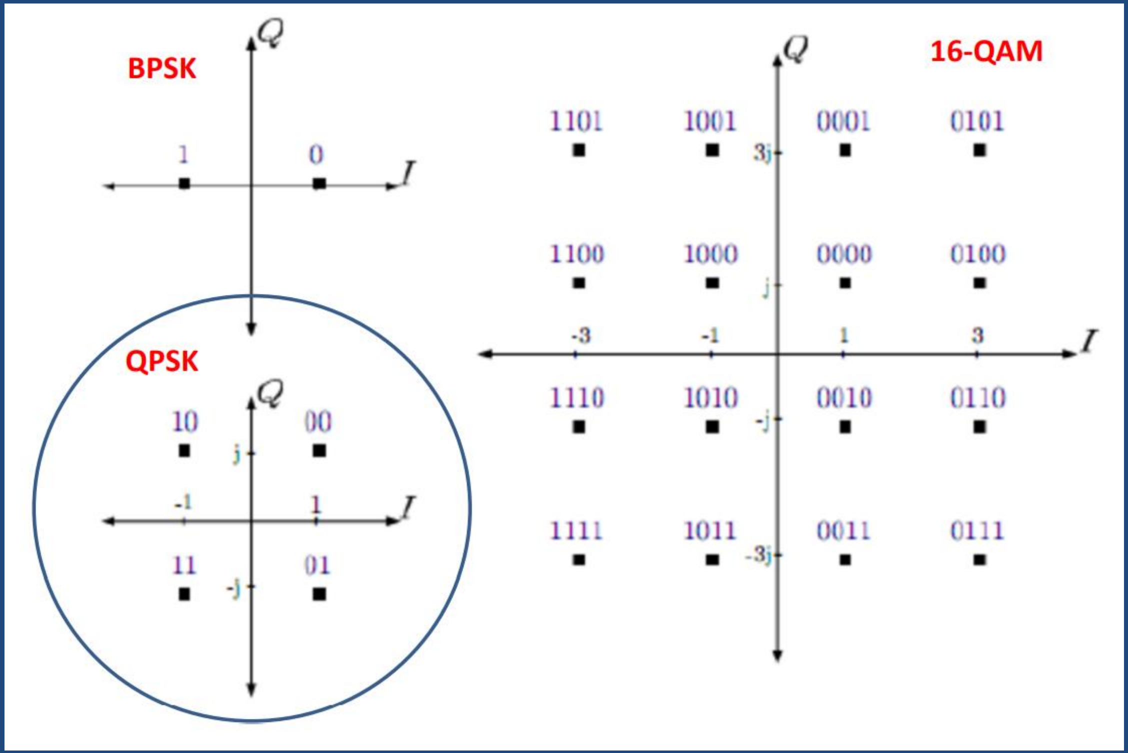
LECTURE 8: BANDPASS TRANSMISSION



ECEN 310 Communication Engineering

VICTORIA UNIVERSITY OF WELLINGTON TE HERENGA WAKA

This lecture

Reading Proakis and Salehi

In the last lecture we studied "baseband" digital signals; that is, the modulating signal $m(t)$ have not been frequency shifted.

However, for wireless and satellite communications we must use higher frequencies to transmit and receive communication signals.

Now we require a modulator and a demodulator – together they form a "**modem**."

There are two basic forms of carrier modulation – they are (1) **amplitude modulation** and (2) **angle modulation** (phase and frequency modulation). We have already studied both under the heading of analog modulation.

Overview

- So far we considered digital signaling in baseband
- The techniques developed, such as MF, ISI and BER analysis, generally apply to bandpass techniques as well
- Today we will look at digital modulation techniques for bandpass systems
- In doing so, we will also look at $M-ary$ communications, that is transmitting multiple bits per symbol
- This will lead to the notion of constallation diagrams, as graphical signal representation

Complex Baseband Representaton of Signal

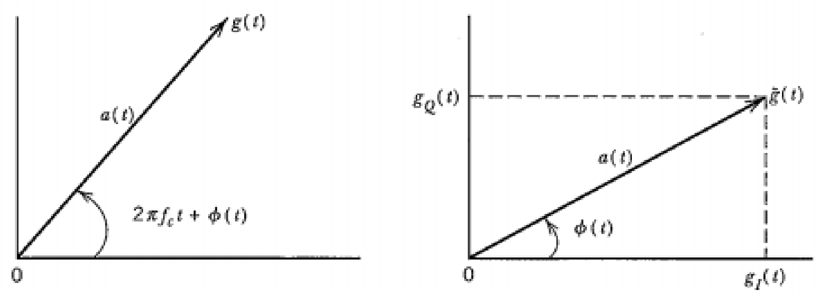
- Recall the complex baseband representation of a bandpass signal
- Consider a signal $g(t)$ modulated by a carrier. We can write

$$g(t) = a(t)\cos(\omega_c t + \phi(t)) = g_I(t)\cos(\omega_c t) - g_Q(t)\sin(\omega_c t)$$

- We have defined $g_I(t) = a(t)\cos\phi(t)$ and $g_Q(t) = a(t)\sin\phi(t)$ as the in-phase and quadrature components of $g(t)$, respectively.
- We can rewrite $g(t)$ as

$$g(t) = \Re\{\tilde{g}(t)e^{j\omega_c t}\}$$

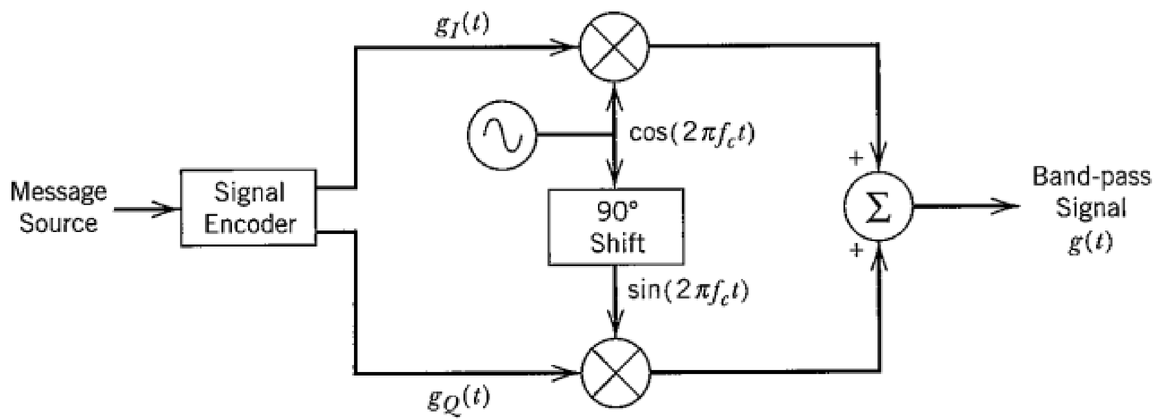
  where we have defined $\tilde{g}(t) = g_I(t) + jg_Q(t)$ as the *complex envelope* of the bandpass signal
- both $g(t)$ and $\tilde{g}(t)$ can be expressed by phasors



- note that the phasor representation of $\tilde{g}(t)$ is obtained by simply suppressing the constant angular rotation $\omega_c t$

Bandpass Transmission Model

- A bandpass signal can be constructed at the transmitter from the low-pass in-phase and quadrature components

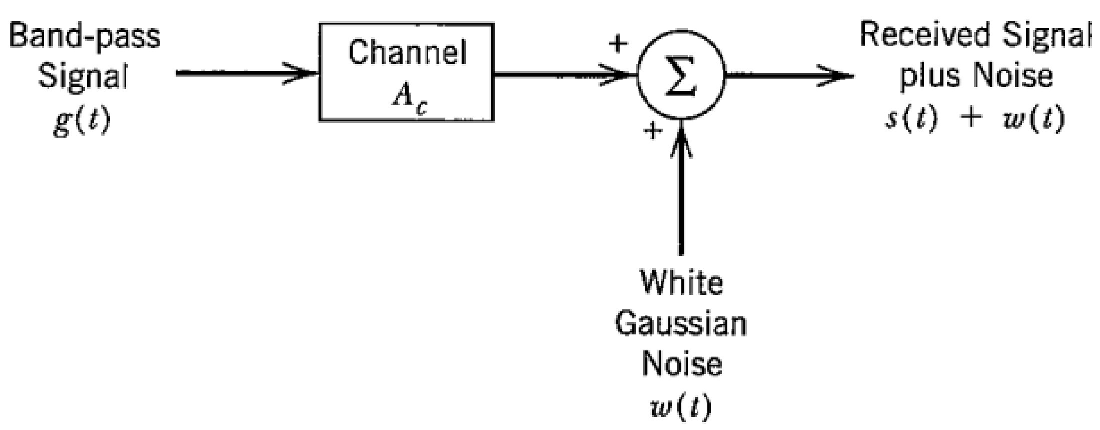


- Signal encoder maps the binary data onto the I and Q components
- Note the two branches are often referred to as *I*- and *Q*- *rails*

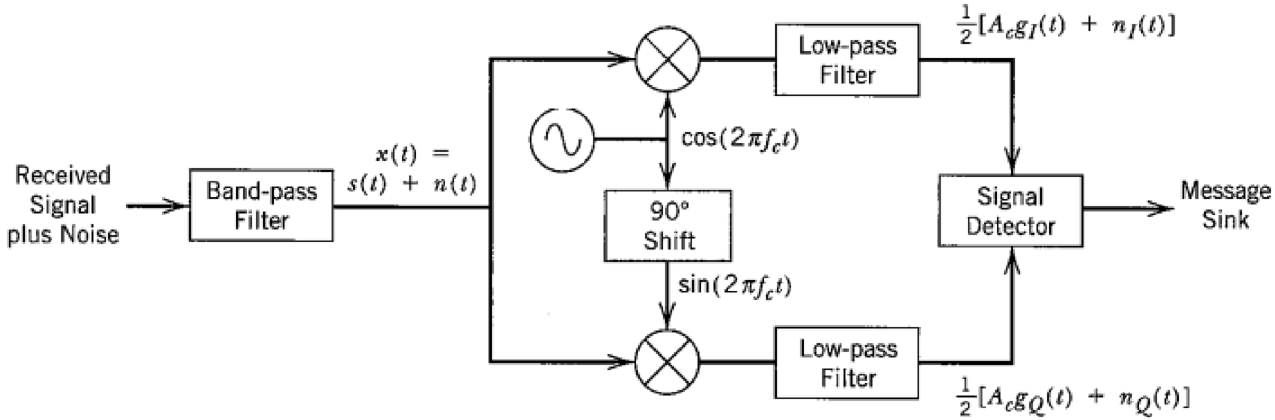
Bandpass Transmission Model

- We will assume that the channel attenuates the signal and perturbs it by AWGN with PSD of $N_0/2$
- For simplicity of the presentation we assume that ISI would be mitigated by appropriate pulse shaping.
- The received signal is given by

$$x(t) = A_c g(t) + w(t)$$



Bandpass Transmission Model



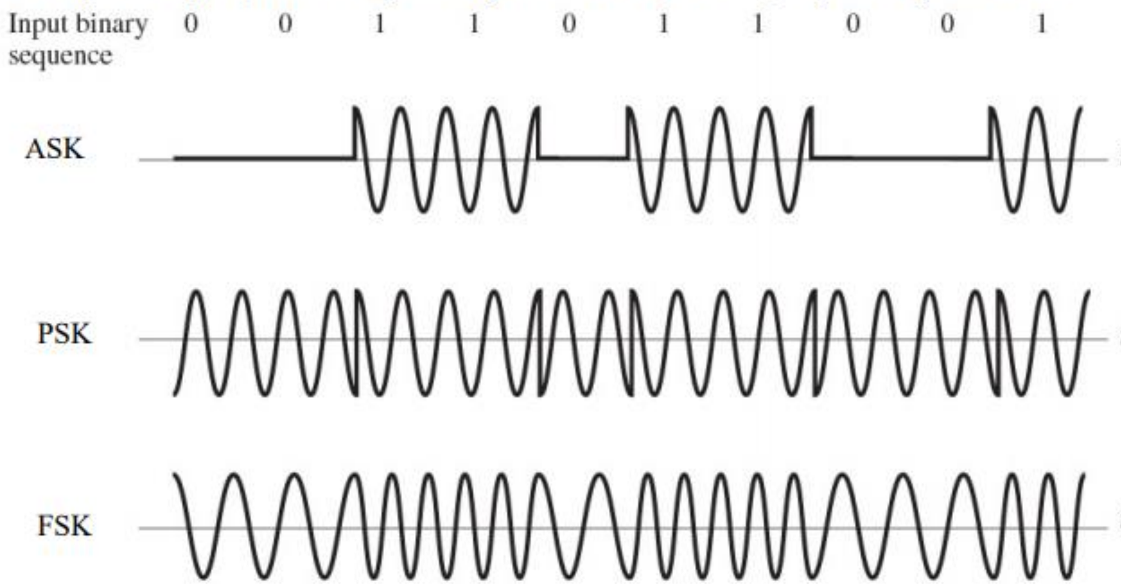
- The receiver includes two coherent detectors on I and Q branches.
- Mixer outputs are filtered to remove double frequency components
- The resulting in-phase and quadrature components are given by

$$\frac{1}{2}[A_c g_I(t) + n_I(t)] \quad \text{and} \quad \frac{1}{2}[A_c g_Q(t) + n_Q(t)]$$

  where $n_I(t)$ and $n_Q(t)$ are lowpass noise random processes
- The Signal Detector observes the complex signal $g_I(t) + n_I(t) + j(g_Q(t) + n_Q(t))$ and estimates the transmitted signal
- The actual structure of the signal detector depends on the type of bandpass signalling scheme employed...

Bandpass Signalling

- The bandpass modulation process involves switching (*keying*) the amplitude (ASK), phase (PSK), or frequency (FSK) of the carrier



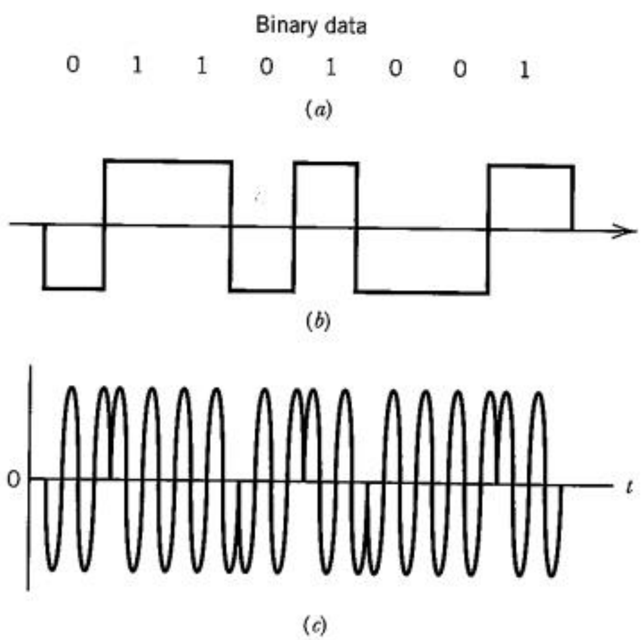
- ASK, FSK and PSK are special cases of AM, FM and PM (where message is a rectangular pulse sequence)
- The shown baseband pulses are rectangular, however in practice pulse shaping would be performed to eliminate ISI and minimize bandwidth

Binary Phase-Shift Keying

- For *binary* Phase Shift Keying (BPSK) the received signal component is given by

$$
\begin{aligned}
s_1(t) &= A_c \cos \omega_c t \quad \text{binary symbol 1} \\
s_0(t) &= -A_c \cos \omega_c t \quad \text{binary symbol 0} \\
&= A_c \cos(\omega_c t + \pi)
\end{aligned}
$$

- In the case of BPSK, $g_Q(t) = 0$, while $g_I(t)$ corresponds to the output of a *bipolar line code*

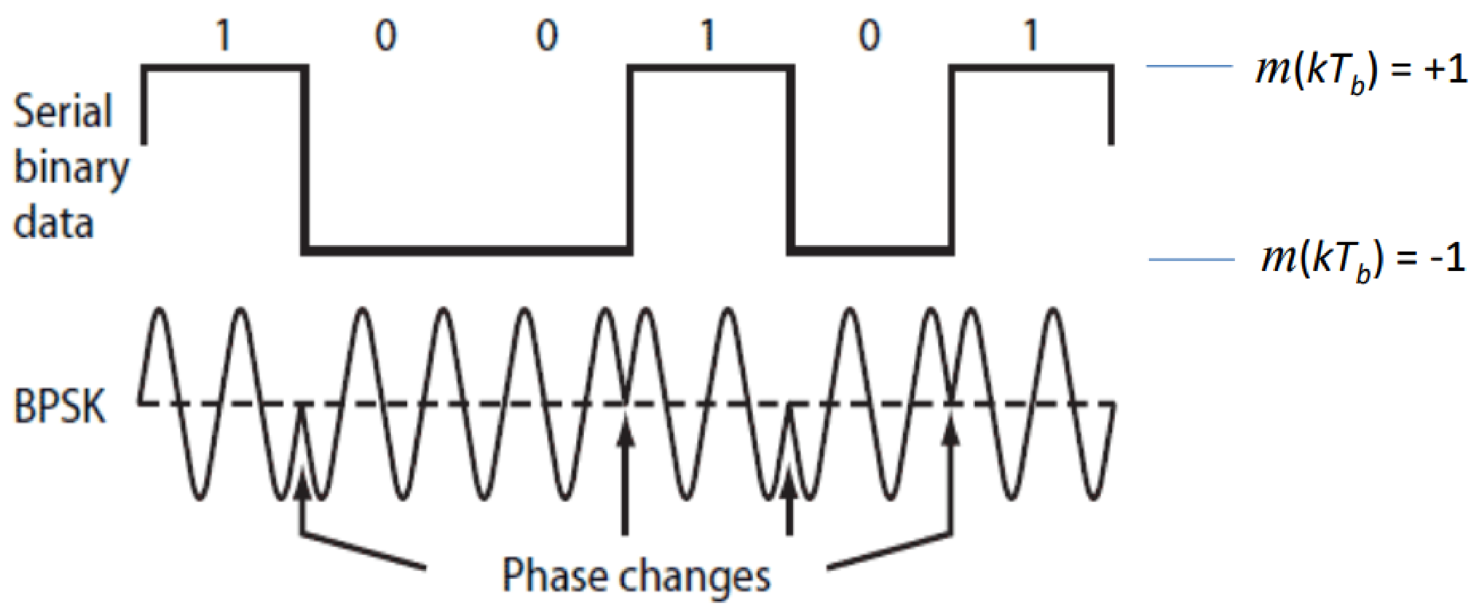


BPSK in picture

Angle modulation gives rise to both phase modulation and frequency modulation.

