Demo 3: Stability Analysis

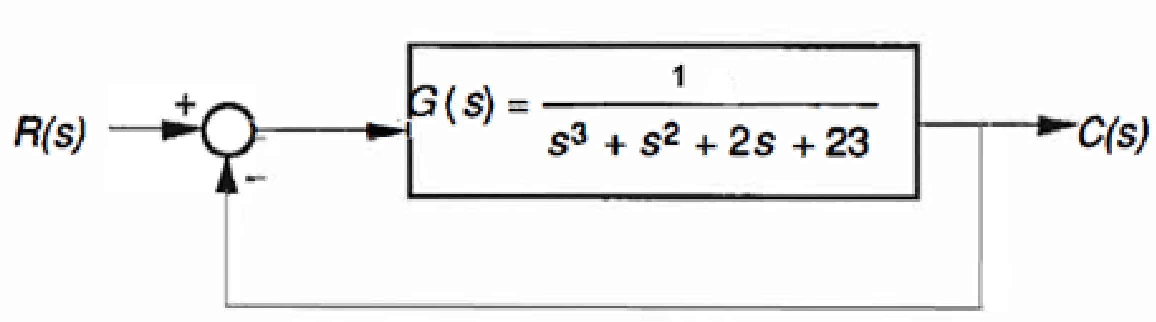
XMUT315 Control Systems Engineering

Topics

1. Stability analysis with Routh-Hurwitz criterion.
2. Stability analysis with other methods (e.g. Nyquist plot, Nichols chart, Bode plots and Root Locus Diagram).

Exercise 1 (Stability Analysis in MATLAB)

Given an open loop system with transfer function of $C(s)/R(s) = 1/(s^3 + s^2 + 2s + 23)$:



The characteristic equation of the given open loop system is given as follows:

$$q(s) = s^3 + s^2 + 2s + 23 = 0$$

Determine the system using Routh-Hurwitz criterion and MATLAB whether it is stable or not?

Solution

Apply the feedback system equation to the system given above, to obtain the closed loop transfer function of the system.

$$\frac{C(s)}{R(s)} = \frac{G(s)H(s)}{1 + G(s)H(s)}$$

Note that for unity gain feedback system like the system given above: $G(s) = 1/(s^3 + s^2 + 2s + 23)$ and $H(s) = 1$.

$$\frac{C(s)}{R(s)} = \frac{1/(s^3 + s^2 + 2s + 23)}{1 + 1/(s^3 + s^2 + 2s + 23)} = \frac{1}{(s^3 + s^2 + 2s + 24)}$$

Apply stability analysis of the system using Routh Hurwitz criterion.

s^3	1	2
s^2	1	24
s^1	-22	0
s^0	24	0

The corresponding Routh-Hurwitz array is shown as above. The two sign changes in the first column indicate that there are two roots of the characteristic polynomial in the right-half plane. Hence the closed-loop system is unstable.

Using MATLAB we can verify the Routh-Hurwitz result by directly computing the roots of the characteristic equation. Using the roots function e.g. it computes the roots of a polynomial.

The MATLAB code for simulating the stability analysis of the system is given as follows:

MATLAB code:

```
% demo31.m

% Stability analysis using root function

numg=[1];

deng=[1 1 2 23];

[num,den]=cloop(numg,deng);

roots(den)
```

The result of the MATLAB simulation is as given below:

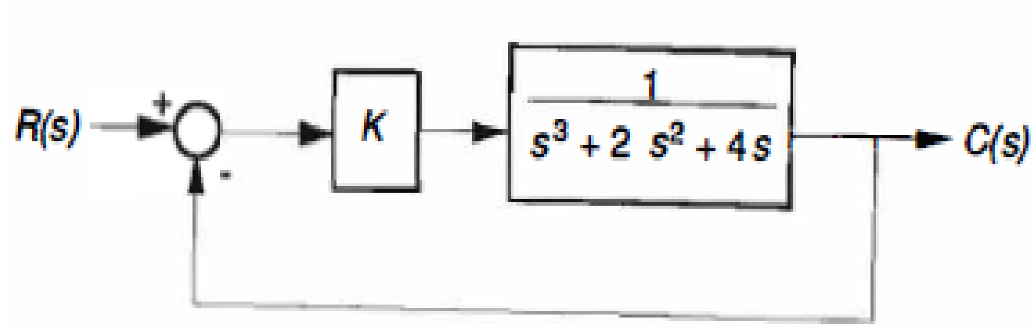
```
ans =

  -3.0000 + 0.0000i
   1.0000 + 2.6458i
   1.0000 - 2.6458i
```

The result shows that there are two complex poles pair e.g. 1 + j2.6458 and 1 - j2.6458 in the right-hand side of the s-plane.

