of filters with steep transition band is difficult
- However, for signals that have little or no energy at low frequencies, DSB-SC will contain an *energy gap*



- This gap enables us to use a filter with a reasonable transition band to obtain an SSB signal



SSB: time domain representation

- Before discussing the second method of generating SSB signal, let's look at the time-domain representation of such signal.
- Let the upper and lower message sidebands be $M_+(\omega)$ and $M_-(\omega)$
- Note that $M_+(\omega) = M(\omega)u(\omega)$ and $M_-(\omega) = M(\omega)u(-\omega)$
- Denote by $m_+(t)$ and $m_-(t)$ the inverse Fourier Transforms of $M_+(\omega)$ and $M_-(\omega)$
  1. Since their spectra are not even, $m_+(t)$ and $m_-(t)$ are complex valued
  2. $M_+(\omega)$ and $M_-(\omega)$ are complex conjugates (for real function, $\phi(-\omega) = -\phi(\omega)$), and consequently, so are $m_+(t)$ and $m_-(t)$.
  3. $m(t) = m_+(t) + m_-(t)$
- Finally, using the above we conclude that

$$
\begin{aligned}
m_+(t) &= \frac{1}{2}\left[m(t) + jm_h(t)\right] \\
m_-(t) &= \frac{1}{2}\left[m(t) - jm_h(t)\right]
\end{aligned} \tag{1}
$$

for some unknown $m_h(t)$.

Hilbert Transform

- Let us solve for $m_h(t)$:
- Note that

$$\begin{aligned} M_+(\omega) &= M(\omega)u(\omega) \\ &= \frac{1}{2}M(\omega)[1+\operatorname{sgn}(\omega)] \\ &= \frac{1}{2}M(\omega) + \frac{1}{2}M(\omega)\operatorname{sgn}(\omega) \end{aligned}$$

- From (1) it thus follows that $jm_h(t) \leftrightharpoons M(\omega)\operatorname{sgn}(\omega)$, or in equivalently

$$M_h(\omega) = -jM(\omega)\operatorname{sgn}(\omega) \tag{2}$$

- Noting that $\frac{1}{\pi t} \leftrightharpoons -j\operatorname{sgn}(\omega)$ and applying the convolution property of the FT, we have from (2)

$$m_h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{m(\alpha)}{t-\alpha}\, d\alpha \tag{3}$$

- (3) is the *Hilbert Transform* of $m(t)$.

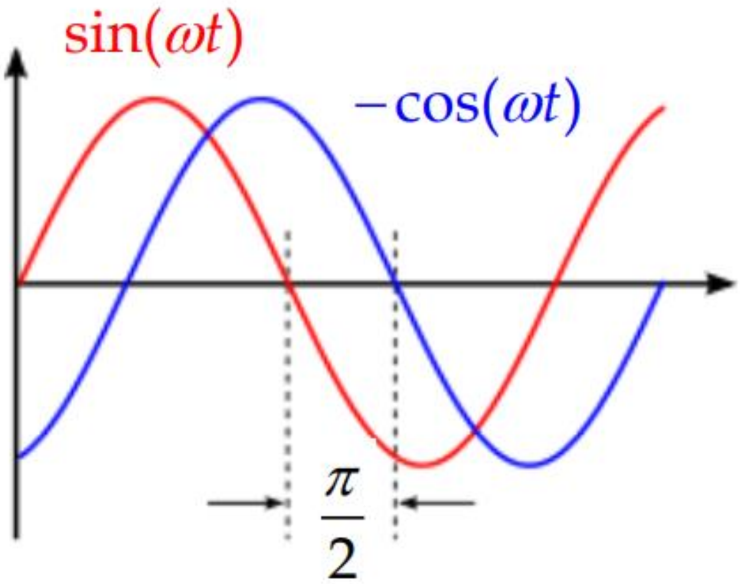
Hilbert Transform

- We have shown that $m(t)$ is passed through system with a transfer function $H(\omega) = -j\,\mathrm{sgn}(\omega)$
- Note that

$$\begin{aligned} H(\omega) &= -j\,\mathrm{sgn}(\omega) \\ &= \begin{cases} -j = e^{-j\pi/2} & \omega > 0 \\ j = e^{j\pi/2} & \omega < 0 \end{cases} \end{aligned}$$

  and so $|H(\omega)| = 1$ and $\theta(\omega) = -\pi/2$ for $\omega > 0$ and $\theta(\omega) = \pi/2$ for $\omega < 0$
- In other words, the Hilbert Transform is equivalent to delaying the phase of $m(t)$ by $\pi/2$

