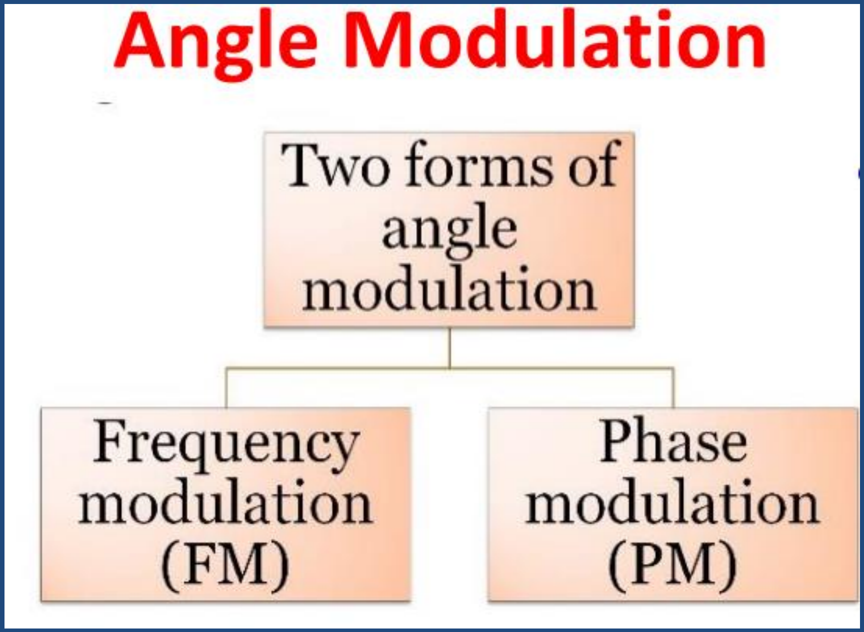
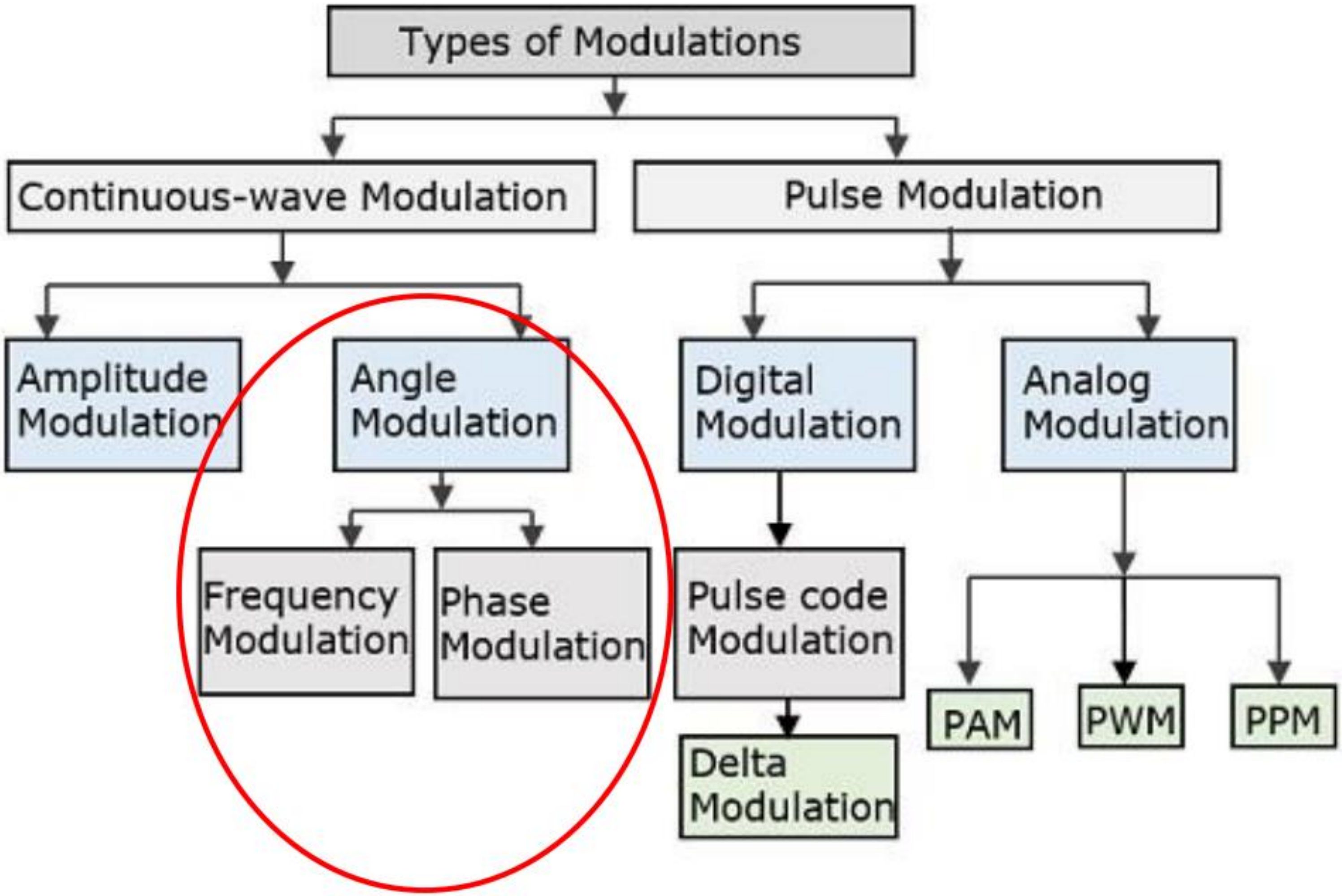
LECTURE 4:
ANGLE MODULATION



Angle modulation



Review from last lecture…

Modulation is the systematic alteration of a carrier wave so that it "carries" the information of the message or data signal $m(t)$.

Modulation allows for the designated frequency bands (with the carrier frequency at the center of the band) to be utilized for communication and allows for signal multiplexing.

Amplitude modulation (AM) is an analog and linear modulation process as opposed to frequency modulation (FM) and phase modulation (PM).

AM involves the variation of the carrier signal's amplitude in direct proportion to the modulating signal $m(t)$.

AM is simple to implement and can be accomplished inexpensively with a small number of components; but AM has a low power efficiency (ratio of power in the message signal relative to the total transmitted power) and is very susceptible to noise and interference.

An amplitude modulation time-varying signal (double sideband with carrier – DSB-WC) is

$$\phi_{AM}(t) = \left[A_C + m(t)\right] \cdot \cos(\omega_C t)$$

AM can be interpreted using phasors where the carrier of the AM signal is a phasor of constant amplitude A_C rotating CCW at frequency f_C and the modulating signal $m(t)$ made up of a collection of slower rotating Fourier components of $m(t)$ attached to the tip of the carrier phasor. The vector sum of the phasors gives the AM phasor.

The corresponding amplitude modulation spectrum is

$$\Phi_{AM}(\omega) = \frac{1}{2}M(\omega - \omega_C) + \frac{1}{2}M(\omega + \omega_C) + \pi A_C\left[\delta(\omega - \omega_C) + \delta(\omega + \omega_C)\right]$$

which is related to the frequency shift property of the Fourier transform.

The AM **modulation index** is defined as $\mu = m_p/A_C$, where m_p is the peak amplitude of $m(t)$. When μ > 100% overmodulation results in an AM waveform (*i.e.*, envelope distortion).

The power efficiency η of AM is defined as

$$\eta = \frac{\text{message power}}{\text{total power}} = \frac{P_m}{A_C^2 + P_m},$$

where P_m is the message power. The power efficiency η is 11.1% when $\mu = 0.5$ and η is 33.3% when $\mu = 1.0$ (best case).

Bandwidth efficiency can be improved is with quadrature amplitude modulation (QAM). QAM involves two data streams: the I-channel and the Q-channel. Bandwidth efficiency is improved because two signals to share the same bandwidth of a channel. But this can only be done if the two modulated signals are orthogonal to each other.

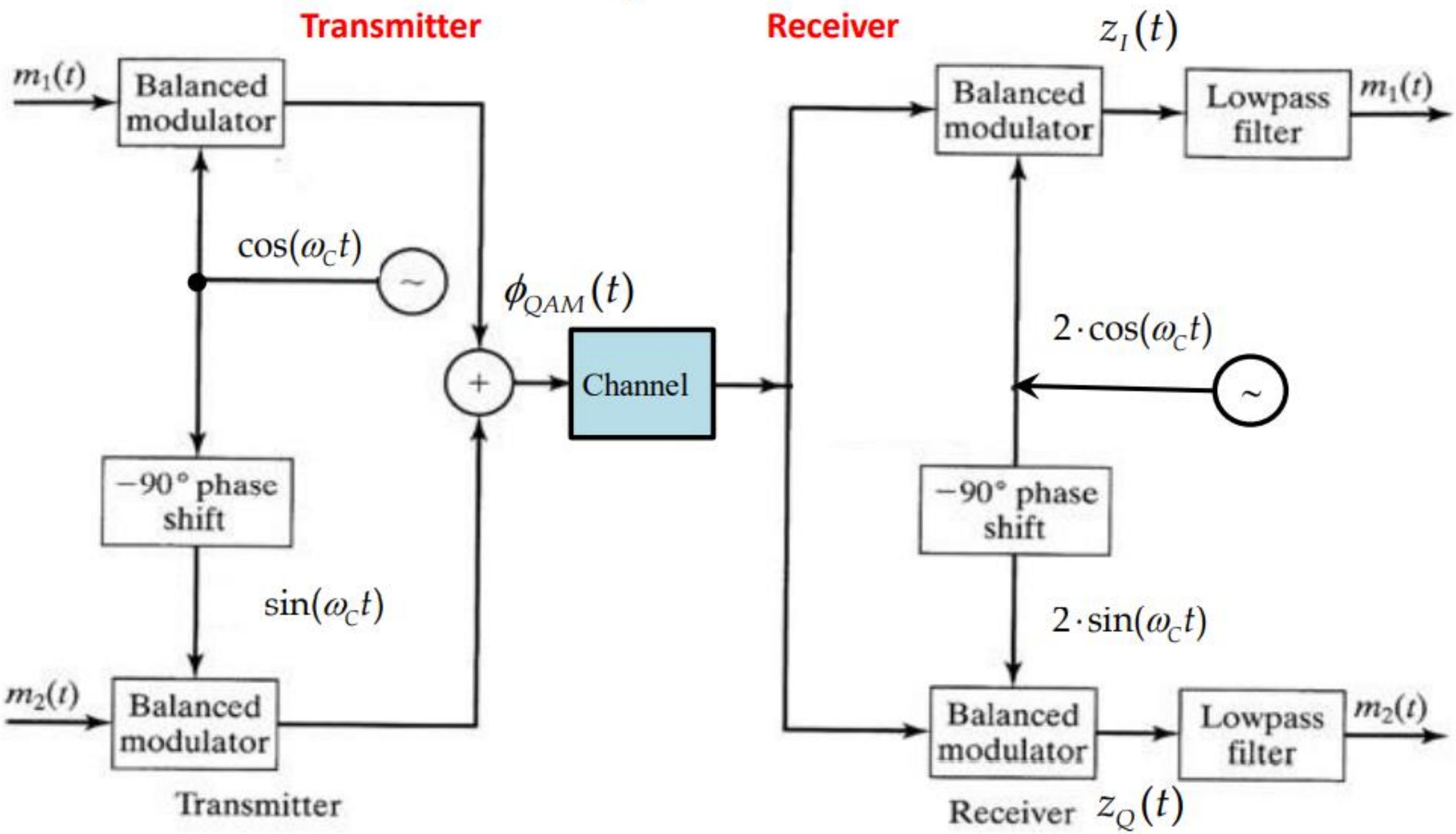
Modulating one message (call it the in-phase message m_I) with $\cos(\omega_C t)$ and another message (call it the quadrature message m_Q) with $\sin(\omega_C t)$ makes the two signals orthogonal to each other. Thus, both messages can be independently modulated and demodulated.

The QAM signal is of the form, $\varphi_{QAM}(t) = m_I(t) \cdot \cos(\omega_C t) + m_Q(t) \cdot \sin(\omega_C t)$

QAM transmits two DSB-SC signals in the bandwidth of one DSB-SC signal. Interference between the two modulated signals of the same frequency is prevented by using two carriers in phase quadrature. The **In-phase** (I-phase) channel modulates the $\cos(\omega_C t)$ signal and the **Quadrature-phase** (Q-phase) channel modulates the sin $(\omega_C t)$ signal. The carriers used in the transmitter and receiver are synchronous with each other. In fact, they must be almost exactly in quadrature with each other; otherwise, they experience cochannel interference. Low-pass filters are used to extract the baseband signals $m_I(t)$ and $m_Q(t)$ in the receiver.

QAM is used extensively as a modulation scheme for digital telecommunication systems, such as in 802.11 Wi-Fi standards.

QAM transmitter and receiver block diagram:



Effect of error $\Delta\omega$ in carrier frequencies between the in-phase and the quadrature channels.

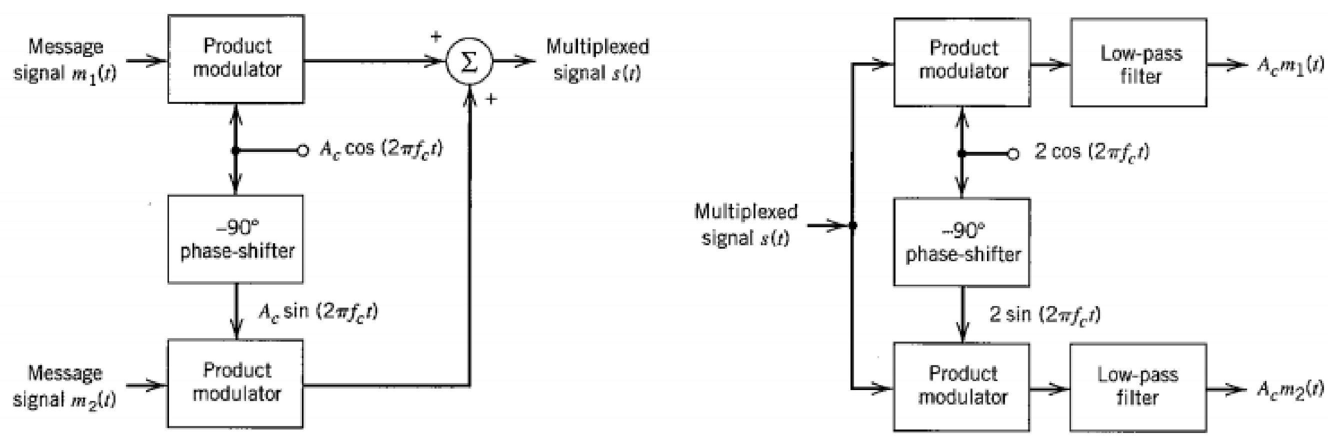
$$y_I(t) = m_I(t)\cos(\Delta\omega t) - m_Q(t)\sin(\Delta\omega t)$$

$$y_Q(t) = m_Q(t)\cos(\Delta\omega t) + m_I(t)\sin(\Delta\omega t)$$

Quadrature Multiplexing

- The orthogonality of the I- and Q- branches of a receiver can be exploited for multiplexing two message signals onto a single frequency band
- Consider a transmitter where $m_1(t)$ and $m_2(t)$ modulate two carriers with equal frequency but in *phase quadrature*

$$s(t) = A_c m_1(t)\cos(2\pi f_c t) + A_c m_2(t)\sin(2\pi f_c t)$$



- at the receiver the composite signal is coherently demodulated by two carriers, again, in phase quadrature
- low-pass filtering removes the double frequency components, recovering the individual message signals
- This scheme is also referred to as *quadrature amplitude modulation*

Complex Baseband Representaton of Signal

- Consider a generic *bandpass* signal with an envelope $a(t)$ and phase $\phi(t)$

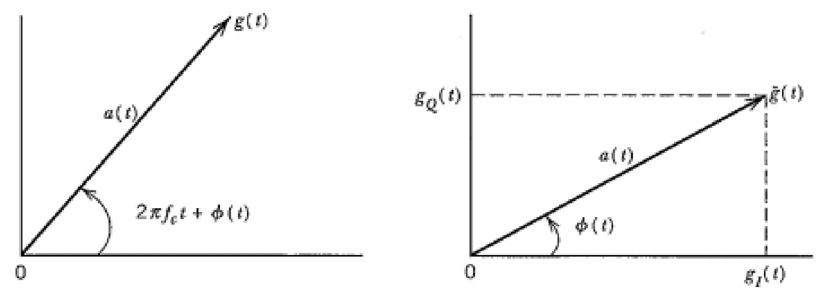
$$g(t) = a(t)\cos(2\pi f_c t + \phi(t)) = g_I(t)\cos(2\pi f_c t) + g_Q(t)\sin(2\pi f_c t)$$

- We have defined $g_I(t) = a(t)\cos\phi(t)$ and $g_Q(t) = a(t)\sin\phi(t)$ as the in-phase and quadrature components of $g(t)$, respectively.
- We can rewrite $g(t)$ as

$$g(t) = \Re\{\tilde{g}(t)e^{j2\pi f_c t}\}$$

where we have defined $\tilde{g}(t) = g_I(t) + jg_Q(t)$ as the *complex envelope* of the bandpass signal

- both $g(t)$ and $\tilde{g}(t)$ can be expressed by phasors



- note that the phasor representation of $\tilde{g}(t)$ is obtained by simply suppressing the constant angular rotation $2\pi f_c t$

Angle Modulation - Motivation

- Recall that in amplitude modulation we vary the *amplitude* of a sinusoidal carrier.
- We now investigate another type of continuous wave modulation: *angle modulation*, where the message is used to vary the *phase* or *frequency* of the carrier.
- advantages include:
    - constant transmit power (constant envelope)
    - can tradeoff bandwidth for noise immunity

Applications of FM

Application	Type of Modulation
AM broadcast radio	AM
FM broadcast radio	FM
FM stereo multiplex sound	DSB (AM) and FM
TV sound	FM
TV picture (video)	AM, VSB
TV color signals	Quadrature DSB (AM)
Cellular telephone	FM, FSK, PSK
Cordless telephone	FM, PSK
Fax machine	FM, QAM (AM plus PSK)
Aircraft radio	AM
Marine radio	FM and SSB (AM)
Mobile and handheld radio	FM
Citizens band radio	AM and SSB (AM)
Amateur radio	FM and SSB (AM)
Computer modems	FSK, PSK, QAM (AM plus PSK)
Garage door opener	OOK
TV remote control	OOK
VCR	FM
Family Radio service	FM

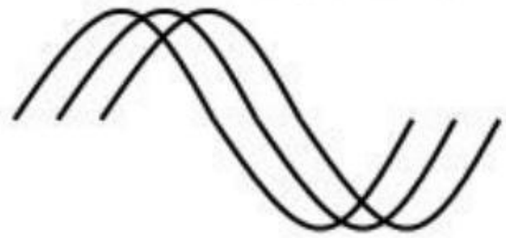
Not a complete list of applications.

We have studied AM, next is FM and PM.