As a result, the system is unstable, and we can verify that the prediction made by Routh-Hurwitz that the stability analysis is correct.

Exercise 2 (Stability Analysis with System Gain)

Given a closed loop control system with transfer function $C(s)/R(s) = K/(s^3 + 2s^2 + 4s + K)$as described in the following figure:



Using a Routh-Hurwitz approach, find the limit of the value of system gain, K for stability. Use MATLAB to verify this.

Solution

Apply the feedback system equation to the system given above, to obtain the closed loop transfer function of the system.

$$\frac{C(s)}{R(s)} = \frac{G(s)H(s)}{1 + G(s)H(s)}$$

Note that for unity gain feedback system like the system given above: $G(s) = K/(s^3 + 2s^2 + 4s)$ and $H(s) = 1$.

$$\frac{C(s)}{R(s)} = \frac{K/(s^3 + 2s^2 + 4s)}{1 + K/(s^3 + 2s^2 + 4s)} = \frac{K}{s^3 + 2s^2 + 4s + K}$$

Use Routh-Hurwitz stability analysis to find that the value of K for stability.

s^3	1	4
s^2	2	K
s^1	$(8-K)/2$	0
s^0	K	0

Using a Routh-Hurwitz approach, we find that we require $K > 0$ and $K < 8$ for maintaining the first column of the Routh-Hurwitz table to be positive.

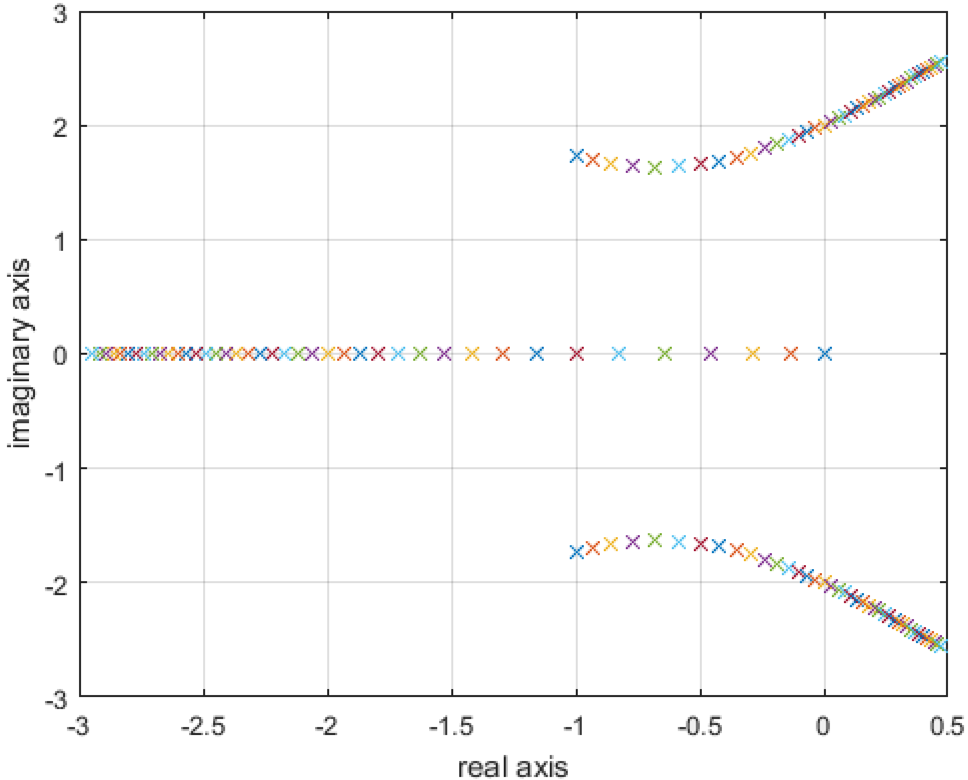
In this example gain of the system: $0 \leq K \leq\ 8$ for stability.

