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Introduction to Bode Plots

XMUT315 Control System Engineering

Topics

- Frequency response methods.
- Foundation of frequency response.
- Introduction to Bode plots.
- Bode plots and transfer functions.
- Examples of Bode plots.
- Resonance response.
- Right-hand plane roots.

Frequency Response Methods

- Frequency response methods are a set of graphical techniques that focus on how the gain and phase of a system change with frequency.
- Frequency responses can be used when we have a good mathematical model for a system (a transfer function).
- Frequency response methods can also be used even when we don’t have a good model of the system (plant) that we are trying to stabilise.
- We just use experimental data in place of a model.

Frequency Response Overview

We will cover in this course, detailed construction of:

- Bode plots (plots of gain and phase vs. frequency),
- Root locus diagrams (plot of real vs imaginary parts of transfer function)
- Nyquist plots (plot of gain vs. phase).
- If time permits we will also look at the Nichols plot, which can be considered an alternate form of the Nyquist plot.

Frequency Response Overview

For each plot type, we will discuss:

- how to assess system stability,
- how to determine the closed-loop characteristics of a control-loop system, and
- how to design compensators (or controllers).

Foundations of Frequency Response

- Consider a system described by the transfer function:

$$G(s) = \frac{N(s)}{D(s)}$$

Where: $N(s)$ and $D(s)$ are polynomials such that $G(s)$ is proper.

- Let us apply a sinusoidal signal $r(t)$ at the input of the system and determine the corresponding output signal $y(t)$.

$$r(t) = A\cos(\omega t)u(t)$$

$$R(s) = A\left(\frac{s}{s^2 + \omega^2}\right)$$

- The Laplace transform of the resulting output signal is then:

$$Y(s) = G(s)R(s) = A\left(\frac{s}{s^2 + \omega^2}\right)G(s)$$

Foundations of Frequency Response

- If we expand this using partial fractions, we obtain:

$$Y(s) = \frac{c}{s - j\omega} + \frac{c^*}{s + j\omega} + \frac{V(s)}{D(s)}$$

- Where: the first two terms arise from the sinusoidal excitation and the last term arises from the poles of the system (as contained in $D(s)$).
- The polynomial $V(s)$ here arises from the polynomial simplification.
- We can use the Heaviside method to find c and c^*.

$$c = (s - j\omega)Y(s)\Big|_{s\to j\omega} = A\left[\frac{s(s - j\omega)}{s^2 + \omega^2}\right]G(\mathrm{s})\Bigg|_{s\to j\omega}$$

Foundations of Frequency Response

- The equation becomes:

$$c = A\left[\frac{s(s-j\omega)}{(s+j\omega)(s-j\omega)}\right]G(s)\Bigg|_{s-j\omega}$$

- As a result:

$$c = A\left(\frac{s}{s+j\omega}\right)G(s)\Bigg|_{s\rightarrow j\omega} = A\left(\frac{j\omega}{j\omega+j\omega}\right)G(j\omega)$$

- Thus

$$c = \left(\frac{A}{2}\right)G(j\omega) \qquad \text{and} \qquad c^{*} = \left(\frac{A}{2}\right)G^{*}(j\omega)$$

Foundations of Frequency Response

- Substituting back into the expression for $Y(s)$, we obtain:

$$Y(s) = \left(\frac{A}{2}\right)\left(\frac{G(j\omega)}{s - j\omega}\right) + \left(\frac{A}{2}\right)\left(\frac{G^*(j\omega)}{s + j\omega}\right) + \frac{V(s)}{D(s)}$$

- We now take the inverse Laplace transform to move back into the time domain.

$$y(t) = \left(\frac{A}{2}\right) G(j\omega)e^{j\omega t} + \left(\frac{A}{2}\right) G^*(j\omega)e^{-j\omega t} + \mathcal{L}^{-1}\left(\frac{V(s)}{D(s)}\right)$$

Where: the term $\mathcal{L}^{-1}\left(\frac{V(s)}{D(s)}\right) = y_{tr}(t)$ is a transient arising from the poles of the system.