Hilbert transform



For a + 90° (or π/2) phase shift:

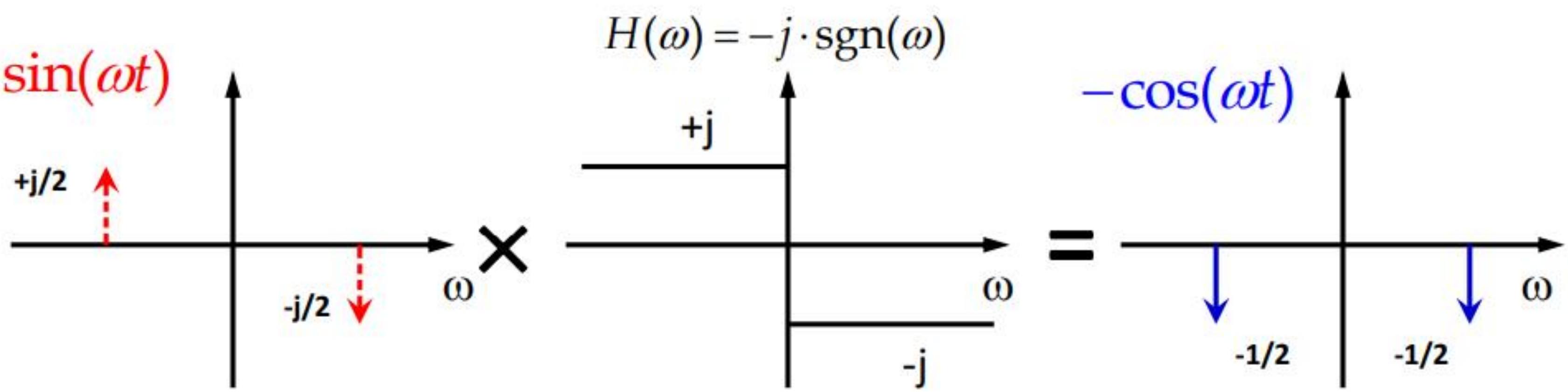
$$\sin(\omega t) \Rightarrow \cos(\omega t)$$

$$\cos(\omega t) \Rightarrow -\sin(\omega t)$$

For a - 90° (or -π/2) phase shift:

