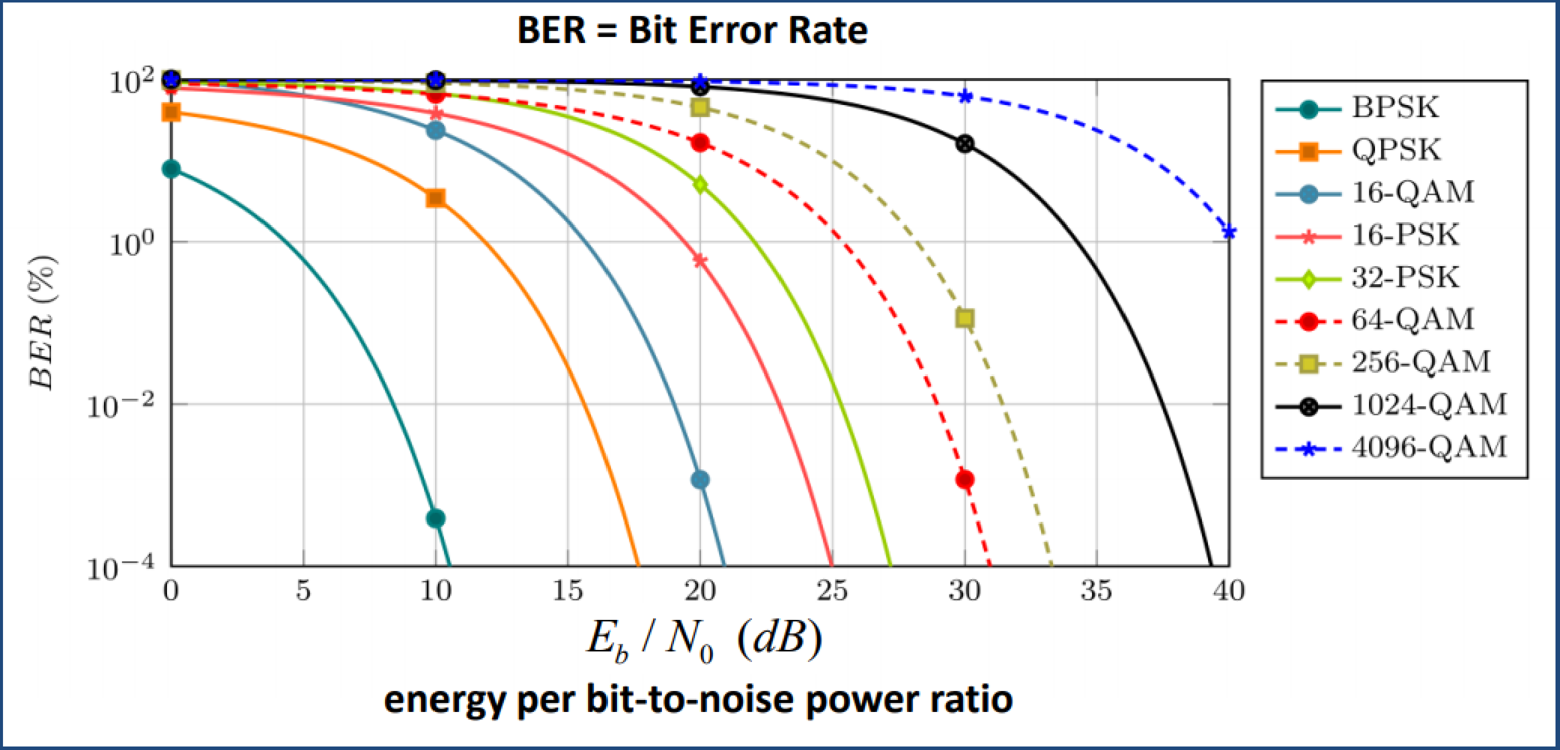
LECTURE 9:
BIT ERROR RATE



ECEN 310

Communication Engineering

VICTORIA UNIVERSITY OF WELLINGTON TE HERENGA WAKA

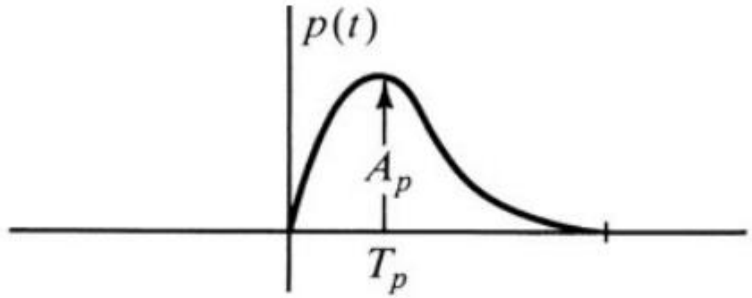
Review

Detection Errors

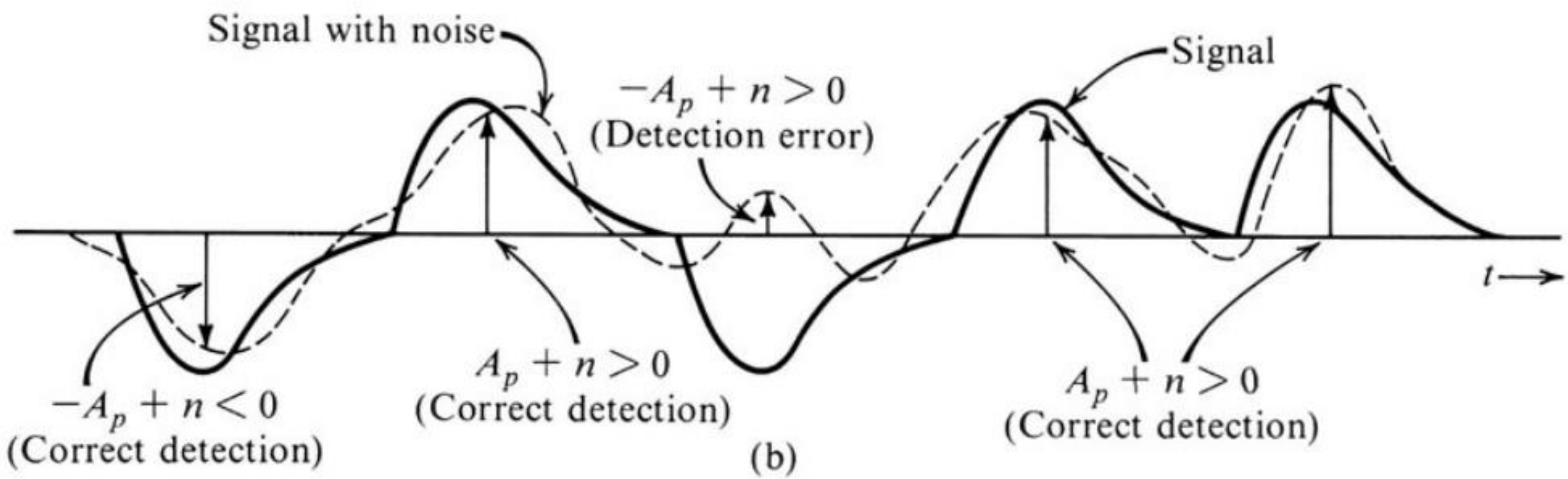
- The received pulse, assuming binary transmission, is $\pm p(t) + n(t)$
- At each sampling instant, sample value is $y_i = \pm A_p + n$ where we let $A_p = p(nT)$ is the pulse amplitude at the time of sampling
- The decision device must estimate the transmitted symbol $\hat{a}_k$ based on the sample value
  - Decide $b_k = 1$, (ie $a_k = 1$) if $y_i > 0$
  - Decide $b_k = 0$, (ie $a_k = -1$) if $y_i < 0$