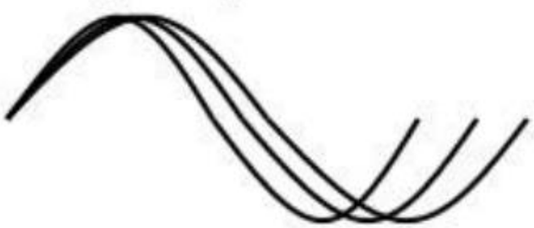
Modulation in picture

Amplitude Modulation

Phase Modulation

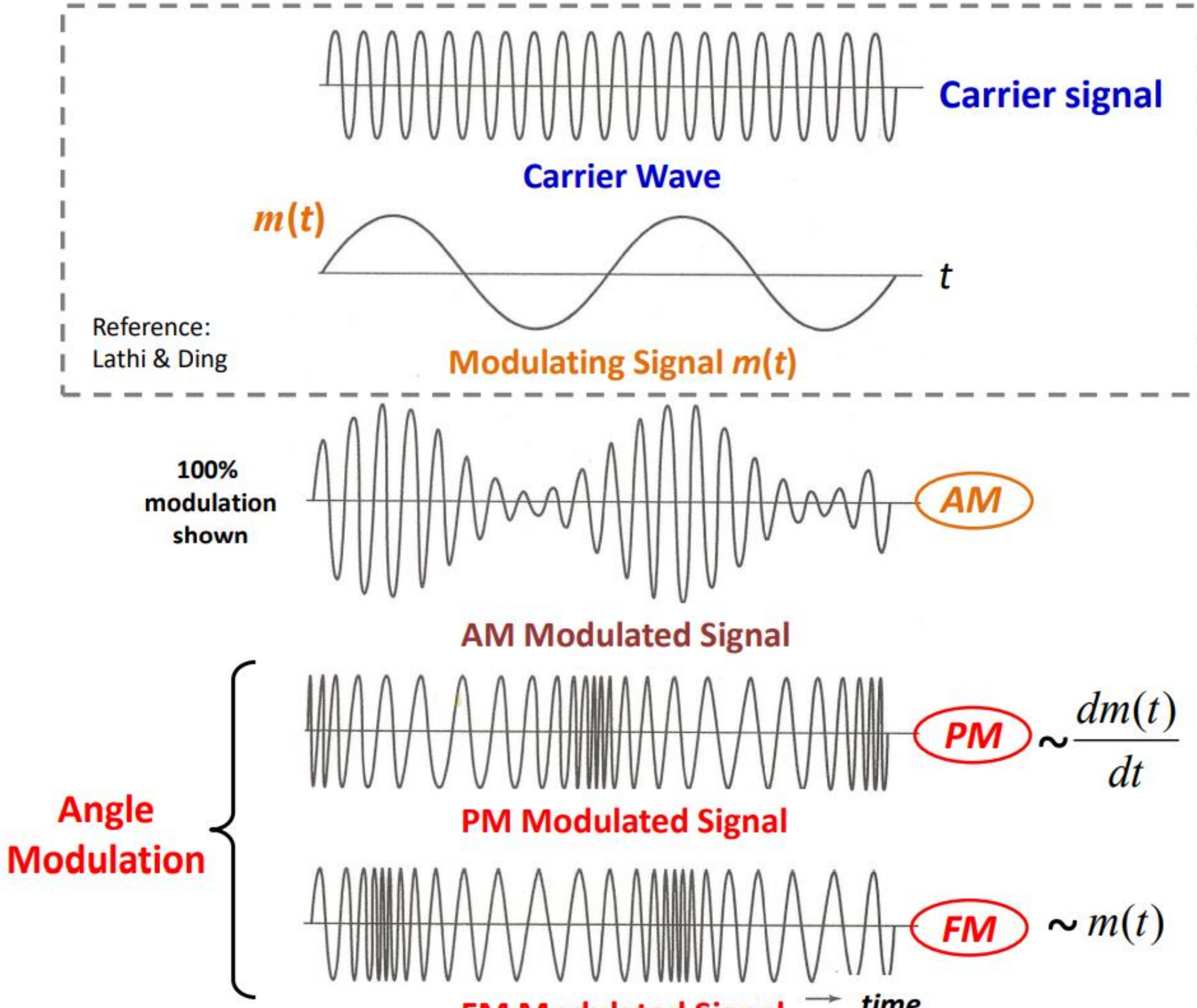


Frequency Modulation

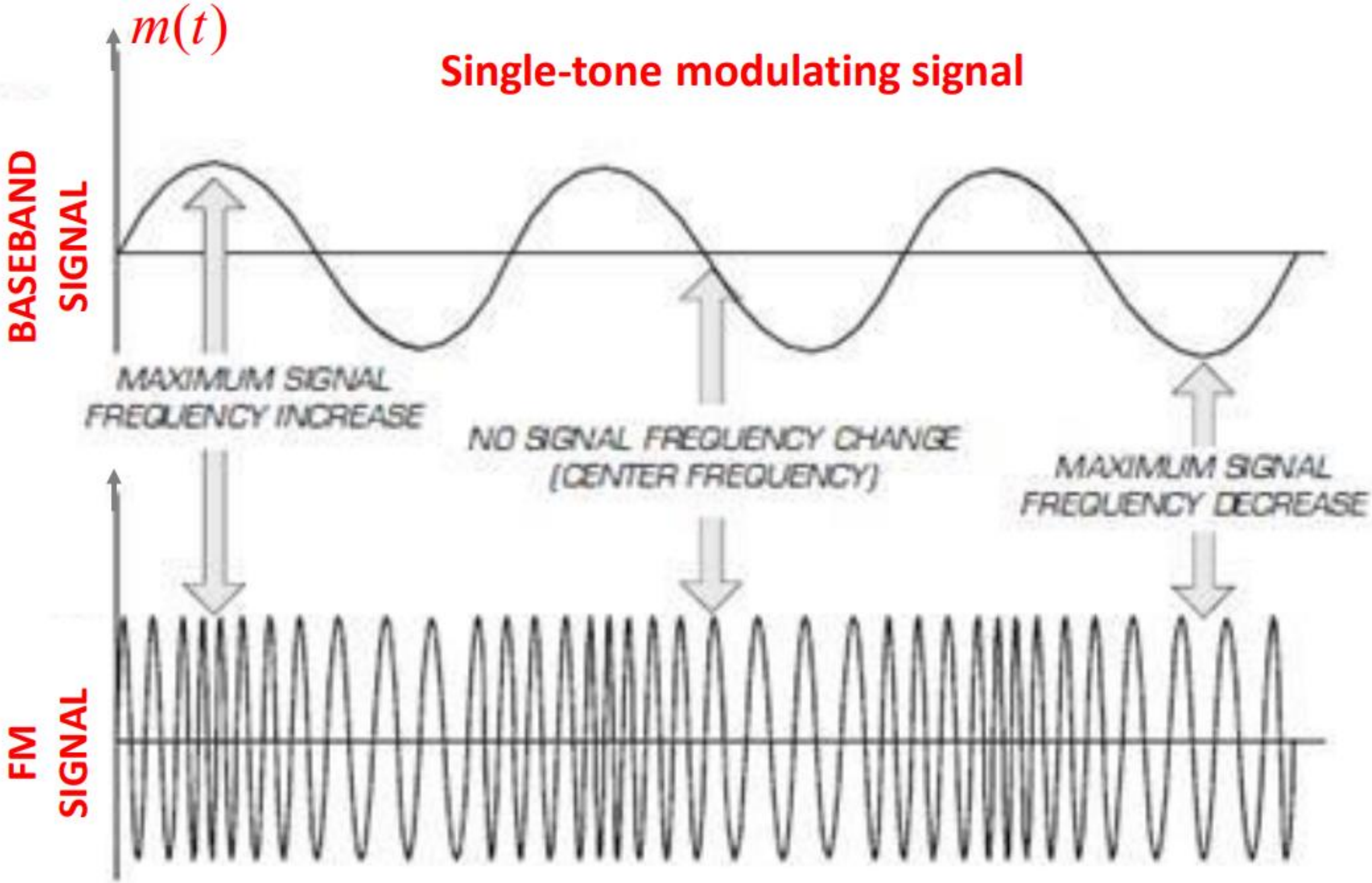


With few exceptions, Phase Modulation (PM) is used predominantly in digital communication

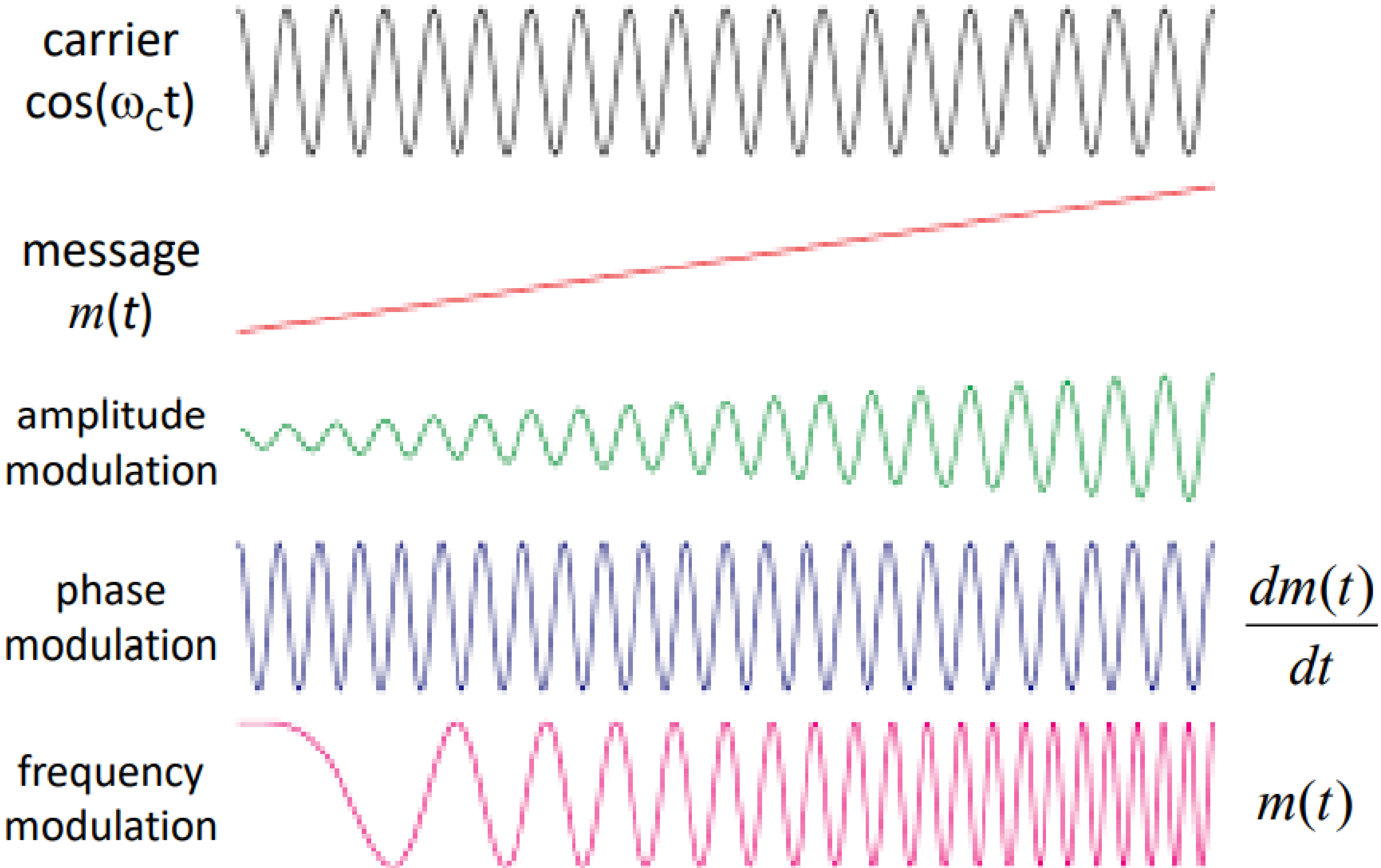
Remember that $f = \frac{d\phi}{dt}$


Carrier signal
Carrier Wave
m(t)
t
Reference:
Lathi & Ding
Modulating Signal m(t)
100%
modulation
shown
AM
AM Modulated Signal
Angle
Modulation
PM
~ dm(t)/dt
PM Modulated Signal
FM
~ m(t)

Single-tone FM



Comparing the different modulations



Some Definitions

- Let $\theta_i(t)$ denote the angle of a modulated sinusoidal carrier at time t; $\theta_i(t)$ is assumed to be some function of the message $m(t)$
- Given a carrier with amplitude A_c, the *angle* modulated wave is

$$s(t) = A_c \cos(\theta_i(t))$$

- A complete oscillation occurs when $\theta_i(t)$ changes by 2π radians. Assuming $\theta_i(t)$ increases monotonically with time, the average frequency in Hertz over an interval $(t, t + \Delta t)$ is

$$f_{\Delta t}(t) = \frac{\theta_i(t + \Delta t) - \theta_i(t)}{2\pi \Delta t}$$

- *instantaneous frequency* of $s(t)$ is defined as the limit $\Delta t \to 0$, that is

$$f_i(t) = \frac{1}{2\pi} \frac{d\theta_i(t)}{dt}$$

Some Basics

- For an unmodulated carrier, the angle is simply

$$\theta_i(t) = 2\pi f_c t + \phi_c$$

where ϕ_c is the value of $\theta_i(t)$ at $t = 0$

- In the case of *Phase Modulation*(PM) the angle $\theta_i(t)$ is varied linearly with the message signal; assuming $\phi_c = 0$, we have

$$\theta_i(t) = 2\pi f_c t + k_p m(t)$$

  - $2\pi f_c t$ - phase of the unmodulated carrier
  - k_p - phase sensitivity

- The phase-modulated signal is thus

$$s(t) = A_c \cos[2\pi f_c t + k_p m(t)]$$

Some Basics

- In the case of *Frequency Modulation*(FM) the instanteous frequency $f_c(t)$ is varied linearly with the message signal

$$f_i(t) = f_c + k_f m(t)$$

  - f_c - frequency of the unmodulated carrier
  - k_f - frequency sensitivity

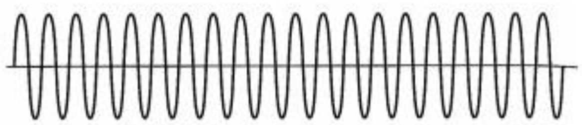
- We can obtain the phase of the FM signal by integrating over time from 0 to t (and multiplying by 2π), giving

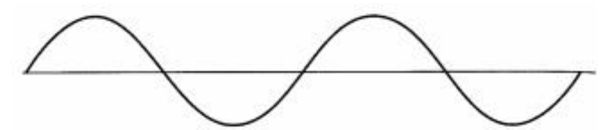
$$\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau$$

- The frequency-modulated signal is thus

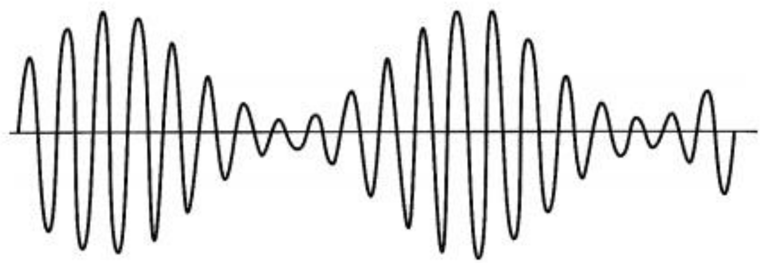
$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau\right]$$

Angle Modulation

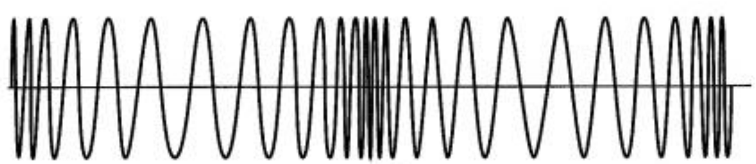




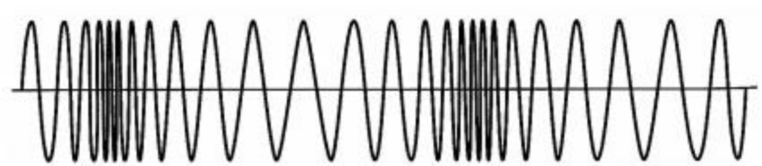
- AM:



- PM:

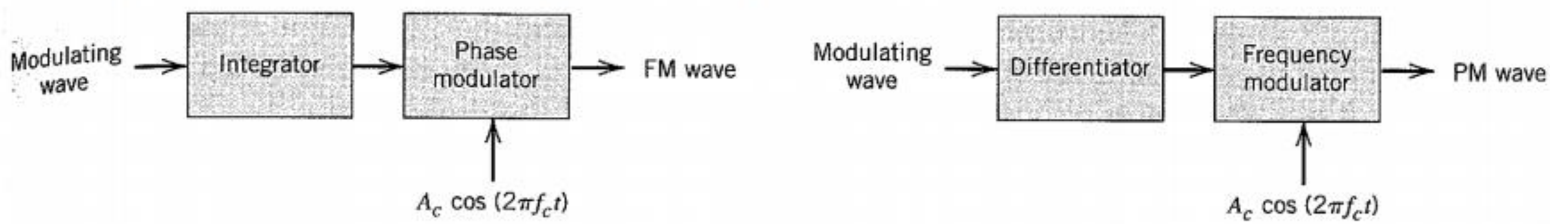


- FM:



Relationship between PM and FM

- From the definitions of FM and PM, we see that an FM wave for $m(t)$ can be viewed as a PM wave for $\int_0^t m(\tau)d\tau$
- Similarly, a PM wave for $m(t)$ is simply an FM wave for $dm(t)/dt$



- Thus, we can limit our analysis to FM, with the understanding that the corresponding properties can be easily deduced for PM

General Observations on FM and PM Waveforms

1. Both FM and PM waveforms are identical except for a time shift, when $m(t)$ is a sinusoidal signal.

2. For FM, the maximum frequency deviation occurs when modulating signal is at its peak values (*i.e.*, at $+m_p$ and $-m_p$).

3. For PM, the maximum frequency deviation takes place at the zero crossings of the modulating signal $m(t)$.

4. It is generally difficult to know from looking at a waveform whether the modulation is FM or PM.

5. The message resides in the zero-crossings alone, provided the carrier frequency is large compared to frequency content of $m(t)$.