MATLAB code:

```
% demo32.m
% Stability analysis of system

K=[0:0.5:20];

for i= 1 :length(K)
    q=[1 2 4 K(i)];
    p(:,i)=roots(q);
end

plot(real(p),imag(p),’x’)
grid
xlabel(‘real axis’)
ylabel(‘imaginary axis’)
```

For the MATLAB code for verifying the stability analysis, we establish vector values for K at which we wish to compute the roots of the characteristic equation.

Then, using the root function, we calculate and plot the roots of the characteristic equation, as shown in figure below. The result of the MATLAB simulation is given below.



It looks like the complex poles pair locations are moved from left-hand side area to the right-hand side area of the s-plane.

- At $K = 8$ the complex poles pair are located on the imaginary axis of the s-plane.

- When $K > 8$, the poles pair are in the right-hand side area of the s-plane, as a result we have the system to be unstable at this condition.

- In the end, if we keep the gain of the system to be less than 8 e.g. $0 < K < 8$, then the system is still stable.

Notice also that another pole of the system is moved from the origin to the left-hand side of the real axis.

The Routh-Hurwitz method allows us to make definitive statements regarding absolute stability of a linear system. The method does not address the issue of relative stability, which is directly related to the location of the roots of the characteristic equation.

Routh-Hurwitz tells us how many poles lie in the right-half plane, but not the specific location of the poles.

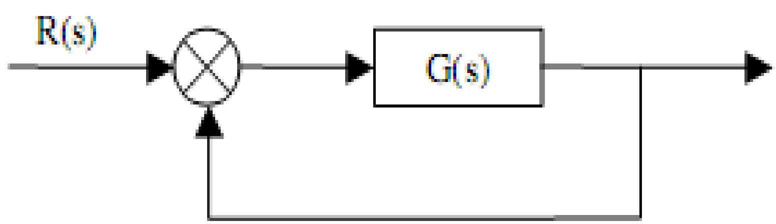
With MATLAB we can easily calculate the poles explicitly, thus allowing us to comment on the system relative stability.

Exercise 3 (Stability Analysis with Range of System Gain)

For the unity feedback system shown in figure below with:

$$G(s) = \frac{K(s+1)}{s(s+1)(s+5)(s+6)}$$

Determine the range of K for stability using Routh-Hurwitz stability analysis and verify this value using MATLAB.



Answer

Use feedbacks system equation to the transfer function of the system given above:

$$\frac{C(s)}{R(s)} = \frac{G(s)H(s)}{1 + G(s)H(s)}$$

The system becomes:

$$G(s) = \frac{K(s+1)}{s(s+1)(s+5)(s+6) + K(s+1)}$$

Rearrange the equation given above, the transfer function of the system becomes:

$$G(s) = \frac{K(s+1)}{s(s+1)(s+5)(s+6) + Ks + K}$$

$$= \frac{K(s+1)}{s^4 + 12s^3 + 41s^2 + (K+30)s + K}$$

Apply Routh Hurwitz on the characteristic equation of the system:

s^4	1	41	K
s^3	12	$K + 30$	0
s^2	$\frac{462 - K}{12}$	K	0
s^1	$K + 30 - \left(\frac{144K}{462 - K}\right)$	0	0
s^0	K	0	0

From the data shown in the Routh-Hurwitz table, $K < 462$ and $K > 0$ for the first column to be made positive and

$$K + 30 - \left(\frac{144K}{462 - K}\right) = \frac{13860 + 288K - K^2}{462 - K}$$

Applying roots of quadratic equation to the equation given above, we obtain $K = 330$ and $K = -42$.

By taking the account all equations for the gain of the system, the gain of the system is $0 < K < 330$ for the stability.

The MATLAB code for determining the range of K for stability for the given feedback system is given as follows.