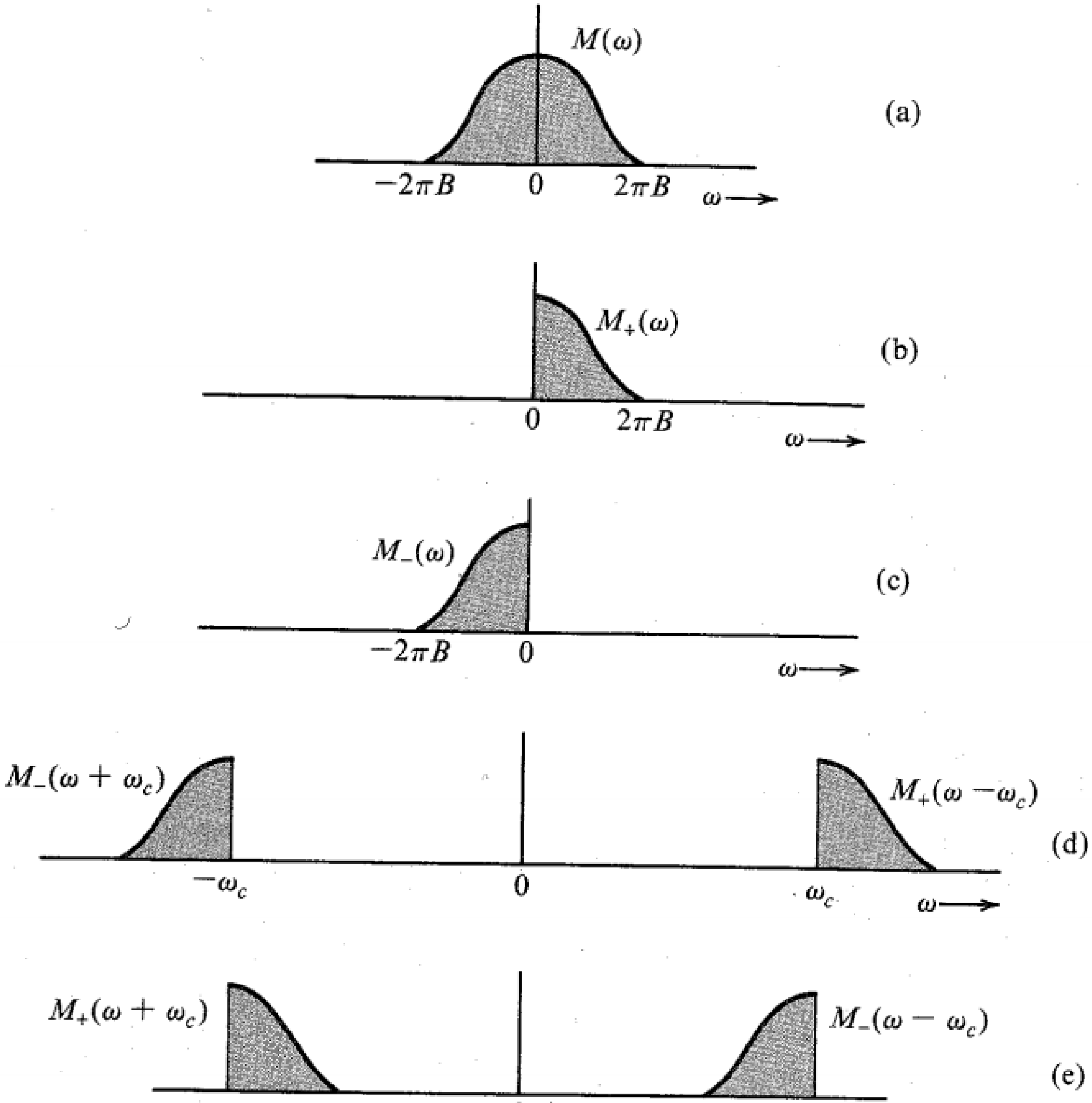
$$\sin(\omega t) \Rightarrow -\cos(\omega t)$$

$$\cos(\omega t) \Rightarrow \sin(\omega t)$$



Hilbert transform

SSB



$M(\omega)$
$-2\pi B$
0
$2\pi B$
$\omega \rightarrow$
(a)
$M_+(\omega)$
0
$2\pi B$
$\omega \rightarrow$
(b)
$M_-(\omega)$
$-2\pi B$
0
$\omega \rightarrow$
(c)
$M_-(\omega + \omega_c)$
$M_+(\omega - \omega_c)$
$-\omega_c$
0
ω_c
$\omega \rightarrow$
(d)
$M_+(\omega + \omega_c)$
$M_-(\omega - \omega_c)$
(e)