6. The modulated waveform doesn't resemble the message waveform.

Advantages of angle modulation

1. Angle modulation is **resistant to propagation-induced selective fading** because the amplitude variations don't contain information.
2. Angle modulation is very efficient in **rejecting interference** (*i.e.*, it minimizes the effect of amplitude noise on the signal transmission).
3. Angle modulation allows for more **efficient use of transmitter power**.
4. Angle modulation can **handle a greater dynamic range** in the modulating signal without distortion (as would occur in AM).
5. Wideband FM gives significant **improvement in the signal-to-noise ratio** at the output and is proportional to the square of the modulation index β, where

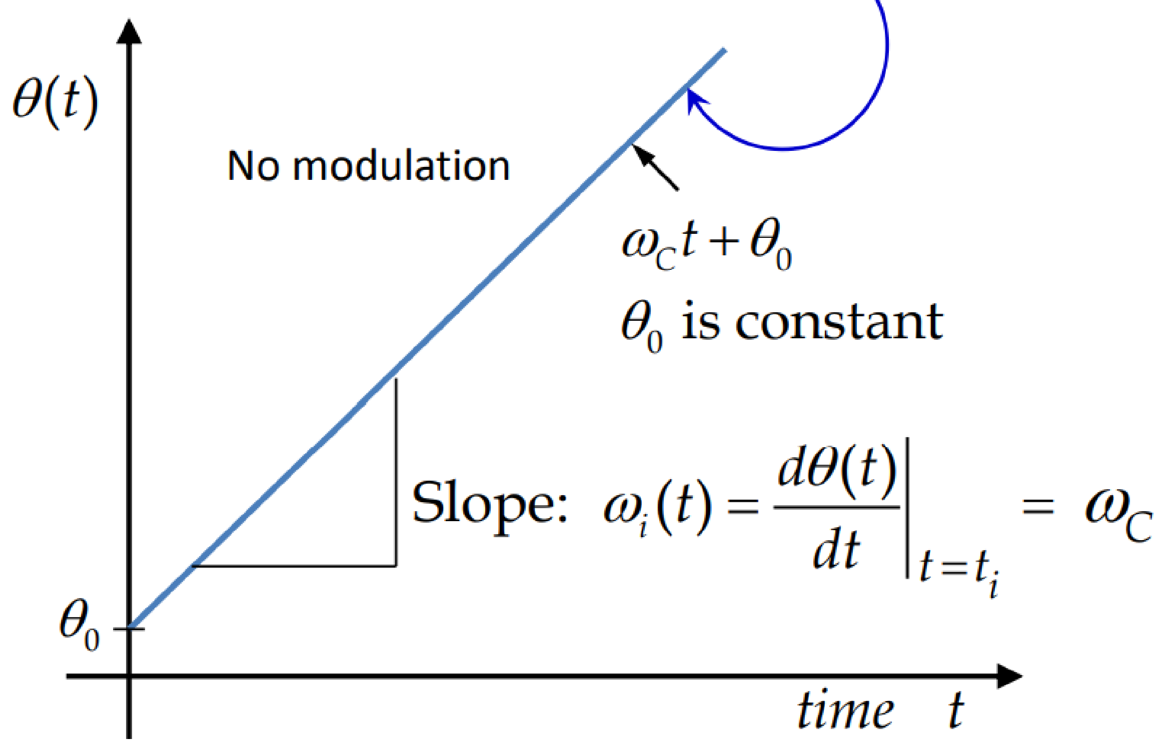
$$\beta = \frac{\Delta f}{B} \qquad \begin{cases} B & \text{Bandwidth} \\ \Delta f & \text{Frequency deviation} \end{cases}$$

Phase-Frequency Relationship When Frequency is Constant

$$\varphi(t) = A_C \cos(\theta(t))$$

$\theta(t)$ is generalized angle

$$\varphi(t) = A_C \cos(\omega_C t + \theta_0)$$



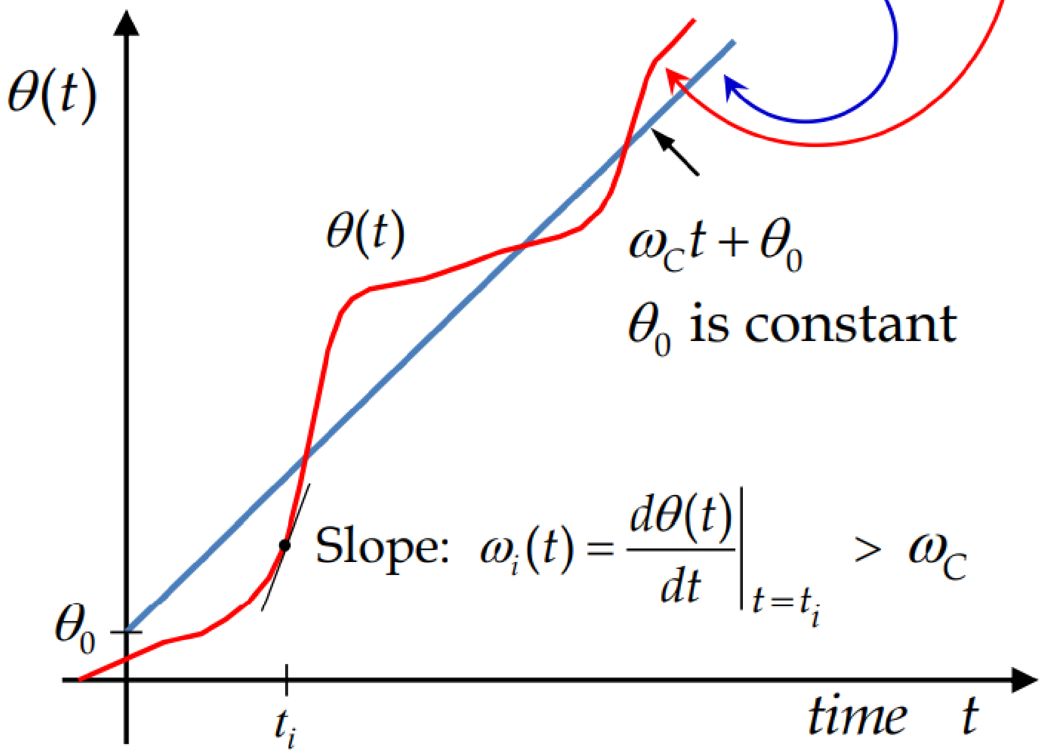
Concept of Instantaneous Frequency

$$\varphi(t) = A_C \cos(\theta(t))$$

$\theta(t)$ is generalized angle

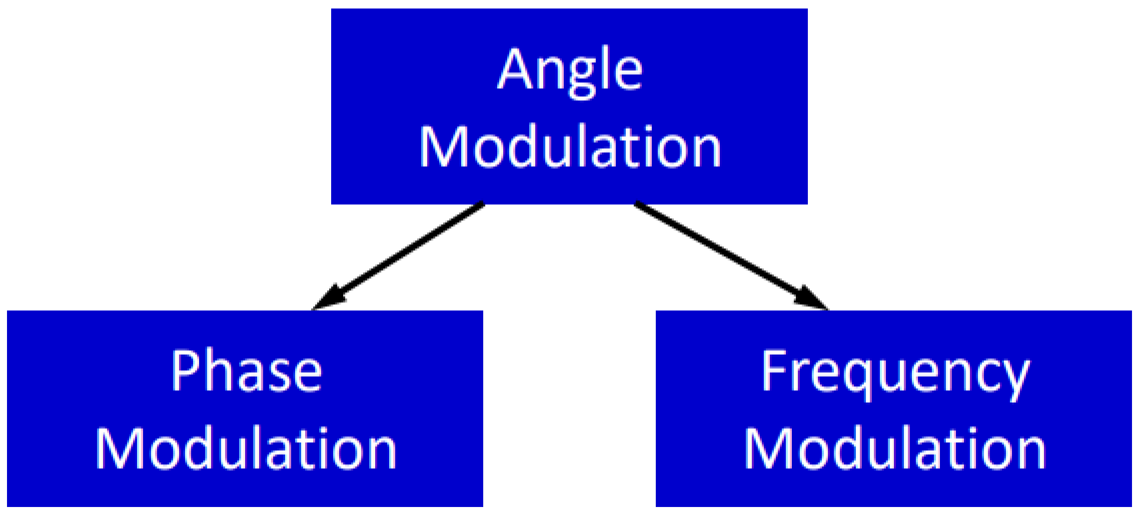
Angle Modulation in red line

$$\varphi(t) = A_C \cos(\omega_C t + \theta_0)$$



Angle Modulation Gives PM and FM

$$\omega_i(t) = \left.\frac{d\theta(t)}{dt}\right|_{t=t_i} \quad \text{and} \quad \theta(t) = \int_{-\infty}^{t} \omega_i(\lambda)\, d\lambda$$



Frequency modulation and phase modulation are closely related!

Comparing Frequency Modulation to Phase Modulation

No.	Frequency Modulation (FM)	Phase Modulation (PM)
1	Frequency deviation is proportional to modulating signal $m(t)$	Phase deviation is proportional to modulating signal $m(t)$
2	Noise immunity is superior to PM (and of course AM)	Noise immunity better than AM, but not FM
3	Signal-to-noise ratio (SNR) is better than PM (and of course AM)	Signal-to-noise ratio (SNR) is not quite as good as with FM
4	FM is widely used for commercial broadcast radio (88 MHz to 108 MHz)	PM is primarily used for mobile radio services
5	Modulation index is proportional to modulating signal $m(t)$ as well as the modulating frequency f_m	Modulation index is proportional to modulating signal $m(t)$

Phase Modulation

$\theta_i(t) = \omega_C t + \theta_0 + k_p m(t)$;   Usually we set $\theta_0 = 0$,

$$\boxed{\varphi_{PM}(t) = A_C \cos(\omega_C t + k_p m(t))}$$

The instantaneous angular frequency (in radians/second) is

$$\omega_i(t) = \frac{d\theta_i(t)}{dt} = \omega_C + k_p \frac{dm(t)}{dt} = \omega_C + k_p m'(t)$$

In phase modulation (PM) the instantaneous angular frequency ω_i varies linearly with the time derivative of the message signal $m(t)$ [denoted here by $m'(t)$].

k_p is the phase-deviation (sensitivity) constant. Units: radians/volt
[Actually it is radians/unit of the parameter $m(t)$.]

Frequency modulation

But in frequency modulation the instantaneous angular frequency ω_i varies linearly with the modulating signal $m(t)$,

$$\omega_i(t) = \omega_C + k_f m(t)$$

$$\theta_i(t) = \int_{-\infty}^{t} \left[\omega_C + k_f m(\lambda)\right] d\lambda = \omega_C t + k_f \int_{-\infty}^{t} m(\lambda)\, d\lambda$$

k_f is frequency-deviation (sensitivity) constant. Units: radians/volt-sec.

Then

$$\boxed{\varphi_{FM}(t) = A_C \cos\left(\omega_C t + k_f \int_{-\infty}^{t} m(\lambda)\, d\lambda\right)}$$

FM and PM are related to each other.
In PM the angle is directly proportional to $m(t)$.
In FM the angle is directly proportional to the integral $\int m(t)dt$.

Summary: PM and FM

Message signal is *m*(*t*)

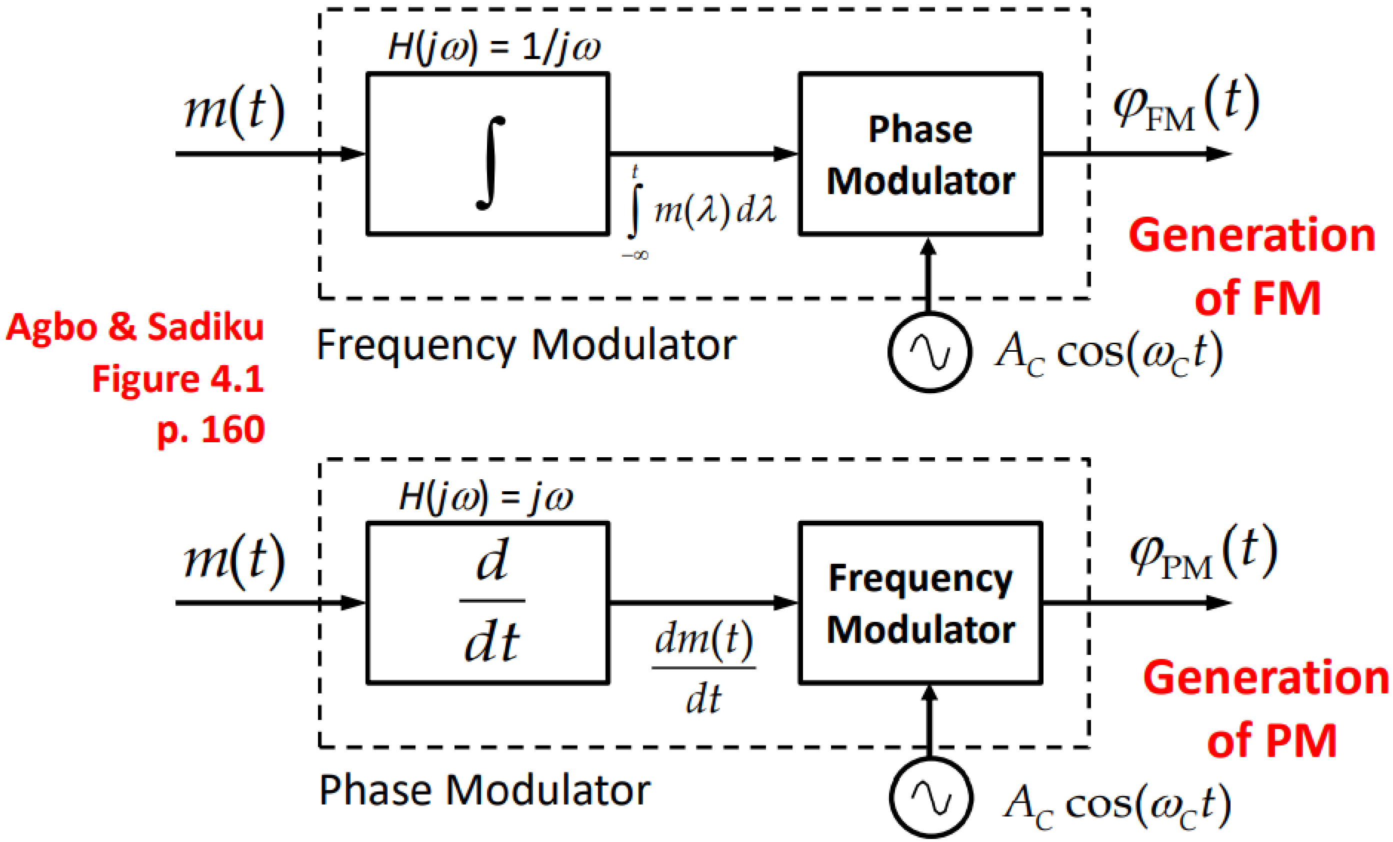
Definition: Instantaneous frequency is $\omega_i(t) = \dfrac{d\theta_i(t)}{dt}$

	Phase Modulation	Frequency Modulation
Angle	$\theta_i(t) = \omega_C t + k_p m(t)$	$\theta_i(t) = \omega_C t + k_f \int_{-\infty}^{t} m(\lambda)\, d\lambda$
Frequency	$\omega_i = \omega_C + k_p \dfrac{dm(t)}{dt}$	$\omega_i = \omega_C + k_f m(t)$

In phase modulation $m(t)$ drives the time variation of phase θ_i.

In frequency modulation $m(t)$ drives the time variation of frequency f_C.

Generation of PM and FM



We require that $H(j\omega)$ be a reversible (or invertible) operation so that $m(t)$ is recoverable.

FM and PM are Nonlinear Processes

Consider a phase modulated signal:

Let $s(t) = A_C \cos\left(\omega_C t + k_p[m_1(t) + m_2(t)]\right)$

If $s_1(t) = A_C \cos\left(\omega_C t + k_p m_1(t)\right)$, and

$s_2(t) = A_C \cos\left(\omega_C t + k_p m_2(t)\right)$

It then holds that

$$s_1(t) + s_2(t) \neq s(t) \quad \Leftarrow \quad \therefore \text{ additivity fails}$$

So PM can't be linear.

The same argument holds for FM.

Note: Linearity requires both additivity and homogeneity to hold.

Modulation Index β for Angle Modulation

Let the peak values of the message signal $m(t)$ and its first derivative $m'(t)$ be represented by

Peak value of $m(t) = m_p = ½(m_{max} - m_{min})$

Peak value of $m'(t)$ [$= dm(t)/dt$] $= m'_p$

Frequency Deviation is the maximum deviation of the instantaneous modulated carrier frequency relative to the unmodulated carrier frequency. It is (symbolically) represented by either $\Delta\omega$ or Δf.

$$\text{FM:} \quad \Delta\omega = k_f m_p \quad \text{or} \quad \Delta f = \frac{k_f\, m_p}{2\pi}$$

$$\text{PM:} \quad \Delta\omega = k_p m'_p \quad \text{or} \quad \Delta f = \frac{k_p m'_p}{2\pi}$$

The ratio of the frequency deviation Δf to the message signal's bandwidth B Is called the Frequency Deviation Ratio or the **Modulation Index**, and is denoted by β (unitless).

$$\boxed{\beta = \frac{\Delta f}{B} = \frac{\Delta\omega}{2\pi B}}$$

Equations for PM and FM

Carrier signal	$A_C \cos(\omega_C t)$ (volts)
Carrier frequency	$\omega_C = 2\pi f_C$ (radians/sec)
Modulating wave $m(t)$	$A_m \cos(\omega_m t)$ Single-tone modulation
Modulating frequency	$\omega_m = 2\pi f_m$ (radians/sec)
Deviation sensitivity	k_f (radians/volt-second)
Frequency deviation	$\Delta\omega = k_f A_m$ (radians/sec)
Modulation Index	$\beta = \dfrac{\Delta f}{f_m} = \dfrac{\Delta\omega}{\omega_m} = \dfrac{k_f A_m}{\omega_m}$ (unitless)
Instantaneous frequency	$f_i = f_C + k_f \dfrac{A_m}{2\pi}\cos(\omega_m t) = f_C + \Delta f \cos(\omega_m t)$
Remember	$\varphi_{FM}(t) = A_C\left[\cos\left(\omega_C t + k_f\left(\int_{-\infty}^{t} m(\lambda)d\lambda\right)\right)\right]$, generally
Tone modulated wave	$\varphi_{FM}(t) = A_C\left[\cos\left(\omega_C t + \dfrac{k_f A_m}{\omega_m}\sin(\omega_m t)\right)\right]$
or	$\varphi_{FM}(t) = A_C\left[\cos(\omega_C t + \beta\sin(\omega_m t))\right]$

Summary of Mathematical Equations for FM and PM

Type of Modulation	Modulating Signal	Angle Modulated Wave
Phase modulation	$m(t)$	$A_C \cdot \cos\left(\omega_C t + k_p m(t)\right)$
Frequency modulation	$m(t)$	$A_C \cdot \cos\left(\omega_C t + k_f \int_{-\infty}^{t} m(\lambda) d\lambda\right)$
Phase modulation	*Tone:* $m(t) = A_m \cdot \cos(\omega_m t)$	$A_C \cdot \cos\left(\omega_C t + k_p A_m \cos(\omega_C t)\right)$
Frequency modulation	*Tone:* $m(t) = A_m \cdot \cos(\omega_m t)$	$A_C \cdot \cos\left(\omega_C t + \frac{k_f A_m}{\omega_m} \sin(\omega_m t)\right)$

$$\beta \triangleq \frac{k_f A_m}{\omega_m}$$

Example:

- A single-tone FM signal is

$$\varphi_{FM}(t) = 10\left[\cos\left(2\pi(10^6)t + 8\sin(2\pi(10^3)t)\right)\right]$$

Determine

a) the carrier frequency f_C

b) the modulation index β

c) the peak frequency deviation Δf

Solution:

Start with the basic FM equation:

$$\varphi_{FM}(t) = A_C\left[\cos\left(2\pi f_C t + \beta\sin(2\pi f_m t)\right)\right]$$

Compare this to

$$\varphi_{FM}(t) = 10\left[\cos\left(2\pi(10^6)t + 8\sin(2\pi(10^3)t)\right)\right]$$

(a) We see that f_C = 1,000,000 Hz & f_m = 1000 Hz.
(b) The modulation index is β = 8.
(c) The peak deviation frequency Δf is

$$\Delta f = \beta \cdot f_m = 8 \cdot 1000 = 8{,}000 \text{ Hz}$$

Note: $\Delta f / f_C$ is 0.008 or 0.8 % deviation frequency to carrier frequency.