MATLAB code:

```
% demo33.m

K=[0:0.2:400];

for i=1: length(K);
    deng=poly([0 -1 -5 -6]);
    dent=deng+[0 0 0 K(i) K(i)];
    R=roots(dent);
    A=real(R);
    B=max(A);
    if B > 0
        R
        K=K(i)
        break
    end
end
```

The result of the MATLAB simulation is given below.

```
R =

 -11.0013 + 0.0000i
   0.0007 + 5.4786i
   0.0007 - 5.4786i
  -1.0000 + 0.0000i


K =

  330.2000
```

The result of the simulation shows that the value of K = 330 for the given system to be marginally stable. This confirms the result of the Routh-Hurwitz stability analysis.

Exercise 4 (Other Stability Analysis of Systems 1)

Consider the system having the following transfer function:

$$G(s) = \frac{1 - 2s}{s(1 + s)}$$

Determine the stability of the system above using the following stability analysis methods:

a. Nyquist plot.
b. Nichols chart.
c. Root locus diagram.
d. Bode plots.

Answer

a. The transfer function of the system is as given below:

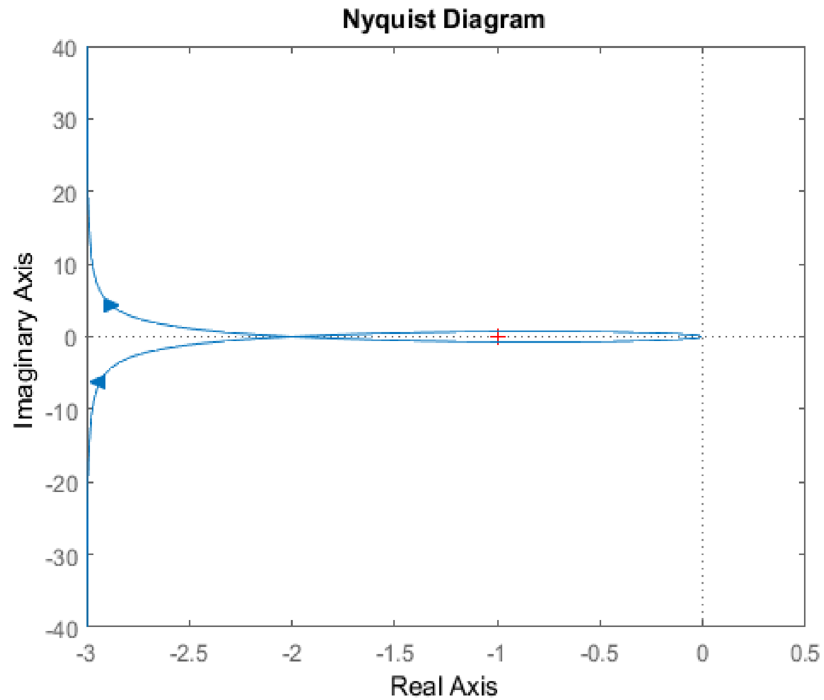
$$G(s) = \frac{1-2s}{s(1+s)} = \frac{-2s+1}{s^2+s}$$

By simulating the system in MATLAB with `nyquist()` function, the Nyquist diagram of the system is given in the figure below.

MATLAB code:

```
% demo34a.m

% Stability analysis using Nyquist plot

H = tf([-2 1],[1 1 0]);

nyquist(H)
```

The result of the MATLAB simulation is given in the graph below.