Starting with phase modulation; this is generally known as "phase shift keying."

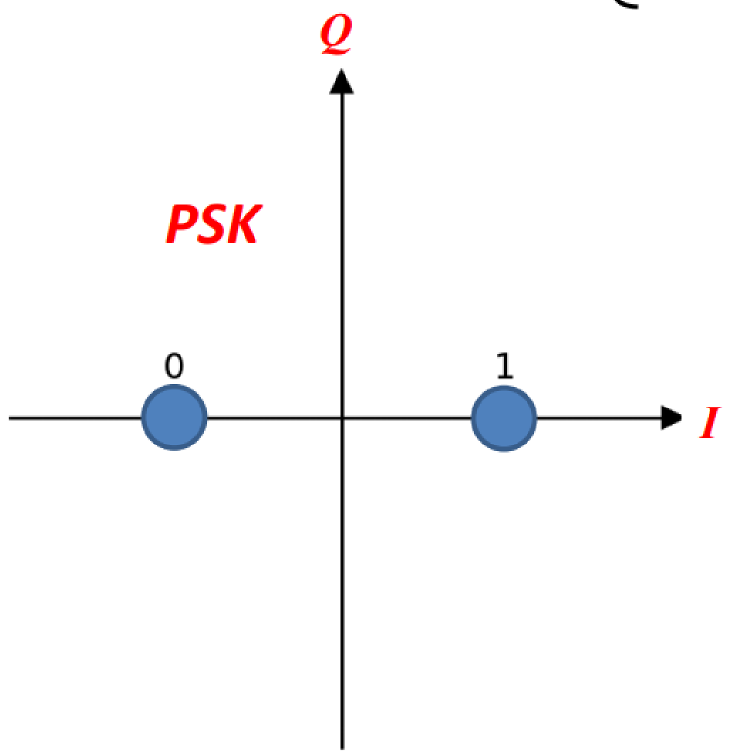
Example:



In binary phase shift keying, note how a binary 0 is 0° while a binary 1 is 180°. The phase changes when the binary state switches so the signal is coherent.

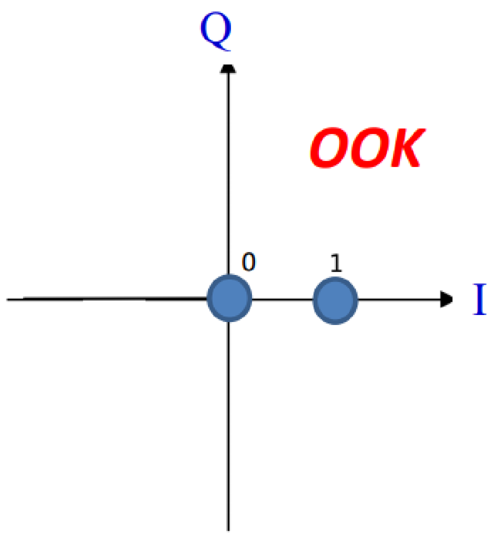
Constellation of PSK

$$A_C \cdot m(t)\cos(\omega_C t) \;=\; \begin{cases} A_C \cdot \cos(\omega_C t) & \text{for} \quad m(kT_b) = +1 \\ A_C \cdot \cos(\omega_C t + \pi) & \text{for} \quad m(kT_b) = -1 \end{cases}$$



We can also express as ***I*** and ***Q*** components.

A special case: on-off keying (*OOK*)



PSK in mathematics

For *PSK* we can write,

$$\varphi_{PSK} = A_C \cos\left(\omega_C t + \theta_k\right) \quad \text{for} \quad kT_b \le t < kT_b + T_b$$

$$\varphi_{PSK} = A_C \cos(\theta_k)\cos\left(\omega_C t\right) - A_C \sin(\theta_k)\sin\left(\omega_C t\right)$$

Therefore,

$$\varphi_{PSK} = I_k \cos\left(\omega_C t\right) + Q_k \sin(\omega_C t) \quad \text{for} \quad kT_b \le t < kT_b + T_b$$

This is in polar form (I and Q)

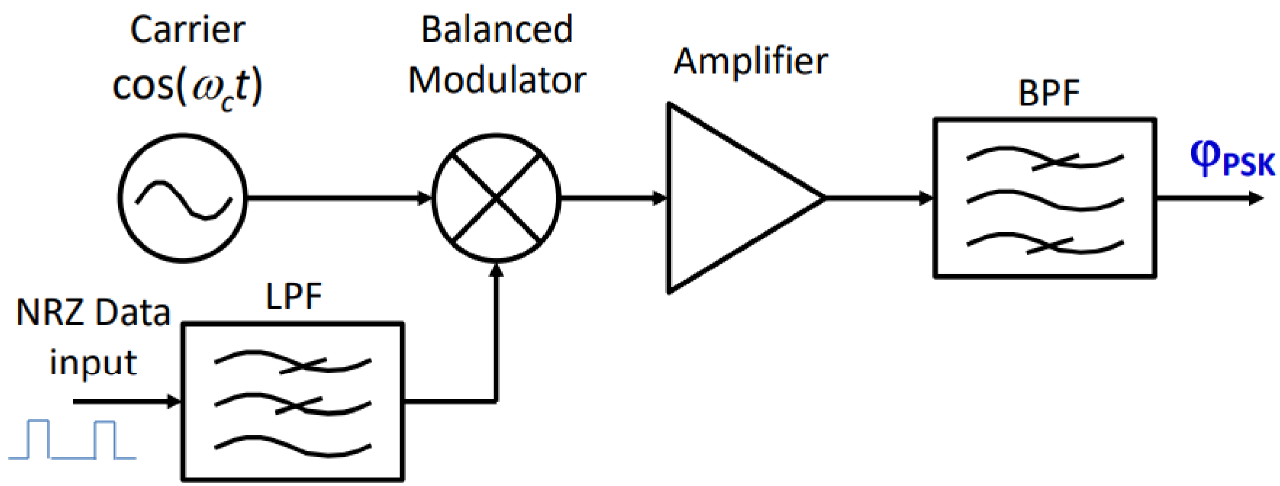
For binary *PSK* we have θ_k = 0 or π radians – there is no Q component.

This is *2-QAM*, but we don't use this terminology for binary *PSK*.

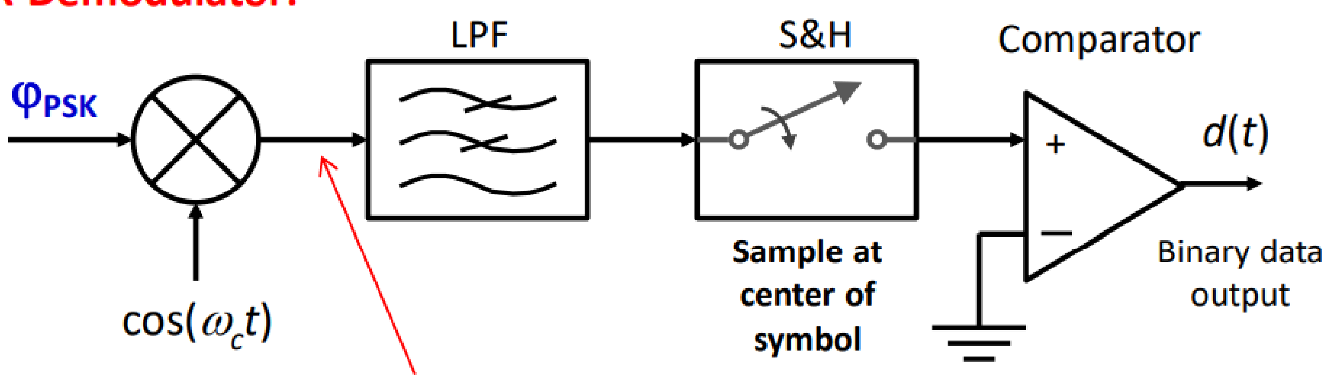
Note: **Quadrature amplitude modulation** (*QAM*) is a mixture of both amplitude modulation and phase modulation.

BPSK transmission

BPSK Modulator:



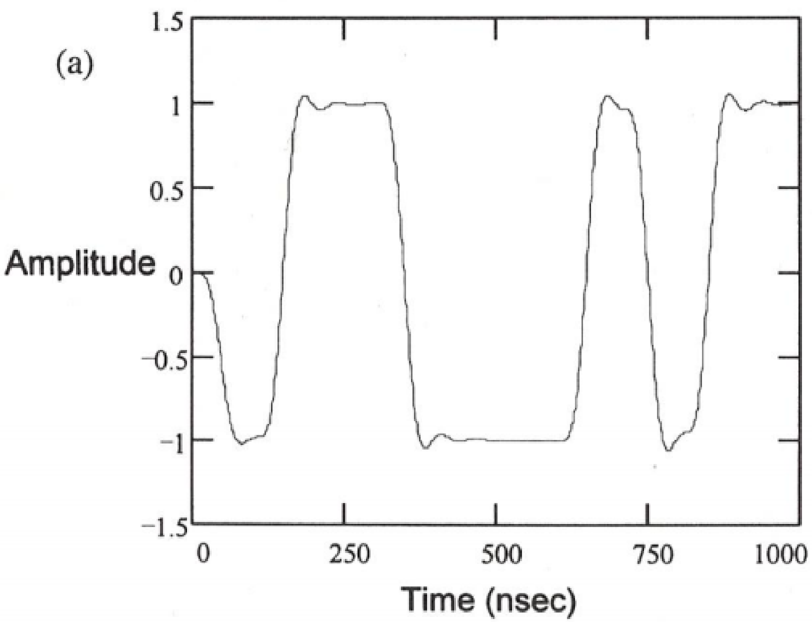
BPSK Demodulator:



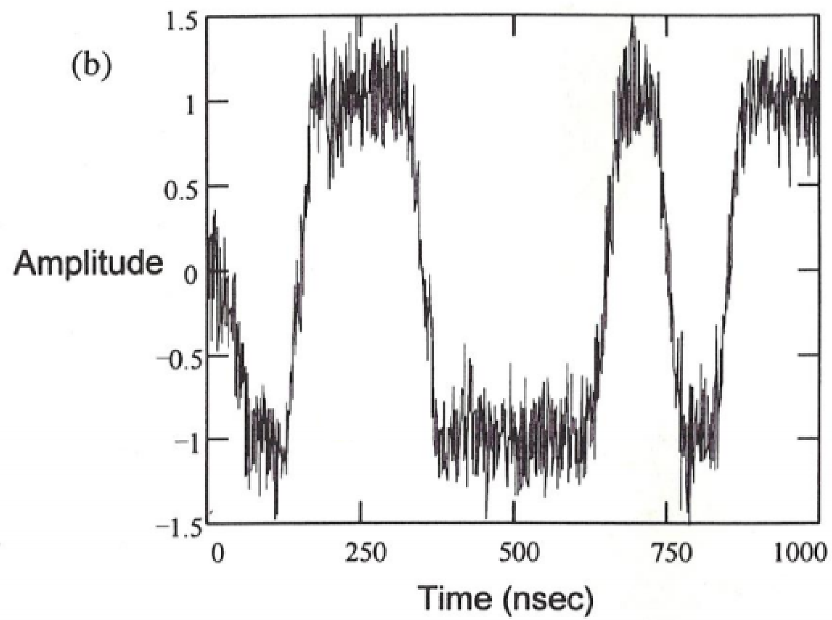
$$r(t) \sim B\cos[2\omega_C t + \theta(t)] + B\cos[\theta(t)]$$

BPSK at the receiver

Without noise

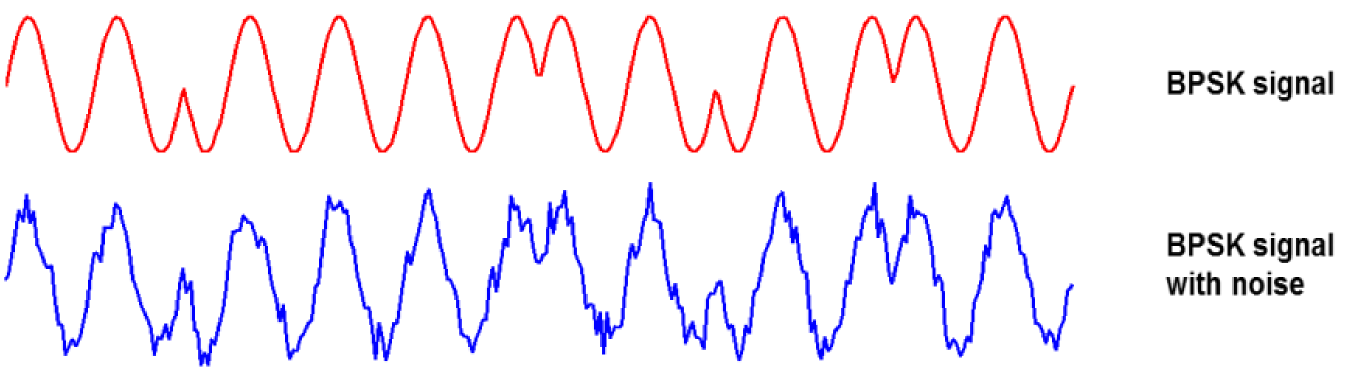


With noise

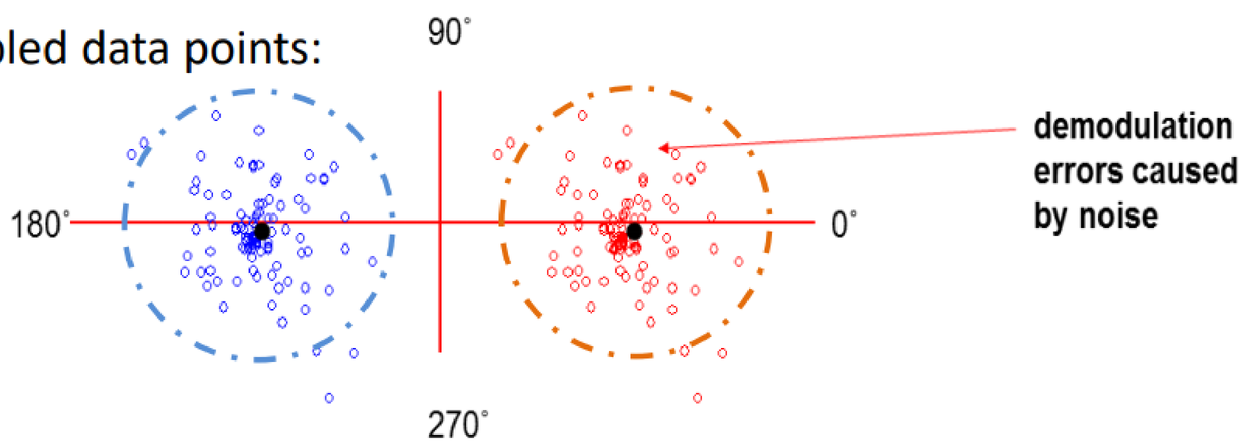


After Lawrence Burns, "Digital Modulation and Demodulation," Chapter 4 in ***RF and Microwave Circuit Design for Wireless Communications***, edited by Lawrence E. Larson, Artech House Publishers, 1996. Pages 99 to 233. Lawrence Burns was an engineer at 3COM.