Average Power of a FM or PM Wave

The amplitude A_C is constant in a phase modulated or a frequency modulated signal. RF power does not depend upon the frequency or the phase of the waveform.

$$\varphi_{FM\ or\ PM}(t) = A_C \cos\left[\omega_C t + g(k_k, m(t))\right]$$

$$\boxed{\text{Average Power} = \frac{A_C^2}{2} \ \text{(always)}}$$

This is a result of FM and PM signals being constant amplitude.

Note: k_k becomes k_f for FM and k_p for PM.

Example:

Problem:

Consider an angle modulated signal given by

$$\phi(t) = 6 \cdot \left[\cos\left(2\pi \times 10^6 t + 2 \cdot \sin(8000\pi t) \right) \right] \text{ volts}$$

What is the average power of this signal?

Solution:

$$\text{Average power} = P_C = \frac{A_C^2}{2} \quad \text{where } A_C = 6 \text{ volts}$$

$$\text{Therefore,} \quad P_C = \frac{6^2}{2} = \frac{36}{2} = 18 \text{ watts (assumes 1 ohm resistance)}$$

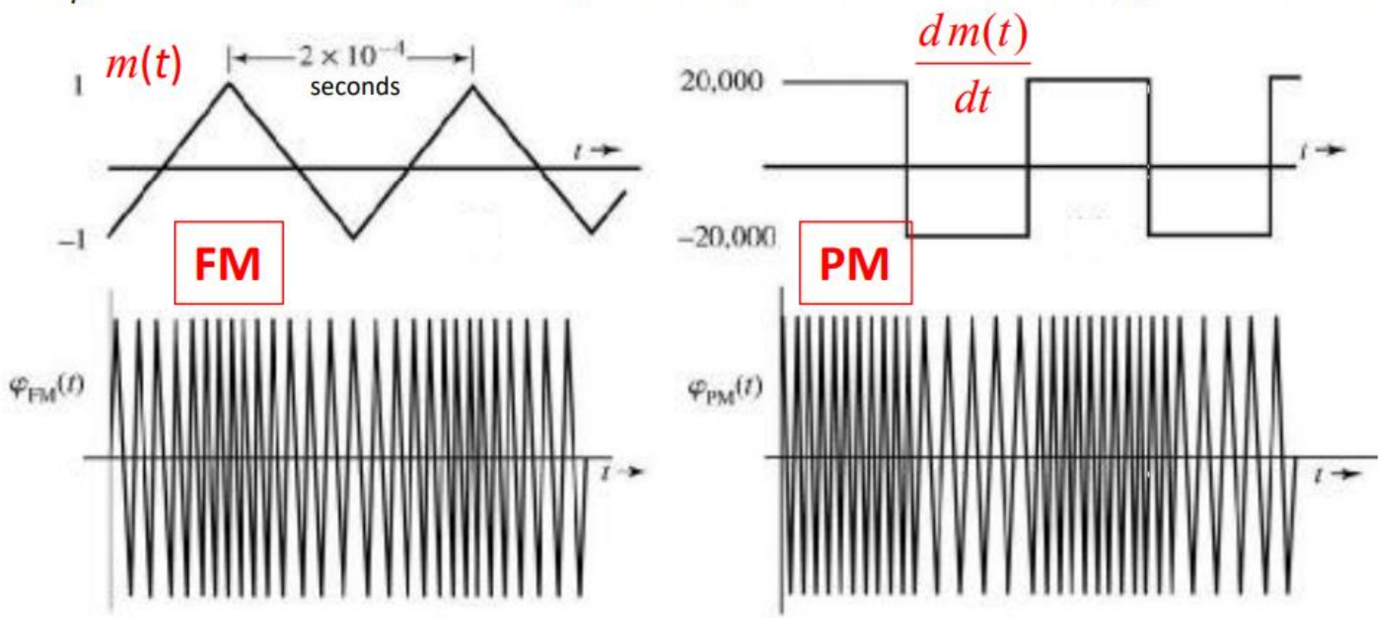
Note that the result does not depend upon it being FM or PM.

Comparison of FM (or PM) to AM

#	Frequency Modulation (FM)	Amplitude Modulation (AM)
1	FM receivers have better noise immunity	AM receivers are very susceptible to noise
2	Noise immunity can be improved by increasing the frequency deviation	The only option in AM is to increase the transmission power
3	Bandwidth requirement is greater and depends upon modulation index	AM bandwidth is less than FM or PM and doesn't depend upon a modulation index
4	FM (or PM) transmitters and receivers are more complex than for AM	AM transmitters and receivers are less complex than for FM (or PM)
5	All transmitted power is useful so FM is very efficient	Power is wasted in transmitting the carrier and double sidebands in DSB (but DSB-SC & SSB addresses this)

Example:

Sketch FM and PM waveforms for the modulating signal $m(t)$. The constants k_f and k_p are $2\pi \times 10^5$ and 10π, respectively. Carrier frequency f_c = 100 MHz.



$$f_i = f_C + \frac{k_f}{2\pi} m(t) = 1\times10^8 + 1\times10^5 \cdot m(t);$$

$$m_{\min} = -1 \text{ and } m_{\max} = 1$$

$$(f_i)_{\min} = 10^8 + 10^5(-1) = 99.9\text{ MHz},$$

$$(f_i)_{\max} = 10^8 + 10^5(+1) = 100.1\text{ MHz}$$

$$f_i = f_C + \frac{k_p}{2\pi} m'(t) = 1\times10^8 + 5\cdot m'(t);$$

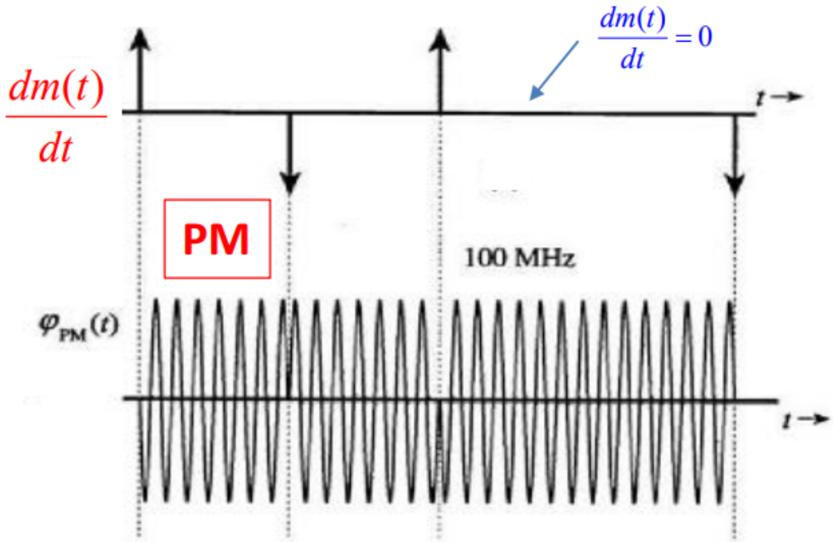
$$m'_{\min} = -20{,}000 \text{ and } m'_{\max} = 20{,}000$$

$$(f_i)_{\min} = 10^8 + 5(-20{,}000) = 99.9\text{ MHz},$$

$$(f_i)_{\max} = 10^8 + 5(+20{,}000) = 100.1\text{ MHz}$$

Example:

Sketch the PM waveform for the modulating signal $m(t)$ from prior slide. The constant k_p equals $\pi/2$. Carrier frequency f_c = 100 MHz.



$$f_i = f_C + \frac{k_p}{2\pi}\frac{dm(t)}{dt} = 1\times 10^8 + \frac{1}{4}\frac{dm(t)}{dt}$$

This is carrier PM by a digital signal – it is **Phase Shift Keying (PSK)** because the digital data is represented by phase of the carrier wave.

Evaluate the instantaneous jumps by considering:

$$\varphi_{PM}(t) = A_C \cos\left[\omega_C t + k_p m_d(t)\right] = A_C \cos\left[\omega_C t + \frac{\pi}{2} m_d(t)\right]$$

$k_p m_d(t) \sim (-\pi, \pi)$

$$\varphi_{PM}(t) = A_C \sin(\omega_C t) \quad \text{when} \quad m(t) = -1$$

$$\varphi_{PM}(t) = -A_C \sin(\omega_C t) \quad \text{when} \quad m(t) = 1$$

where jump in $m_d(t) = (1) - (-1) = 2$ or $(-1) - (1) = -2$

Instantaneous jumps in phase by π radians.

FM bandwidth

- Since FM is a nonlinear operation, the analysis of the modulated signal is significantly more complex
- We will study a case of single tone modulation, considering two cases, depending on the bandwidth of the message
- consider a sinusoidal message

$$m(t) = A_m \cos(2\pi f_m t)$$

so that the instantenous frequency of the FM signal is

$$\begin{aligned} f_i(t) &= f_c + k_f A_m \cos(2\pi f_m t) \\ &= f_c + \Delta f \cos(2\pi f_m t) \end{aligned}$$

where we defined a *frequency deviation* $\Delta f = k_f A_m$. This is the maximum departure of the frequency from f_c.

FM bandwidth

- the angle of the FM signal is thus

$$\theta_i(t) = 2\pi \int_0^t f_i(t)dt$$
$$= 2\pi f_c t + \beta \sin(2\pi f_m t)$$

- where $\beta = \frac{\Delta f}{f_m}$ is commonly referred to as the *modulation index* of the FM signal
- it represents the maximum departure of the angle $\theta_i(t)$ from $2\pi f_c t$
- for the case of a single tone FM modulation, the overall signal is

$$s(t) = A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)$$

- We distinguish two cases
  1. *narrow-band FM* - $\beta << 1$
  2. *wide-band FM* - β is large compared to 1

Parameter β is the **modulation index** for **angle modulation**.

β is used to differentiate between **narrowband angle modulation** and **wideband angle modulation**.

Narrowband angle modulation requires $\beta << 1$ (Typically < 0.3)
Wideband angle modulation requires $\beta >> 1$ (Typically > 5.0)

Equivalently,

Narrowband angle modulation requires $\Delta f << B$
Wideband angle modulation requires $\Delta f >> B$

Comments:

1. Narrowband FM has about the same bandwidth as that of AM.
2. Commercial (broadcast) FM is wideband FM (required due to its superior noise performance).
3. Why even consider narrowband FM? Two reasons:
   a. NBFM is easier to generate that WBFM.
   b. It is commonly used as the first step in generating WBFM.

Narrowband FM bandwidth

- Expanding $s(t)$ for a sinusoidal message we have

$$s(t) = A_c \cos(2\pi f_c t)\cos[\beta \sin(2\pi f_m t)] - A_c \sin(2\pi f_c t)\sin[\beta \sin(2\pi f_m t)]$$

- for $\beta << 1$ we have

$$\cos[\beta \sin(2\pi f_m t)] \approx 1$$
$$\sin[\beta \sin(2\pi f_m t)] \approx \beta \sin(2\pi f_m t)$$

which gives

$$\begin{aligned} s_{FM}(t) &\approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t)\sin(2\pi f_m t) \\ &\approx A_c \cos(2\pi f_c t) + \frac{1}{2}\beta A_c\{\cos[2\pi(f_c + f_m)t)] - \cos[2\pi(f_c - f_m)t)]\} \end{aligned}$$

- compare this expression to the case of sinusoid-modulated AM signal

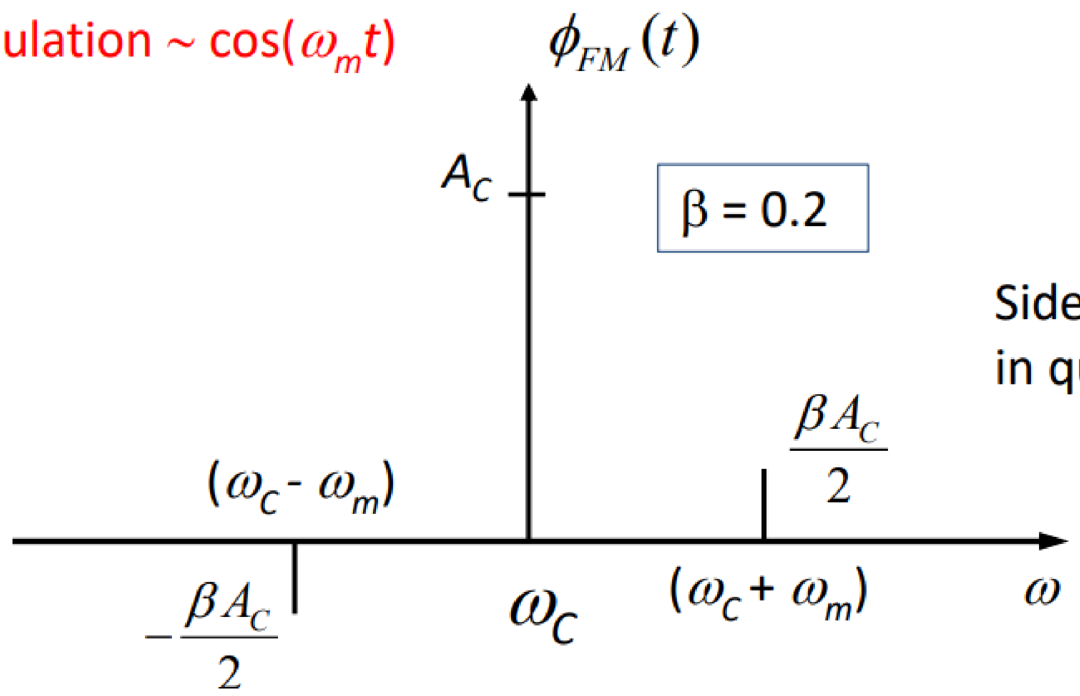
$$s_{AM}(t) \approx A_c \cos(2\pi f_c t) + \frac{1}{2}\mu A_c\{\cos[2\pi(f_c + f_m)t)] + \cos[2\pi(f_c - f_m)t)]\}$$

- the only difference is the sign of the lower frequency component.
- This implies that the bandwidth of a *narrowband* FM is the same as the corresponding AM signal, ie $2f_m$

Spectrum of NBFM

$$\phi_{FM}(t) \cong A_C\cos(\omega_C t) + \frac{1}{2}\beta A_C\cos\left((\omega_C+\omega_m)t\right) - \frac{1}{2}\beta A_C\cos\left((\omega_C-\omega_m)t\right)$$

Tone modulation ~ $\cos(\omega_m t)$



Sidebands are in quadrature.

NBPM requires $\beta << 1$ radian
(generally less than 0.3 radian)

Narrowband Angle Modulation

- Another way to look at the case of narrowband FM is to note that the signal can be expressed as

$$s(t) = \Re\{A_c e^{j\phi(t)} e^{j2\pi f_c t}\}$$

where in the case of single tone message, $\phi(t) = \beta \sin(2\pi f_m t)$

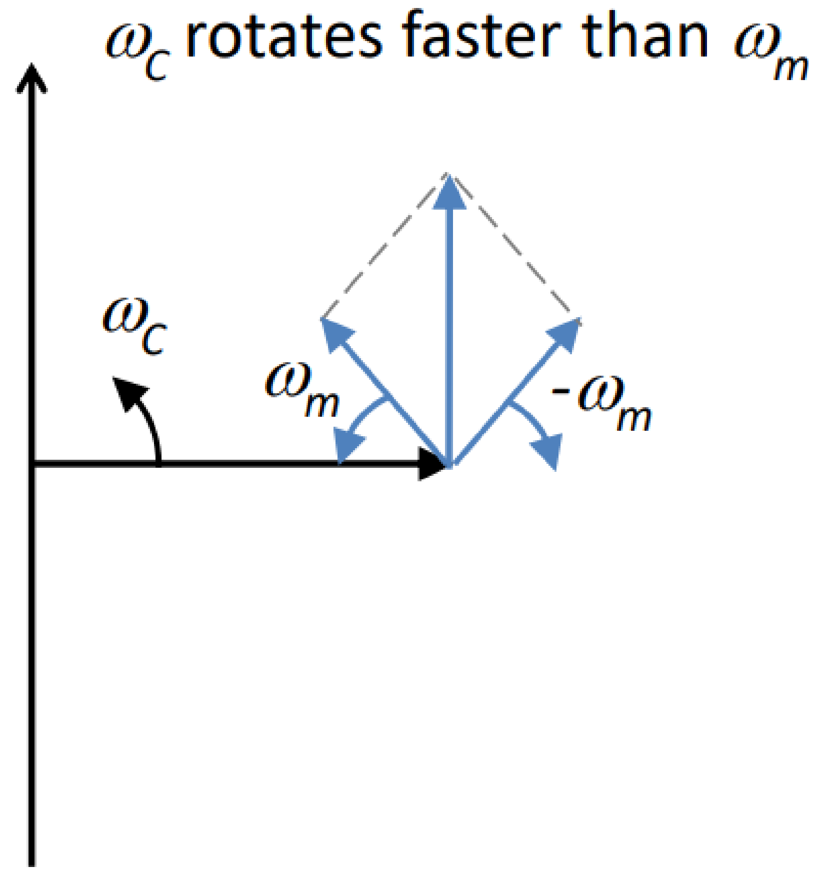
- Expanding $e^{j\phi}$ in a power series gives

$$s(t) = \Re\left\{A_c\left[1 + j\phi(t) - \frac{\phi^2(t)}{2!} - \ldots\right] e^{j2\pi f_c t}\right\}$$

- if the maximum value of $|\phi(t)| << 1$, we can approximate

$$\begin{aligned} s(t) &\approx \Re\{A_c e^{j2\pi f_c t} + A_c \phi(t) j e^{j2\pi f_c t}\} \\ &= A_c \cos(2\pi f_c t) - A_c \phi(t) \sin(2\pi f_c t) \end{aligned}$$

Phasor view of NBFM



Phasor lengths adjust to keep constant A_c.

Example:

Exercise: The message signal input to a modulator is $m(t) = 4\cdot\cos(2\pi\times 10^4 t)$ and the carrier is $10\cdot\cos(\pi\times 10^8 t)$. If frequency modulation is performed with $k_f = 1000\pi$, verify that the modulated signal meets the criteria of being narrowband FM. Also, obtain an expression for its spectrum and sketch this spectrum.