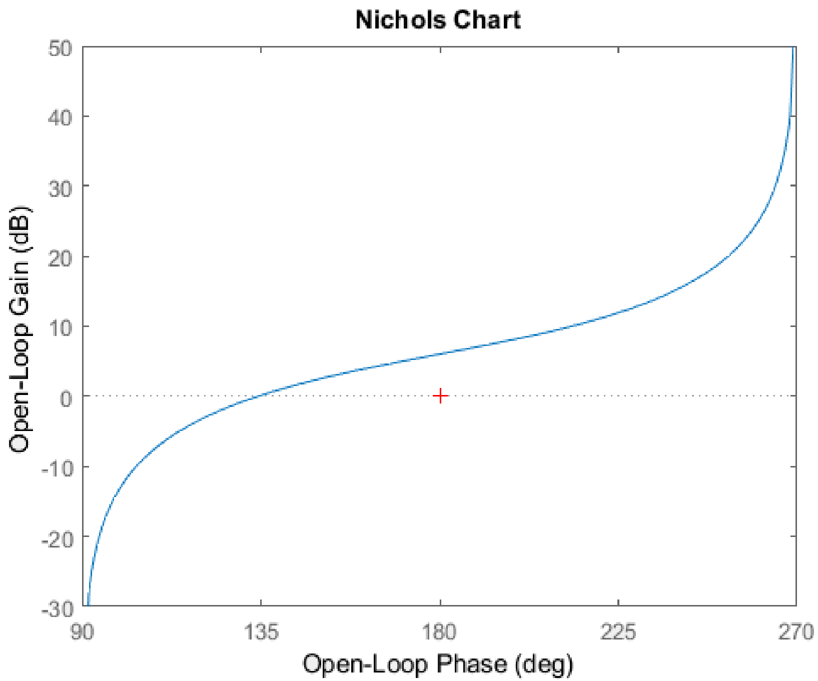
Based on the graph given above, it was found that the contour of the plot encircle the unity gain point (-1, j0). As a result, the system is found to be unstable.

b. The MATLAB code for stability analysis of the system using the Nichols chart is given as follows.

MATLAB code:

```
% demo34b.m

% Stability analysis using Nichols chart

H=tf([-2 1],[1 1 0]);

nichols(H)

% ngrid
```

The result of the MATLAB simulation is given in the graph below.



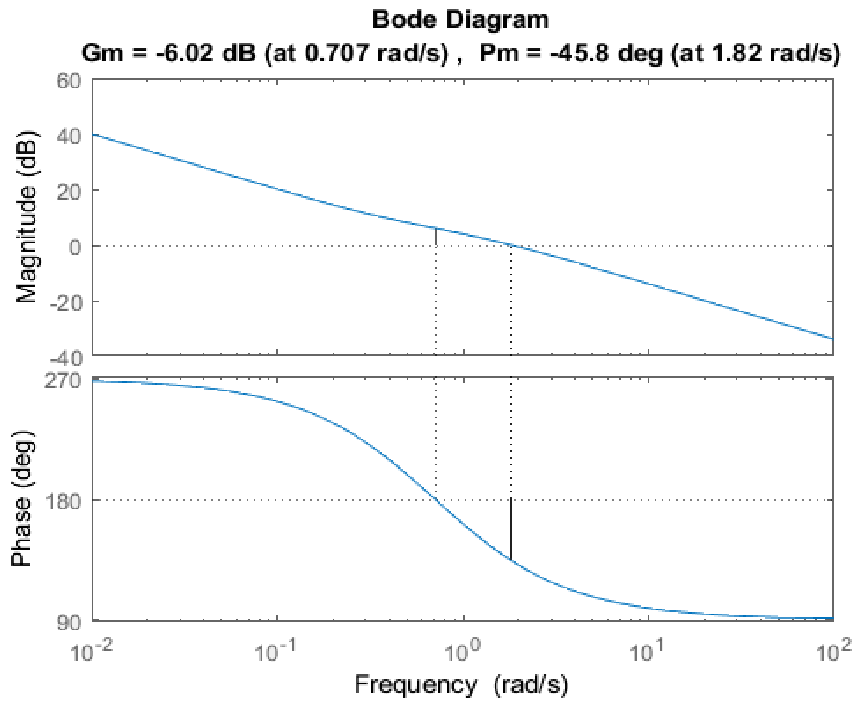
The result of the simulation shows that the gain margin of the system at 180° is negative. So, the system is unstable.

c. By simulating the system in the MATLAB with `bode()` and `margin()` functions, the Bode plot of the system is given in the figure below.

MATLAB code:

```
% demo34c.m

% Stability analysis using Bode plots

H=tf([-2 1],[1 1 0]);

bode(H)

margin(H)
```

The result of the MATLAB simulation is given in the graph below.



