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Demo 6: Root Locus Analysis

XMUT315 Control Systems Engineering

Topics

1. Poles location of system.
2. Poles location of feedback system.
3. Root locus plot of second order system.
4. Root locus plot of complex system.
5. Root locus of feedback system.
6. Root locus plot for stability analysis.

Exercise 1 (Poles Location of System)

Determine the pole locations for the system shown below using MATLAB.

$$\frac{C(s)}{R(s)} = \frac{s^3 - 6s^2 + 7s + 15}{s^5 + s^4 - 5s^3 - 9s^2 + 11s - 12}$$

Solution

The following MATLAB code determine the pole locations of the system given above.

MATLAB code:

```
den=[1 1 -5 -9 11 -12];

A=roots(den)
```

The outcome of the simulation is given below.

```
A =

  -2.1586 + 1.2396i
  -2.1586 - 1.2396i
   2.3339 + 0.0000i
   0.4917 + 0.7669i
   0.4917 - 0.7669i
```

Exercise 2 (Poles Location of Feedback System)

Determine the pole locations for the unity feedback system shown below using MATLAB.

$$G(s) = \frac{150}{(s+5)(s+7)(s+9)(s+11)}$$

Solution

The following MATLAB code determines the pole locations of the feedback system given above.

MATLAB code:

```
numg=150
deng=poly([-5 -7 -9 -11]);

'G(s)'
G=tf(numg,deng)

'Poles of G(s)'
pole(G)

'T(s)'
T=feedback(G,1)

pole(T)
```

The outcome of the simulation is given below.

```
numg =

   150


ans =

    'G(s)'


G =

                    150
  ----------------------------------------
  s^4 + 32 s^3 + 374 s^2 + 1888 s + 3465

Continuous-time transfer function.
```

```
ans =

    'Poles of G(s)'


ans =

  -11.0000
   -9.0000
   -7.0000
   -5.0000


ans =

    'T(s)'
```

```
T =

                      150
  ---------------------------------------
  s^4 + 32 s^3 + 374 s^2 + 1888 s + 3615

Continuous-time transfer function.


ans =

 -10.9673 + 1.9506i
 -10.9673 - 1.9506i
  -5.0327 + 1.9506i
  -5.0327 - 1.9506i
```

Exercise 3 (Root Locus Plot of Second Order System)

A plant to be controlled is described by a transfer function

$$G(s) = \frac{s+5}{s^2+7s+25}$$

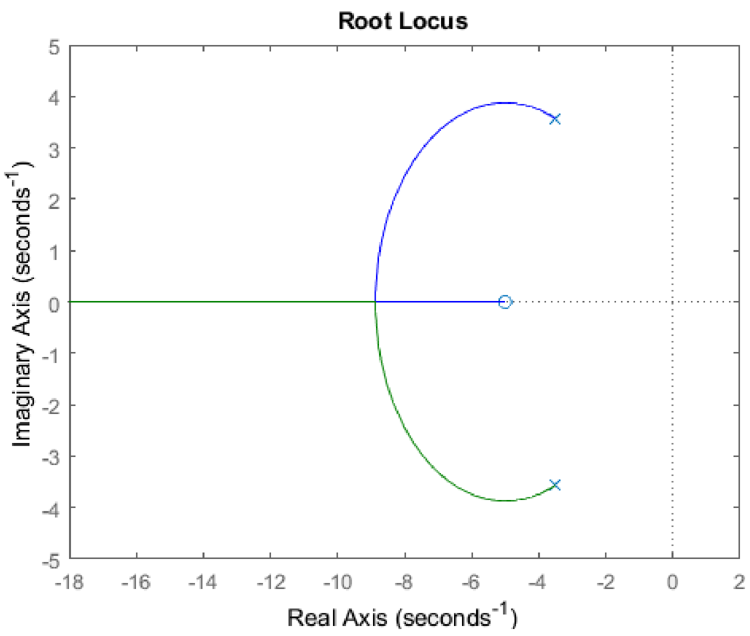
Obtain the root locus plot using MATLAB.

Solution

The following MATLAB code the root locus diagram of the system given above is obtained.

```
clf

num=[1 5];
den=[1 7 25];

rlocus(num,den);
```

Computer response of the simulation is shown in the figure below.



The root locus diagram shows the complex pole on the top is settling to the zero and the complex pole pair on the bottom is to $-\infty$.