Solution:

First we calculate the modulation index β

$$\beta = k_f \frac{A_m}{\omega_m} = 1000\pi\left(\frac{4}{2\pi\times 10^4}\right) = 0.2;\;\; \beta < 0.3 \;\Rightarrow\; \text{NBFM}$$

$$A_C = 10 \quad \text{thus,} \quad \frac{1}{2}\beta A_C = \frac{1}{2}(0.2)(10) = 1$$

Using the equation from slide #49:

$$\phi_{FM}(t) \cong A_C \cos(\omega_C t) + \frac{1}{2}\beta A_C \cos\left((\omega_C + \omega_m)t\right) - \frac{1}{2}\beta A_C \cos\left((\omega_C - \omega_m)t\right)$$

$$\phi_{FM}(t) \cong 10\cdot\cos(\omega_C t) + \cos\left((\omega_C + \omega_m)t\right) - \cos\left((\omega_C - \omega_m)t\right)$$

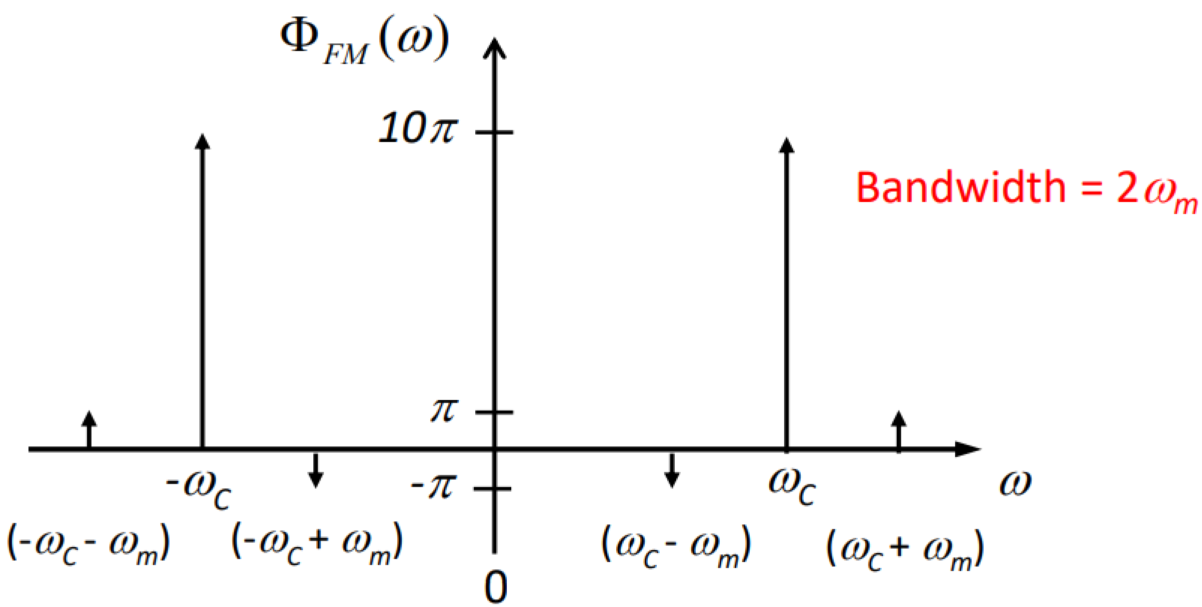
Solution:

$$\phi_{FM}(t) \cong 10 \cdot \cos(\omega_C t) + \cos\left((\omega_C + \omega_m)t\right) - \cos\left((\omega_C - \omega_m)t\right)$$

The corresponding expression for the specturm becomes

$$\Phi_{FM}(\omega) = 10\pi\left[\delta(\omega+\omega_C) + \delta(\omega-\omega_C)\right] + \pi\left[\delta(\omega+\omega_C+\omega_m) + \delta(\omega-\omega_C-\omega_m)\right]$$
$$-\pi\left[\delta(\omega+\omega_C-\omega_m) + \delta(\omega-\omega_C+\omega_m)\right]$$

where $\omega_C = 10^8\pi$ radians/sec and $\omega_m = 2\pi\times 10^4$ radians/sec



Spectrum of Wideband FM

- To study the case of wideband FM, we once again consider a sinusoidal message signal such that

$$\phi(t) = \beta \sin(2\pi f_m t)$$

where β is not necessarily smaller than 1

- The resulting signal is

$$\begin{aligned} s(t) &= A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \\ &= \Re\{A_c e^{j\beta \sin(2\pi f_m t)} e^{j2\pi f_c t}\} \\ &= \Re\{\tilde{s}(t) e^{j2\pi f_c t}\} \end{aligned}$$

where $\tilde{s}(t) = A_c e^{j\beta \sin(2\pi f_m t)}$ is the complex envelope of $s(t)$.

- Note that $\tilde{s}(t)$ is a periodic function with a fundamental frequency f_m.

Spectrum of Wideband FM

- Since $\tilde{s}(t)$ is periodic, we can expand it as a complex Fourier Series

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_m t}$$

where the FS coefficients are given by

$$\begin{aligned} c_n &= f_m \int_{-1/(2f_m)}^{1/(2f_m)} \tilde{s}(t) e^{-j2\pi n f_m t} dt \\ &= f_m A_c \int_{-1/(2f_m)}^{1/(2f_m)} e^{j\beta \sin(2\pi f_m t) - j2\pi n f_m t} dt \end{aligned}$$

- if we define $x = 2\pi f_m t$, we can rewrite the FS coefficients as

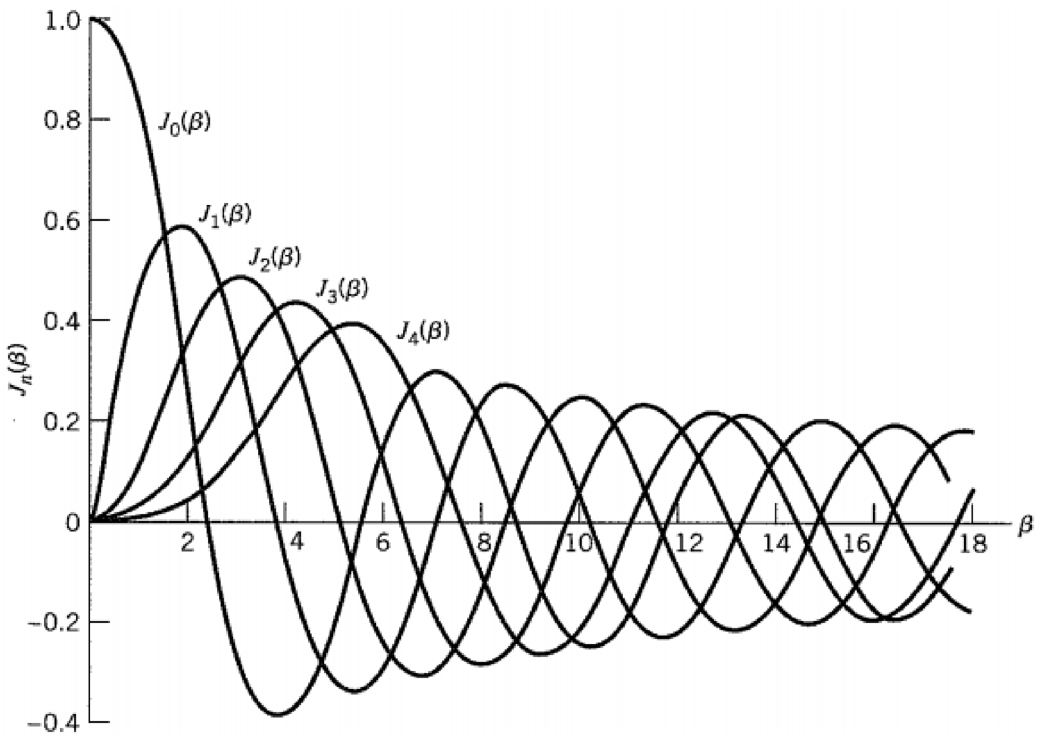
$$\begin{aligned} c_n &= \frac{A_c}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} dx \\ &= A_c J_n(\beta) \end{aligned}$$

where $J_n(\beta)$ is *nth order Bessel function of the first kind*

Bessel functions

- an nth order Bessel function of first kind is defined by

$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin x - nx)}dx$$



Spectrum of Wideband FM

- Substituting, we have that the *complex envelope* of the FM signal is

$$\tilde{s}(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t}$$

  and thus the FM signal itself is

$$s(t) = \Re\left\{ A_c \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi(f_c + n f_m)t} \right\}$$

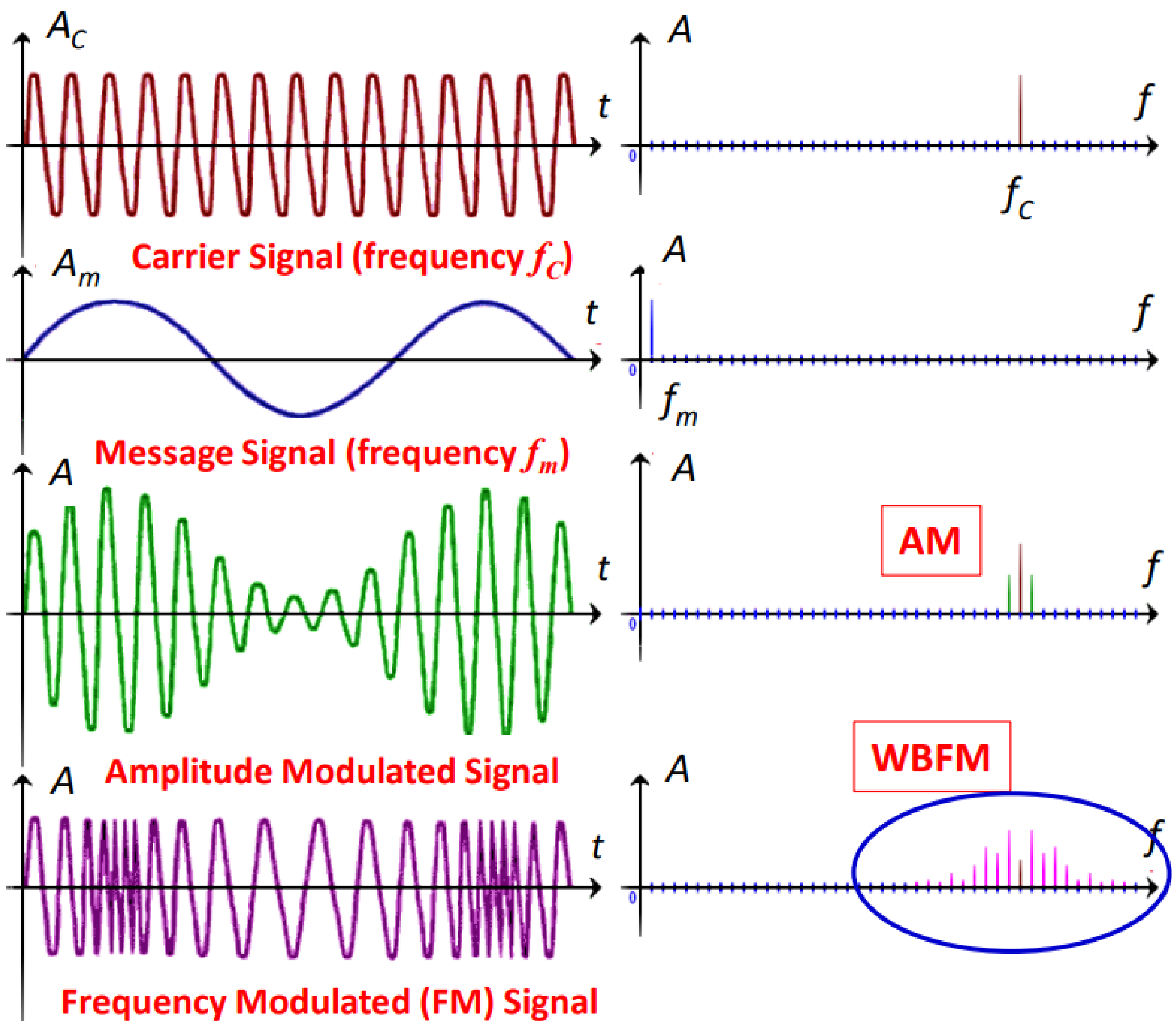
- taking the constant amplitude A_c outside, and exchanging the order of summation and $\Re\{\cdot\}$, we can simplify to

$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos[2\pi(f_c + n f_m)t]$$

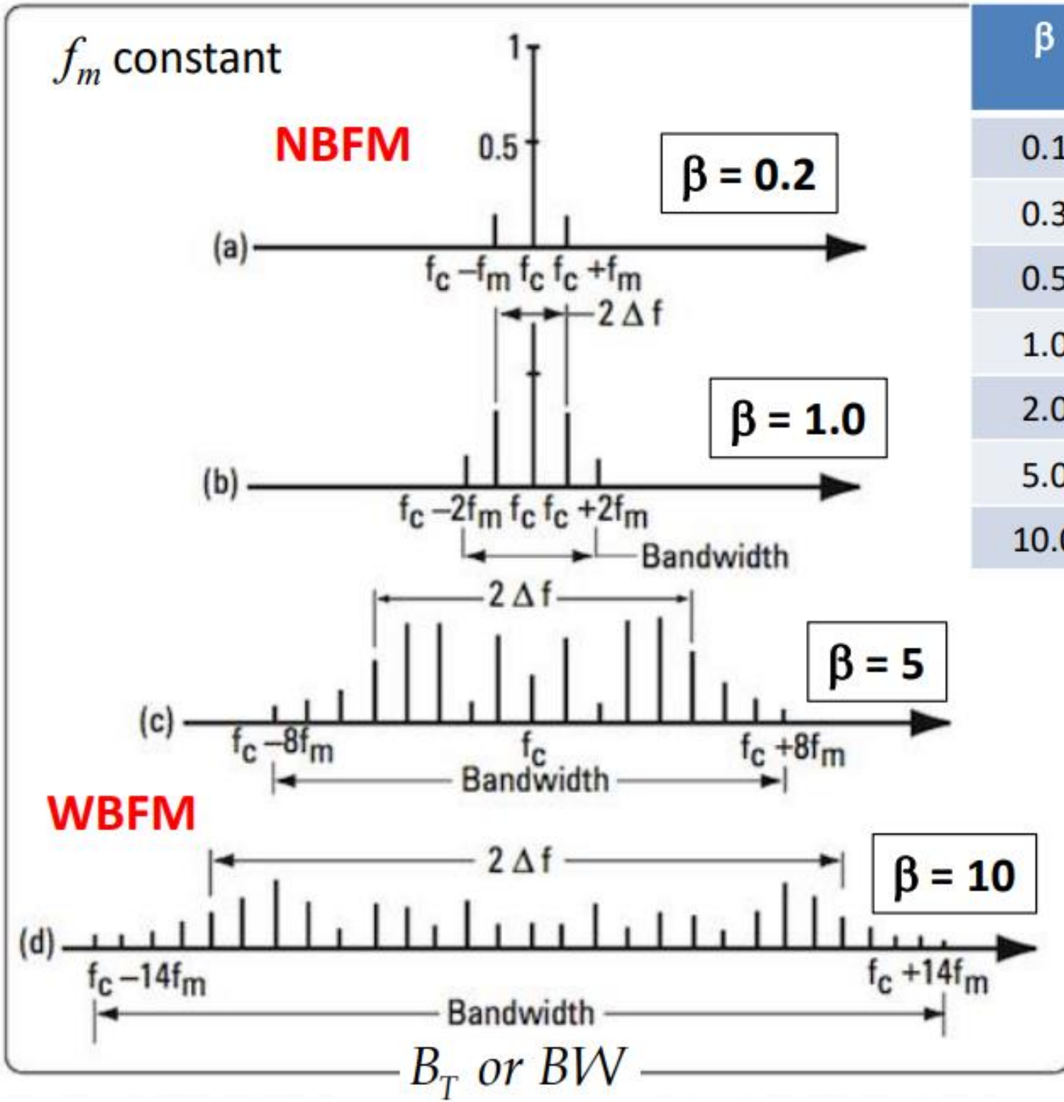
- the above is a FS expansion of *single-tone* FM for arbitrary modulation index β.
- At last, Fourier Transform of both sides gives the spectrum of $s(t)$

$$S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - n f_m) + \delta(f + f_c + n f_m)]$$

WBFM requires more bandwidth



Single-Tone FM Spectra



β	Number of Sidebands¶	Bandwidth
0.1	2	$2\,f_m$
0.3	4	$4\,f_m$
0.5	4	$4\,f_m$
1.0	6	$6\,f_m$
2.0	8	$8\,f_m$
5.0	16	$16\,f_m$
10.0	28	$28\,f_m$

¶Both upper and lower sidebands about f_c.

Single-tone Modulation Index

$$\beta = \frac{\Delta f}{f_m} = \frac{\Delta\omega}{\omega_m}$$

Spectra of FM

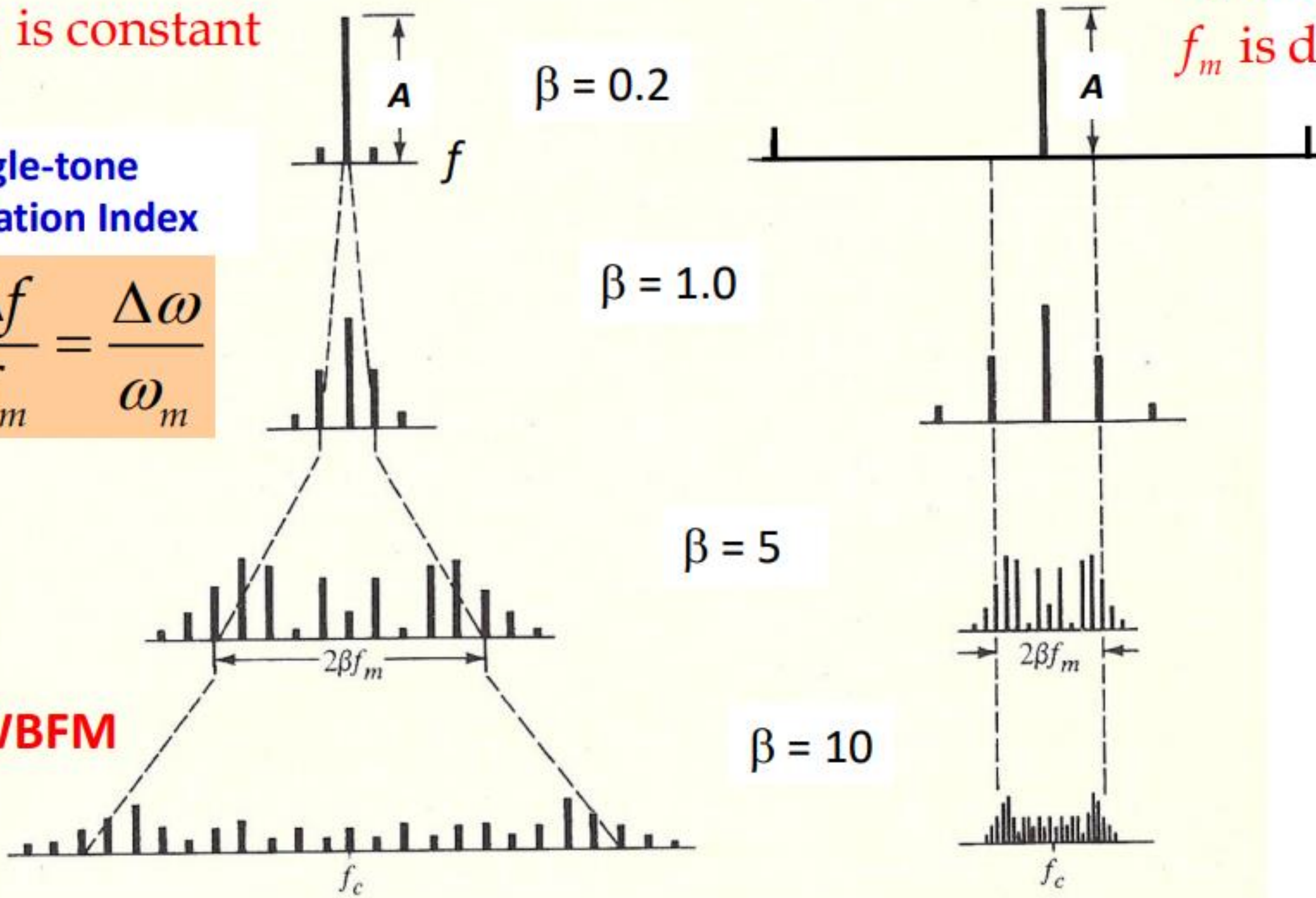
Δf increasing &
f_m is constant

Δf is constant &
f_m is decreasing

Single-tone Modulation Index

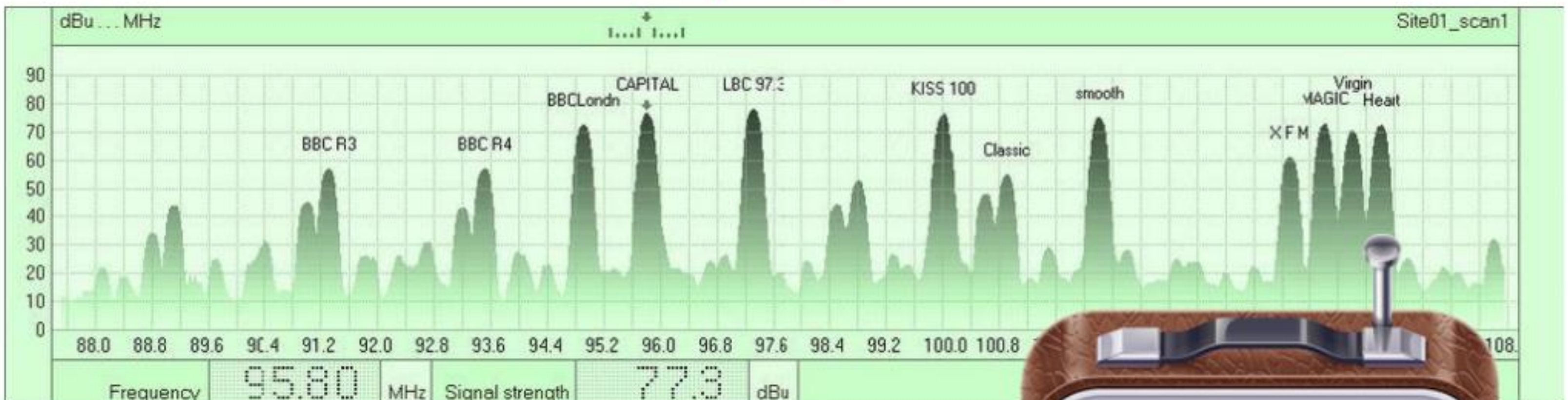
$$\beta = \frac{\Delta f}{f_m} = \frac{\Delta\omega}{\omega_m}$$