BPSK waveform and noise



Sampled data points:



Constellation Diagram

Binary Amplitude-Shift Keying

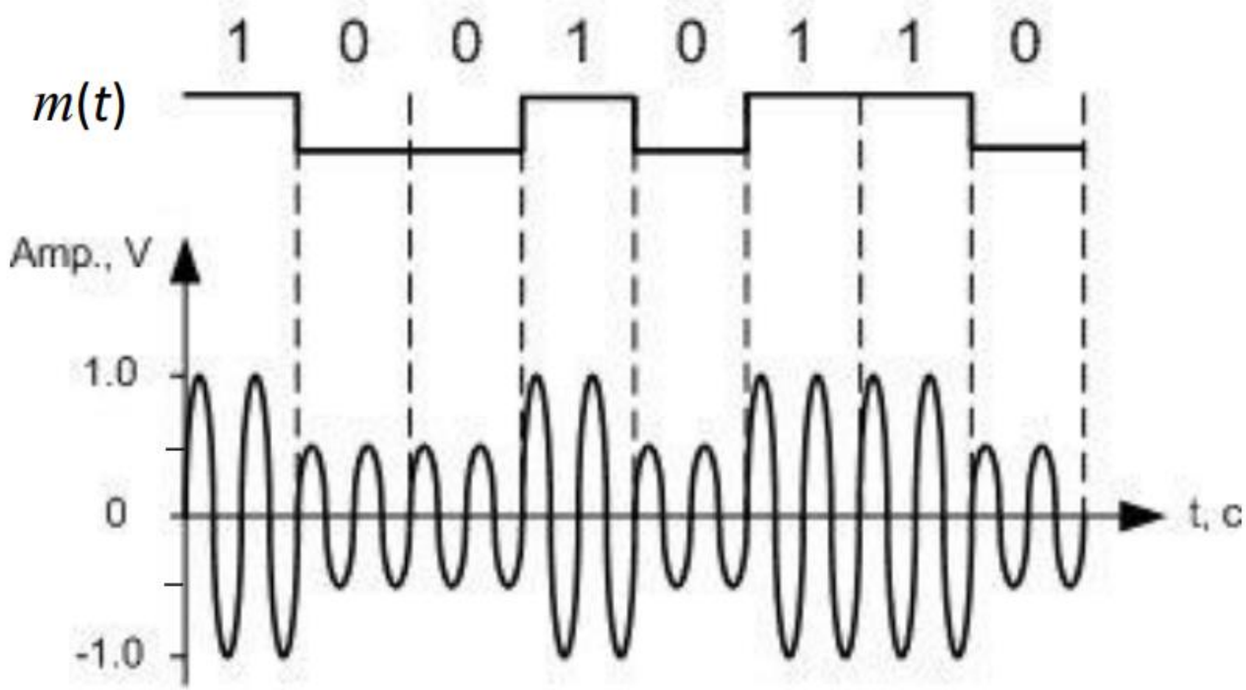
- Binary Amplitude-Shift Keying (BASK) is similar to BPSK, except that a binary 0 is represented by switching off the signal

$$s_1(t) = A_c \cos \omega_c t \quad \text{binary symbol 1}$$
$$s_0(t) = 0$$

- In the case of APSK, $g_Q(t) = 0$, while $g_I(t)$ corresponds to the output of a *unipolar* line code

BASK in picture

This is **binary amplitude shift keying (*BASK*).**



Amplitude shift keying (ASK)

$$\varphi_{ASK} = m(t)\cos\left(\omega_C t\right)$$

A more realistic ASK picture

This is the more realistic case for *ASK* communication systems.
In fact, all waveforms are softened by bandwidth limitations of transceiver equipment and the channel itself.

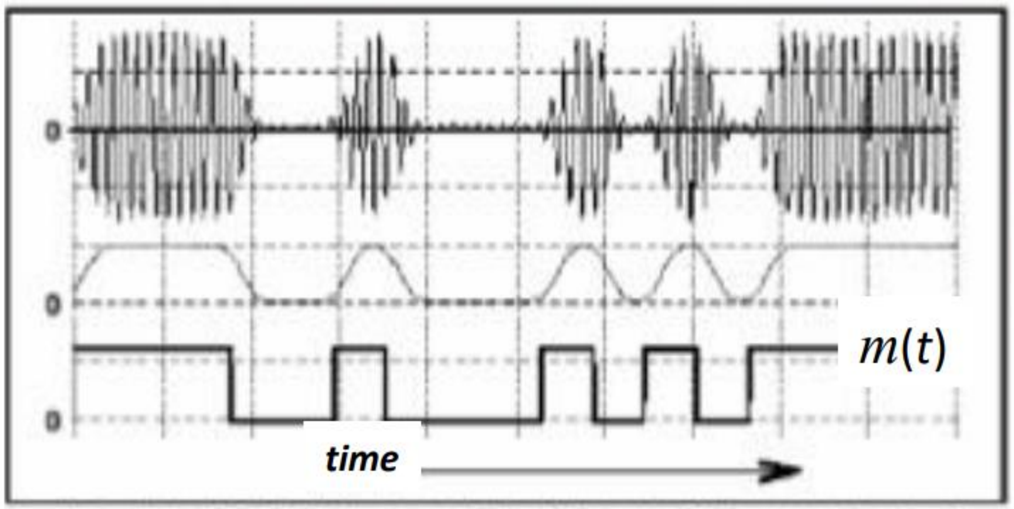


Figure 3: original TTL message (lower), bandlimited message (center), and ASK (above)

Notice the similarity between *ASK* and analog AM because the amplitude of the modulated signal is proportional to $m(t)$.

Binary Frequency-Shift Keying

- For *binary* Frequency-Shift Keying (BFSK) the received signal component is given by

$$
\begin{aligned}
s_1(t) &= A_c \cos \omega_1 t \quad \text{binary symbol 1} \\
s_0(t) &= -A_c \cos \omega_0 t \quad \text{binary symbol 0}
\end{aligned}
$$

- In the case of BFSK $g_Q(t)$ and $g_I(t)$ cannot be expressed as a simple line code. Instead, BFSK is defined by

$$
\begin{aligned}
g_I(t) + j g_Q(t) &= A_c e^{-j2\pi\Delta f t} \quad \text{binary symbol 1} \\
g_I(t) + j g_Q(t) &= A_c e^{+j2\pi\Delta f t} \quad \text{binary symbol 0}
\end{aligned}
$$

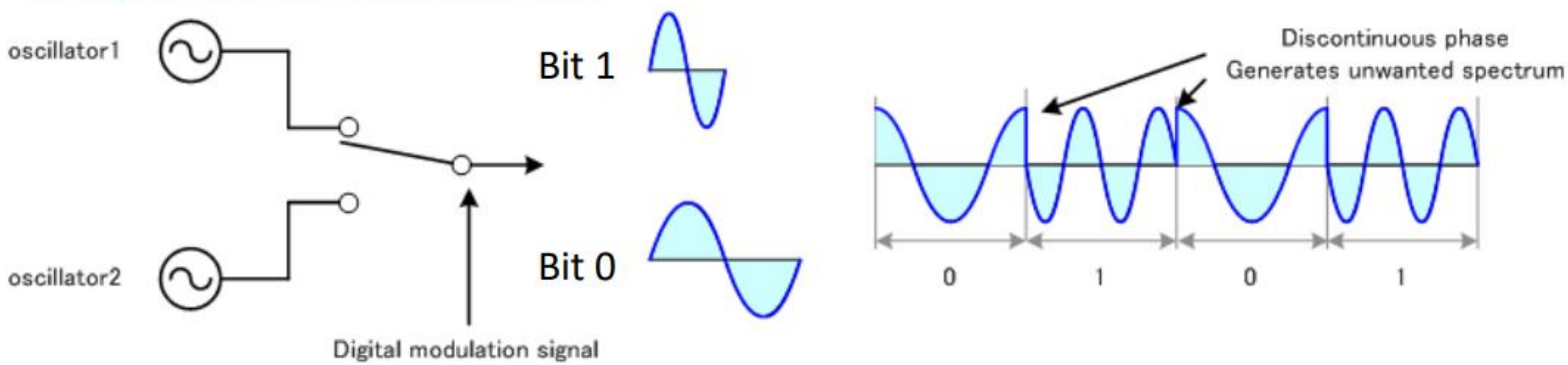
  where we have defined $\Delta f = (f_1 - f_0)/2$.

- In doing so, we expressed

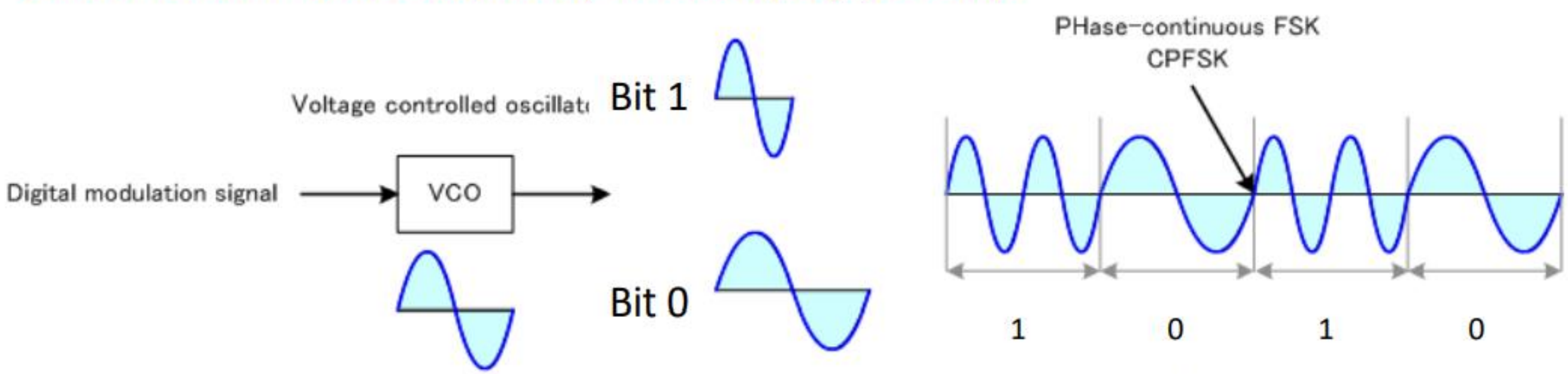
$$
\begin{aligned}
s_1(t) &= A_c \cos[2\pi(f_c + \Delta f)t] = \Re\{A_c e^{j2\pi(f_c+\Delta f)}\} \\
s_0(t) &= A_c \cos[2\pi(f_c - \Delta f)t] = \Re\{A_c e^{j2\pi(f_c-\Delta f)}\}
\end{aligned}
$$

FSK in picture

Frequency Shift Keying (FSK):



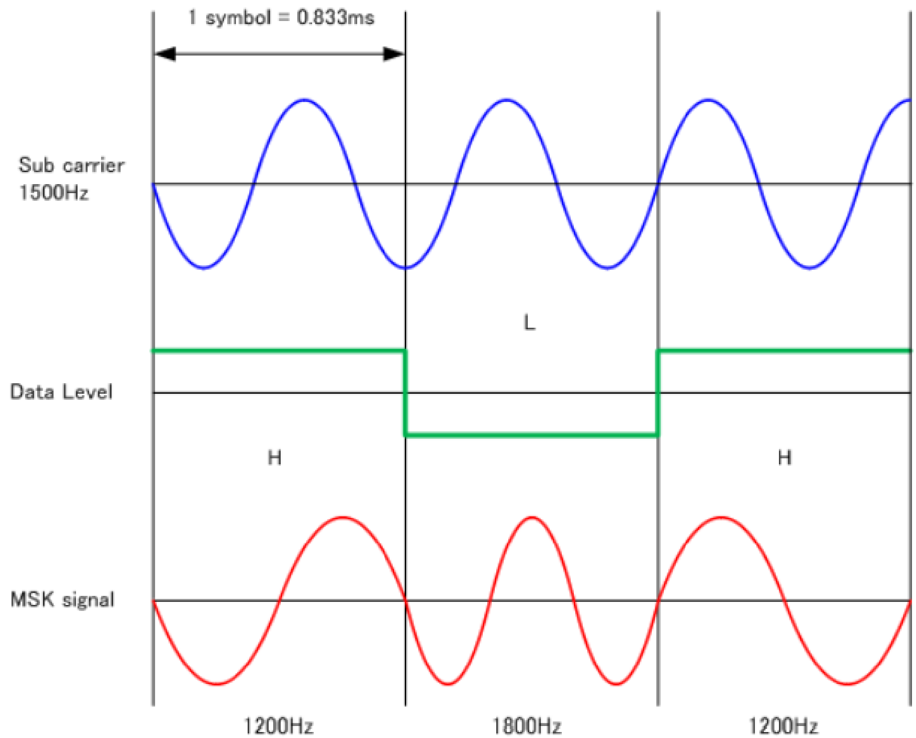
Continuous-Phase Frequency Shift Keying (CPFSK):



aka *MSK*

MSK… in picture

Minimum shift keying (*MSK*) is a form frequency modulation based on a system called continuous-phase frequency-shift keying (*CPFSK*).
MSK advantages are (1) better spectral efficiency as compared to other modes, and (2) it allows power amplifiers to operate in saturation enabling higher levels of efficiency.



MSK… in mathematics

For Bit 0 (frequency f_0) we write,

$$s_0(t)=\sqrt{\frac{2E_b}{T_b}}\cos\left(2\pi f_0 t\right) \quad \text{over interval} \quad 0 \le t < T_b$$

For Bit 1 (frequency f_1) we write,

$$s_1(t)=\sqrt{\frac{2E_b}{T_b}}\cos\left(2\pi f_1 t\right) \quad \text{over interval} \quad 0 \le t < T_b$$

For orthogonality we require $\int_0^{T_b} s_1(t)s_2(t)dt=0$

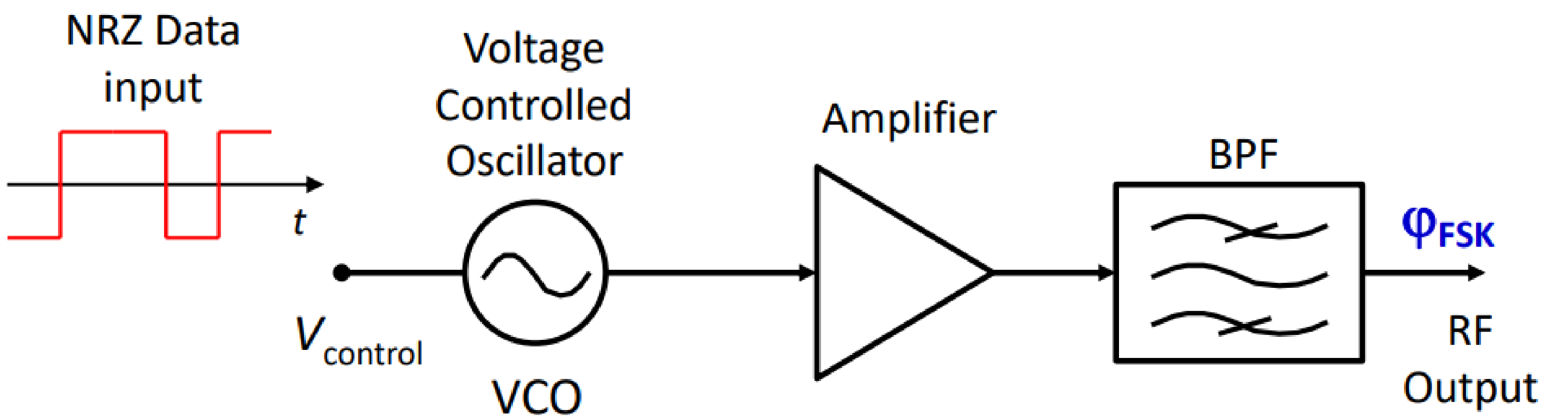
Define the frequency separation: $\Delta f = f_1 - f_2 = \dfrac{1}{2T_b}$

is the minimum separation to ensure orthogonality to hold.