Hilbert Transform

- The USB spectrum can be expressed as

$$\Phi_{\text{USB}}(\omega) = M_+(\omega - \omega_c) + M_-(\omega + \omega_c)$$

which, after inverse FT gives

$$\phi_{\text{USB}}(t) = m_+(t)e^{j\omega_c t} + m_-(t)e^{-j\omega_c t}$$

- Finally, substituting (1) gives the *USB modulated* signal

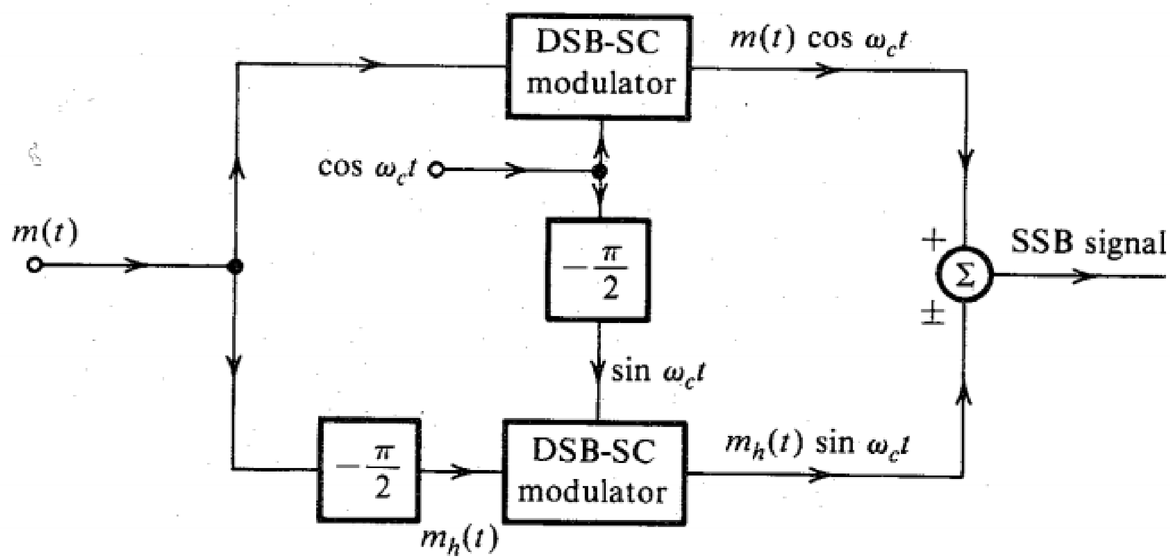
$$\phi_{\text{USB}}(t) = m(t)\cos\omega_c t - m_h(t)\sin\omega_c t$$

- Similar approach for LSB leads to

$$\phi_{\text{LSB}}(t) = m(t)\cos\omega_c t + m_h(t)\sin\omega_c t$$

SSB generation: phase-shift method

- Thus, we have the following implementation of an SSB modulator using the phase-shift method



- Note however, that an ideal phase shifter is not realizable - we can only approximate it over a *finite* frequency range.

SSB: coherent demodulation

- It is simple to verify that SSB (suppressed carrier) can be coherently demodulated

$$\phi_{\mathsf{SSB}}(t) = m(t)\cos\omega_c t \mp m_h(t)\sin\omega_c t$$

- so multiplying by the carrier leads to

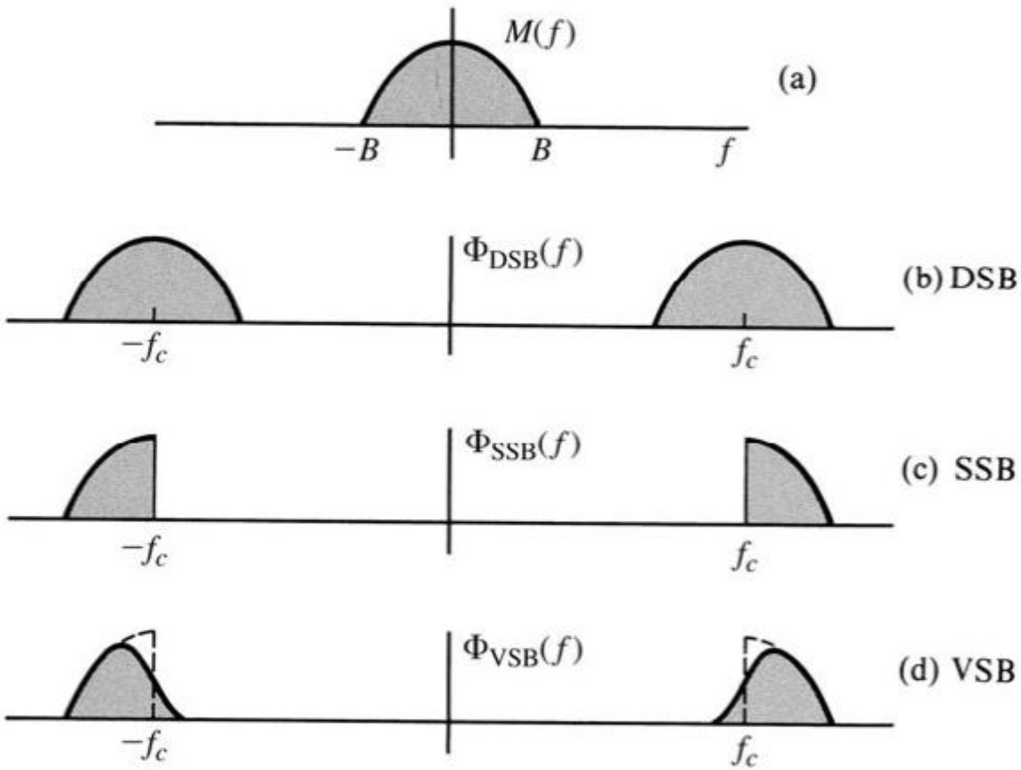
$$\begin{aligned} \phi_{\mathsf{SSB}}(t)\cos\omega_c t &= \frac{1}{2}m(t)[1+\cos 2\omega_c t] \mp \frac{1}{2}m_h(t)\sin 2\omega_c t \\ &= \frac{1}{2}m(t) + \frac{1}{2}[m(t)\cos 2\omega_c t \mp m_h(t)\sin 2\omega_c t] \qquad (4) \end{aligned}$$