WBFM



Selecting an FM radio station

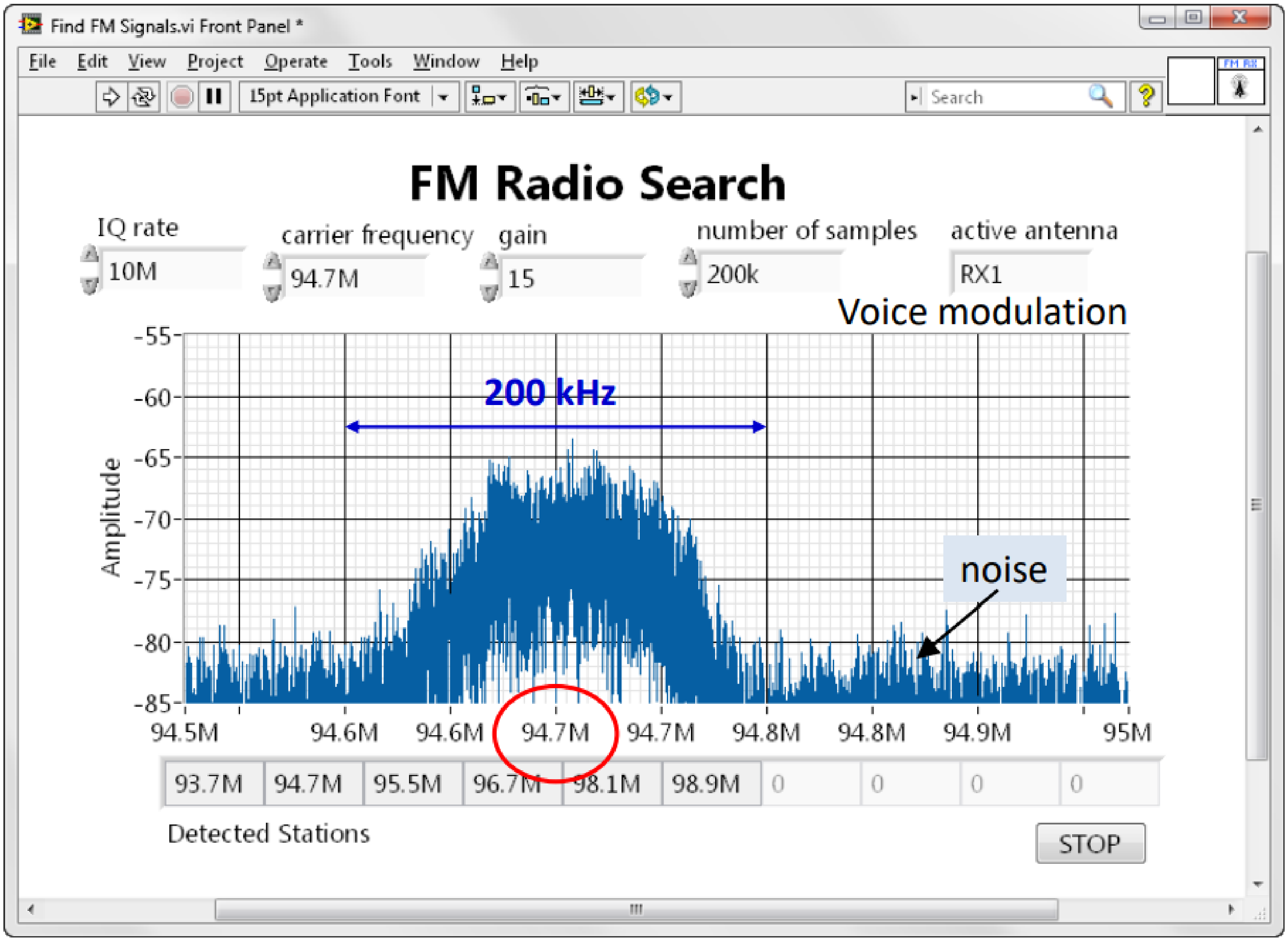
Broadcast FM Radio covers from 88 MHz to 108 MHz
100 stations – 200 kHz spacing between FM stations



Note: 0 **dBu** = 0.775 volt into 600 ohms
(which is equivalent to 1 mW power
delivered into the 600 ohm resistor)

Service Type	Frequency Band	Channel Bandwidth	Maximum Deviation	Highest Audio
Commercial FM Radio Broadcast	88.0 to 108.0 MHz	200 kHz	±75 kHz	15 kHz

Measured spectrum of an FM station



Three important parameters:

The <u>three important frequencies</u> in FM and PM are

1. **Carrier frequency** f_C (or ω_C)
2. **Maximum modulation frequency** f_m (or ω_m), and
3. **Peak frequency deviation** Δf (or $\Delta\omega$)

<u>Two Definitions of importance</u>:

1. **Modulation index** β

$$\beta = \frac{\Delta f}{B_m} = \frac{\Delta f}{f_m} = \frac{\Delta\omega}{\omega_m} = \frac{\Delta\omega}{2\pi B_m} \quad \text{(can be a very large number)}$$

2. **Deviation ratio** D

$$D = \frac{\Delta f}{f_C} = \frac{\Delta\omega}{\omega_C} \quad \text{(always much less than unity)}$$

Remember: For FM $\Delta\omega = k_f m_p$ & for PM $\Delta\omega = k_p m'_p$

FM Bandwidth and the Modulation Index β

A. Narrowband FM (NBFM) – $\beta \ll 1$ radian

$$B_{FM}^{NB} \cong 2B_m \quad \text{where } B_m \text{ is the bandwidth of } m(t)$$

B. Wideband FM (WBFM) – $\beta \gg 1$ radian

$$B_{FM}^{WB} \cong 2(\beta+1)B_m\,, \quad \text{where} \quad \beta = \frac{\Delta f}{B_m} = \frac{\Delta f}{f_m} = \frac{\Delta\omega}{\omega_m}$$

Δf is the peak frequency deviation $\Delta f = \max\left[k_f m(t)\right]$

$$\boxed{B_{FM}^{WB} \cong 2(\Delta f + B_m) = 2(\beta+1)B_m} \Leftarrow \text{Carson's Rule}$$

For PM we have analogous equation,

$$B_{PM}^{WB} \cong 2(\beta+1)B_m$$

Example:

The message signal input to a modulator is $10 \cdot \cos(2\pi \times 10^4 t)$. If frequency modulation with frequency deviation constant $k_f = 10^4 \pi$ is performed, find the bandwidth of the resulting FM signal.

Solution:

$$\beta = \frac{1}{2\pi} \frac{k_f A_m}{f_m} = \frac{10^4 \pi \times (10)}{2\pi \times 10^4} = 5$$

$$B_{FM} = 2(\beta + 1) f_m = 2(5+1) \cdot 10 \text{ kHz} = 120 \text{ kHz},$$

using Carson's rule to calculate bandwidth B_{FM}

Example:

If phase modulation is performed using the message signal $10 \cdot \cos(2\pi \times 10^4 t)$ used in the previous slide, find the phase deviation constant k_p giving the PM signal the same bandwidth, namely, 120 kHz.

Solution:

For both the FM and PM signals to have the same bandwidth, β and Δf must be the same. For FM, $\Delta\omega = k_f A_m$; but for PM, $\Delta\omega = k_p m'_p$.
Expressing the message signal $m(t) = A_m \cdot \cos(\omega_m t)$ gives

$$m'(t) = \frac{d}{dt}\left(A_m \cos(\omega_m t)\right) = -\omega_m A_m \cdot \sin(\omega_m t) \quad \Rightarrow \quad m'_p = -\omega_m A_m$$

Thus,

$$k_f A_m = -k_p \omega_m A_m \quad \rightarrow \quad k_p = \frac{k_f}{\omega_m} = \frac{10^4 \pi}{2\pi \times 10^4} = \frac{1}{2} \quad \Leftarrow$$

$$\text{Check:} \quad \beta = \frac{k_p m'_p}{\omega_m} = \frac{k_p \omega_m A_m}{\omega_m} = k_p A_m = \frac{1}{2}(10) = 5$$

Example:

For commercial FM radio, the audio message signal has a spectral range of 30 Hz to 15 kHz, and the FCC allows a frequency deviation of 75 kHz. Estimate the transmission bandwidth for commercial FM using Carson's Rule.

Solution:

We start by calculating β

$$\beta = \frac{\Delta f}{B_m} = \frac{75 \text{ kHz}}{15 \text{ kHz}} = 5$$

Using Carson's rule gives

$$B_{FM} = 2(\beta + 1)B_m = 2(5+1)15 \text{ kHz} = 180 \text{ kHz}$$

The allowed bandwidth for commercial FM is 200 kHz.

Note that Carson's rule slightly underestimates the bandwidth.

Why does FM need more bandwidth?

Observation: The bandwidth required for AM and NBFM are the same.

However, WBFM (wideband FM) requires much more bandwidth. Why?

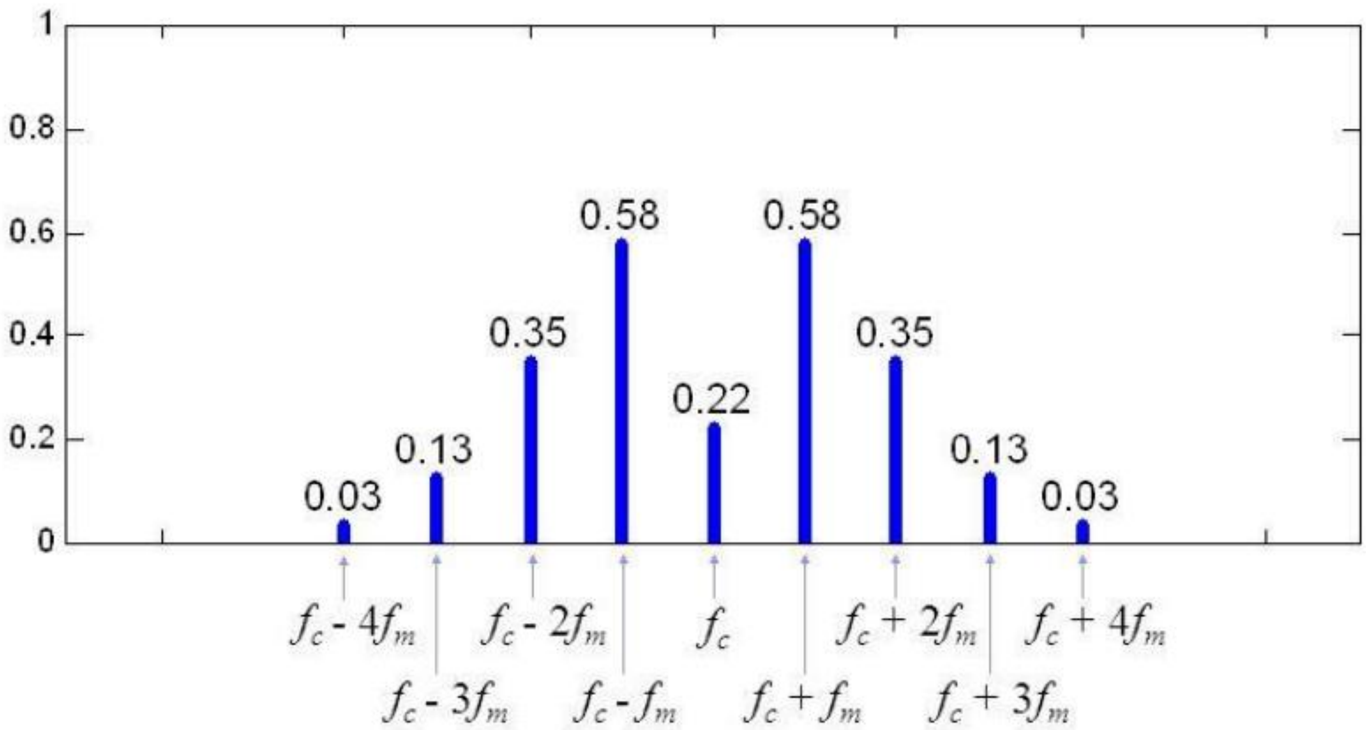
A Fourier spectrum of an FM signal shows that to keep the amplitude constant of an FM signal that many components are required to represent the FM waveform. The frequency spectrum of an actual FM signal has components extending infinitely, although their amplitude decreases for sufficiently higher frequencies. Sufficiently higher frequencies applies to frequencies above the Carson bandwidth rule.

$$B_{FM}^{WB} \cong 2(\Delta f + B_m) = 2(\beta + 1)B_m \quad \Leftarrow \quad \text{Carson's Rule}$$

Next we examine the Fourier components this using phasors.

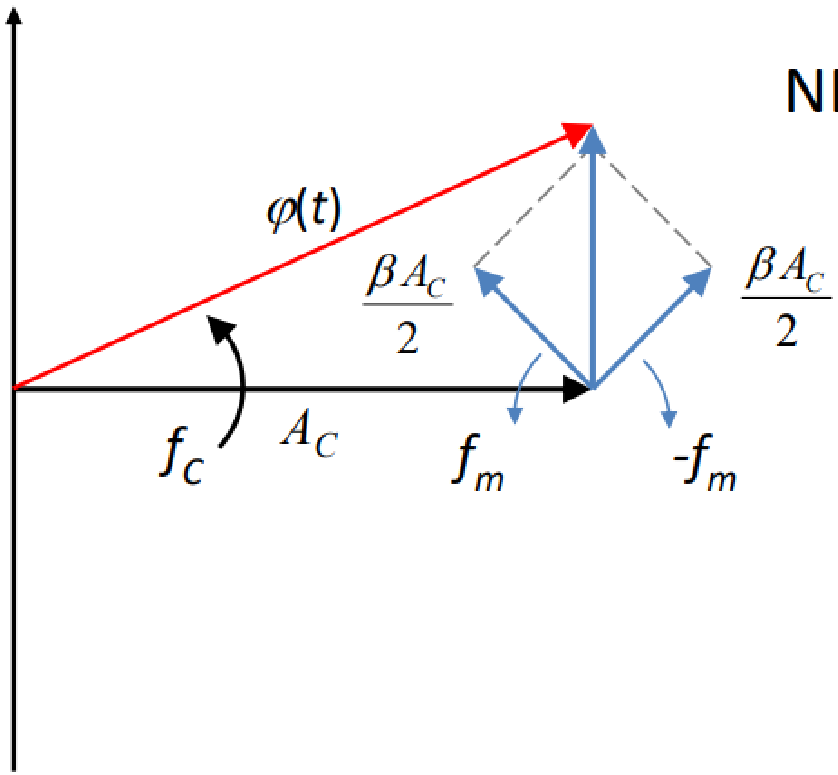
Remember:

For $\beta = 2.0$

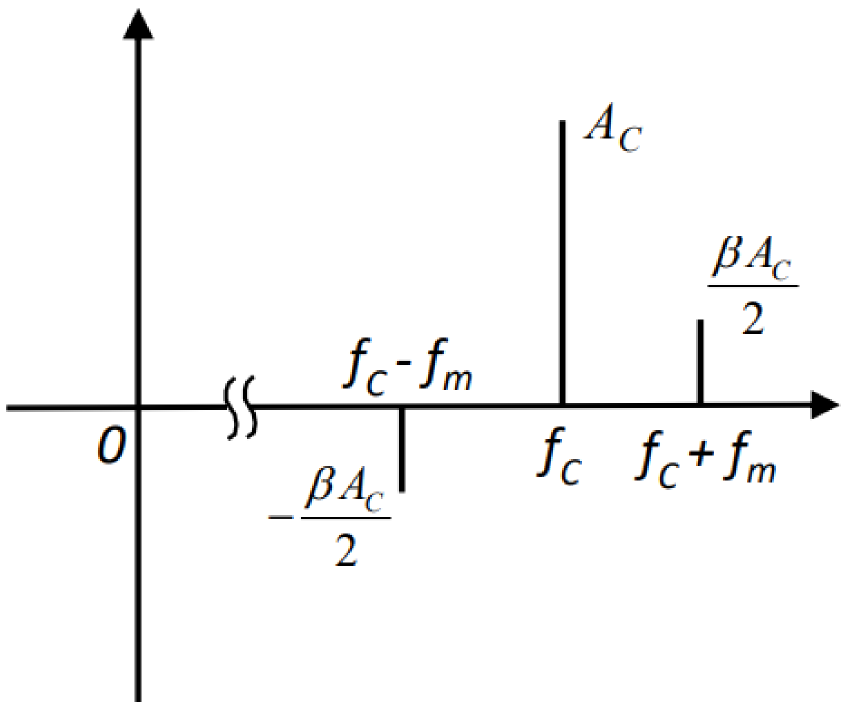


Note: Only magnitudes of spectral lines shown.