Foundations of Frequency Response

- If the system is stable, then the transient will (eventually) decay to zero and we will be left with just the first two terms.
- After the decay of the transient, we have:

$$y(t) = \left(\frac{A}{2}\right) G(j\omega)e^{j\omega t} + \left(\frac{A}{2}\right) G^*(j\omega)e^{-j\omega t}$$

$$= \left(\frac{A}{2}\right)\left(|G(j\omega)|e^{j\theta}e^{j\omega t} + |G^*(\omega t)|e^{-j\theta}e^{-j\omega t}\right)$$

$$= \left(\frac{A}{2}\right)\left(|G(j\omega)|e^{j(\omega t+\theta)} + |G^*(\omega t)|e^{-j(\omega t+\theta)}\right)$$

Foundations of Frequency Response

- Applying exponential form to trigonometry identity:

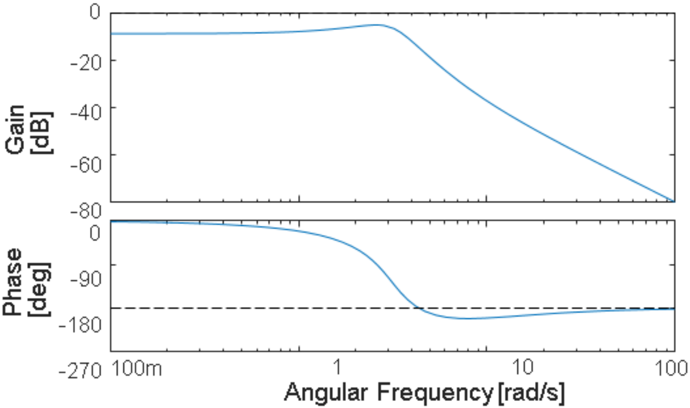
$$y(t) = \left(\frac{A}{2}\right)|G(j\omega)|2\cos(\omega t + \theta)$$

$$= A|G(j\omega)|\cos(\omega t + \theta) \quad \text{where: } \theta = \angle G(j\omega)$$

- If we compare this with the input signal $r(t)$, we can see that the effect of the system is therefore to multiply the magnitude of the input signal by $|G(j\omega)|$ and phase shift it by $\angle G(j\omega)$

Bode Plots

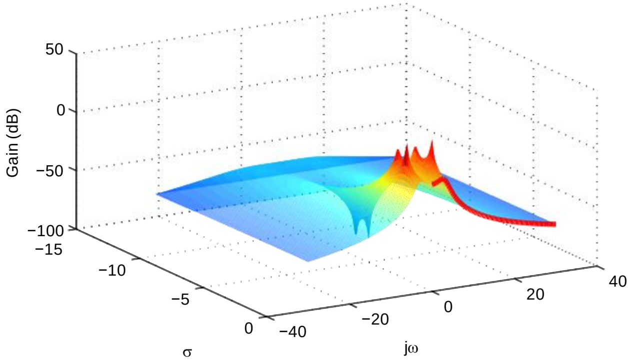
- A Bode plot is a pair of plots showing the variation of gain (in dB) and phase (normally in degrees) against the logarithm of frequency.



- We can use either angular or linear frequency on the x-axis, although angular frequency (in rad/s) is usually more convenient.

Magnitude and Phase Response

- We wish to determine the gain and phase response as a function of ω for a system having transfer function $G(s)$.
- We can find this by plotting $|G(j\omega)|$ and $\angle G(j\omega)$ as we vary ω.



Bode Approximations

- If we need to plot an accurate frequency response, then we could solve the magnitude and phase responses at many points.
- Alternatively, we could find explicit expressions for $|G(j\omega)|$ and $\angle G(j\omega)$.
- Tools like Matlab make plotting an accurate Bode plot very easy – bode(tf([num],[den])) and bode(zpk([z],[p],[k])) produce Bode plots.
- Read the Matlab help file for details and options.

Bode Approximations

- However, much of the time, we only need an approximate frequency response for control design.
- In fact, in many cases, the approximations are easier to work with than the accurate curves would be.
- Bode developed a set of straight-line approximations to the real response curves.
- Using these approximations makes it simple to plot a frequency response by hand.

Bode Plots - The Big Picture

- Bode plots are intended to be quick and easy to draw:
  - They produce reasonably accurate gain and phase responses that are adequate for many purposes.
  - The approximations become less accurate for systems containing lightly damped oscillatory modes.
- The basic idea of the Bode plot is to break a transfer function into smaller simple parts, each of which has a known Bode plot.
- We will build the Bode plot of an arbitrarily complex transfer function by adding the constituent plots graphically.