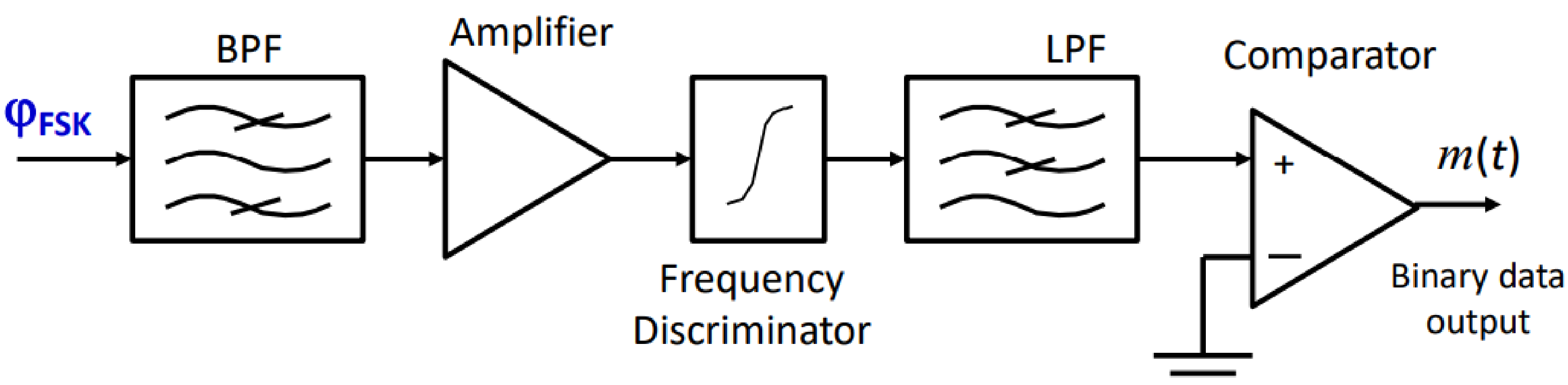
The carrier frequency f_C is defined to be $f_C = \dfrac{f_0 + f_1}{2}$

MSK transmission

MSK Modulator:



MSK Demodulator:

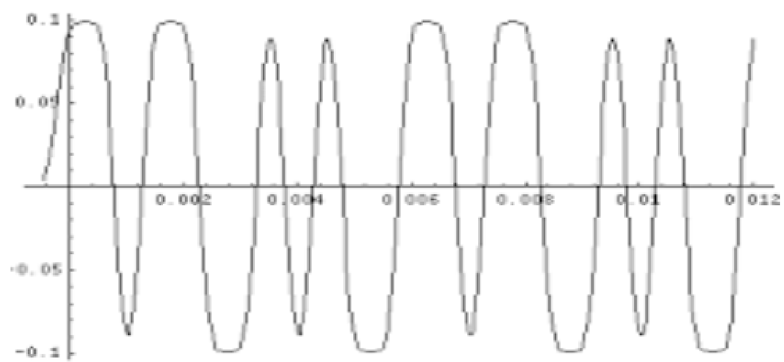


Gaussian MSK

Gaussian Minimum Shift Keying (*GMSK*) modulation is a modified version of the **Minimum Shift Keying** (*MSK*) modulation where the phase is further filtered through a **Gaussian filter** to smooth the transitions from one point to the next in the constellation of states.

A Gaussian filter's impulse response is a Gaussian function (or an approximation to it, because a true Gaussian response is physically unrealizable). Gaussian filters have no overshoot in responding to a step function input and minimize the rise and fall times of pulses.

GMSK waveform



GMSK

Advantages:

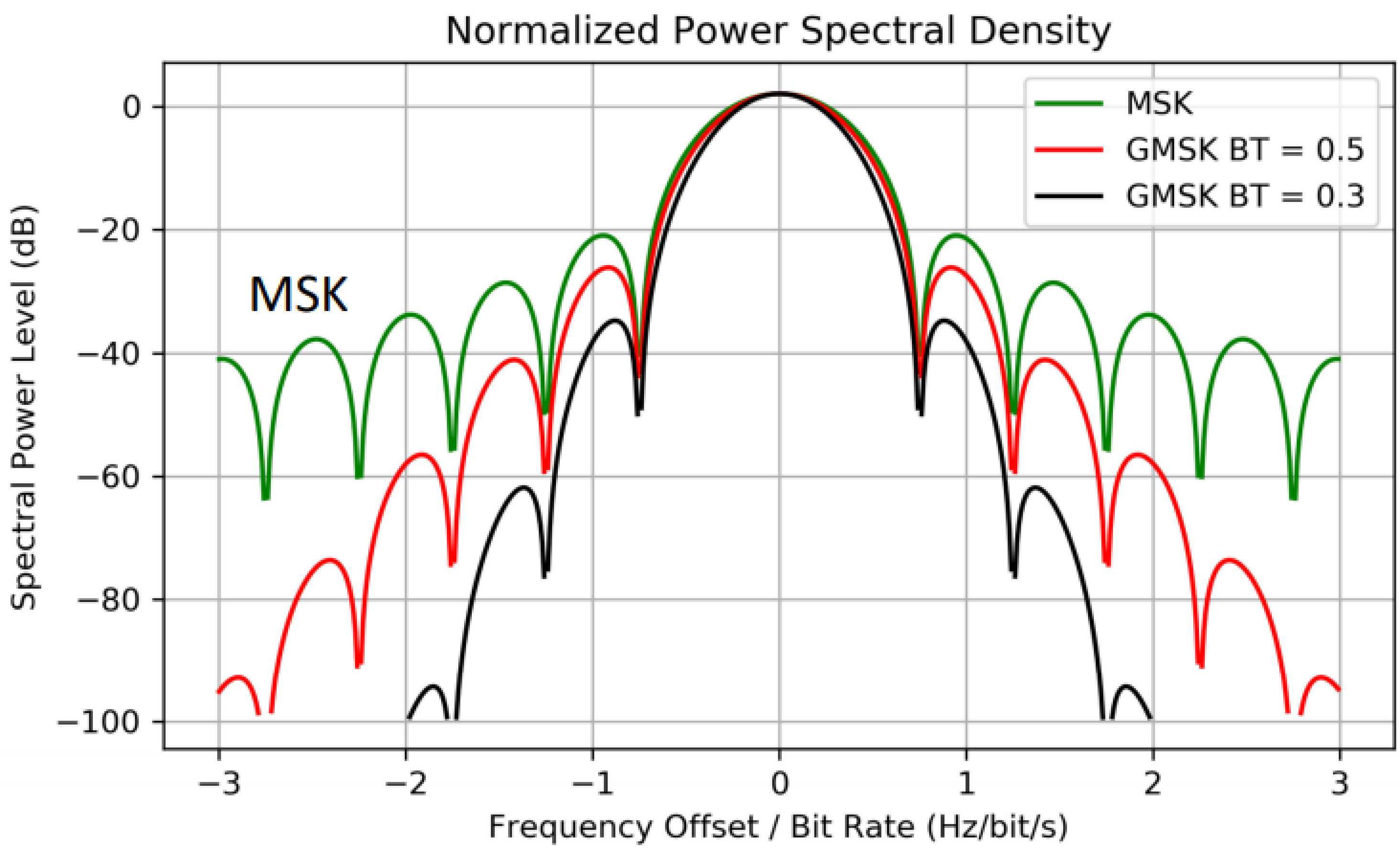
(1) Constant amplitude (better noise resistance)
(2) Spectrally efficient (spectrum falls as 4th power of frequency)
(3) Good Bit Error Rate performance
(4) Self-synchronizing capability

Applications:

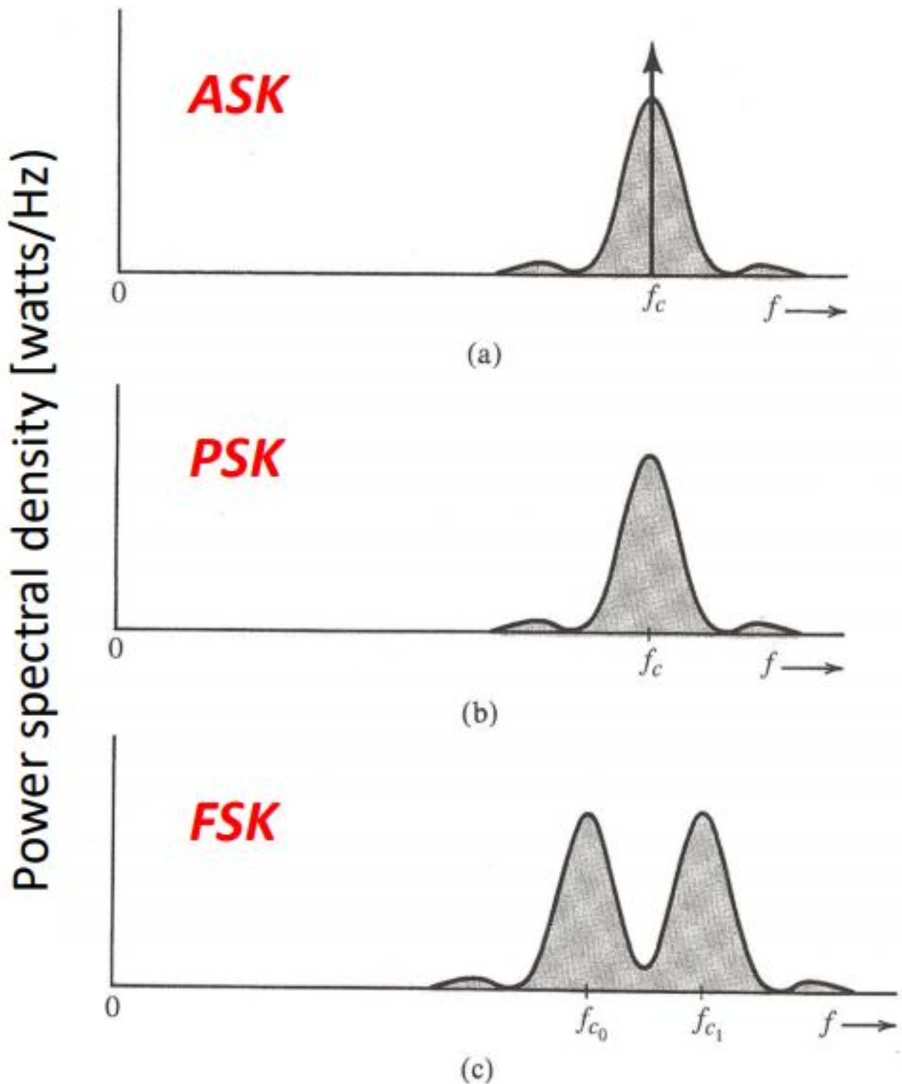
(1) Automatic Identification System in maritime navigation
(2) Bluetooth headsets
(3) Standard GSM cellular (usage is > 40% of cell phones in world)
Uses $BT_b = 0.3$ for Gaussian filter giving 99% of transmission power in bandwidth of 250 kHz. The GSM bandwidth per channel is 200 kHz and GSM transmits at 270 kb/sec (limited by ISI problems). $B = 81.3$ kHz and $T_b = 3.7$ microseconds.

<u>Tradeoff</u>: Smaller BT_b product gives faster spectrum falloff at expense of longer pulse duration leading to increased ISI.

Comparing Spectrums of *MSK* with *GMSK* (BT_b = 0.5 and 0.3)



Comparing PSDs For Binary *ASK*, *PSK* and *FSK*

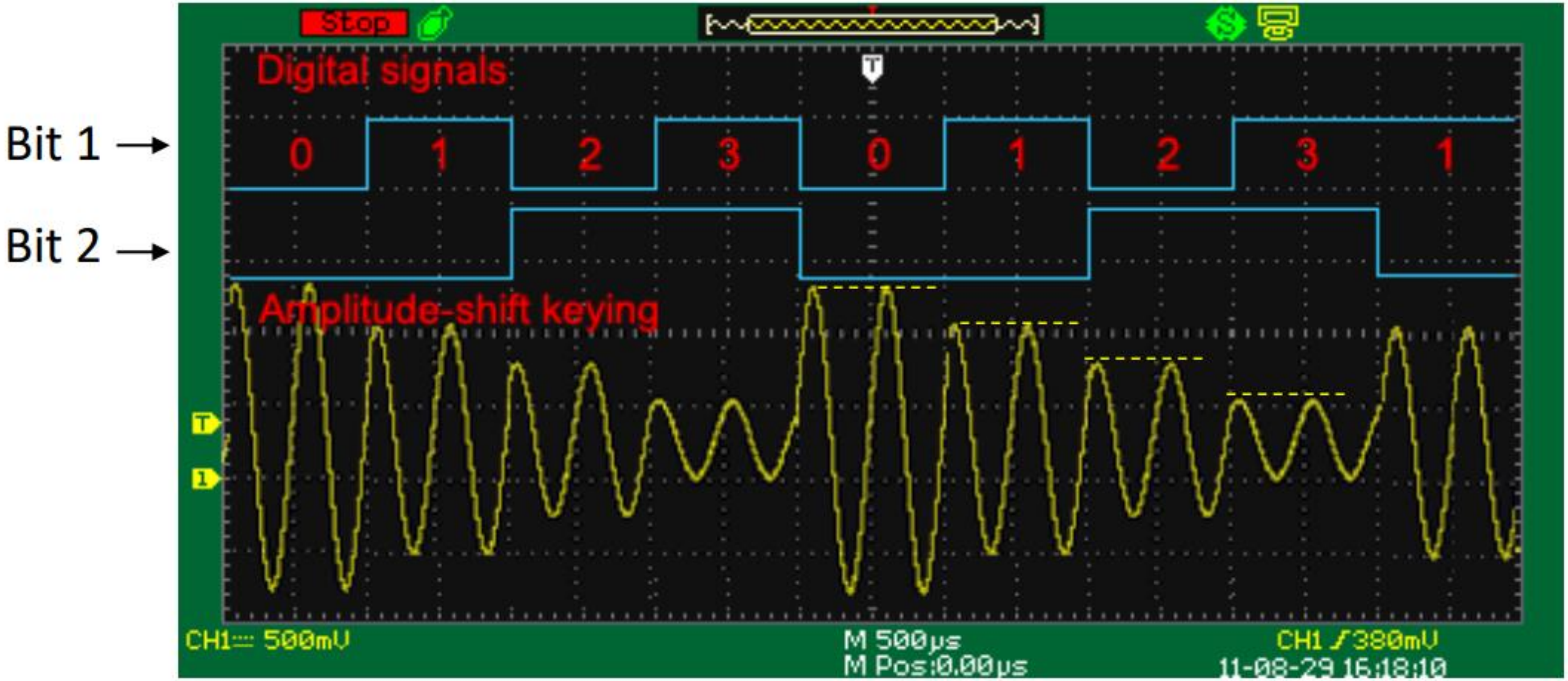


M-ary Transmission

- In *Binary* transmission we send one of two possible signals $s_0(t)$ and $s_1(t)$ (representing one of two data bits) during each *bit* interval T_b.
- In an *M-ary* data transmission system, we can send M possible signals $s_i(t)$ for $0 \leq i \leq M - 1$ during each *signalling* interval T.
- In most cases, the number of possible signals $M = 2^m$, where n is an integer, and the signalling interval is $T = nT_b$.
- Each of $s_i(t)$ is called a data *symbol* (as opposed to data bit)
- The rate of symbol transmission is referred to as the *baud rate* (as opposed to bit rate for binary signalling)
- Each symbol represents a sequence of m bits, which are *Gray encoded* - such that 'adjacent' symbols differ only in one bit.

M-ASK

This is **multilevel amplitude shift keying.**

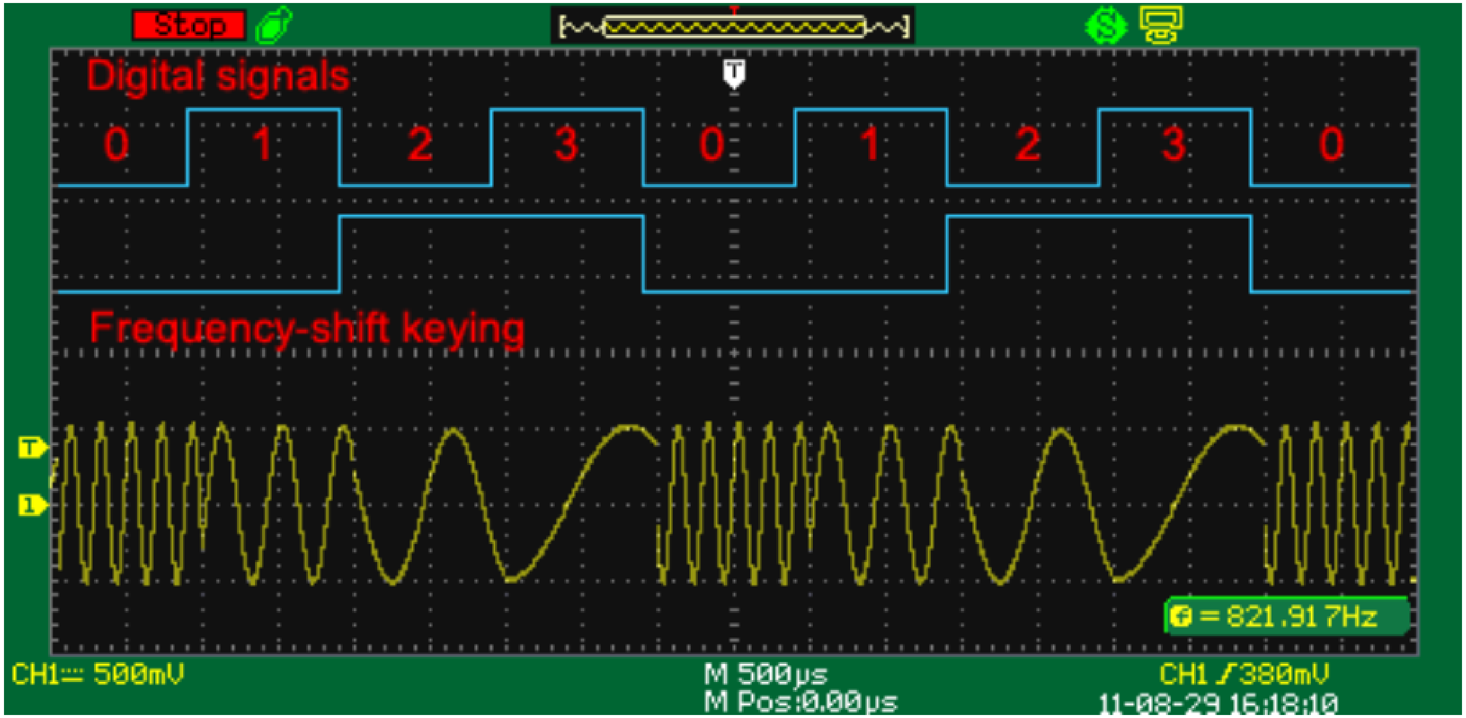


Symbols **00, 01, 10** & **11** translate into four amplitude levels.