Read Proakis and Salehi

Detection Errors



(a)



(b)

Detection Errors

- The decision error will occur if either
  - $b_k = 1$ was sent but $y_i = A_p + n < 0$
  - $b_k = 0$ was sent but $y_i = -A_p + n > 0$
- In the second case, the probability of error is

$$P(\epsilon|b_k = 0) = P(n > A_p) = \frac{1}{\sqrt{2\pi}\sigma}\int_{A_p}^{\infty} e^{-y^2/2\sigma^2}\, dy = Q\left(\frac{A_p}{\sigma}\right)$$

- By symmetry, the error given 1 is

$$P(\epsilon|b_k = 1) = P(n < -A_p) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{-A_p} e^{-y^2/2\sigma^2}\, dy = Q\left(\frac{A_p}{\sigma}\right)$$

- In the above we performed change of variable $\tilde{y} = y/\sigma$ to obtain the standard form for the Q function

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_{x}^{\infty} e^{-y^2/2} dy$$

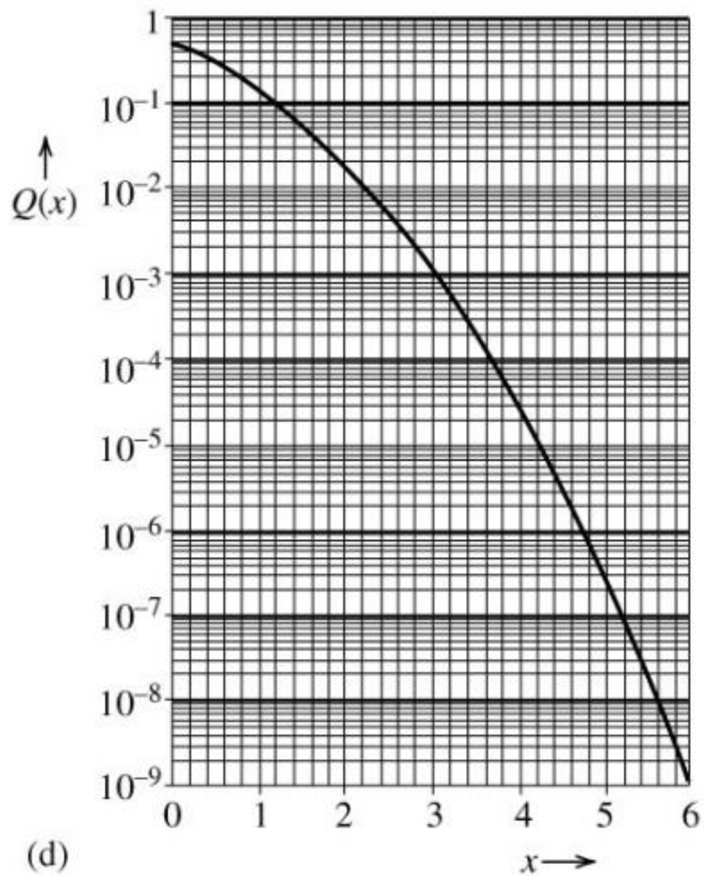
Detection Errors

- Thus, the total probability error is the average of the two

$$
\begin{aligned}
P(\epsilon) &= P(\epsilon|b_k = 1)P(b_k = 1) + P(\epsilon|b_k = 0)P(b_k = 0) \\
&= 2\frac{1}{2}Q\left(\frac{A_p}{\sigma}\right) = Q\left(\frac{A_p}{\sigma}\right)
\end{aligned}
$$

where we assumed equal probability of the input bits $P(b_k = 0) = P(b_k = 1) = \frac{1}{2}$

Detection Errors



Matched Filter

- We then derived the optimum receive filter shape from the point of view of bit error rate.
- We found that it was given by a matched filter - one 'matched' to the incoming pulse shape

$$h(t) = p(T_0 - t)$$
$$H(f) = P(-f)e^{-j2\pi f T_0}$$

Matched Filter

- One can show (see Lathi) that for a MF receiver, the pulse amplitude A_p and the filtered noise variance σ_n^2 are given by in terms of the pulse energy E_p

$$A_p = \frac{1}{2\pi} \int_{-\infty}^{\infty} \|P(\omega)\|^2 \, d\omega = E_p$$

and

$$\sigma_n^2 = \frac{\mathcal{N}}{4\pi} \int_{-\infty}^{\infty} \|P(\omega)\|^2 \, d\omega = \frac{\mathcal{N} E_p}{2}$$

and thus the probability of error is equal to

$$P(\epsilon) = Q\left(\frac{A_p}{\sigma}\right) = Q\left(\sqrt{\frac{2E_p}{\mathcal{N}}}\right)$$

- Note that the performance is independent of the pulse shape used, only the SNR!