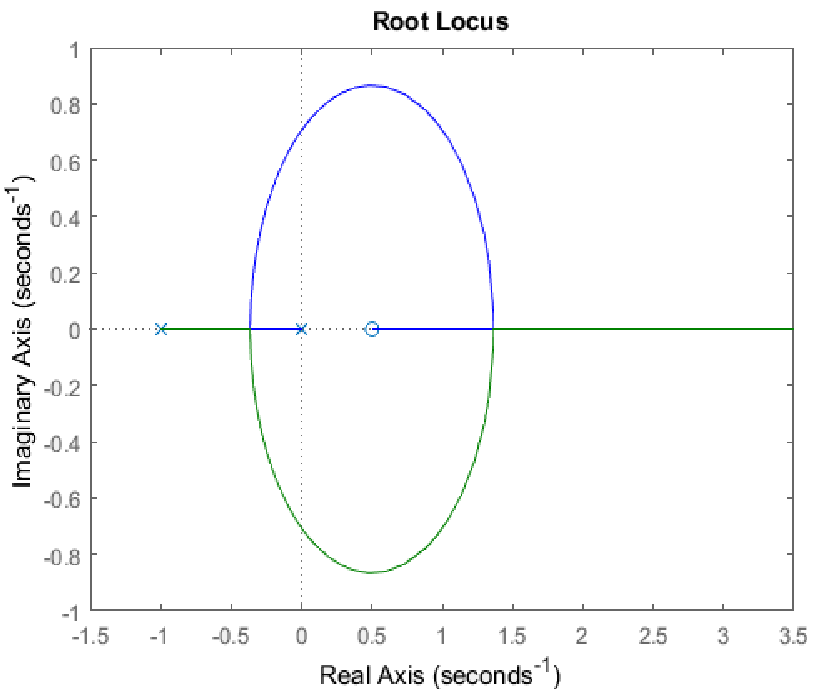
From the graph given above, the gain and phase margins of the system are -6.02 dB (at 0.707 rad/s) and -45.8 degree (at 1.82 rad/s). Since both the margins are negative values, as a result the system is found to be unstable.

d. By simulating the system in MATLAB with `rlocus()` function, the root-locus diagram of the system is given in the figure below.

MATLAB code:

```
% demo34d.m

% Stability analysis using root locus diagram

H = tf([-2 1],[1 1 0]);

rlocus(H)
```

The result of the MATLAB simulation is given in the graph below.



Analysing the graph given above, at low value of gain of the system, K, there is pole simple pole and a pole at origin then the system is found to be marginally stable.

Looking at the root locus that at high value of K, there are poles that exist in the right-hand side region in the s-plane before settling down to the zero and (plus) infinity. As a result, the system is found to be unstable.

Exercise 5 (Other Stability Analysis of Systems 2)

Consider the system having the following transfer function:

$$G(s) = \frac{s^2 + 2s + 16}{s(s-1)(s+1)}$$

Determine the stability of the system using:

a. Nyquist plot.
b. Nichols chart.
c. Bode plots.
d. Root locus diagram.

Answer

a. The transfer function of the system is as given below:

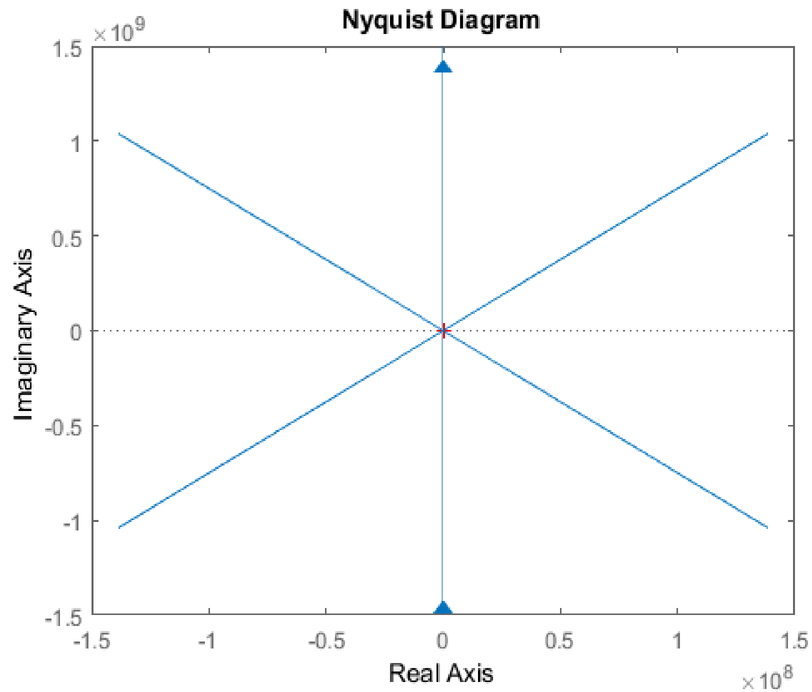
$$G(s)=\frac{s^2+2s+16}{s(s-1)(s+1)}=\frac{s^2+2s+16}{s(s^2-1)}=\frac{s^2+2s+16}{s^3-s}$$