M-FSK

This animation shows frequency shift keying of the sinusoidal carrier signal. A two-digit code modulates the carrier signal frequency into four frequencies

Symbol	Binary code	Frequency
"0"	**00**	4 kHz
"1"	**01**	3 kHz
"2"	**10**	2 kHz
"3"	**11**	1 kHz



M-ary Phase-Shift Keying

- in M-ary PSK (MPSK), the available 2π radians are equally divided to represent M data symbols.

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left(\omega_c t + \frac{2\pi}{M} i\right) \quad i = 0, 1, \ldots, M-1 \quad 0 \le t \le T$$

where we scaled the signal by $\sqrt{2}$ and noted that signal energy is $E = A_c^2 T$.

- We can decompose $s_i(t)$ into quadrature and in-phase components, noting that

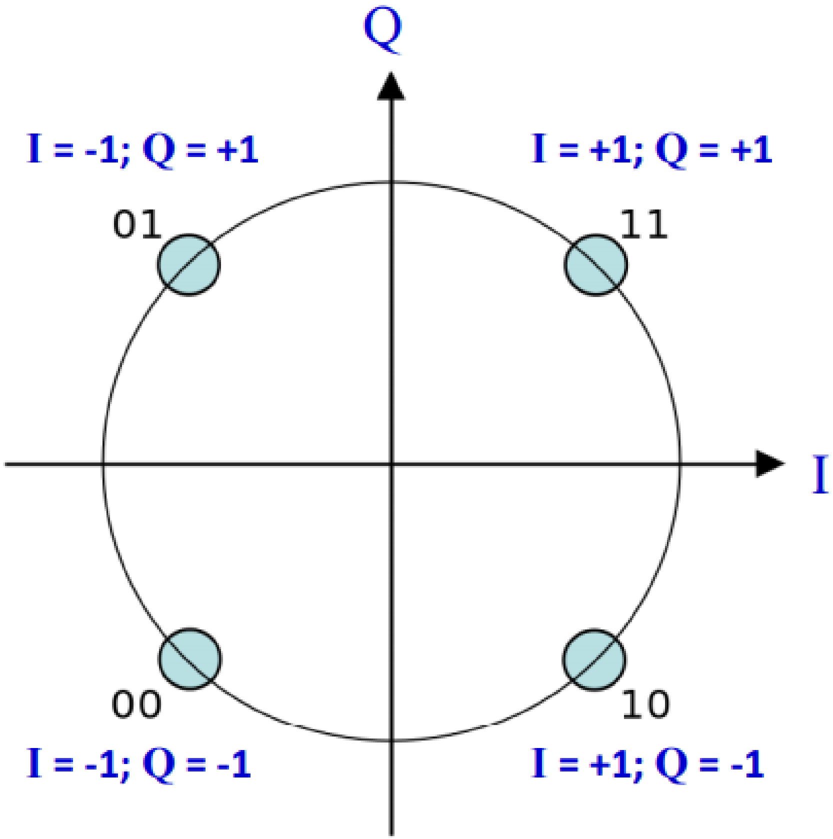
$$s_i(t) = \left[\sqrt{E} \cos\left(\frac{2\pi}{M} i\right)\right] \left[\sqrt{\frac{2}{T}} \cos(\omega_c t)\right] - \left[\sqrt{E} \sin\left(\frac{2\pi}{M} i\right)\right] \left[\sqrt{\frac{2}{T}} \sin(\omega_c t)\right]$$

- We have that $g_Q(t) = \sqrt{E} \sin\left(\frac{2\pi}{M} i\right)$ and $g_I(t) = \sqrt{E} \cos\left(\frac{2\pi}{M} i\right)$

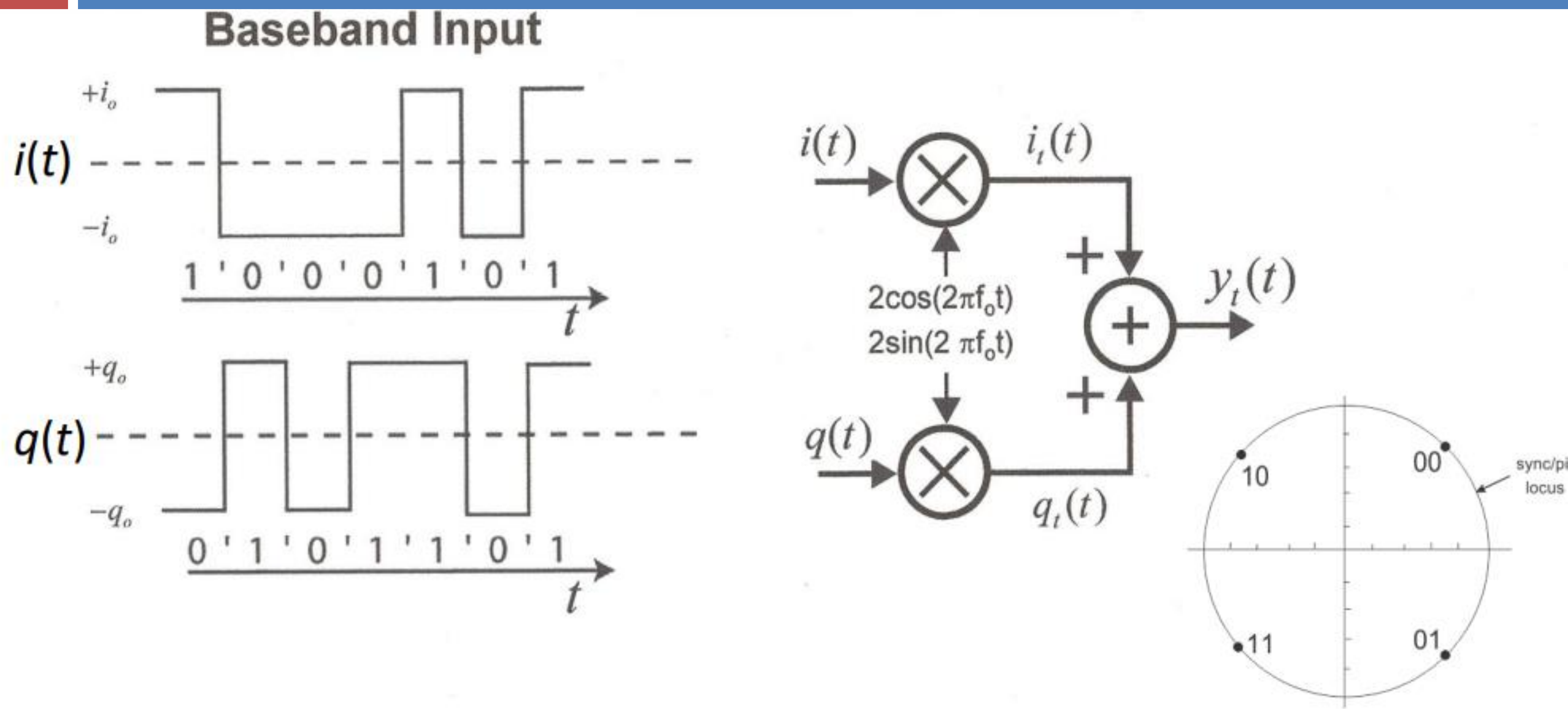
M=4, QPSK

Sometimes this is known as *quadrature-phase PSK*, 4-*PSK*, or ***4-QAM***.
QPSK uses a circle of four points on the constellation diagram, equi-spaced around a circle. With four states, *QPSK* can encode two bits per symbol,

$$\varphi_{QPSK} = I \cdot \cos(\omega_C t) + Q \cdot \sin(\omega_C t)$$



QPSK generation

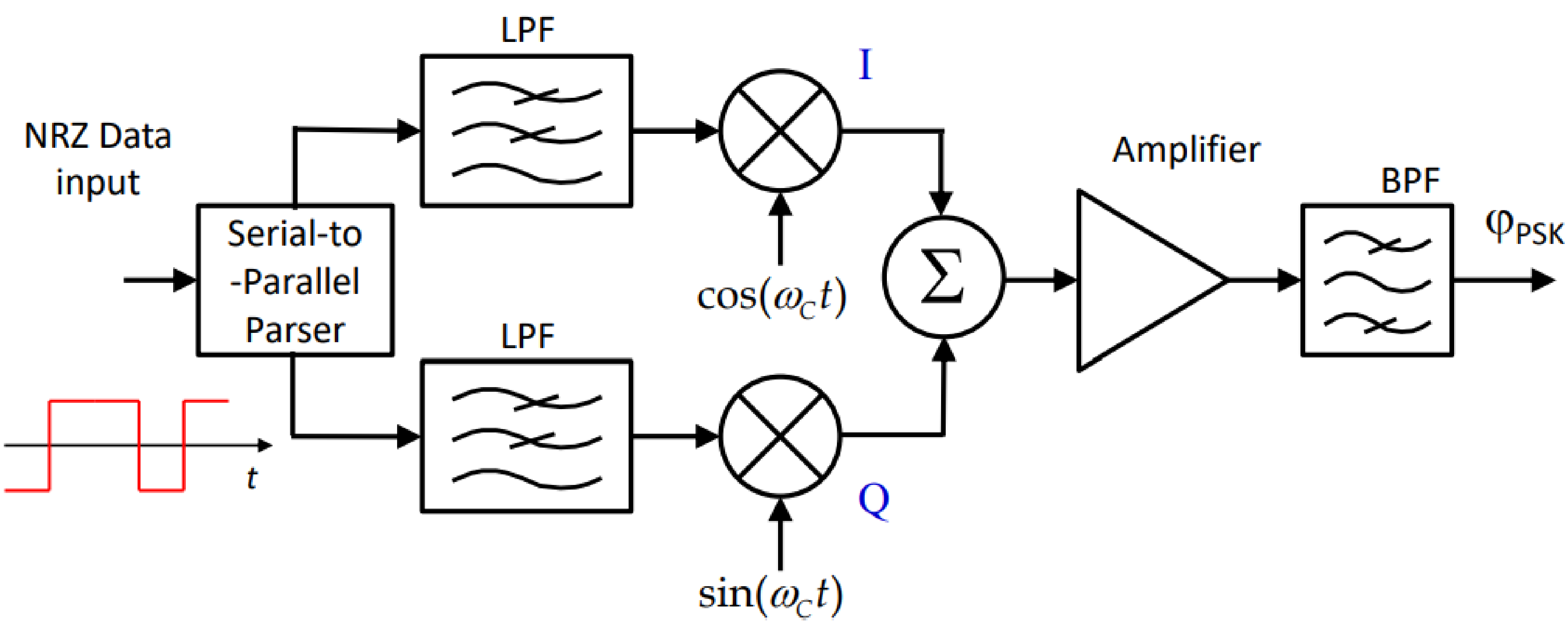


4-QAM Symbol Constellation

I/Q signals take on discrete values at discrete time instants corresponding to digital data

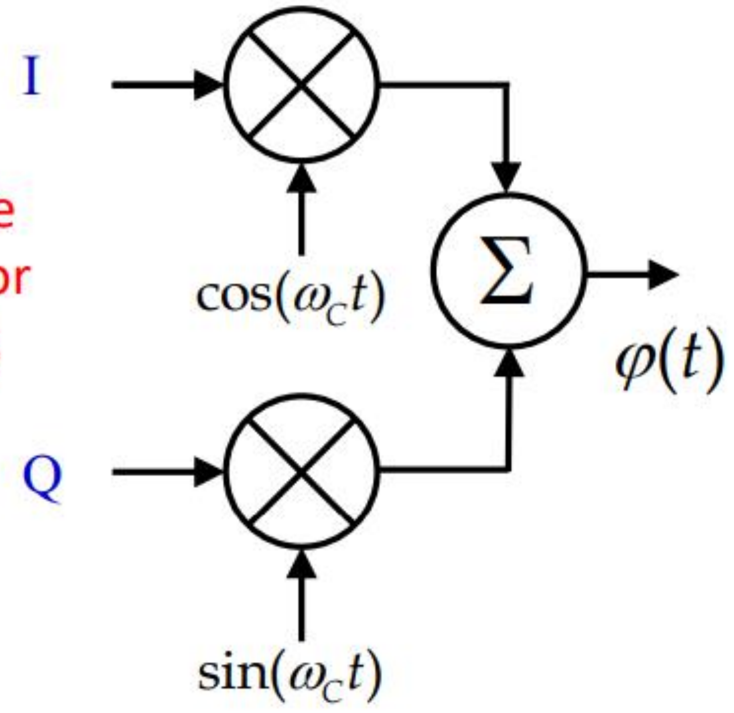
A typical QPSK modulator

QPSK Modulator



… a typical QPSK demodulator

I and Q can be either analog or digital signals



$$\varphi(t) = \sqrt{I^2 + Q^2} \cdot \cos(\omega_C t + \theta(t))$$

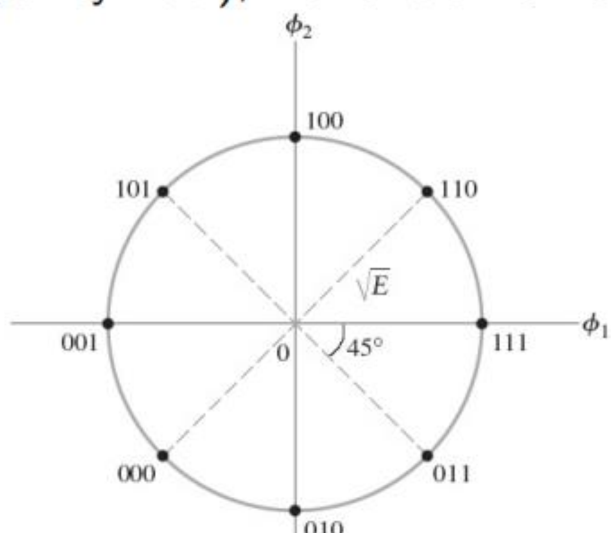
where $\theta(t) = \tan^{-1}\left[\frac{Q(t)}{I(t)}\right]$

Constellation Diagrams

- The in-phase and quadrature components derived lead to a geometrical potrayal of complex baseband signals, referred to as *constellation diagram* or a *signal space diagram*.
- A signal constallation is a diagram of all possible signal points where axis are *orthonormal functions*. In the case of MPSK, these correspond to the two quadrature carriers $\cos \omega_c t$ and $\sin \omega_c t$ (scaled to unity energy).

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos \omega_c t, \quad \phi_2(t) = \sqrt{\frac{2}{T}} \sin \omega_c t$$

- For $M = 8$ ($m = 3$ bits per symbol), we have an 8-PSK constellation:



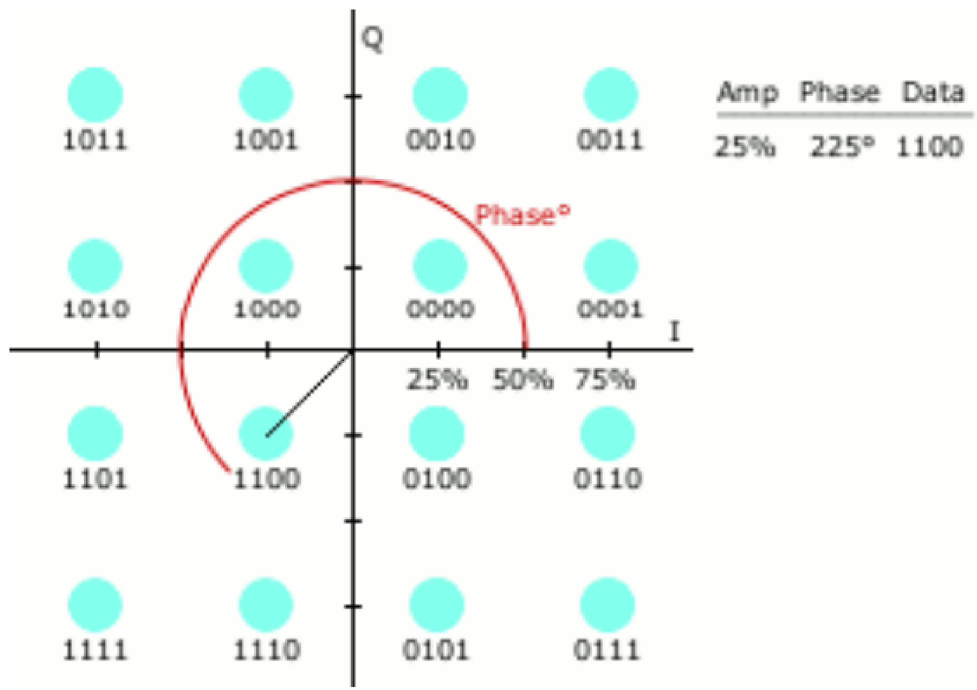
16-QAM

APSK definition

Definition: Amplitude and Phase-Shift Keying, ***APSK***, is a digital modulation scheme that uses both the amplitude and the phase changes of on the carrier signal to provide the data transport mechanism for the information. Also called *QAM*.