- which upon low-pass filtering the double frequency components results in the desired signal.

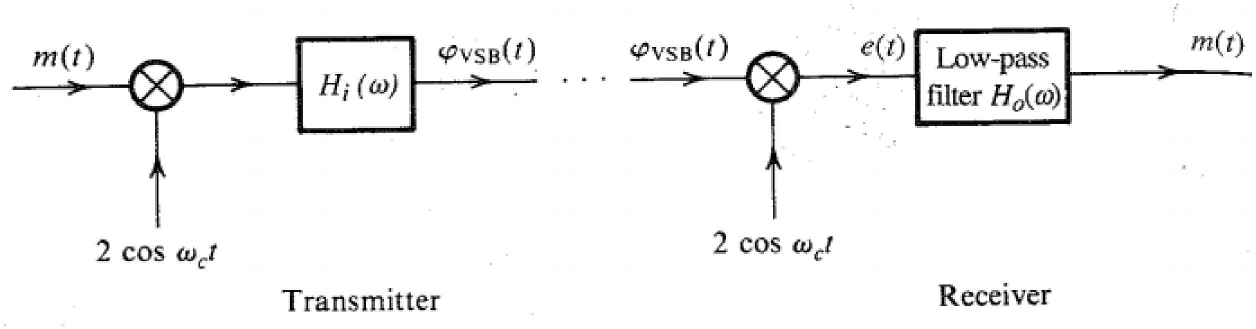
Vestigial Side Band (VSB) modulation

- Signals that do not contain an energy gap near DC are cannot be easily SSB modulated
- Another modulation technique, *Vestigial Side Band* (VSB), transmits a single side band, plus a small amount (or *vestige*) of the other sideband.
- the transmission bandwidth for VSB is $B_T = f_v + W$, where f_v is the vestige bandwidth and W is the signal bandwidth
- A filter to obtain a VSB signal from DSB-SC is allowed to have a non-zero transition band

Vestigial Side Band (VSB) modulation



VSB: filtering



- Let $H_i(f)$ denote the filter transfer function to obtain VSB from DSB-SC
- After applying the filter $H_i(f)$, we have

$$\Phi_{\mathsf{VSB}}(f) = [M(f+f_c) + M(f-f_c)]H_i(f) \tag{5}$$

- We want to recover $m(t)$ synchronously from $\phi_{\mathsf{VSB}}(t)$. Multiplying by the $2\cos(2\pi f_c t)$ we have

$$e(t) = 2\phi_{\mathsf{VSB}}(t)\cos(2\pi f_c t) \rightleftharpoons [\Phi_{\mathsf{VSB}}(f+f_c) + \Phi_{\mathsf{VSB}}(f-f_c)]$$

- The signal $e(t)$ is then passed through a low-pass equalising filter $H_0(f)$ which removes the double frequency components and reshapes the spectrum. The output is required to be

$$M(f) = [\Phi_{\mathsf{VSB}}(f+f_c) + \Phi_{\mathsf{VSB}}(f-f_c)]H_0(f) \tag{6}$$