By simulating the system in MATLAB, the Nyquist diagram of the system is shown as follow.

MATLAB code:

```
% demo35a.m

% Stability analysis using Nyquist diagram

H=tf([1 2 16],[1 0 1 0]);

nyquist(H)
```

The result of the MATLAB simulation is given in the graph below.



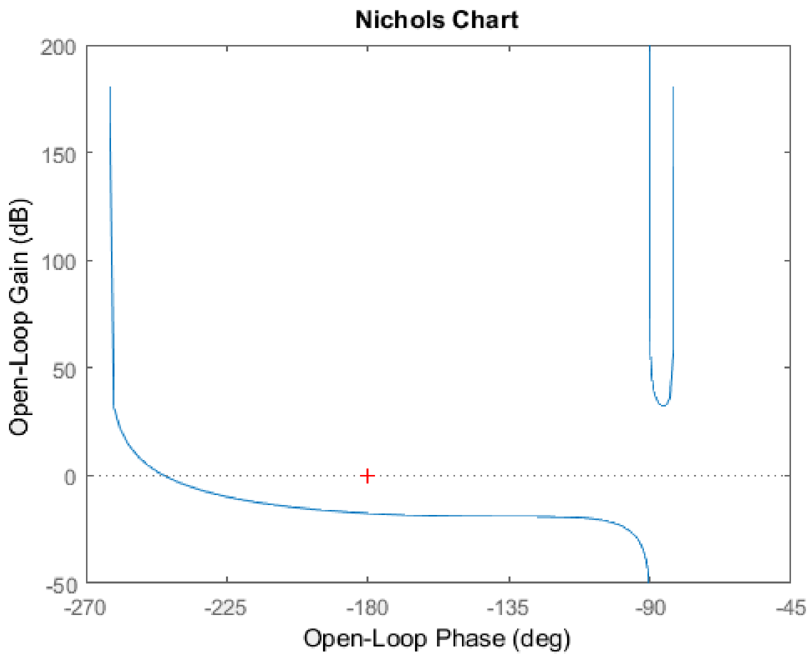
By referring to the graph given above, it was found that the contour of the plot encircle the unity gain point (-1, j0). As a result, the system is found to be unstable.

b. The MATLAB code for stability analysis of the system using the Nichols chart is given as follows.

MATLAB code:

```
% demo35b.m

% Stability analysis using Nichols chart

H=tf([1 2 16],[1 0 1 0]);

nichols(H)

% ngrid
```

The result of the MATLAB simulation is given in the graph below.