Dividing a Transfer Function into its Parts

- For example, consider a system having transfer function:

$$G(s) = \frac{K(s+z_1)(s+z_2)}{s(s^3 + d_2 s^2 + d_1 s + d_0)} \qquad \text{for} \quad K \in R$$

- We may be able to break this down into a set of simpler elemental transfer functions, say:

$$G(s) = K(s+z_1)(s+z_2)\left(\frac{1}{s}\right)\left(\frac{1}{s+p_1}\right)\left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right)$$

- Usually, a transfer function is broken down into these terms:
  - Poles and zeros at dc: s^n for $n \in Z$
  - Simple poles and zeros: $(s+a)^{\pm 1}$
  - Complex pairs of poles and zeros: $(s + 2\zeta\omega s + \omega_n^2\ )^{\pm 1}$

Magnitude Response of an Arbitrary TF

- Let's derive an equation for the gain and phase of an arbitrary transfer function $G(s)$, where:

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_k)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

- That is, the transfer function has k zeros at $-z_i \in \ C$ and n poles at $s = -p_i \in \ C$.
- The z_i and p_i are not necessarily distinct.
- The magnitude of $G(s)$ at $s = j\omega$ is:

$$|G(j\omega)| = \frac{|K||j\omega+z_1||j\omega+z_2|\ldots|j\omega+z_k|}{|j\omega+p_1||j\omega+p_2|\ldots|j\omega+p_n|}$$

Magnitude Response of an Arbitrary TF

- Converting to dB, we write this as:

$$
\begin{aligned}
20\log|G(j\omega)| = 20\log|K| \\
&+20\log|j\omega + z_1| + \cdots + 20\log|j\omega + z_k| \\
&-20\log|j\omega + p_1| - \cdots - 20\log|j\omega + p_n|
\end{aligned}
$$

- To find the magnitude response of our overall function in dB, we can find the magnitude responses arising from each pole and zero separately and then add them.
- This is why we use a dB scale, because otherwise we would have to go to the bother of multiplying the individual responses.
- We shall see shortly that adding the plots graphically is trivial.

Phase Response of an Arbitrary TF

- Recall that a complex number $a = bc/de$ will have a phase given by:

$$\angle a = \angle b \ + \angle c \ - \angle d \ - \angle e$$

- Similarly, our transfer function:

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_k)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

- This will have a phase response of:

$$\begin{aligned}\angle G(s) = \angle K \\ + \angle(s+z_1) + \ \angle(s+z_2) + \cdots + \ \angle(z_k) \\ - \angle(s+p_1) - \ \angle(s+p_2) - \cdots - \ \angle(s+p_n)\end{aligned}$$

Phase Response of an Arbitrary TF

- Remember that $K \in R$, so $\angle K = 0$ if $K > 0$, or $180°$ if $K < 0$.
- If we calculate the phase responses for our family of prototype pole/zero combinations, we will be able to add them to determine the overall phase response of an arbitrary transfer function.

Transfer Functions for Bode Plots

- It is easier to draw a Bode plot if each term in TF has unity gain at DC. Normally we write a transfer function as:

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_k)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

- But we will find it easier for Bode plotting if we first place it in the equivalent form:

$$G(s) = \frac{K\left(1+\frac{s}{z_1}\right)\left(1+\frac{s}{z_2}\right)\ldots\left(1+\frac{s}{z_k}\right)}{\left(1+\frac{s}{p_1}\right)\left(1+\frac{s}{p_2}\right)\ldots\left(1+\frac{s}{p_n}\right)}$$

- Converting to this form is accomplished by dividing through by the constant in each term and adjusting overall gain to compensate.

Example of TF Modification

Convert the transfer function given below into a suitable modified form for Bode plots.

$$G(s) = \frac{s+10}{s(s+2)(s^2+3s+9)}$$

a. Calculate the form manually. [4 marks]

b. Use simulation in MATLAB. [5 marks]

Example of TF Modification

- Manually, the modified equation is:

$$G(s) = \frac{10s + 2\left(\frac{s}{10} + 1\right)}{s \times 2\left(\frac{s}{2} + 1\right) \times 9\left(\frac{s^2}{3^2} + \frac{s}{3} + 1\right)}$$

$$= \frac{5}{9}\left[\frac{\left(\frac{s}{10} + 1\right)}{s\left(\frac{s}{2} + 1\right)\left(\left(\frac{s}{3}\right)^2 + \frac{s}{3} + 1\right)}\right]$$

- Note the change in the constant (dc) gain term and also the form change for the complex pair of poles, from:

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

Example of TF Modification

- Thus, the equation becomes:

$$\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1$$

- Matlab does not distinguish internally between the two forms of the transfer function that we have discussed.
- However, you can specify which form Matlab uses to present the transfer function.
- This is useful to convert between the two.