VSB: filtering

- substituting (5) into (6) and removing double frequency components (due to LPF nature of $H_0(f)$), we have

$$M(f) = M(f)[H_i(f + f_c) + H_i(f - f_c)]H_0(f)$$

- And so the receiver filter $H_0(f)$ must satisfy the condition

$$H_0(f) = \frac{1}{H_i(f + f_c) + H_i(f - f_c)} \quad -B < f < B$$

where we have assumed $M(f)$ is zero outside some bandwidth $-W < f < W$

Analog Television

- video signal:
    - large bandwidth (4.5 MHz)- don't want DSB
    - significant low-frequency component - no SSB
- the demodulation of the tv signal must be simple (cost) - envelope detector; this implies the addition of the carrier to the VSB modulated wave

Analog Television

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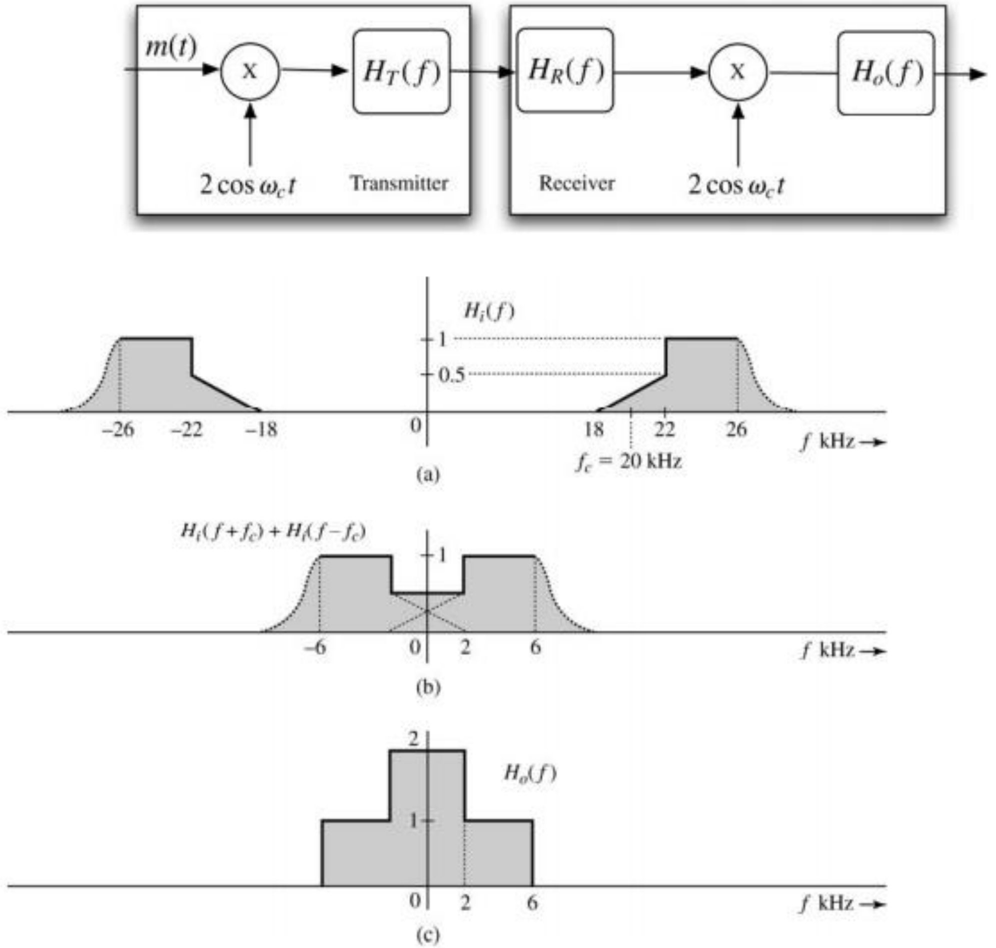
    - large bandwidth (4.5 MHz)- don't want DSB
    - significant low-frequency component - no SSB
- the demodulation of the tv signal must be simple (cost) - envelope detector; this implies the addition of the carrier to the VSB modulated wave
- however: since the transmitted power is large, it would be expensive to carefully control the shape of the filter sideband.
- instead the VSB filtering is applied at the receiver, where the power level is lower
- the transmitter includes a filter to simply limit the bandwidth of the signal