Quadrature Amplitude Modulation, *QAM* is a form of modulation that is a combination of phase modulation and amplitude modulation. The *QAM* scheme represents bits as points in a quadrant grid know as a constellation map.

16-ary *QAM*

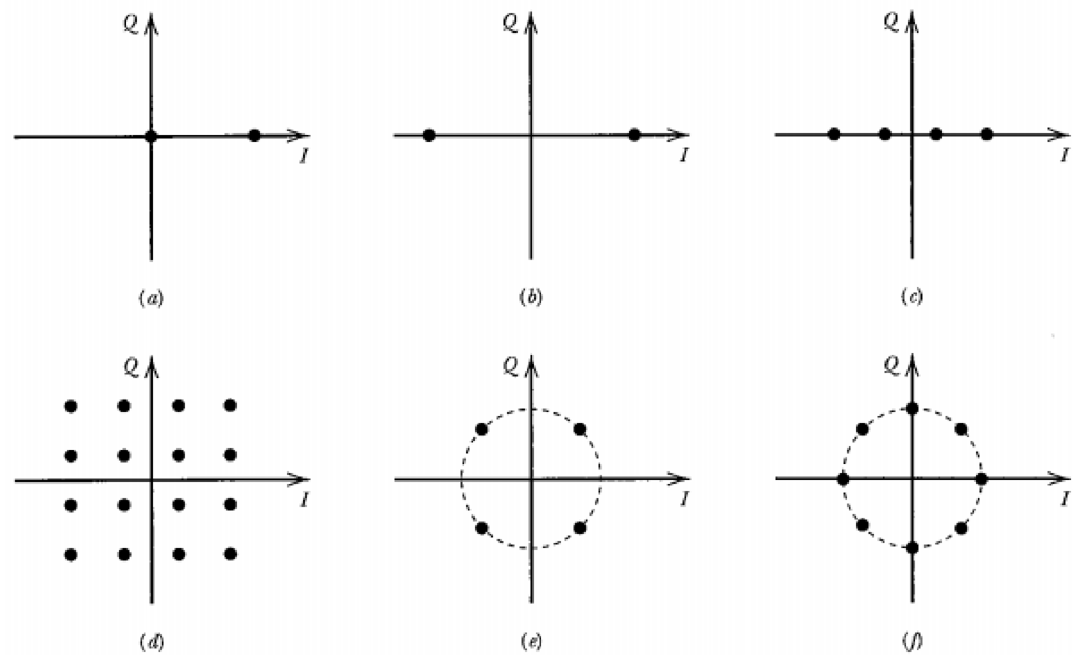


MPSK Constellation Diagram

- Observations regarding MPSK constallation diagram
  1. It is a geometric representation of M signal points uniformly distributed on a circle of radius $\sqrt{E}$
  2. Each signal point corresponds to the signal $s_i(t)$ for a particulare index i
  3. The squared length from the origin to each signal point equals to the signal energy E
- Note the Gray encoding of the bit sequences - where adjacent bit sequences differ only in one position

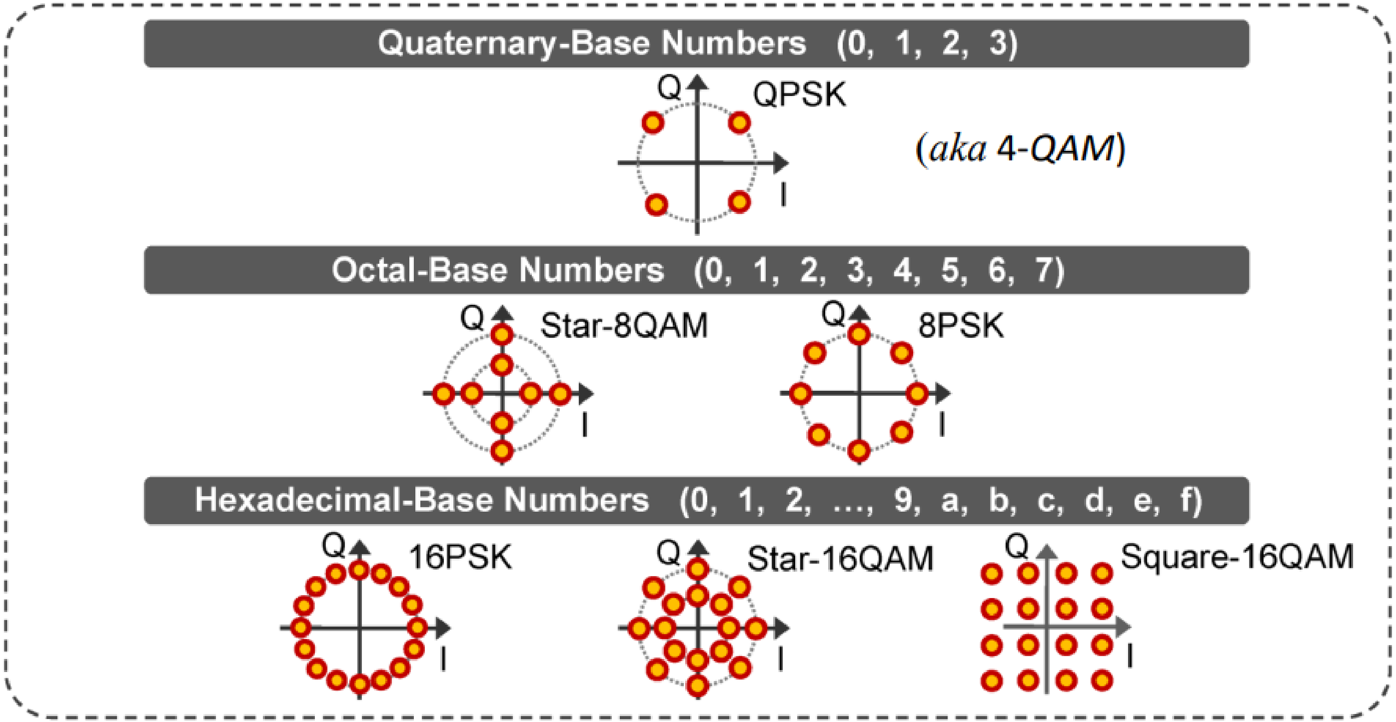
Constellation Diagrams

- Examples of other signal constellations



- BASK, BPSK, 4-ASK, QAM, QPSK, 8-PSK

Number bases in constellation



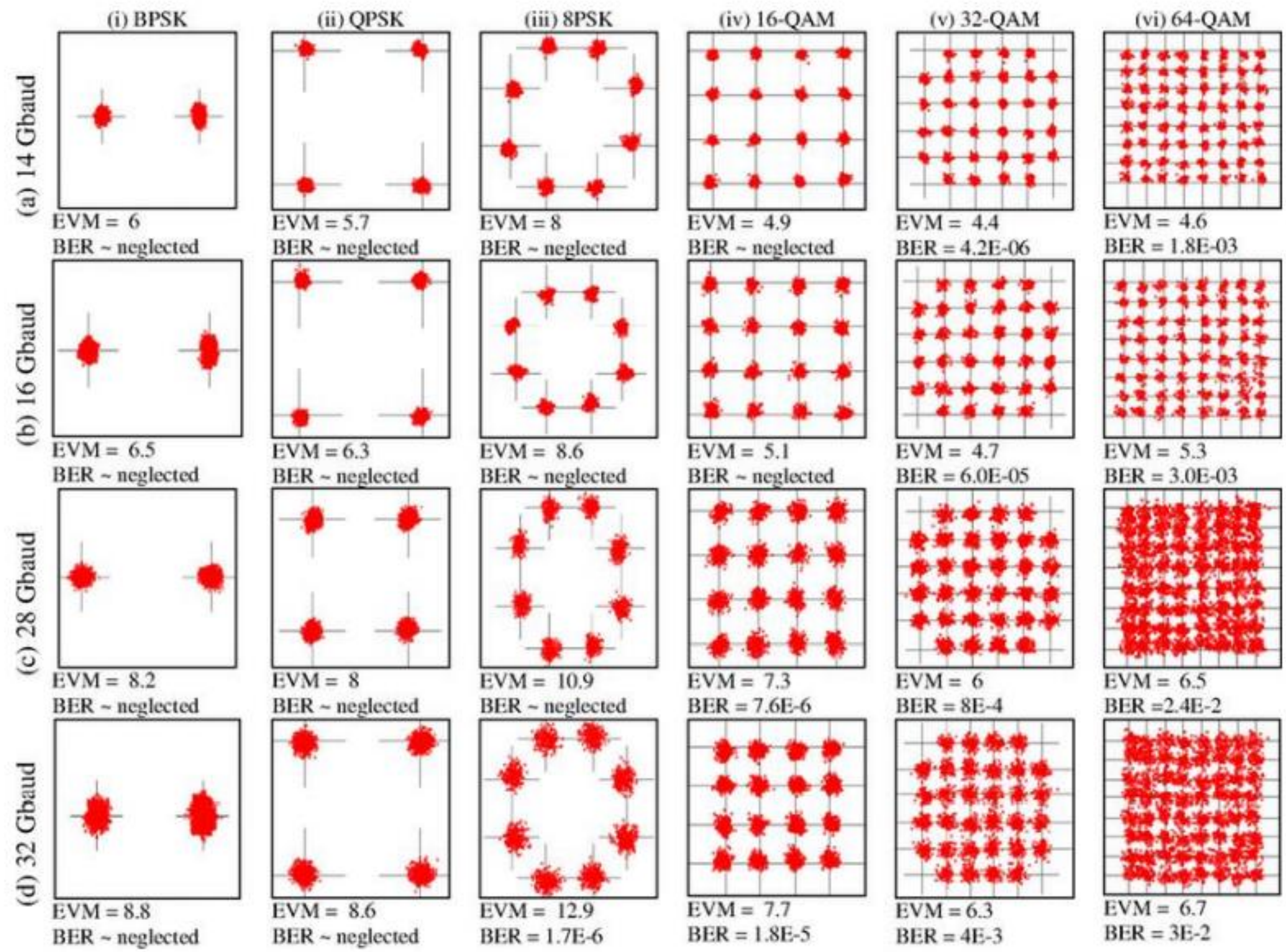
Variants of *QAM* are also used for many wireless and cellular technology applications. In addition, 64-*QAM* and 256-*QAM* are commonly used in digital cable television and cable modem applications. In the US, 64-*QAM* and 256-*QAM* are the mandated modulation schemes for digital cable as standardized by the SCTE in the standard ANSI/SCTE 07 2000.

SCTE = Society of Cable Telecommunications Engineers

Bits/Symbol and Symbol Rates

Modulation	Bits per Symbol	Symbol Rate
BPSK	1	1 × bit rate
QPSK	2	1/2 bit rate
8-*PSK*	3	1/3 bit rate
16-*QAM*	4	1/4 bit rate
32-*QAM*	5	1/5 bit rate
64-*QAM*	6	1/6 bit rate

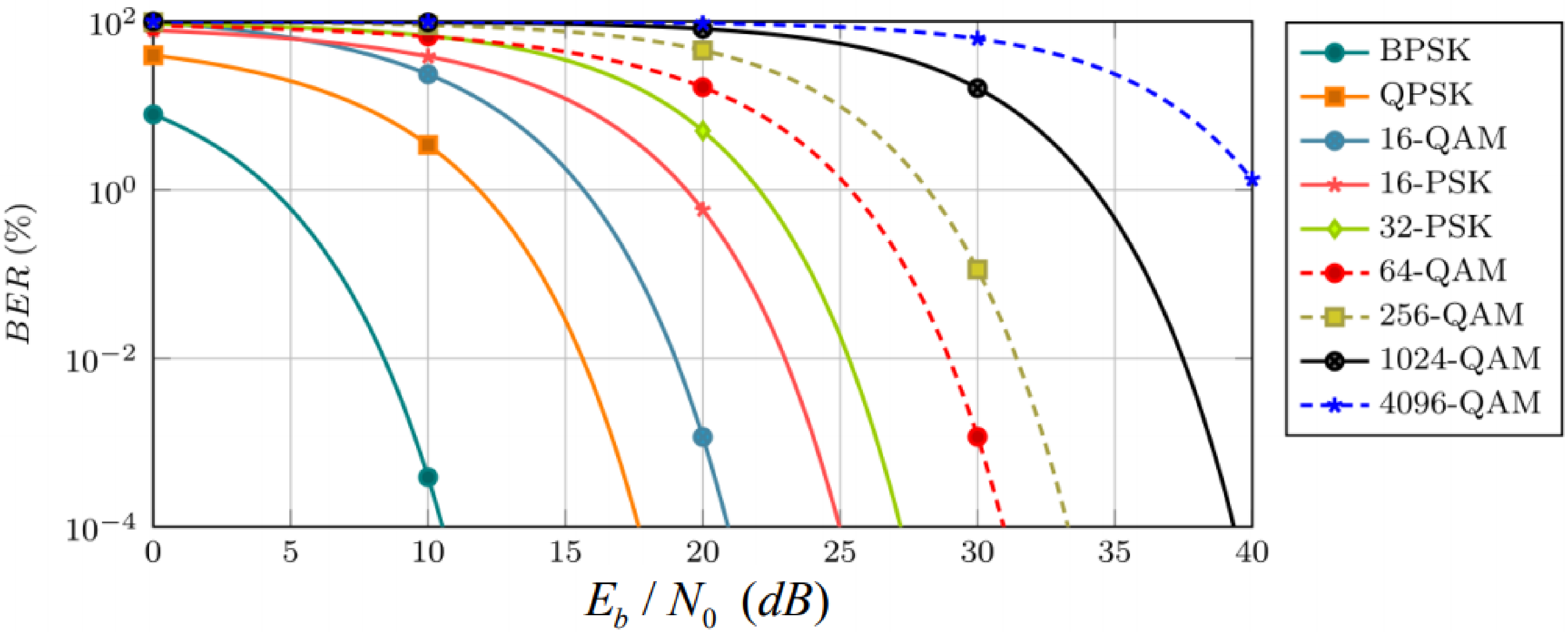
Noise…



(i) BPSK
(ii) QPSK
(iii) 8PSK
(iv) 16-QAM
(v) 32-QAM
(vi) 64-QAM
(a) 14 Gbaud
EVM = 6
BER ~ neglected
EVM = 5.7
BER ~ neglected
EVM = 8
BER ~ neglected
EVM = 4.9
BER ~ neglected
EVM = 4.4
BER = 4.2E-06
EVM = 4.6
BER = 1.8E-03
(b) 16 Gbaud
EVM = 6.5
BER ~ neglected
EVM = 6.3
BER ~ neglected
EVM = 8.6
BER ~ neglected
EVM = 5.1
BER ~ neglected
EVM = 4.7
BER = 6.0E-05
EVM = 5.3
BER = 3.0E-03
(c) 28 Gbaud
EVM = 8.2
BER ~ neglected
EVM = 8
BER ~ neglected
EVM = 10.9
BER ~ neglected
EVM = 7.3
BER = 7.6E-6
EVM = 6
BER = 8E-4
EVM = 6.5
BER =2.4E-2
(d) 32 Gbaud
EVM = 8.8
BER ~ neglected
EVM = 8.6
BER ~ neglected
EVM = 12.9
BER = 1.7E-6
EVM = 7.7
BER = 1.8E-5
EVM = 6.3
BER = 4E-3
EVM = 6.7
BER = 3E-2

Bit Error Rate versus Energy/Noise Ratio

BER = Bit Error Rate



energy per bit-to-noise power ratio

What modulation does Wi-Fi use?

WiFi systems use two primary radio transmission techniques.

802.11b (≤ 11 Mbps) – The 802.11b radio link uses a direct sequence spread spectrum technique (DSSS) called **complementary coded keying** (*CCK*). The bit stream is processed and then modulated using **Quadrature Phase Shift Keying (*QPSK*).**

802.11a and 802.11g (≤ 54 Mbps) – The 802.11a and g systems use **64-channel orthogonal frequency division multiplexing** (***OFDM***). The transmitter encodes the bit streams onto 64 subcarriers using **Binary Phase Shift Keying** (***BPSK***), **Quadrature Phase Shift Keying** (***QPSK***), or one of two levels of **Quadrature Amplitude Modulation** (**16-*QAM*, or 64-*QAM***).