Correlation Receiver

- The matched filter can also be realised using a *correlation receiver*.
- The MF output $r(t)$ to the input $y(t)$ is

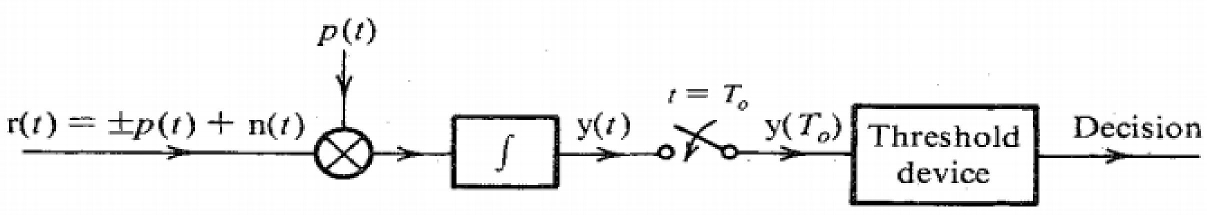
$$r(t) = \int_{-\infty}^{\infty} y(x)h(t-x)\ dx = \int_{-\infty}^{\infty} y(x)p(x+T_0-t)\ dx$$

where $h(t) = p(T_0 - t)$ and $h(t-x) = p(x + T_0 - t)$.

- At the decision making instant $t = T_0$ we have

$$r(T_0) = \int_{-\infty}^{\infty} y(x)p(x)\ dx = \int_{0}^{T_0} y(x)p(x)\ dx$$

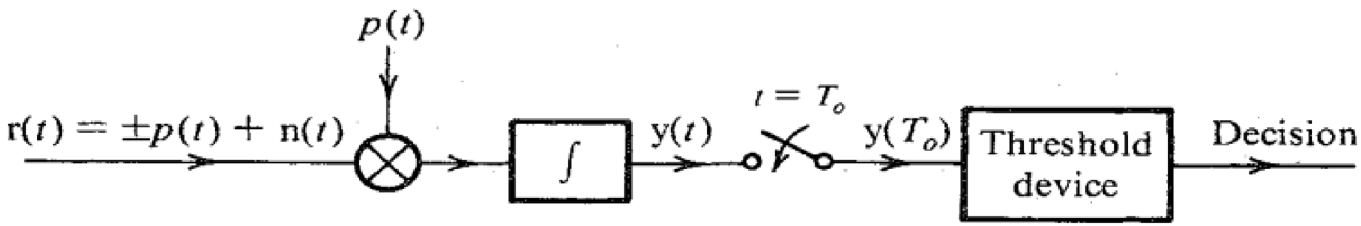
since the pulse is assumed to be confined to $[0, T_0]$.



Correlation Receiver

$$r(T_0) = \int_{-\infty}^{\infty} y(x)p(x)\ dx = \int_{0}^{T_0} y(x)p(x)\ dx$$

- The above is the cross-correlation of the received pulse with $p(t)$, and thus the detector measures the similarity of the received signal with the pulse $p(t)$



- The correlation receiver is equivalent to the MF receiver *only at the sampling instant* $t = T_0$

BER - General Analysis

- So far we considered the case of binary signalling using a single pulse shape $p(t)$ ($\pm p(t)$), with the resulting BER P_e given by

$$P_e = Q\left(\sqrt{\frac{2E}{N_0}}\right)$$

  where E is the energy of the pulse used.

- Similar analysis can be generalised for binary signalling using two arbitrary pulse shapes $p(t)$ and $q(t)$, representing 1 and 0, respectively.
- The resulting BER is (see *Lathi 'Modern Analog and Digital Communication Systems' 4e*, Sec 10.2.1)

$$P_e = Q\left(\sqrt{\frac{E_p + E_q - 2E_{pq}}{2N_0}}\right)$$

  where $E_p = \int_0^{T_b} p^2(t)\ dt$ and $E_q = \int_0^{T_b} q^2(t)\ dt$ are the energies of pulses $p(t)$ and $q(t)$, respectively, and E_{pq} is defined as

$$E_{pq} = \int_0^{T_b} p(t)q(t)\ dt$$

Examples

1. (Sanity check) Consider polar (antipodal) signaling, which we analysed last class, that is $p(t) = -q(t)$: We have

$$E_{pq} = -\int_0^{T_b} p^2(t)\, dt = -E_p$$
$$E_p = E_q$$

and so $P_e = Q\left(\sqrt{\frac{2E_p}{N_0}}\right)$. Since the *average energy per bit* $E_b = (E_p + E_q)/2 = E_b$, we have $P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

2. Now, for On-Off (unipolar) signaling, that is $q(t) = 0$: We have

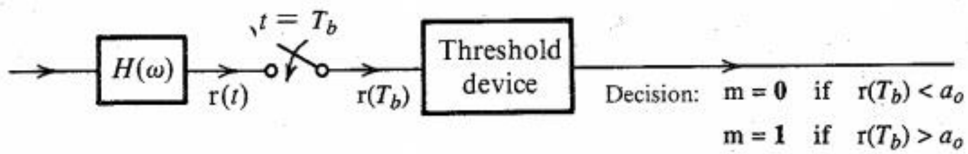
$$E_{pq} = 0 \quad E_q = 0$$

and so $P_e = Q\left(\sqrt{\frac{E_p}{2N_0}}\right)$. Note that $E_b = E_p/2$, and so $P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$. Thus we require twice as much energy per bit to achieve the same BER as polar signaling.

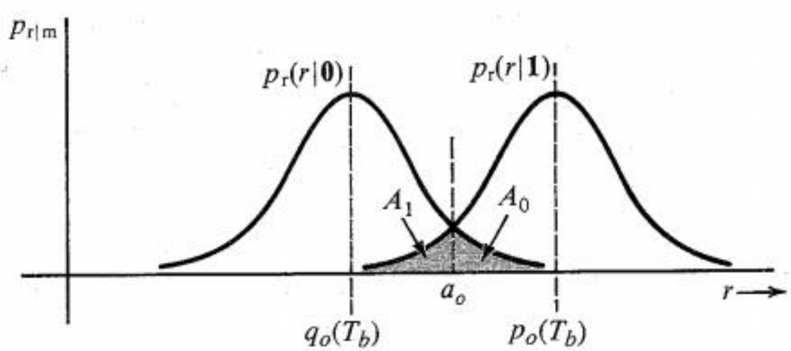
BER - Proof

The proof is similar to the polar case (details: Lathi 4e 10.2.1):