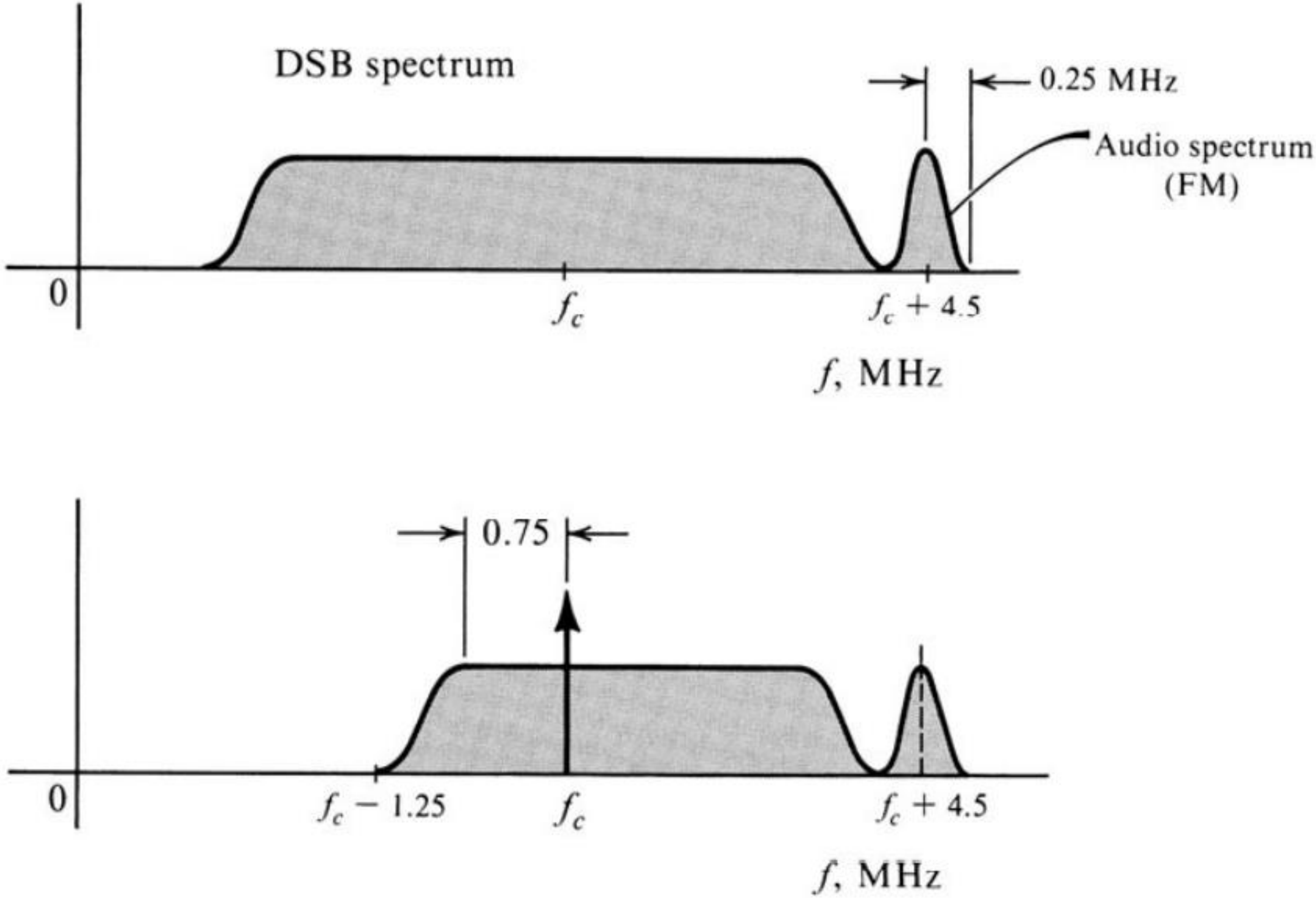
Analog Television

- VSB filter is split between Tx and Rx, ie $H_i(f) = H_T(f)H_R(f)$



Analog Television

Resulting VSB spectrum is 6 MHz



Next Lecture

- *Angle Modulation. Read Proakis & Salehi Chapter 4*