Bandwidth Efficiency (*aka* Spectral Efficiency)

Given: E_b = energy per bit (joules or ergs)
R_b = bit rate (bits/second)
B = bandwidth of baseband signal (Hz)
N_0 = noise spectral density (watts/Hz)
N = noise power = $N_0 B$ (watts)

Therefore, $E_b R_b$ **= total signal power**

We define the Bandwidth Use Efficiency as

$$\frac{R_b}{B} \quad \left[\frac{\text{bits/second}}{\text{Hz}}\right]$$

In general,

$$\frac{R_b}{B} = \log_2\left(1 + \frac{E_b R_b}{NB}\right)$$

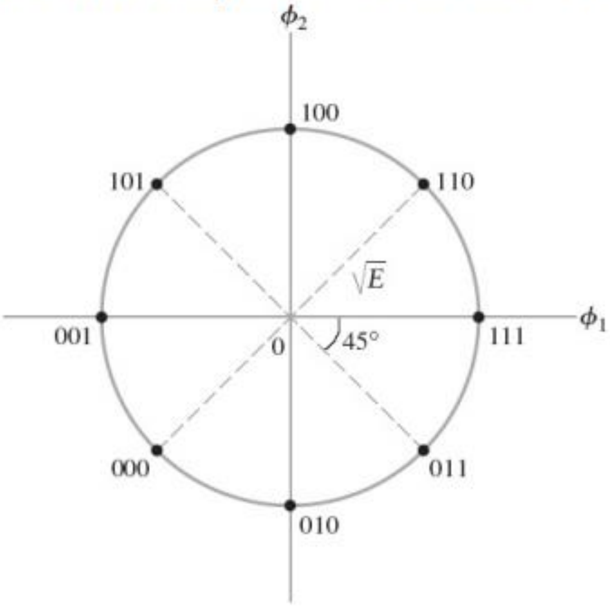
Example:
GSM Digital Cellular
Data rate = 270 kb/s
B = 200 kHz, thus
Bandwidth efficiency =
1.35 bits/sec/Hz

MPSK

- The in-phase and quadrature components derived lead to a geometrical potrayal of complex baseband signals, referred to as *constellation diagram* or a *signal space diagram*. This space is spanned by an basis set, which for PSK is simply the quadrature carriers

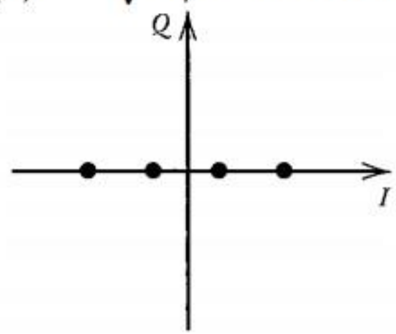
$$\phi_1(t) = \sqrt{\frac{2}{T}}\cos\omega_c t, \quad \phi_2(t) = \sqrt{\frac{2}{T}}\sin\omega_c t$$

- For $M = 8$ ($m = 3$ bits per symbol), we have an 8-PSK constellation:



MASK

- An M-ary extention of ASK is obtained by transmitting a single pulse with M amplitudes, that is $\pm p(t), \pm 3p(t), \pm 5p(t), \ldots, \pm(M-1)p(t)$ where $p(t) = \sqrt{2/T}\cos\omega_c t$ (assuming rectangular baseband pulse)
- The resulting constallation diagram is simply a set of points on the x-axis, which corresponds to $\phi_1(t) = \sqrt{2/T}\cos\omega_c t$



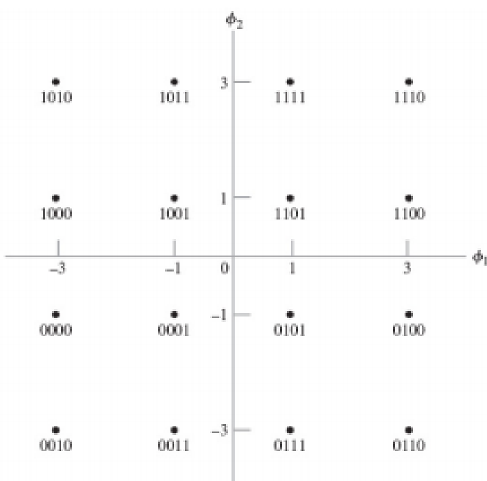
- Note that unlike in MPSK the symbol energies are not equal

QAM

- A common signaling scheme is Quadriture Amplitude Modulation (QAM)
- It is a hybrid of MPSK and MASK, which can be described as

$$s_i(t) = \sqrt{\frac{2E_0}{T}} a_i \cos \omega_c t - \sqrt{\frac{2E_0}{T}} b_i \sin \omega_c t \quad i = 0, 1, \ldots, M-1; \quad 0 \le t \le T$$

- The constellation diagram has the same basis function as MPSK, that is $\phi_1(t) = \sqrt{\frac{2}{T}} \cos \omega_c t$ and $\phi_2(t) = \sqrt{\frac{2}{T}} \sin \omega_c t$
- For example, 16-QAM:



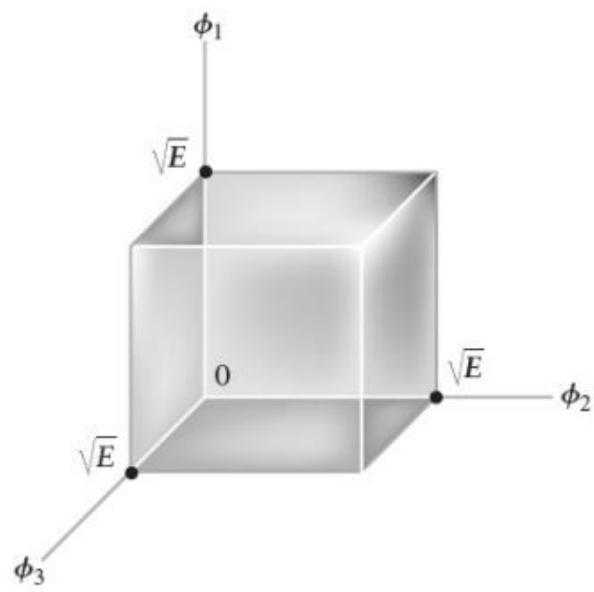
- Similarly to MASK, the symbols have different energy levels

MFSK

- In M-ary FSK, each of the M symbols is given by

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left[\frac{\pi}{T}(n+i)t\right] \quad i = 0, 1, \ldots, M-1; \quad 0 \leq t \leq T$$

- Since the symbols is orthogonal to each other, M-ary FSK occupies M-dimensional signal space, with basis corresponding to the normalized signals $\phi_i(t) = \frac{1}{\sqrt{E}} s_i(t)$



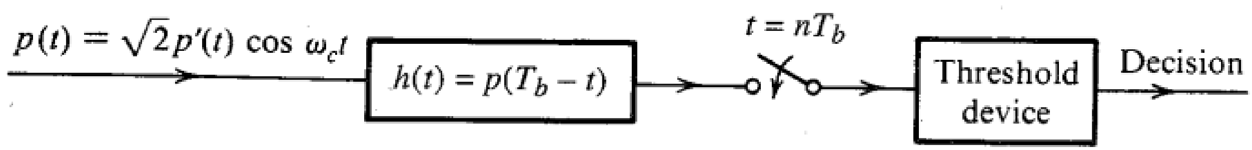
Bit Error Performance

Coherent Bandpass Receiver Techniques

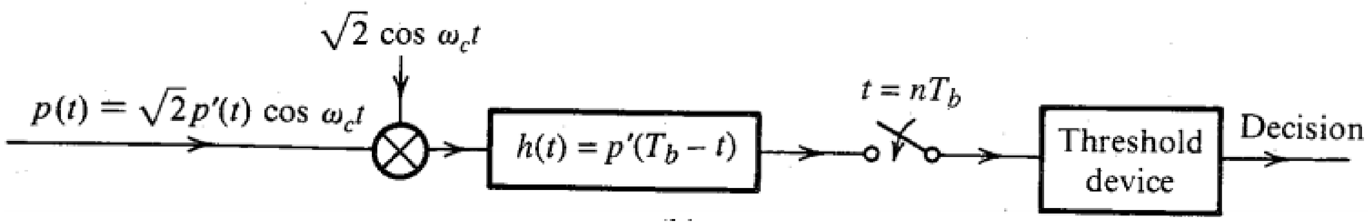
- As with baseband systems, the error probability of digital bandpass systems with optimum receivers depends only on the pulse energy, not the shape.
- As before, the incoming pulses can be demodulated either coherently (synchronously) or noncoherently (envelope detection).
- Coherent detection is optimum, but necessitates carrier phase synchronisation.

Coherent Bandpass Receiver Techniques

- Let the modulated pulse be $p(t) = \sqrt{2}p'(t)\cos\omega_c t$, where $p(t)$ is the baseband pulse
- There are two equivalent formulations of coherent bandpass detection
  1. a filter matched to the bandpass pulse



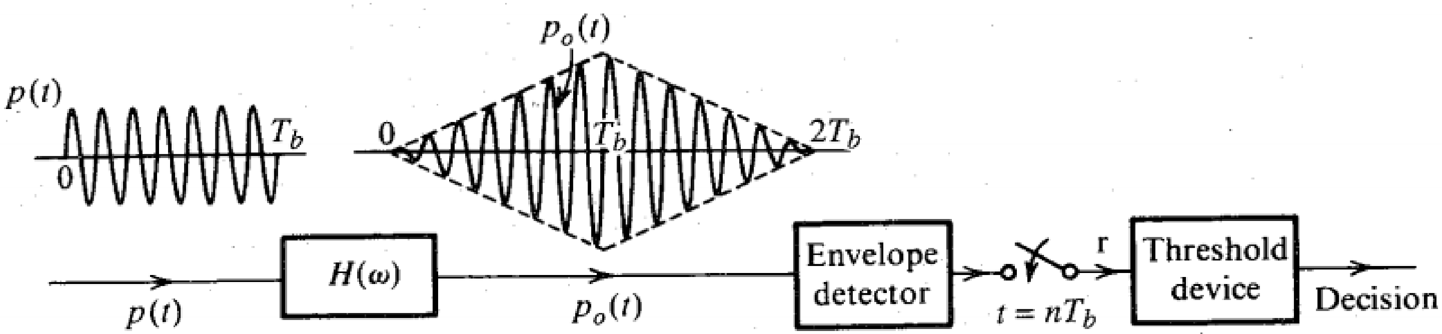
  2. a filter matched to the baseband pulse, preceded by frequency downconversion



- Note, that in both cases we require a synchronised carrier phase: either for the MF response or for the coherent demodulation.

Non-Coherent Bandpass Receiver Techniques

- If the phase θ of the incoming pulse $p(t) = \sqrt{2}p'(t)\cos\omega_c t_\theta$ we must employ noncoherent detection
- Note: this does not apply to PSK, since the information carried by PSK signal is actually contained in the phase
- In order to apply noncoherent detection to PSK, we must use *differential encoding* (which is beyond our scope)
- it can be shown that the optimum noncoherent detector is a filter matched to the RF pulse (with a phase error), followed by an envelope detector



- We now analyse the error performance, focusing on coherent detection

BPSK and BASK BER

- Recall that Binary PSK is equivalent to applying polar signalling to a carrier wave
- Since multiplying by a sinusoid does not effect the average pulse energy, the performance of binary PSK is that of polar signalling

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- Similarly for Binary ASK, where the underlying line code is on-off signalling (unipolar), the performance is

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- As with the baseband probabilities of the equivalent modulations, BASK requires 3 dB more power than BPSK.

FSK BER

- To evaluate the BER for binary FSK, we need to evaluate the crosscorrelation term E_{pq} in

$$P_e = Q\left(\sqrt{\frac{E_p + E_q - 2E_{pq}}{2N_0}}\right)$$

- Recall that the two (normalized) pulses can be expressed as

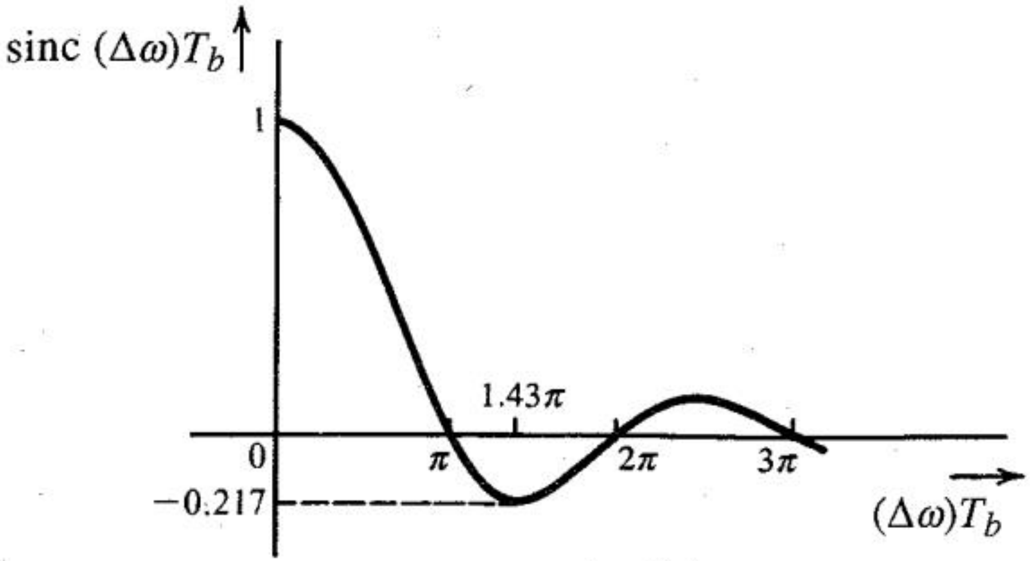
$$q(t) = \sqrt{2}A\cos[2\pi(\omega_c + \Delta\omega)t]$$
$$p(t) = \sqrt{2}A\sin[2\pi(\omega_c - \Delta\omega)t]$$

- Evaluating E_{pq}, we can show that

$$\begin{aligned} E_{pq} &= \int_0^{T_b} p(t)q(t) \\ &= A^2\left[\frac{\sin\Delta\omega T_b}{\Delta\omega T_b} + \frac{\sin 2\omega_c T_b}{2\omega_c T_b}\right] \\ &\approx A^2 T_b\,\mathrm{sinc}(\Delta\omega T_b) \quad \text{since } \omega_c T_b >> 1 \end{aligned}$$

- In order to minimize P_e, we want to minimize E_{pq}...

FSK BER



(a)

- The minimum value of E_{pq} is $E_{pq,\text{min}} = -0.217A^2T_b$, which occurs at $\Delta\omega T = 1.43\pi$, or $\Delta f = \frac{0.715}{T_b} = 0.715R_b$ (where R_b is the symbol rate).
- In this case, $E_b = E_p = E_q = A^2T_b$, $E_{pq} = -0.217A^2T_b$, and thus the error probability is

$$P_e = Q\left(\sqrt{\frac{1.217A^2T_b}{N_0}}\right) = Q\left(\sqrt{\frac{1.217E_b}{N_0}}\right)$$

MSK

- Note that the separation Δf impacts on the transmission bandwidth of FSK.
- In order to minimize the bandwidth (without sacrificing the error rate performance), we can lower Δf to the first zero crossing of $\mathrm{sinc}(\Delta\omega T_b)$, that is $\Delta f = 1/2T_b$, such that $E_{pq} = 0$.
- (Signaling where $E_{pq} = 0$ is referred to as *orthogonal* signaling)
- This FSK scheme is referred to as *minimum-shift keying* (MSK) or *fast-frequency-shift keying*

M-ary Error Performance

MASK

- So far we have analysed the *bit error rate* (BER) performance for *binary* ASK/PSK/FSK
- Our disussion of constallation diagrams lead us to the notion of M-ary schemes, where a *symbol* represents $\log_2 M$ bits of data. This results in increased bit rate.
- The question is: in terms of the error performance, what is the price we pay as a result of this throughput increase?
- Or stated another way: what increase in power (MASK/MPSK) or bandwidth (MFSK) is required to maintain a given performance level
- We first determine the M-ary *symbol* error rate (SER) P_{eM}, and then convert SER to bit error rate (BER) P_e

MASK SER

- Recall that in M-ary ASK we transmit M pulses $\pm p(t), \pm 3p(t), \ldots \pm (M-1)p(t)$.
- In order to find SER of MASK we take the following approach

1. By examining the conditional PDFs of received MASK, we derive the SER P_{eM} as a function of the energy of $p(t)$ E_p:

$$P_{eM} = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{2E_p}{N_0}}\right)$$

2. We then express the average pulse energy E_{pM} as a function of E_p, which will in turn let us express E_p as a function of the *average energy per bit* E_b:

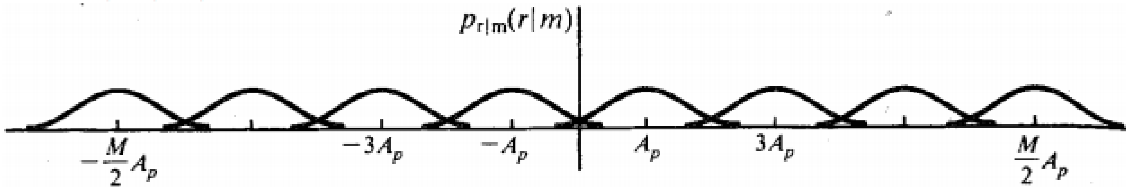
$$E_b = \frac{E_{pM}}{\log_2 M} = \frac{M^2-1}{3\log_2 M} E_p$$

3. The substitution for E_p gives the SER

$$P_{eM} = 2\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_b}{N}}\right)$$

MASK SER (step 1)

- In order to find P_{eM} we note that similarly to the polar signaling, the PDFs corresponding to each transmitted symbol are $\mathcal{N}(kE_p, N_0E_p/2)$ where $k = 1, 3, 5, \ldots M-1$