1. Compute the PDF of received signal samples conditioned on the value of bit sent
2. Choose the intersection of the PDFs as the decision threshold (or set $dP_e/da_0 = 0$)
3. Find P_e as a function of pulse shape samples
4. Minimize P_e, which is equivalent to maximizing the argument of $Q()$, which is related to SNR
5. Find filter shape that maximizes SNR, using Schwartz's Inequality



(a)



(b)

BER - Proof

- Let the output of the receive filter for $p(t)$ and $q(t)$ be $p_0(t)$ and $q_0(t)$, respectively
- The PDFs of the received signals sampled at $t = T_b$, given 0 and 1 sent, are

$$p_{r|0} = \frac{1}{\sigma_n\sqrt{2\pi}} e^{-[r-q_0(T_b)]^2/2\sigma_n^2}$$
$$p_{r|1} = \frac{1}{\sigma_n\sqrt{2\pi}} e^{-[r-p_0(T_b)]^2/2\sigma_n^2}$$

- The optimum decision threshold is

$$a_0 = \frac{p_0(T_b) - q_0(T_b)}{2}$$

- Assuming equiprobable data, the symmetry gives $P_e = P_{e0} = P_{e1}$ can then be shown to be

$$P_e = Q\left(\frac{\beta}{2}\right)$$

where $\beta = \frac{p_0(T_b) - q_0(T_b)}{\sigma_n}$

BER - Proof

- We can expressing β^2 as a function of the filter spectra $P(\omega)$, $Q(\omega)$ and $H(\omega)$, and show using Schwartz's Inequality that it is maximized when

$$H(\omega) = (P(-\omega) - Q(-\omega))e^{-j\omega T_b}$$

- note that maximising β minimizes $Q(\beta/2)$, thus minimizing P_e
- The resulting optimum filter $h(t)$ is then

$$h(t) = p(T_b - t) - q(T_b - t)$$

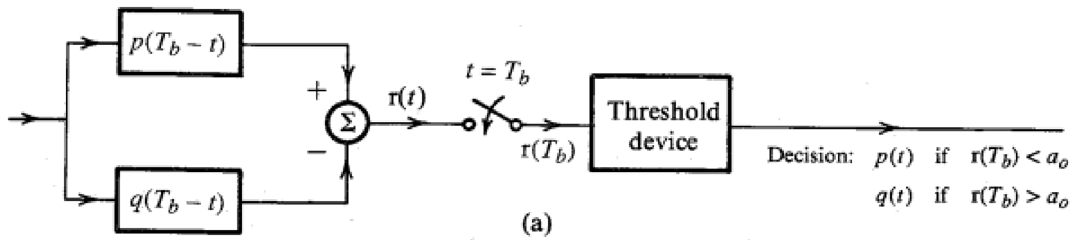
  ie, **matched to** $p(t) - q(t)$
- The optimum $\beta^2 = \beta^2_{\text{max}}$ can be shown to be

$$\beta^2_{\text{max}} = \frac{E_p + E_q - 2E_{pq}}{N_0/2}$$

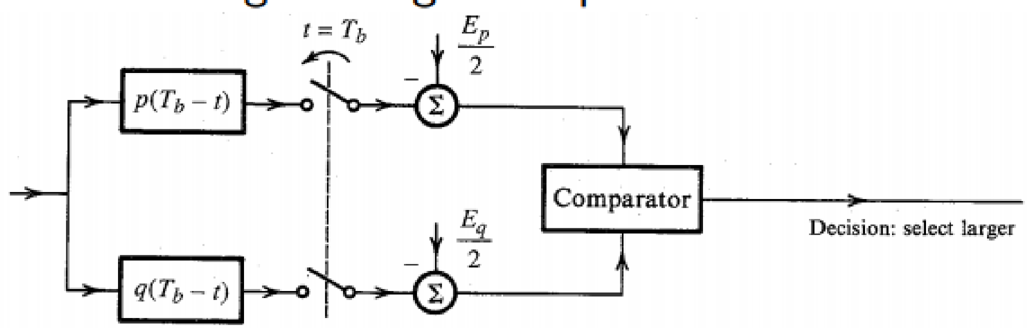
- Finally, we can show that the decision threshold is equivalent to $a_0 = (E_p - E_q)/2$

Optimum Receiver

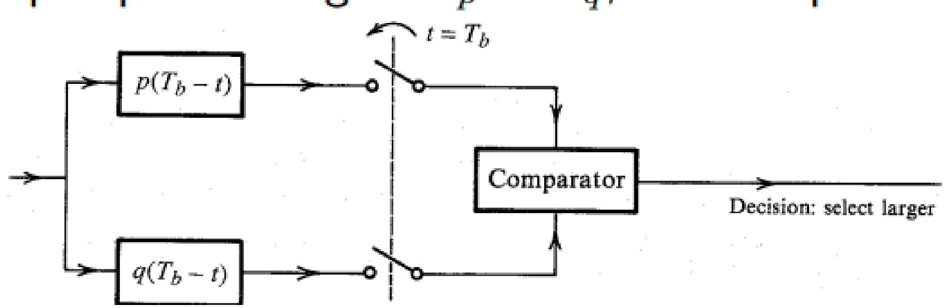
- The resulting receiver structure is thus



- decision threshold is $a_0 = (E_p - E_q)/2$
  and so this is equiv. to choosing the largest output of the individual matched filters