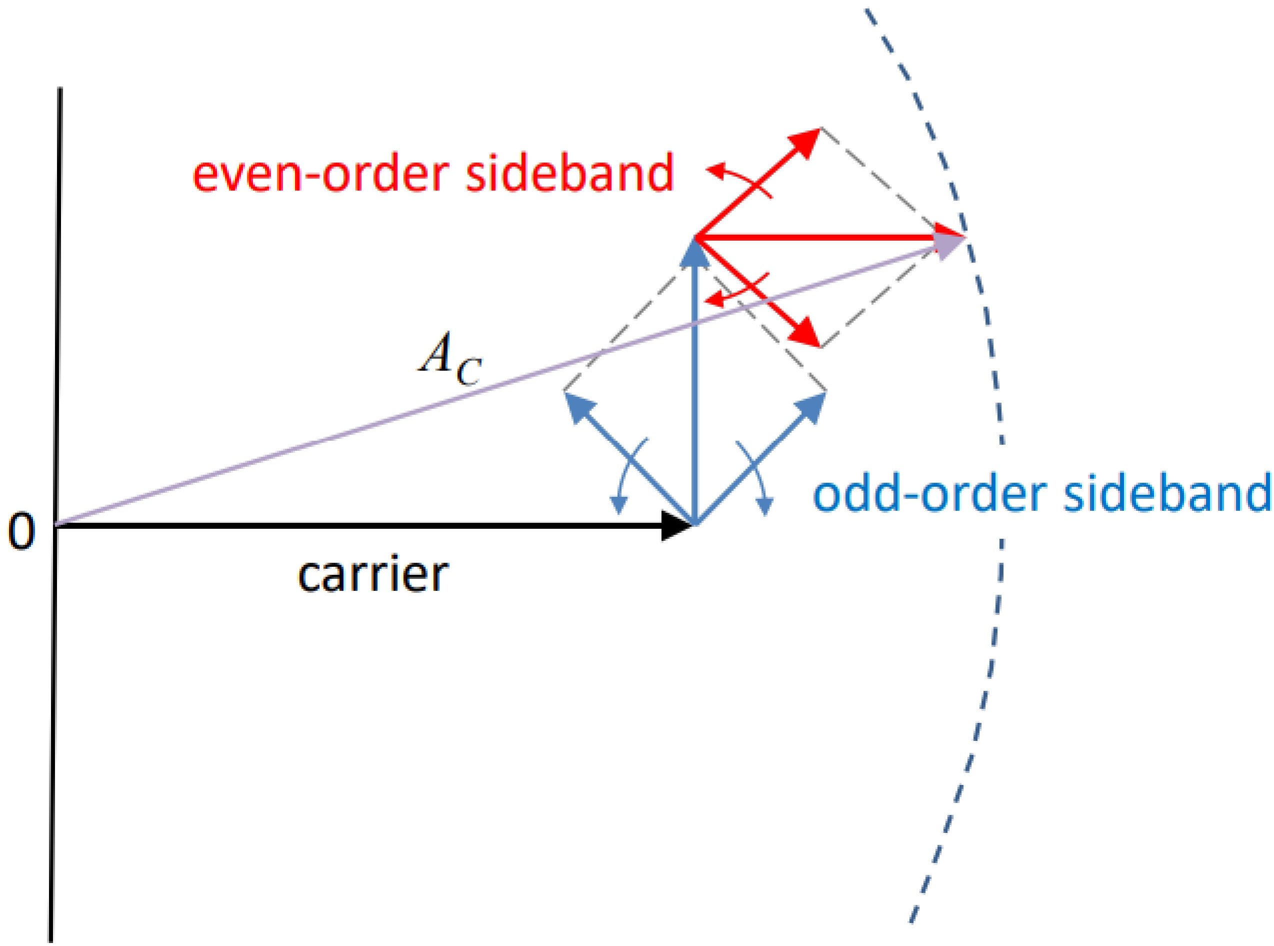
NBFM in phasor



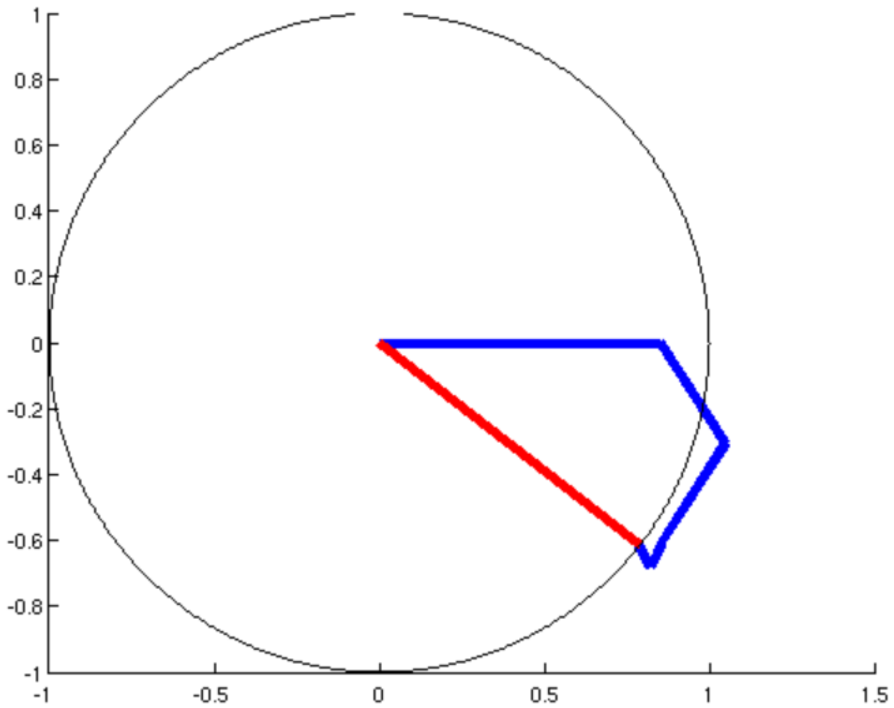
NBFM with tone modulation



WBFM in phasor



Animation

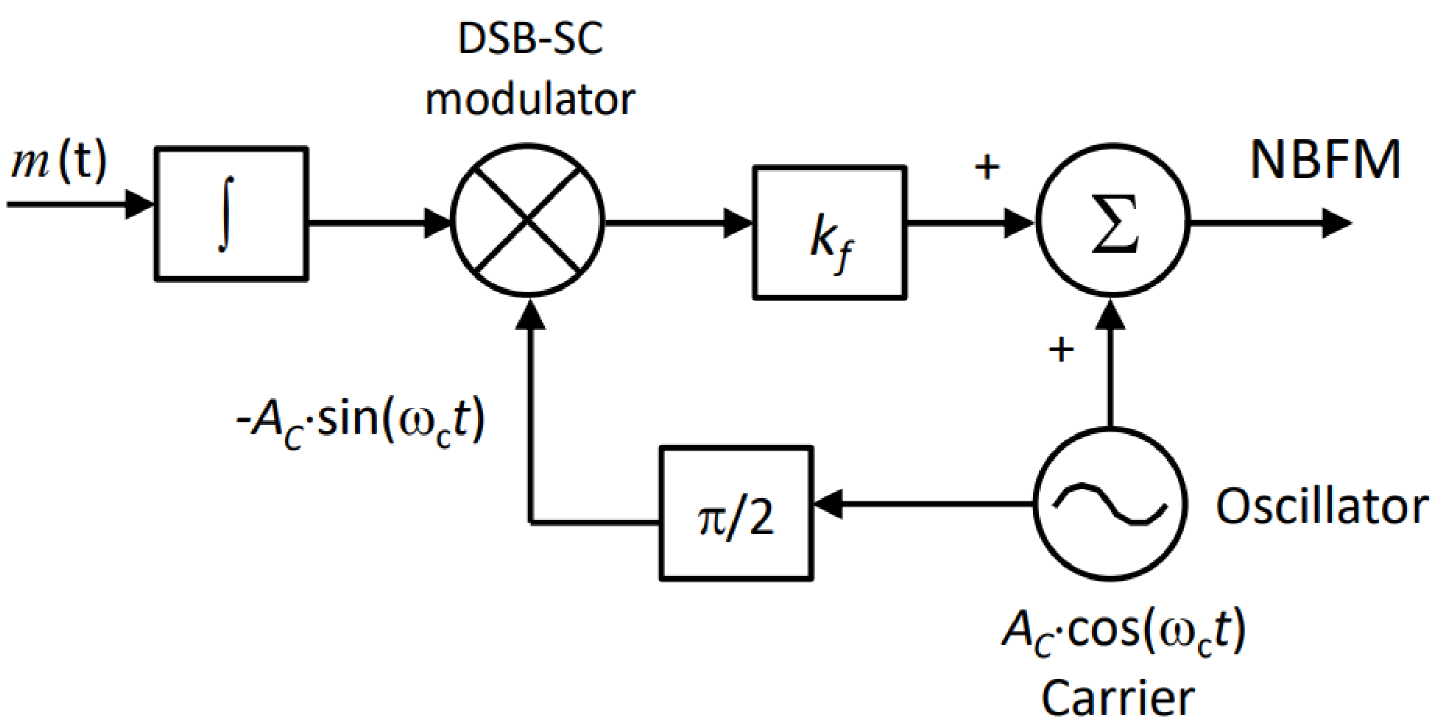


Go to this site to see the animation of a PM wave generation
https://nibot-lab.livejournal.com/114405.html

Generation of NBFM

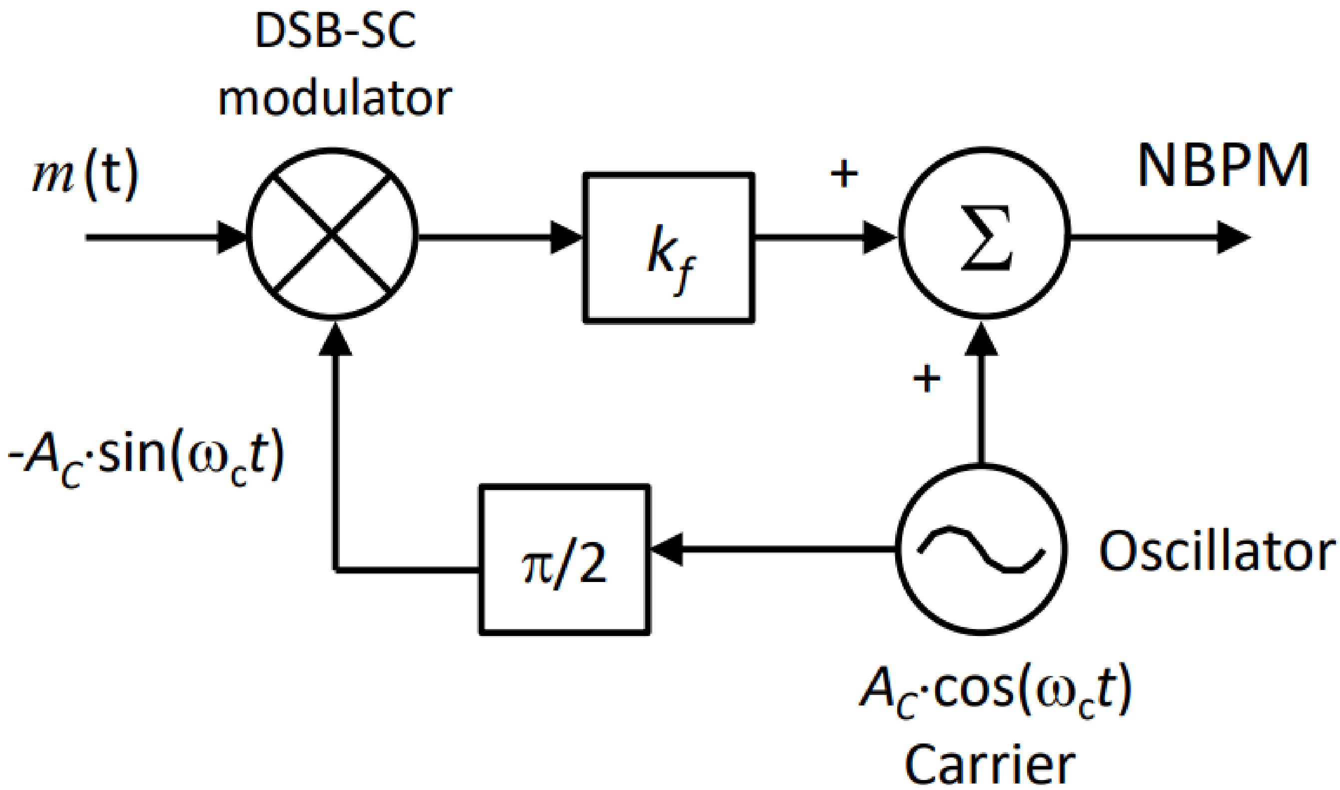
$$\varphi_{FM}(t) = A_C \cos(\omega_C t) - A_C \left(k_f \int_{-\infty}^{t} m(\alpha)\, d\alpha \right) \sin(\omega_C t)$$

NBFM is limited to $\beta \ll 1$ radian

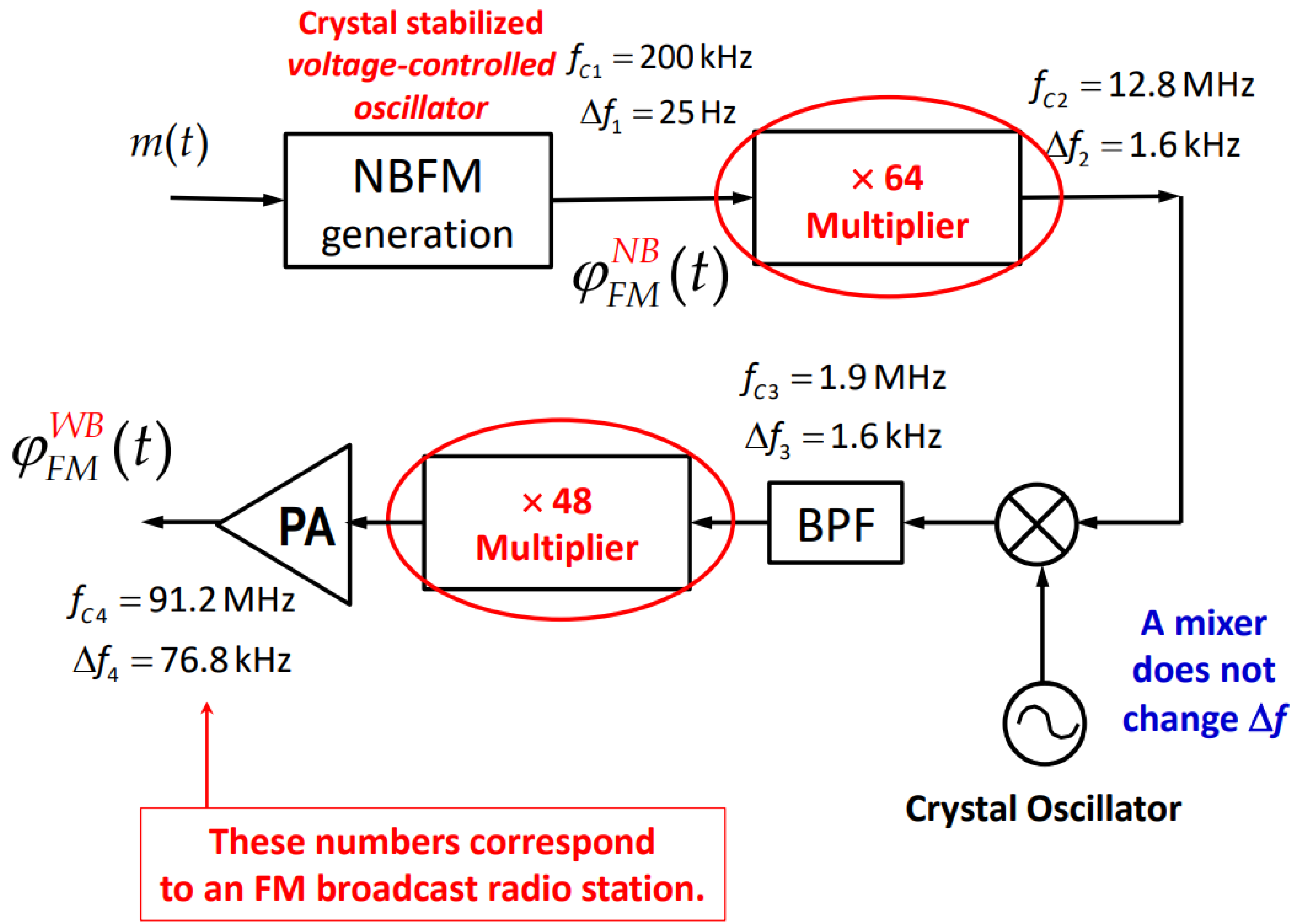


Generation of NBPM

$$\varphi_{FM}(t) = A_C \cos\left(\omega_C t\right) - A_C k_p m(t) \cdot \sin(\omega_C t)$$



Generation of WBFM



Advantages of frequency modulation

1. Resilient to noise: The main advantage of frequency modulation is a reduction in noise. As most noise is amplitude based, this can be removed by running the received signal through a limiter so that only frequency variations remain.

2. Resilient to signal strength variations: In the same way that amplitude noise can be removed, so too can signal variations due to channel degradation because it does not suffer from amplitude variations as the signal level varies. This makes FM ideal for use in mobile applications where signal levels constantly vary.

3. Does not require linear amplifiers in the transmitter: As only frequency changes contain the information carried, amplifiers in the transmitter need not be linear.

4. Enables greater efficiency : The use of non-linear amplifiers (*e.g.*, class C and class D/E amplifiers) means that transmitter efficiency levels can be higher. This results from linear amplifiers being inherently inefficient.

Disadvantages of frequency modulation

1. Requires a more complicated demodulator: One of the disadvantages is that the demodulator is a more complicated, and hence more expensive than the very simple diode detectors used in AM.

2. Sidebands extend to infinity: The sidebands for an FM transmission theoretically extend out to infinity. To limit the bandwidth of the transmission, filters are used, and these introduce some distortion of the signal.

FM differentiator demodulator

The ideal FM detector converts the FM signal's instantaneous frequency ω_i to an amplitude that is proportional to ω_i.