Exercise 4 (Root Locus Plot of Complex System)

Plot the root-locus diagram using MATLAB for a system whose open-loop transfer function $G(s)H(s)$ is given by:

$$G(s)H(s) = \frac{K(s+3)}{(s^2+3s+4)(s^2+2s+7)}$$

Solution

The transfer function of the given system becomes:

$$G(s)H(s) = \frac{K(s+3)}{(s^2+3s+4)(s^2+2s+7)}$$

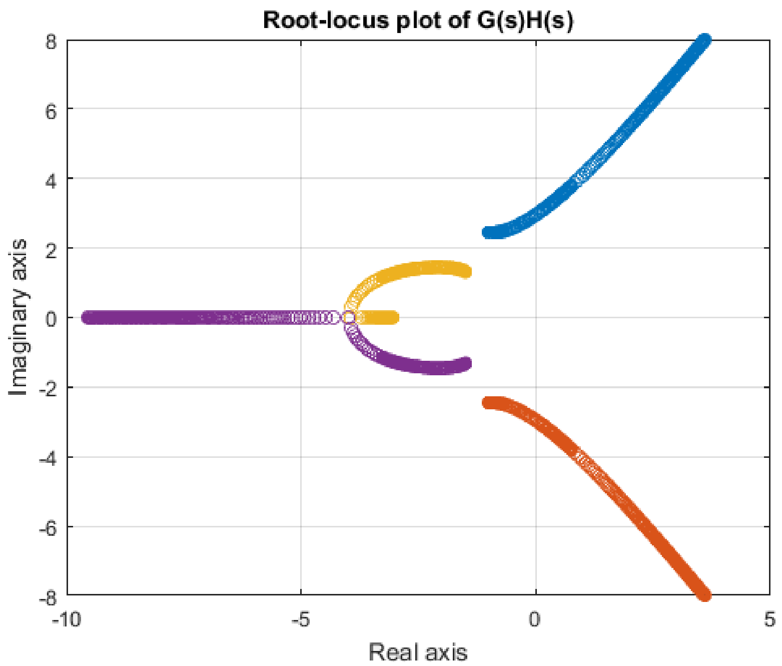
$$= \frac{K(s+3)}{(s^4+5s^3+17s^2+29s+28)}$$

MATLAB code:

```
num=[0 0 0 1 3];
den=[1 5 17 29 28];

K1=0:0.1:2;
K2=2:0.02:2.5;
K3=2.5:0.5:10;
K4=10:1:50;
K5=50:5:800;
K=[K1 K2 K3 K4 K5];

r=rlocus(num,den,K);

plot(r,'o')
```

```
v=[-10 5 -8 8]; axis(v)
grid

title('Root-locus plot of
G(s)H(s)')
xlabel('Real axis')
ylabel('Imaginary axis')
```

The result of the root locus simulation of the system is given below.



The root locus shows various range of gain of the systems and their relevant locus in the diagram.

Exercise 5 (Root Locus of Feedback System)

A unity-feedback control system is defined by the following feedforward transfer function:

$$G(s) = \frac{K}{s(s^2 + 5s + 9)}$$

a. Determine the location of the closed-loop poles, if the value of gain is equal to 3.

b. Plot the root loci for the system using MATLAB.

Solution

a. The following MATLAB code simulate the root locus diagram of the system given above.

MATLAB code:

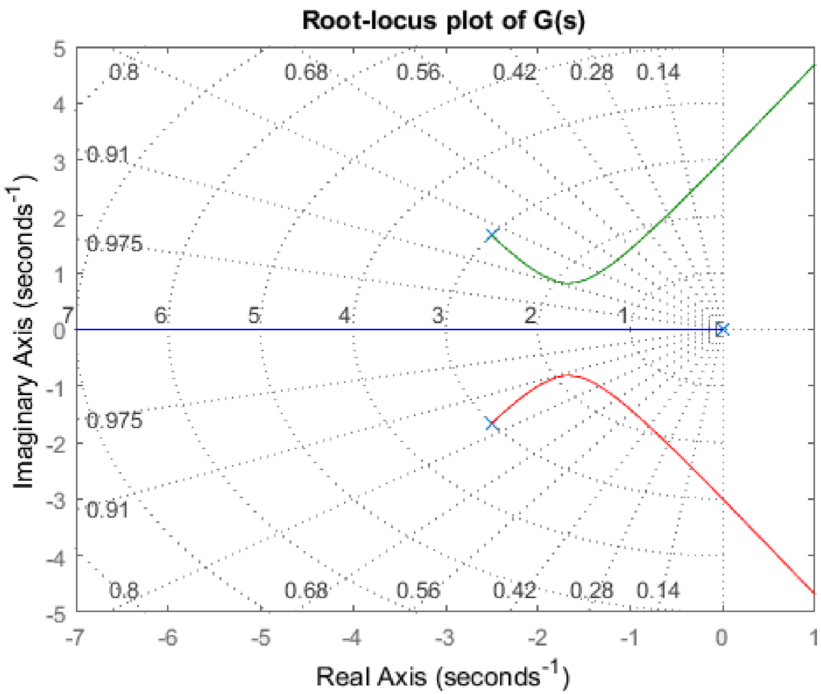
```
p=[1 5 9 3];
r=roots(p)

% Plot the root-loci

num=[0 0 0 1];
den=[1 5 9 0];

rlocus(num,den);

axis('square')

grid

title('Root-locus plot of G(s)')
```

The results of the finding roots of the equation.

```
r =

  -2.2874 + 1.3500i
  -2.2874 - 1.3500i
  -0.4253 + 0.0000i
```

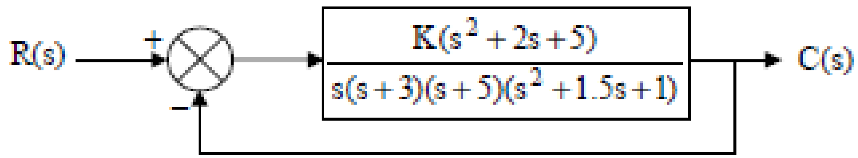
b. The root locus plot simulation is given below.

The root locus diagram of the system shows that the system is conditionally stable with value of gain, K, determine the stability of the system (e.g. crossing points when the locus pass the imaginary axis).



Exercise 6 (Root Locus Plot for Stability Analysis 1)

For the closed-loop control system shown in the figure below, obtain the range of gain K for stability and plot a root-locus diagram for the system.



Solution

The range of gain K for stability is obtained by first plotting the root loci and then finding critical points (for stability) on the root loci. The open-loop transfer function G(s) is:

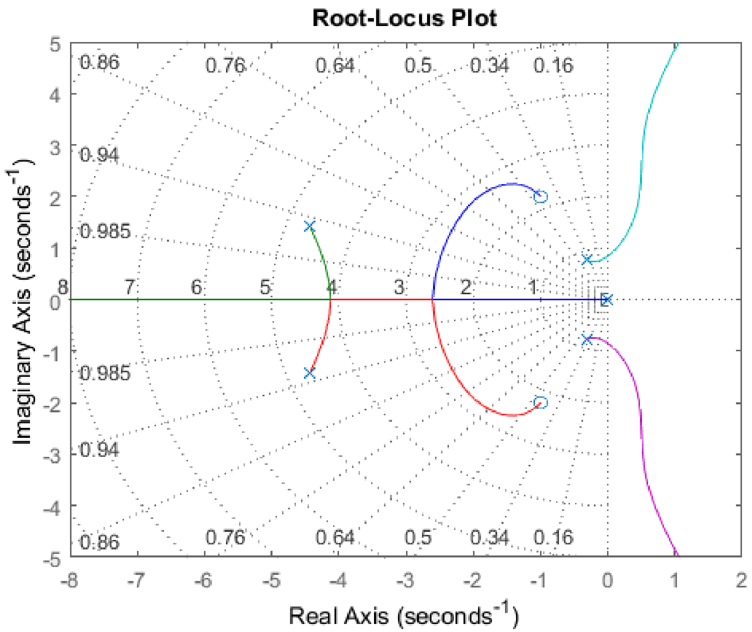
$$G(s) = \frac{K(s^2+2s+5)}{s(s+3)(s+5)(s^2+1.5s+1)} = \frac{K(s^2+2s+5)}{s^5+9.5s^4+28s^3+20s^2+15s}$$

A MATLAB program to generate a plot of the root loci for the system is given below.

MATLAB code:

```
num=[0 0 0 1 2 5];
den=[1 9.5 28 20 15 0];

rlocus(num,den)
v=[-8 2 -5 5];

axis(v);
axis('square')
grid
title('Root-Locus Plot')
```

The result of the plotting root locus diagram simulation is given below.