- ...and if we assume equal pulse energies $E_p = E_q$, this is equivalent to

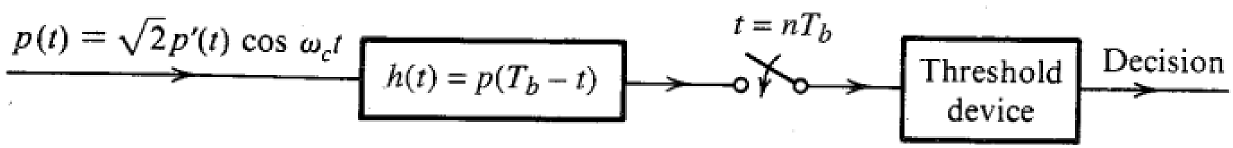


Coherent Bandpass Receiver Techniques

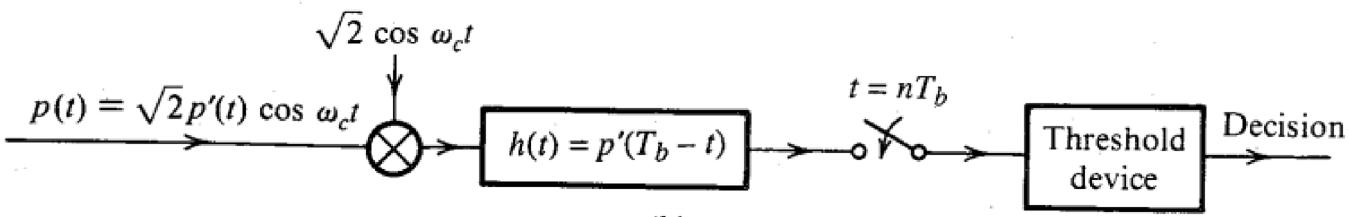
- The preceding analysis is also applicable to bandpass signalling (ASK,PSK,FSK)
- As with baseband systems, the error probability of digital bandpass systems with optimum receivers depends only on the pulse energy, not the shape.
- As before, the incoming pulses can be demodulated either coherently (synchronously) or noncoherently (envelope detection).
- Coherent detection is optimum, but necessitates carrier phase synchronisation.

Coherent Bandpass Receiver Techniques

- Let the modulated pulse be $p(t) = \sqrt{2}p'(t)\cos\omega_c t$, where $p'(t)$ is the baseband pulse
- There are two equivalent formulations of coherent bandpass detection
  1. a filter matched to the bandpass pulse



  2. a filter matched to the baseband pulse, preceded by frequency downconversion



- Note, that in both cases we require a synchronised carrier phase: either for the MF response or for the coherent demodulation.

BPSK and BASK BER

- Recall that Binary PSK is equivalent to applying polar signalling to a carrier wave
- Since multiplying by a $\sqrt{2}\cos\omega_c t$ does not effect the average pulse energy, the performance of binary PSK is that of polar signalling

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- Similarly for Binary ASK, where the underlying line code is on-off signalling (unipolar), the performance is

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- As with the baseband probabilities of the equivalent modulations, BASK requires 3 dB more power than BPSK.

FSK BER

- To evaluate the BER for binary FSK, we need to evaluate the crosscorrelation term E_{pq} in

$$P_e = Q\left(\sqrt{\frac{E_p + E_q - 2E_{pq}}{2N_0}}\right)$$

- Recall that the two (normalized) pulses can be expressed as

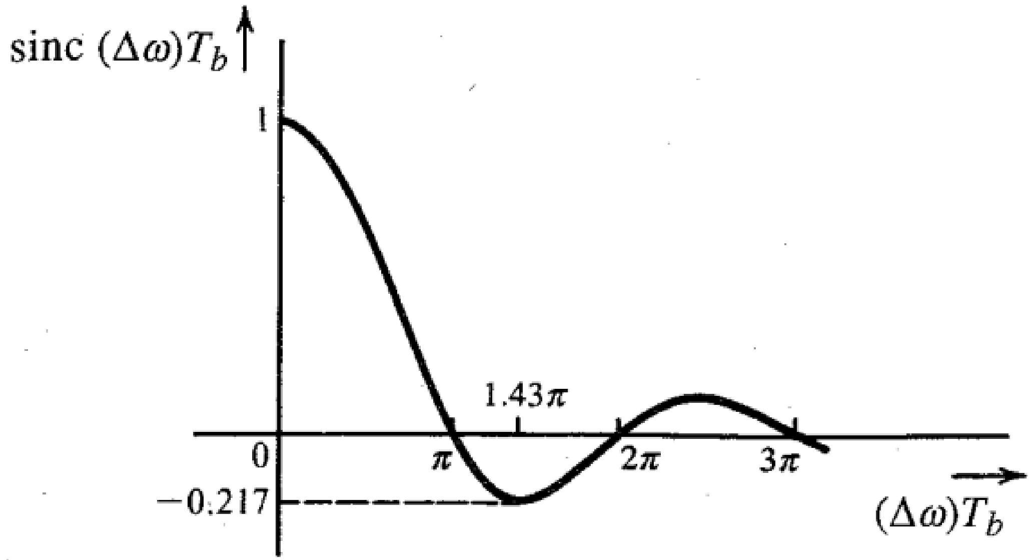
$$q(t) = \sqrt{2}A\cos[2\pi(\omega_c + \Delta\omega)t]$$
$$p(t) = \sqrt{2}A\sin[2\pi(\omega_c - \Delta\omega)t]$$

- Evaluating E_{pq}, we can show that

$$\begin{aligned} E_{pq} &= \int_0^{T_b} p(t)q(t) \\ &= A^2\left[\frac{\sin \Delta\omega T_b}{\Delta\omega T_b} + \frac{\sin 2\omega_c T_b}{2\omega_c T_b}\right] \\ &\approx A^2 T_b \operatorname{sinc}(\Delta\omega T_b) \quad \text{since } \omega_c T_b >> 1 \end{aligned}$$

- In order to minimize P_e, we want to minimize E_{pq}...

FSK BER



(a)

- The minimum value of E_{pq} is $E_{pq,\text{min}} = -0.217A^2T_b$, which occurs at $\Delta\omega T{=}1.43\pi$, or $\Delta f = \frac{0.715}{T_b} = 0.715R_b$ (where R_b is the symbol rate).
- In this case $E_{pq} = -0.217A^2T_b$, and thus the error probability is $(E_b = E_p = E_q = A^2T_b)$

$$P_e = Q\left(\sqrt{\frac{1.217A^2T_b}{N_0}}\right) = Q\left(\sqrt{\frac{1.217E_b}{N_0}}\right)$$

MSK

- Note that the separation Δf impacts on the transmission bandwidth of FSK.
- The case where $E_{pq} = 0$ is referred to as *orthogonal* signaling.
- In order to minimize the bandwidth of orthogonal FSK, we can lower Δf to the first zero crossing of $\mathrm{sinc}(\Delta\omega T_b)$, that is $\Delta f = 1/2T_b$
- This FSK scheme is referred to as *minimum-shift keying* (MSK) or *fast-frequency-shift keying*