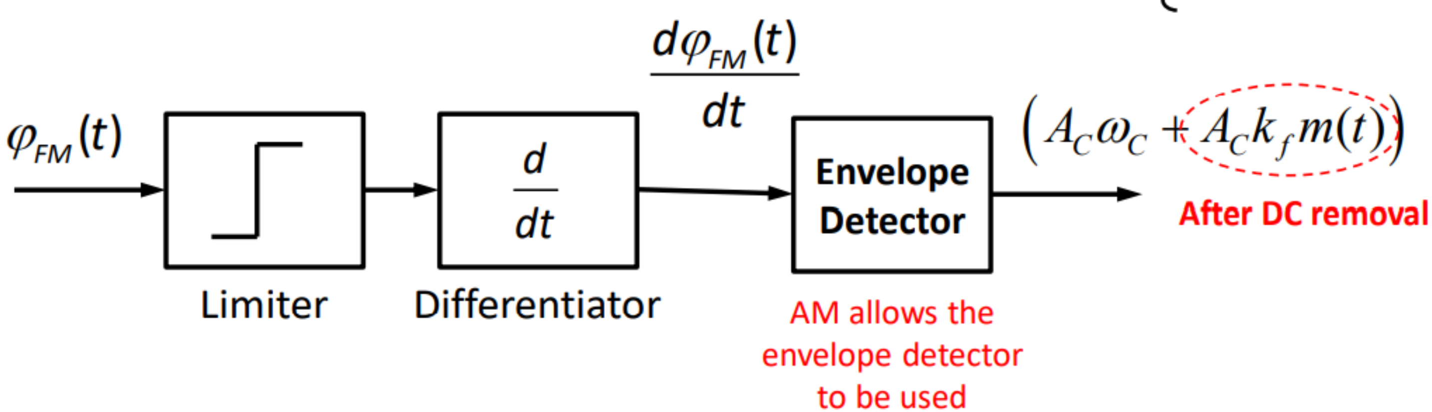
Differentiation performs FM to AM conversion

$$\text{Input:}\quad \varphi_{FM}(t) = A_C\cos\left(\omega_C t + \theta(t)\right) = A_C\cos\left(\omega_C t + k_f\int_{-\infty}^{t} m(\lambda)\,d\lambda\right)$$

$$\text{Output:}\quad \frac{d\varphi_{FM}(t)}{dt} = \frac{d}{dt}\left[A_C\cos\left(\omega_C t + k_f\int_{-\infty}^{t} m(\lambda)\,d\lambda\right)\right]$$

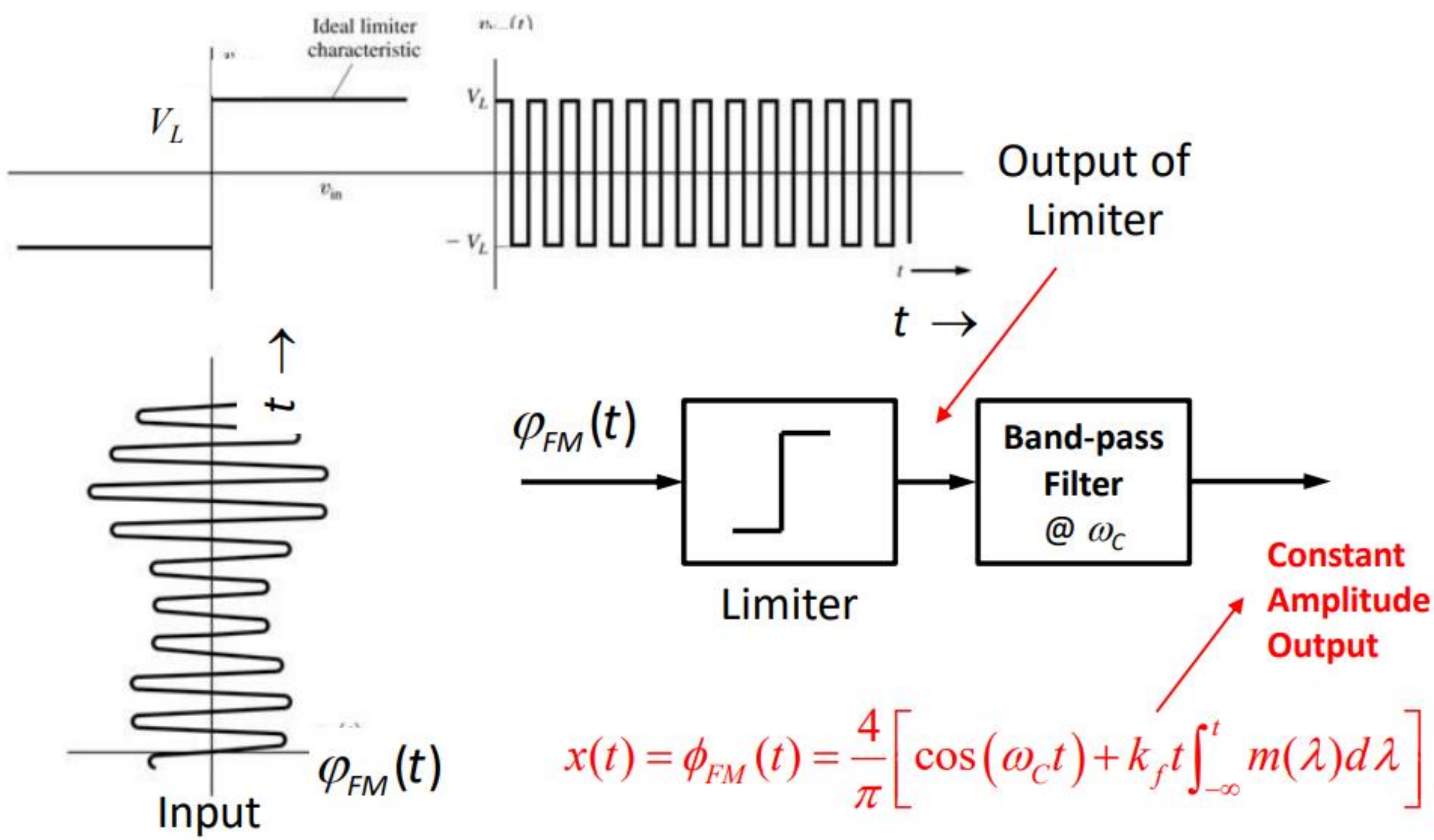
$$\frac{d\varphi_{FM}(t)}{dt} = -A_C\left(\omega_C + k_f m(t)\right)\cdot\sin\left(\omega_C t + k_f\int_{-\infty}^{t} m(\lambda)\,d\lambda\right)$$

Both AM and FM included

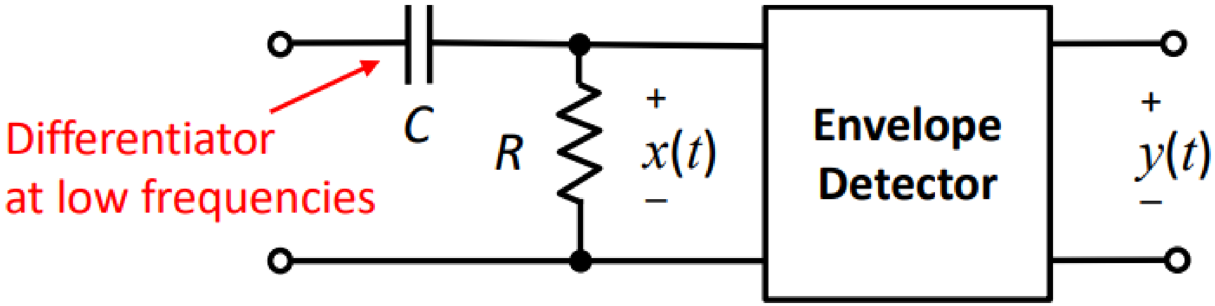


Limiter

For an envelope detector to work well the FM signal's amplitude should be constant or flat. We can accomplish with a "hard limiter." Factors such as channel noise, interference and channel fading result in amplitude variations in an FM signal's amplitude at the receiver.



Differentiator



$$H(j\omega)=\frac{j\omega RC}{1+j\omega RC}=\frac{j\left(\omega/\omega_{3dB}\right)}{1+j\left(\omega/\omega_{3dB}\right)}; \quad \text{where} \quad \omega_{3dB}=\frac{1}{RC}$$

For $\omega << \omega_{3dB}=\dfrac{1}{RC}$; then $H(j\omega)\cong j\omega RC$

Multiplication by $j\omega$ in the frequency domain is equivalent to differentiation in time domain! The high-pass filter acts as a differentiator for an FM signal. Therefore,

$$y(t) = A_c\omega_c RC + A_c\omega_c RCk_f m(t)$$

1G Handphones

1G analog cellular telephone (1983) – AMPS (Advanced Mobile Phone Service)

- First use of cellular concept . . .
- Used 30 kHz channel spacing (but voice BW was B = 3 kHz)
  - Peak frequency deviation Δf = 12 kHz, and
  - $B_T = 2(\Delta f + B) = 2(12 \text{ kHz} + 3 \text{ kHz}) = 30 \text{ kHz}$
- Two channels (30 kHz each); one for uplink and one for downlink
- Used **FM for voice** and **FSK** (next slide) for **data comm**unication
- No protection from eavesdroppers!

Successor to AMPS was GSM (Global System for Mobile) in early 1990s

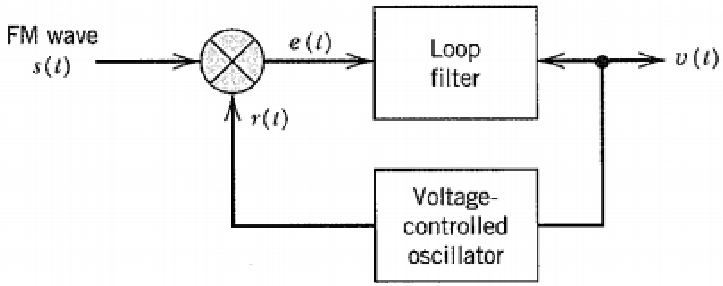
- GSM is **2G** cellular telephone
- Still used by nearly 50% of world's population (as of 2017)
- GSM was a digital communication system
  - Modulating signal is a bit stream representing voice signal
- Uses Gaussian Minimum Shift Keying (GMSK)
- Channel bandwidth is 200 kHz (simultaneously shared by 32 users
  - This is 4.8 times improvement over AMPS

Phase-Locked Loop (PLL)

- A Phase-Locked Loop (PLL) is a feedback system commonly used in *carrier synchronisation* and *frequency demodulation*
- It consists of
  1. Voltage-controlled oscillator (VCO) - an oscillator whose frequency varies linearly with its input voltage $v(t)$, centered around some *free running frequency* f_0

$$f(t) = f_0 + cv(t)$$

  2. Multiplier - serves as a phase detector
  3. Lowpass loop filter - remove the high-frequency components of the multiplier output



PLL - carrier synchronisation

- Lets look at the carrier synchronisation application of a PLL
- Let the input to the PLL be $A\sin(\omega_c t + \theta_i)$, and the output of the VCO be $B\cos(\omega_c t + \theta_o)$
- the output of the multiplier is

$$
\begin{aligned}
x(t) &= AB\sin(\omega_c t + \theta_i)\cos(\omega_c t + \theta_o) \\
&= \frac{AB}{2}[\sin(\theta_i - \theta_o) + \sin(2\omega_c t + \theta_i + \theta_o)]
\end{aligned}
$$

the loop filter will suppress the double frequency component, leaving

$$
v(t) = \frac{AB}{2}\sin(\theta_e) \quad \theta_e = \theta_i - \theta_o
$$

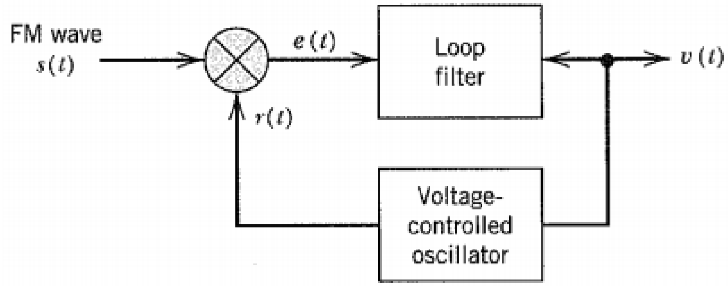
where θ_e is the phase error

PLL - carrier synchronisation

- suppose the PLL is *locked*, that is the frequencies of the input and output sinusiods are equal
- that is θ_i, θ_o and θ_e are constant
- suppose that the input sinusoid frequecy increases to $\omega_c + k$, that is the incoming signal is $A\sin(\omega_c t + \hat{\theta}_i)$, where $\hat{\theta}_i = kt + \theta_i$
- this will increase the error signal $v(t)$, which in turn will increase the frequency of the VCO output to match the increase in the input frequency
- thus the PLL tracks the input sinusoid - the signals are *phase coherent*, or in *phase lock*
- the PLL can track the incoming frequency over a finite range of frequency shift - called the *hold-in* or *lock* range.
- if initially the input and output frequencies are not close enough (not in the *pull-in* range), the PLL may not acquire lock.

PLL - FM demodulation

- now we will examine the application of PLL to frequency demodulation



- Let the incoming FM signal be $s(t) = A_c \sin[2\pi f_c t + \phi_1(t)]$, where the $\phi_1(t) = 2\pi k_f \int_0^t m(\tau) d\tau$
- Let the VCO output be, for some amplitude A_v,

$$r(t) = A_v \cos[2\pi f_c t + \phi_2(t)]$$

with

$$\phi_2(t) = 2\pi k_v \int_0^t v(\tau) d\tau$$

where k_v is the *sensitivity factor* of the VCO

- the function of the feedback loop is to force $\phi_2(t)$ to equal $\phi_1(t)$

PLL - nonlinear model

- The output of the multiplier produces two components
  1. $k_m A_c A_v \sin[4\pi f_c t + \phi_1(t) + \phi_2(t)]$ (suppressed by LPF)
  2. $k_m A_c A_v \sin[\phi_1(t) - \phi_2(t)]$

  where k_m is a multiplier gain
- Disregarding the double frequency component, we have that the error signal $e(t)$

$$e(t) = k_m A_c A_v \sin[\phi_e(t)]$$

where the *phase error* $\phi_e(t)$ is given by

$$\begin{aligned} \phi_e(t) &= \phi_1(t) - \phi_2(t) \\ &= \phi_1(t) - 2\pi k_v \int_0^t v(\tau) d\tau \end{aligned}$$

PLL - model 2

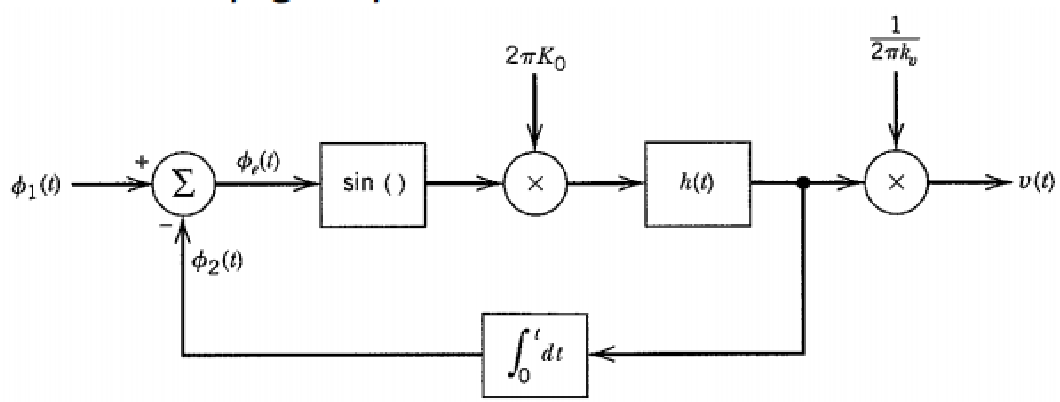
- denoting the loop filter impulse response by $h(t)$, its output is given by

$$v(t) = \int_{-\infty}^{\infty} e(\tau)h(t-\tau)d\tau$$

- The PLL is thus described by the following differential equation

$$\frac{d\phi_e(t)}{dt} = \frac{d\phi_1(t)}{dt} - 2\pi K_0 \int_{-\infty}^{\infty} \sin(\phi_e(\tau))h(t-\tau)d\tau$$

where we defined the *loop-gain parameter* $K_0 = k_m A_c A_v$



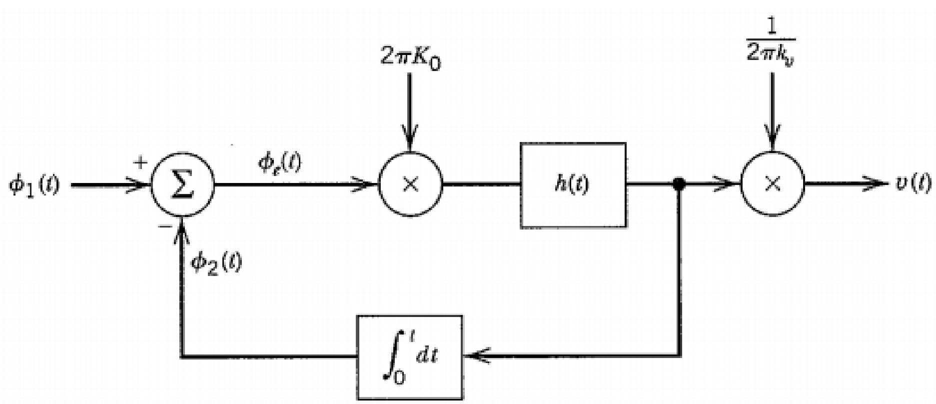
PLL - Linear Model

- when $\phi_e(t) = 0$ we say the PLL is in *phase-lock*
- when $\phi_e(t) << 1$, the PLL is *near-phase-lock*, and we can approximate

$$\sin[\phi_e(t)] \approx \phi_e(t) \quad e(t) \approx \frac{K_0}{k_v}\phi_e(t)$$

- this allows us to *linearize* the model

$$\frac{d\phi_e(t)}{dt} + 2\pi K_0 \int_{-\infty}^{\infty} \phi_e(\tau)h(t-\tau)d\tau = \frac{d\phi_1(t)}{dt}$$



PLL - Linear Model

- Let us define the Fourier transform pairs $\phi_e(t) \rightleftharpoons \Phi_e(f)$, $\phi_1(t) \rightleftharpoons \Phi_1(f)$, $h(t) \rightleftharpoons H(f)$
- Transforming the linearized loop equation into the frequency domain, we can show that the loop is described by

$$\Phi_e(f) = \frac{1}{1+L(f)}\Phi_1(f)$$

  where $L(f) = K_0\frac{H(f)}{jf}$ is the *open-loop transfer function* of the PLL
- Note that if $L(f) >> 1$, $\Phi_e(f) = 0$, that is phase lock is established (our objective)

PLL - Linear Model

- We also note that

$$\begin{aligned} V(f) &= \frac{K_0}{k_v} H(f)\Phi_e(f) \\ &= \frac{jf}{k_v} L(f)\Phi_e(f) \\ &= \frac{(jf/k_v)L(f)}{1+L(f)}\Phi_1(f) \end{aligned}$$

- Once again, making $|L(f)| >> 1$ we have an approximation

$$V(f) \approx \frac{jf}{k_v}\Phi_1(f)$$

which after inverse FT, results in

$$v(t) \approx \frac{1}{2\pi k_v}\frac{d\phi_1(t)}{dt}$$

- thus, provided $|L(f)| >> 1$, the phase-locked loop acts as a differentiator

PLL - Linear Model

- at last, noting the relation of $\phi_1(t)$ to $m(t)$, we have that the output of the PLL is
$$v(t) \approx \frac{k_f}{k_v} m(t)$$
- The complexity of the PLL is determined by the transfer function of the loop filter, i.e. $H(f)$
- A *first order* PLL is one for which $H(f) = 1$, that is no loop filter
- for higher orders, the order of the PLL is the order of the denominator of the *closed loop transfer function*
- the main limitation of the first order PLL is the *hold-in frequency range*, that is the range of frequencies for which the loop stays phase-locked
- first order PLLs are rarely used in practice