From the figure given above, we notice that the system is conditionally stable. All critical points for stability lie on the $j\omega$ axis.

To obtain the crossing points of the root loci with the $j\omega$ axis, we substitute $s = j\omega$ into the characteristic equation.

$$s^5 + 9.5s^4 + 28s^3 + 20s^2 + 15s + K(s^2 + 2s + 5) = 0$$

Or

$$(j\omega)^5 + 9.5(j\omega)^4 + 28(j\omega)^3 + (20 + K)(j\omega)^2 + (15 + 2K)(j\omega) + 5K = 0$$

Or

$$[9.5\omega^4 - (20 + K)\omega^2 + 5K] + j[\omega^5 - 28\omega^3 + (15 + 2K)\omega] = 0$$

Equating the real part and imaginary part equal to zero, respectively, we get (Eq.1):

$$9.5\omega^4 - (20 + K)\omega^2 + 5K = 0$$

And (Eq. 2)

$$\omega^5 - 28\omega^3 + (15 + 2K)\omega = 0$$

Eq. (2) can be written as:

$$\omega = 0$$

Or (Eq. 3)

$$\omega^4 - 28\omega^2 + 15 + 2K = 0$$

And we can find values of K from (Eq. 4):

$$K = \frac{-\omega^4 + 28\omega^2 - 15}{2}$$

Substituting Eq. 4 into Eq. 1, we obtain:

$$9.5\omega^4 - \left[20 + \frac{-\omega^4 + 28\omega^2 - 15}{2}\right]\omega^2 + 5\left[\frac{-\omega^4 + 28\omega^2 - 15}{2}\right]$$

Or

$$0.5\omega^6 - 2\omega^4 + 57.5\omega^2 - 37.5$$

The roots of the above equation can be obtained by MATLAB program given below.

MATLAB code:

```
% demo76b.m

% Determining roots of equation

a=[0.5 0 -2 0 57.5 0 -37.5];

r=roots(a)
```

Output of the simulation is given below:

```
r =

   2.4786 + 2.1157i
   2.4786 - 2.1157i
  -2.4786 + 2.1157i
  -2.4786 - 2.1157i
  -0.8155 + 0.0000i
   0.8155 + 0.0000i
```

The root-locus branch in the upper half plane that goes to infinity crosses the jω axis at $\omega = 0.8155$.

Applying this value to Eq. 4, the gain values at these crossing points are given by:

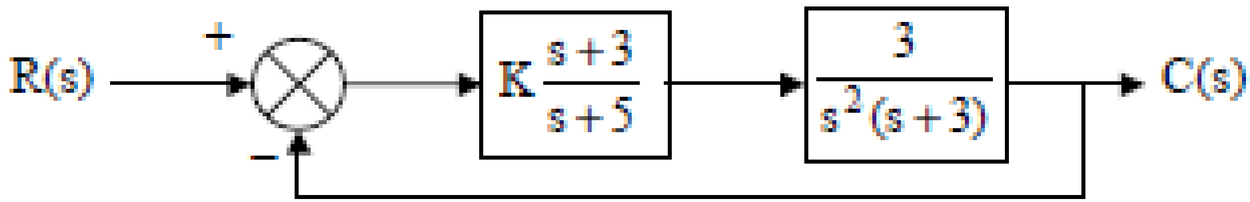
$$K = \frac{-(0.8155)^4 + 28 \times (0.8155)^2 - 15}{2} = 1.5894 \qquad for \quad \omega = 0.8155$$

For this K value, we obtain the range of gain K for stability as: $1.5894 > K > 0$

Exercise 7 (Root Locus for Stability Analysis 2)

For the control system shown in the figure below:

a. Plot the root loci for the system.
b. Find the range of gain K for stability.