M-ary Error Performance

MASK

- So far we have analysed the *bit error rate* (BER) performance for *binary* ASK/PSK/FSK
- Our disussion of constallation diagrams lead us to the notion of M-ary schemes, where a *symbol* represents $\log_2 M$ bits of data. This results in increased bit rate.
- The question is: in terms of the error performance, what is the price we pay as a result of this throughput increase?
- Or stated another way: what increase in power (MASK/MPSK) or bandwidth (MFSK) is required to maintain a given performance level
- We first determine the M-ary *symbol* error rate (SER) P_{eM}, and then convert SER to bit error rate (BER) P_e

MASK SER

- Recall that in M-ary ASK we transmit M pulses $\pm p(t), \pm 3p(t), \ldots \pm (M-1)p(t)$.
- In order to find SER of MASK we take the following approach

1. By examining the conditional PDFs of received MASK, we derive the SER P_{eM} as a function of the energy of $p(t)$, E_p:

$$P_{eM} = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{2E_p}{N_0}}\right)$$

2. We then express the *average pulse energy* E_{pM} as a function of E_p, which will in turn let us express E_p as a function of the *average energy per bit* E_b:

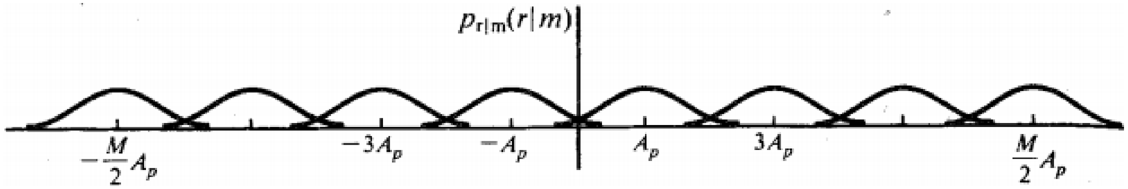
$$E_b = \frac{E_{pM}}{\log_2 M} = \frac{M^2-1}{3\log_2 M} E_p$$

3. The substitution for E_p gives the SER

$$P_{eM} = 2\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{6\log_2 M}{M^2-2}\frac{E_b}{N_0}}\right)$$

MASK SER (step 1)

- In order to find P_{eM} we note that similarly to the polar signaling, the PDFs corresponding to each transmitted symbol are $\mathcal{N}(kE_p, N_0E_p/2)$ where $k = 1, 3, 5, \ldots M-1$



- We express the probability of error P_{eM} as a function of probabilities conditional on transmitting symbol m_i

$$P_{eM} = \sum_{i=1}^{M} P(m_i)P_{e,m_i} = \frac{1}{M}\sum_{i=1}^{M} P_{e,m_i} \qquad \text{(if equiprobable symbols)}$$

- Noting that the end PDF contain only one neighbor, while the middle ones two (and thus their P_{e,m_i} is doubled) we have

$$P_{eM} = \frac{1}{M}\left[Q\left(\sqrt{\frac{2E_p}{N_0}}\right) + Q\left(\sqrt{\frac{2E_p}{N_0}}\right) + (M-2)2Q\left(\sqrt{\frac{2E_p}{N_0}}\right)\right]$$

$$= \frac{2(M-1)}{M}Q\left(\sqrt{\frac{2E_p}{N_0}}\right)$$

MASK SER (step 2)

- The average transmitted pulse energy E_{pM} as a function of E_p is

$$
\begin{aligned}
E_{pM} &= \frac{2}{M}\left[E_p + 9E_p + 25E_p + \ldots + (M-1)^2 E_p\right] \\
&= \frac{2E_p}{M}\sum_{k=0}^{(M-2)/2}(2k+1)^2 \\
&= \frac{M^2-1}{3}E_p
\end{aligned}
$$

- Since each M-ary symbol carries $\log_2 M$ bits of information, we have the energy per bit is

$$
E_b = \frac{E_{pM}}{\log_2 M} = \frac{M^2-1}{3\log_2 M}E_p
$$

MASK SER (step 3)

- Substitution for E_p in the SER expression from step 1, we have

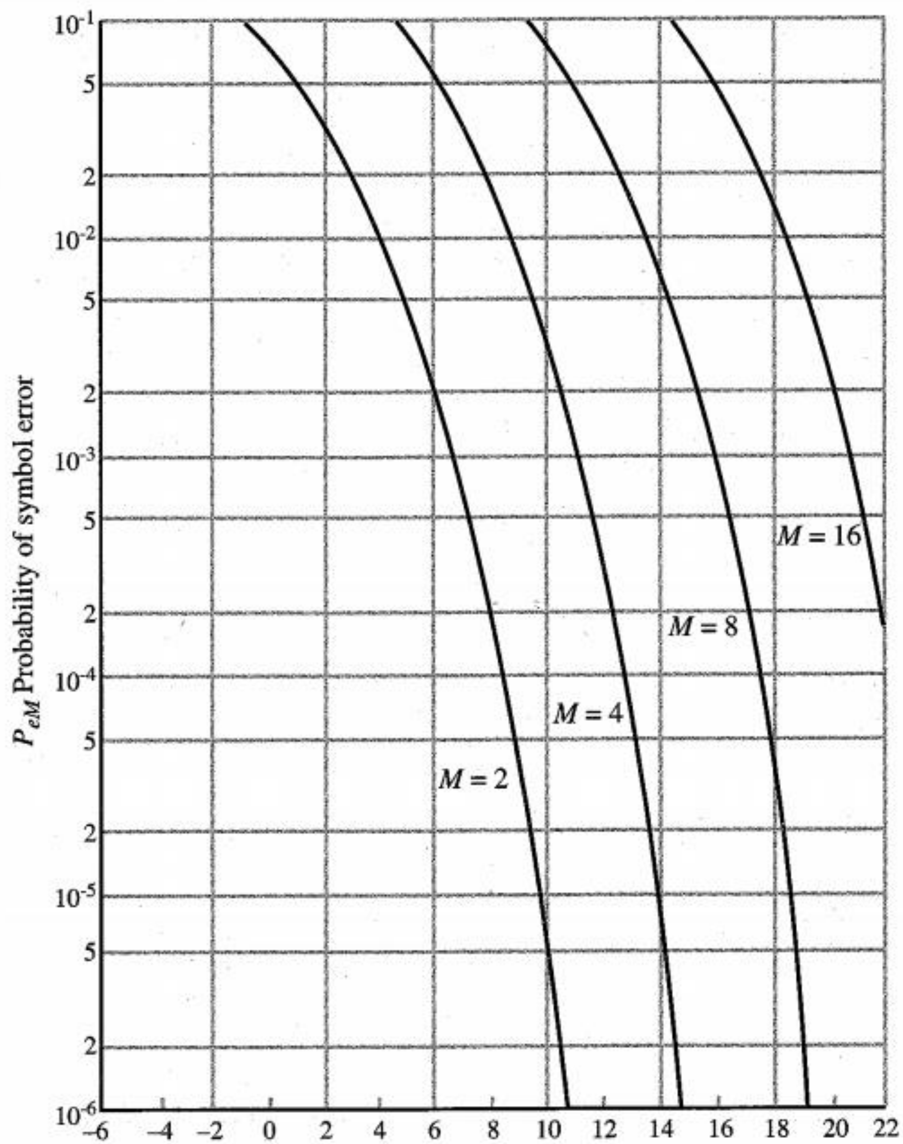
$$P_{eM} = 2\left(\frac{M-1}{M}\right) Q\left(\sqrt{\frac{6\log_2 M}{M^2-1}\frac{E_b}{N_0}}\right)$$