- We express the probability of error P_{eM} as a function of probabilities conditional on transmitting symbol m_i

$$P_{eM} = \sum_{i=1}^{M} P(m_i)P_{e,m_i} = \frac{1}{M}\sum_{i=1}^{M} P_{e,m_i} \qquad \text{(if equiprobable symbols)}$$

- Noting that the end PDF contain only one neighbor, while the middle ones two (and thus their P_{e,m_i} is doubled) we have

$$P_{eM} = \frac{1}{M}\left[Q\left(\sqrt{\frac{2E_p}{N_0}}\right) + Q\left(\sqrt{\frac{2E_p}{N_0}}\right) + (M-1)2Q\left(\sqrt{\frac{2E_p}{N_0}}\right)\right]$$

$$= \frac{2(M-1)}{M}Q\left(\sqrt{\frac{2E_p}{N_0}}\right)$$

MASK SER (step 2)

- The average transmitted pulse energy E_pM as a function of E_p is

$$E_{pM} = \frac{2}{M}\left[E_p + 9E_p + 25E_p + \ldots + (M-1)^2 E_p\right]$$
$$= \frac{2}{M}\sum_{k=0}^{(M-2)/2}(2k+1)^2$$
$$= \frac{M^2-1}{3}E_p$$

- Since each M-ary symbol carries $\log_2 M$ bits of information, we have the energy per bit is

$$E_b = \frac{E_{pM}}{\log_2 M} = \frac{M^2-1}{3\log_2 M}E_p$$

MASK SER (step 3)

- Substitution for E_p in the SER expression from step 1, we have

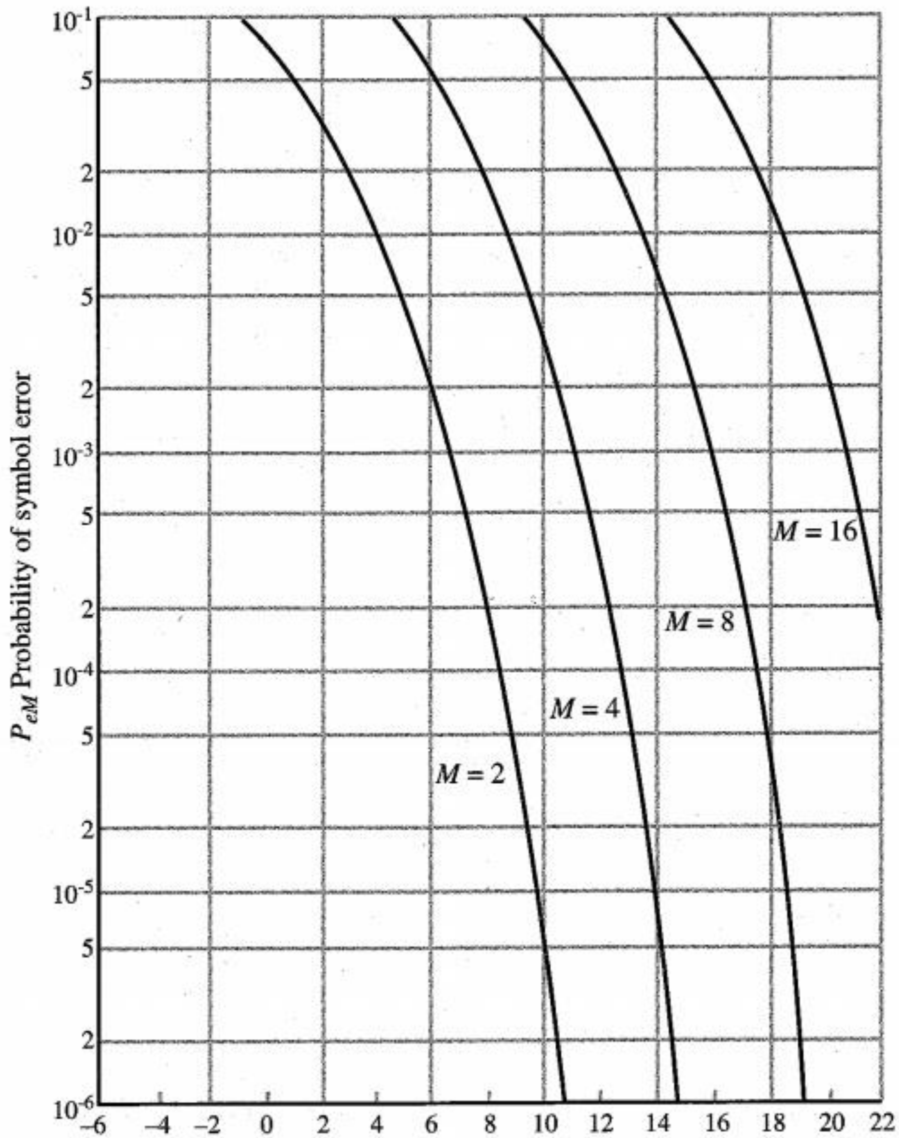
$$P_{eM} = 2\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_b}{N}}\right)$$

- for large M, this can be approximated by

$$P_{eM} \approx 2Q\left(\sqrt{\frac{6\log_2 M}{M^2}\frac{E_b}{N}}\right) \quad M >> 1$$

- Finally, we note that assuming Gray coding , where adjacent symbols differ by only 1 bit, each symbol error will cause only one error in a group of $\log_2 M$. And thus the BER $P_b = P_{eM}/\log_2 M$

MASK

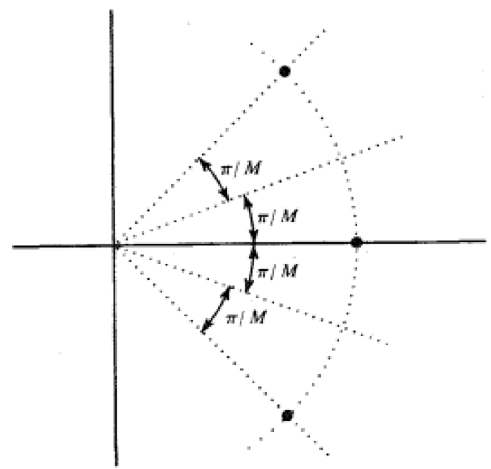


P_{eM} Probability of symbol error
10^{-1}
5
2
10^{-2}
5
2
10^{-3}
5
2
10^{-4}
5
2
10^{-5}
5
2
10^{-6}
−6 −4 −2 0 2 4 6 8 10 12 14 16 18 20 22
M = 2
M = 4
M = 8
M = 16

MPSK SER

- In M-ary PSK the pulses of neighbouring symbols are separated by $2\pi/M$ radians, that is

$$p'(t) = \cos\left[\omega_c t + \frac{2\pi}{M} i\right]$$



- Thus a symbol error occurs when due to noise the phase of any pulse deviates by more than π/M

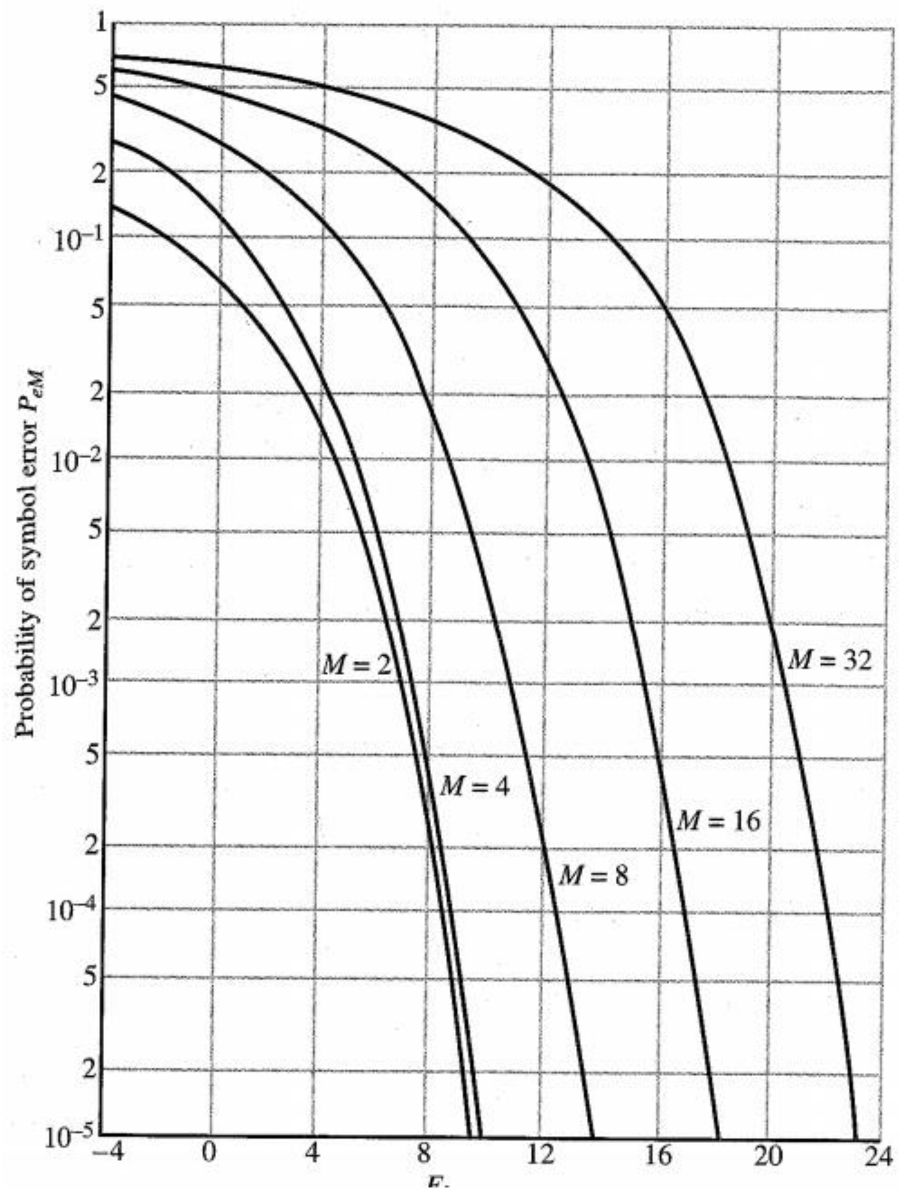
MPSK SER

- The SER analysis is involved, since it requires the derivation of the PDF of phase θ of a sinusoid plus gaussian noise
- The resulting SER is

$$P_{eM} \approx 2Q\left(\sqrt{\frac{2E_b \log_2 M}{N_0}} \sin\frac{\pi}{M}\right)$$
$$\approx 2Q\left(\sqrt{\frac{2\pi^2 E_b \log_2 M}{M^2 N_0}}\right)$$

- Similarly to MASK if we assume Gray encoding of bits onto symbols $P_b = P_{eM}/\log_2 M$

MPSK SER


Probability of symbol error P_{eM}
1
5
2
10^{-1}
5
2
10^{-2}
5
2
10^{-3}
5
2
10^{-4}
5
2
10^{-5}
−4
0
4
8
12
16
20
24
M = 2
M = 4
M = 8
M = 16
M = 32

MFSK SER

- The analysis for M-ary FSK, which is again rather involved, gives

$$P_{eM} = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(y-\sqrt{2E_b \log_2 M/N_0}\right)^2/2} \left[1 - Q(y)\right]^{M-1} \; dy$$

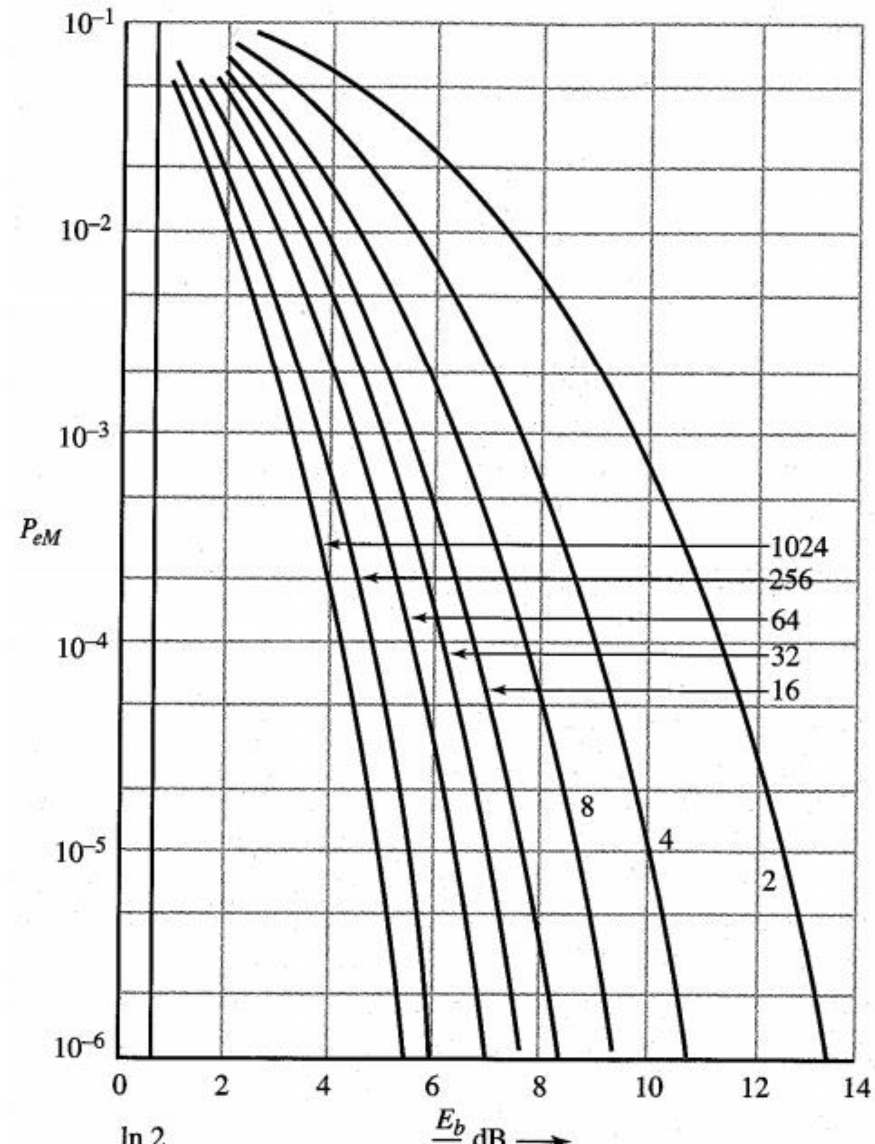
- An interesting observation can be made when evaluating the limit for $M \to \infty$

$$\lim_{M\to\infty} P_{eM} = \begin{cases} 1 & E_b/N_0 \le \log_e 2 \\ 0 & E_b/N_0 \ge \log_e 2 \end{cases}$$

- Since the signal power is $S_i = E_b R_b$, where R_b is the bit rate, we deduce that we can achieve error-free transmission if

$$R_b \le \log_e 2 \frac{S_i}{N_0} = 1.44 \frac{S_i}{N_0} \text{bits/s}$$

MFSK SER



10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
P_{eM}
1024
256
64
32
16
8
4
2
0
2
4
6
8
10
12
14
ln 2
$\frac{E_b}{}$ dB →

MFSK SER -¿ BER

- For MASK and MPSK, we had $P_b = P_{eM}/\log_2 M$.
- In MFSK, since the dimension of the signal space is M, a symbol is equally likely to be mistaken for any of the neighboring $M-1$ symbols
- It can be shown that for $M = 2^m$ FSK, the BER is related to SER by

$$P_b = \frac{2^{m-1}}{2^m - 1} P_{eM} \approx \frac{P_{eM}}{2} \quad m >> 1$$

- Proof: The probability P_{em} of mistaking one M-ary symbol for another is

$$P_{em} = \frac{P_{eM}}{M-1} = \frac{P_{eM}}{2^m - 1}$$

- If an symbol differs by 1 bit from N_1 symbols, by 2 from N_2 symbols... then $\bar{N}_e$, the avg number of bits in error in reception of a symbol is

$$\bar{N}_e = \sum_{n=1}^{m} n N_n P_{em} = \sum_{n=1}^{m} n N_n \frac{P_{eM}}{2^m - 1}$$

$$= \frac{P_{eM}}{2^m - 1} \sum_{n=1}^{m} n \binom{m}{n} = m 2^{m-1} \frac{P_{eM}}{2^m - 1}$$

From Signals to Vectors

- In depicting the various signaling schemes by means of constellation diagrams, we saw that the axes of in the signal space represented an *orthogonal basis set* for the bandpass pulses
- In the case of ASK, PSK, FSK the basis set is somewhat obvious
  - for MASK it is simply the carrier $\cos \omega_c t$
  - for MPSK the basis set is formed by the two quadrature carriers $\cos \omega_c t$ and $\sin \omega_c t$
  - for FSK it is the full set of M carriers $\cos \omega_i t$ $(0 \leq i \leq M-1)$
- For general signals, we can find an appropriate basis set using a process of *Gram-Schmidt Orthogonalization*
- Once we do so we can express any pulse as a vector in the signal space
- This will enable us to use some powerful (and elegant) techniques for system analysis