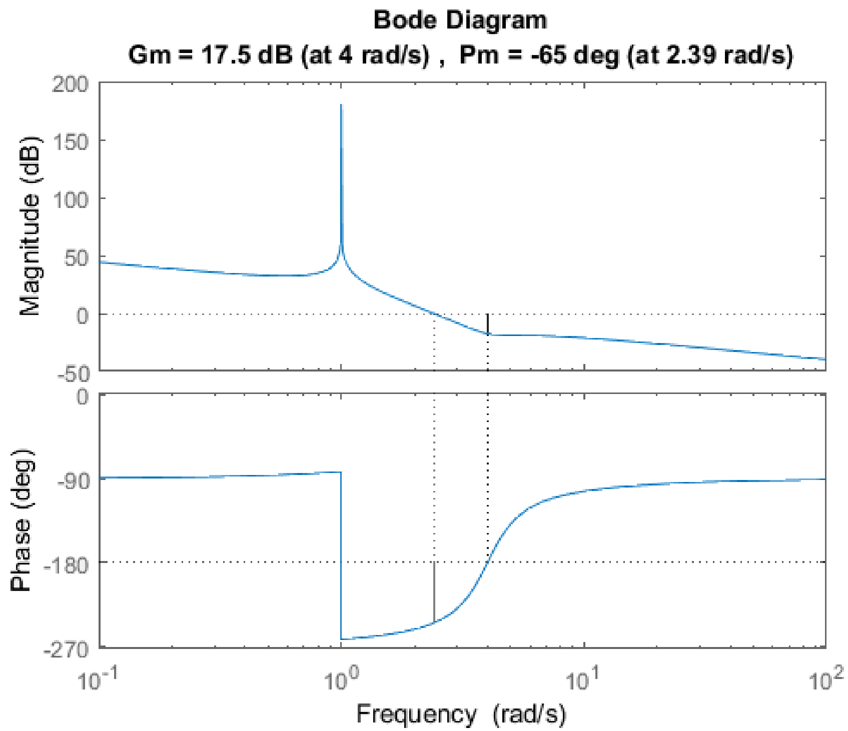
The result of the simulation shows that the gain margin of the system at 180° is negative. So, the system is unstable.

c. By simulating the system in MATLAB, the Bode plot of the system is given below.

MATLAB code:

```
% demo35c.m

% Stability analysis using Bode plot

H=tf([1 2 16],[1 0 1 0]);

bode(H)

margin(H)
```

The result of the MATLAB simulation is given in the graph below.



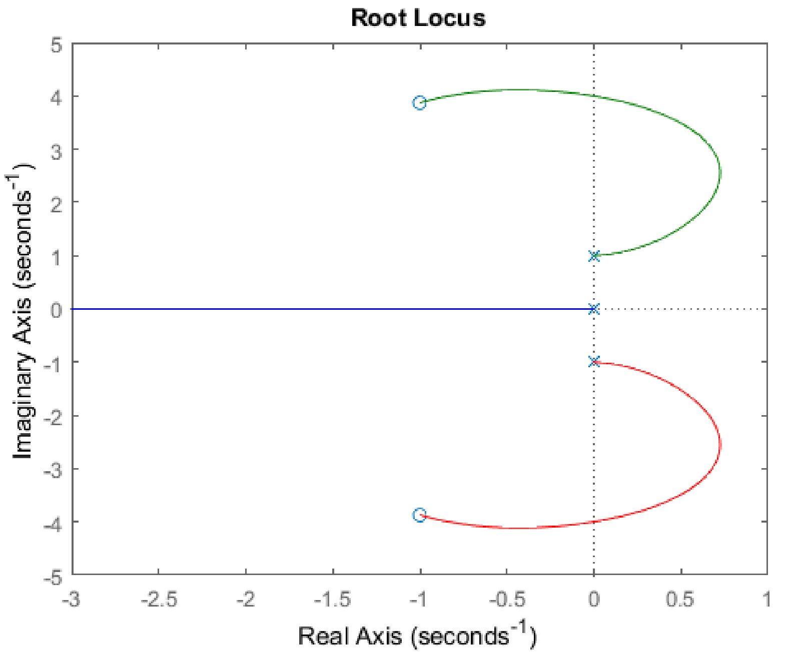
As given on the graph above, the gain and phase margins of the system are 17.5 dB (at 4 rad/s) and -65° (at 2.39 rad/s). Since one of the margins is found to be negative, as a result the system is found to be unstable.

d. By simulating the system in MATLAB, the root locus diagram of the system is shown in the figure below.

MATLAB code:

```
% demo35d.m

% Stability analysis using root locus diagram

H=tf([1 2 16],[1 0 1 0]);

rlocus(H)
```

The result of the MATLAB simulation is given in the graph below.



It looks like in the graph given above, at low value of gain of the system, K there are two complex poles and a pole at the origin, the system is found to be marginally stable.

At higher value of K i.e. at specific range of K, there are poles that exist in the right-hand side region in the s-plane, before the loci of the complex poles settling down to their respective zeros . As a result, the system is found to be unstable for a certain range of values of K.