- for large M, this can be approximated by

$$P_{eM} \approx 2Q\left(\sqrt{\frac{6\log_2 M}{M^2}\frac{E_b}{N_0}}\right) \quad M >> 1$$

- Finally, we note that assuming Gray coding , where adjacent symbols differ by only 1 bit, each symbol error will cause only one error in a group of $\log_2 M$. And thus the BER $P_b = P_{eM} / \log_2 M$

MASK

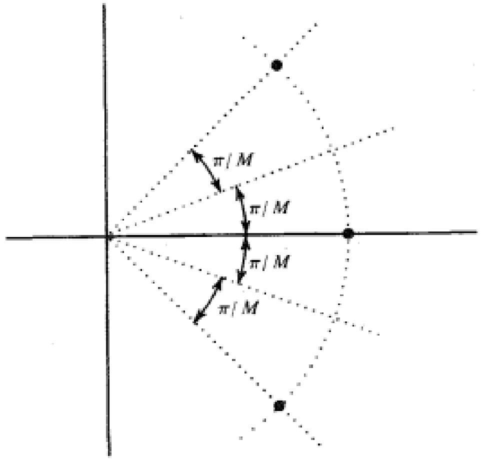


10^{-1}
5
2
10^{-2}
5
2
10^{-3}
5
2
10^{-4}
5
2
10^{-5}
5
2
10^{-6}
P_{eM} Probability of symbol error
–6 –4 –2 0 2 4 6 8 10 12 14 16 18 20 22
$M = 2$
$M = 4$
$M = 8$
$M = 16$

MPSK SER

- In M-ary PSK the pulses of neighbouring symbols are separated by $2\pi/M$ radians, that is

$$p(t) = \cos\left[\omega_c t + \frac{2\pi}{M} i\right]$$



- Thus a symbol error occurs when due to noise the phase of any pulse deviates by more than π/M

MPSK SER

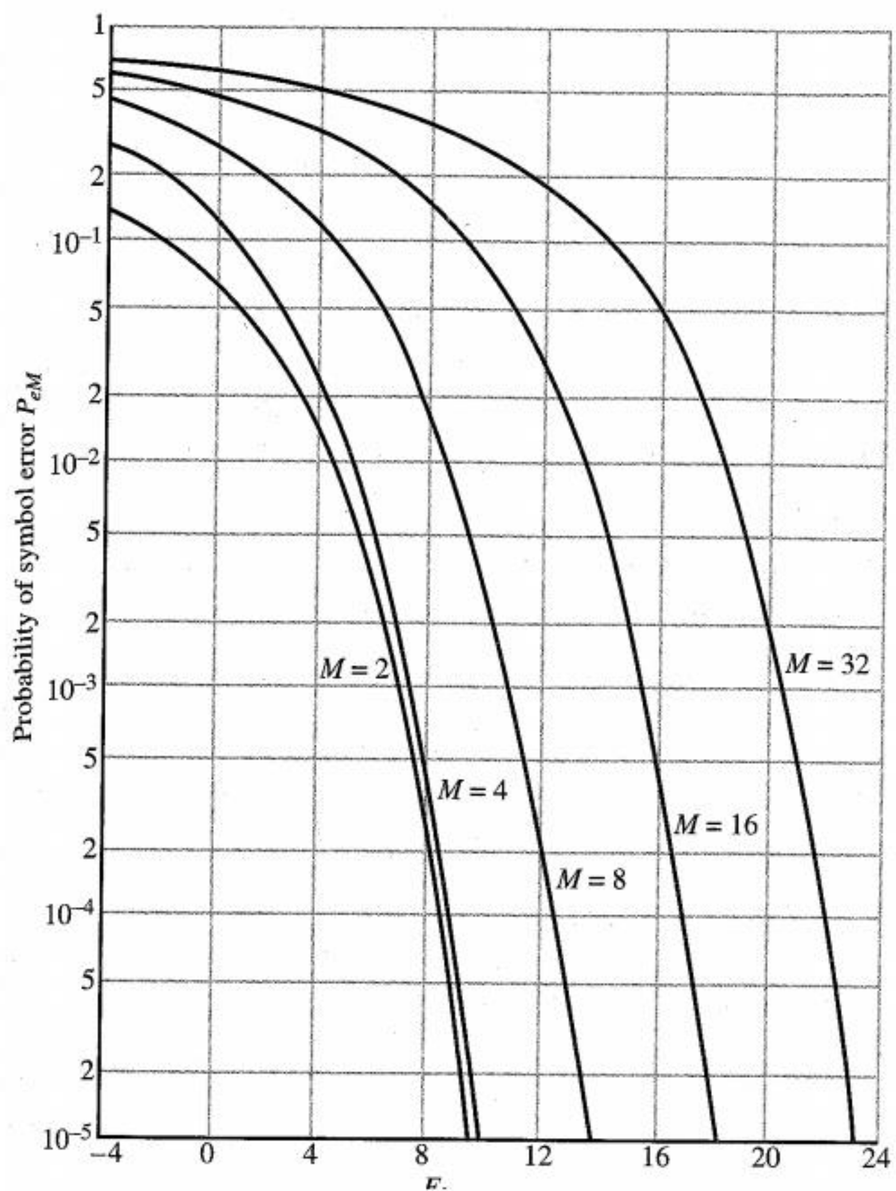
- The SER analysis is involved, since it requires the derivation of the PDF of phase θ of a sinusoid plus gaussian noise
- The resulting SER is

$$P_{eM} \approx 2Q\left(\sqrt{\frac{2E_b \log_2 M}{N_0}} \sin \frac{\pi}{M}\right)$$

$$\approx 2Q\left(\sqrt{\frac{2\pi^2 E_b \log_2 M}{M^2 N_0}}\right) \quad \text{large } M$$