Solution

a. The open-loop transfer function G(s) is given by:

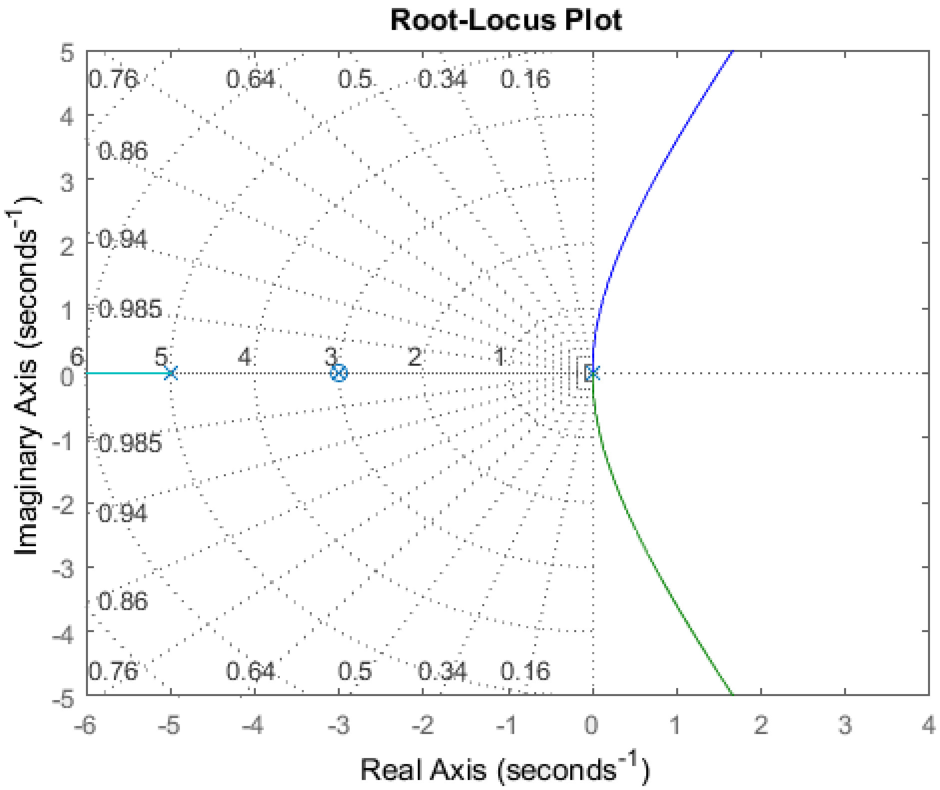
$$G(s) = K\left[\frac{s+3}{s+5}\right]\left[\frac{3}{s^2(s+3)}\right] = \frac{3K(s+3)}{s^4+8s^3+15s^2}$$

A MATLAB program to generate the root-locus plot is given below.

MATLAB code:

```
num=[0 0 0 1 3];
den=[1 8 15 0 0];

rlocus(num,den)

v=[-6 4 -5 5];
axis(v);
axis('square)
grid
title('Root-Locus Plot')
```

The resulting root locus plot of the system is shown in the figure below.



The system is found to be conditionally stable (e.g. depending on the value of the gain of the system, K, it could become unstable).

b. From the figure given above, we notice that the critical value of gain K for stability corresponds to the crossing point of the root locus branch that goes to infinity and the imaginary axis.

Therefore, we first find the crossing frequency and then find the corresponding gain value. The characteristic equation is:

$$s^4 + 8s^3 + 15s^2 + 3Ks + 9K = 0$$

Substituting $s = j\omega$ into the characteristic equation, we get:

$$(j\omega)^4 + 8(j\omega)^3 + 15(j\omega)^2 + 3K(j\omega) + 9K = 0$$

Or

$$(\omega^4 - 15\omega^2 + 9K) + j\omega(-8\omega^2 + 3K) = 0$$

Equating the real part and imaginary part of the above equation to zero, respectively, we obtain (Eq. 1):

$$\omega^4 - 15\omega^2 + 9K = 0$$

And (Eq. 2)

$$\omega(-8\omega^2 + 3K) = 0$$

Eq. 2 can be rewritten as:

$$\omega = 0$$

Or (Eq. 3)

$$-8\omega^2 + 3K = 0$$

Substituting the value of K in Eq. 1, we get:

$$\omega^4 - 15\omega^2 + 9\left(\frac{8}{3}\omega^2\right) = 0 \qquad \text{or} \qquad \omega^4 + 9\omega^2 = 0$$

Which gives

$$\omega = 0 \qquad \text{and} \qquad \omega = \pm j3$$

Since $\omega = j3$ is the crossing frequency with the $j\omega$ axis, by substituting $\omega = 3$ into Eq. 3 we obtain the critical value of gain K for stability as:

$$K = \frac{8}{3}\omega^2 = \frac{8}{3}(3)^2 = 24$$

Therefore, the stability range for K is: $0 < K < 24$