- Similarly to MASK if we assume Gray encoding of bits onto symbols $P_b = P_{eM} / \log_2 M$

MPSK SER


Probability of symbol error P_{eM}
1
5
2
10^{-1}
5
2
10^{-2}
5
2
10^{-3}
5
2
10^{-4}
5
2
10^{-5}
−4
0
4
8
12
16
20
24
$M = 2$
$M = 4$
$M = 8$
$M = 16$
$M = 32$

MFSK SER

- The analysis for M-ary FSK, which is again rather involved, gives

$$P_{eM} = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(y-\sqrt{2E_b \log_2 M/N_0}\right)^2/2} \left[1 - Q(y)\right]^{M-1} \, dy$$

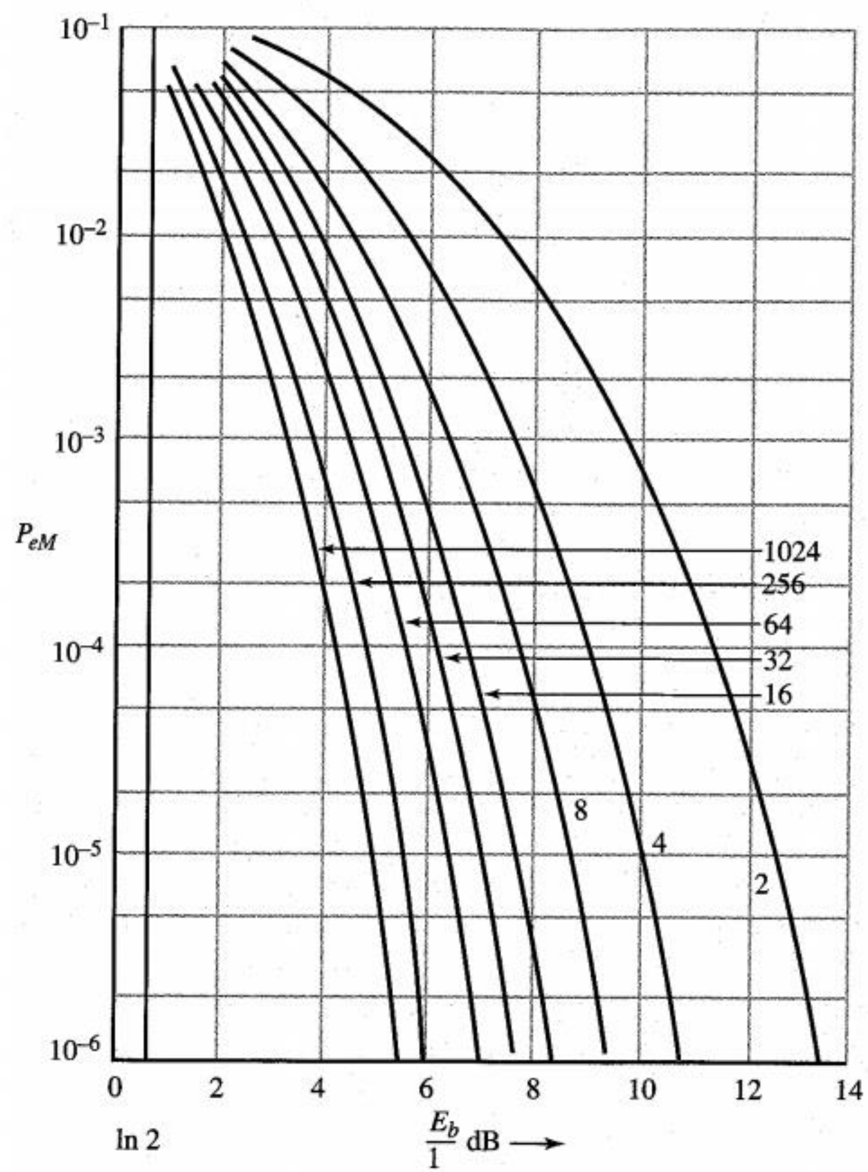
- An interesting observation can be made when evaluating the limit for $M \to \infty$

$$\lim_{M\to\infty} P_{eM} = \begin{cases} 1 & E_b/N_0 \le \log_e 2 \\ 0 & E_b/N_0 \ge \log_e 2 \end{cases}$$

- Since the signal power is $S_i = E_b R_b$, where R_b is the bit rate, we deduce that we can achieve error-free transmission if

$$R_b \le \log_e 2 \frac{S_i}{N_0} = 1.44 \frac{S_i}{N_0} \text{bits/s}$$

MFSK SER



10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
P_{eM}
1024
256
64
32
16
8
4
2
0
2
4
6
8
10
12
14
ln 2
$\frac{E_b}{1}$ dB ⟶

MFSK SER $\rightarrow$ BER

- For MASK and MPSK, we had $P_b = P_{eM}/\log_2 M$.
- In MFSK, since the dimension of the signal space is M, a symbol is equally likely to be mistaken for any of the neighboring $M-1$ symbols
- It can be shown that for $M = 2^m$ FSK, the BER is related to SER by

$$P_b = \frac{2^{m-1}}{2^m - 1} P_{eM} \approx \frac{P_{eM}}{2} \quad m >> 1$$

- Proof: The probability P_{em} of mistaking one M-ary symbol for another is

$$P_{em} = \frac{P_{eM}}{M-1} = \frac{P_{eM}}{2^m - 1}$$

If an symbol differs by 1 bit from N_1 symbols, by 2 from N_2 symbols... then $\bar{N}_e$, the avg number of bits in error in reception of a symbol is

$$\bar{N}_e = \sum_{n=1}^{m} n N_n P_{em} = \sum_{n=1}^{m} n N_n \frac{P_{eM}}{2^m - 1}$$

$$= \frac{P_{eM}}{2^m - 1} \sum_{n=1}^{m} n \binom{m}{n} = m 2^{m-1} \frac{P_{eM}}{2^m - 1}$$