Example of TF Modification

```
>> G= zpk(-6,[-1+j -1-j -2],2)

      2(s+6)                    This is the default,
  ---------------               with Display Format
(s+2)(s^2 + 2s + 2)             = 'roots'

>> G.DisplayFormat='frequency';G

                 3 (1+s/6)
   ------------------------------------
(1+s/2)(1+ 1.414(s/1.414)+(s/1.414)^2)
```

Transfer Functions of Form Ks^n

- The magnitude of a transfer function $G(s) = Ks^n$ is given by:

$$|G(j\omega)| = K\omega^n$$

- So, in dB:

$$|G(j\omega)| = 20\log(K\omega)^n = 20\log K + 20n\log\omega$$

- Thus, the transfer magnitude response is a straight line with a slope of $20n$ dB/decade and is equal to $20\log K$ at $\omega = 1$.

$$G(j\omega) = K(j\omega)^n = j^n K\omega^n$$

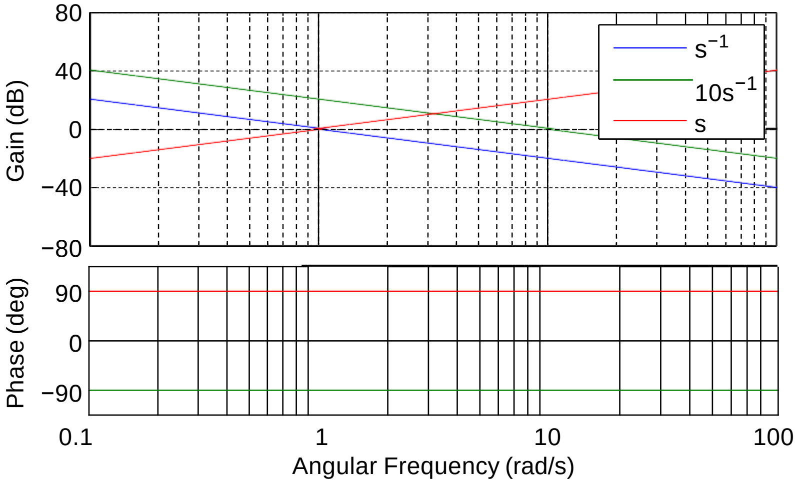
- Thus

$$\angle G(j\omega) = \angle j^n = 90n^\circ \qquad \text{if} \qquad K > 0$$

- The phase of $G(s) = Ks^n$ is constant at $90n$ degrees.

Plot of a TF with Form Ks^n

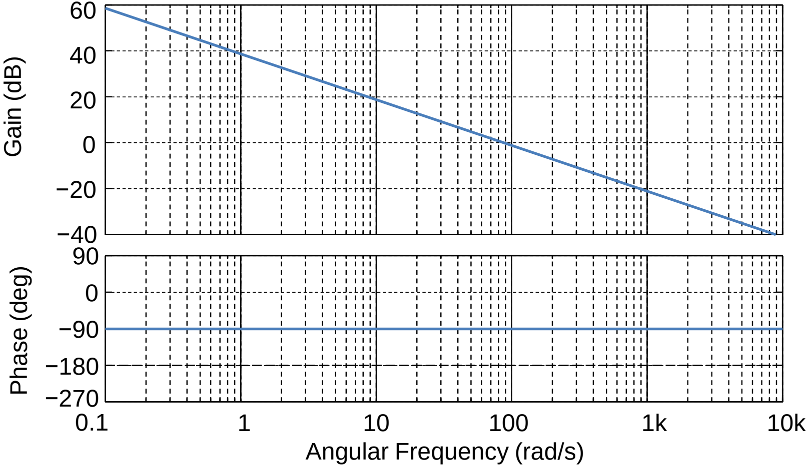
- For example, an integrator (which has $G(s) = 1/s = s^{-1}$) has a magnitude response of $1/\omega$ and a constant phase of −90°.



Example of Gain and Pole

- Sketch Bode plots of system with gain 100 and pole at origin:

$$G(s) = 100/s$$



Example of Negative Gain and Pole

- The gain of the system is:

$$|G(j\omega) = 100/j\omega| = 20\log(100) - 20\log(\omega)$$

$$= 40 - 20\log(\omega)\ \text{dB/dec}$$

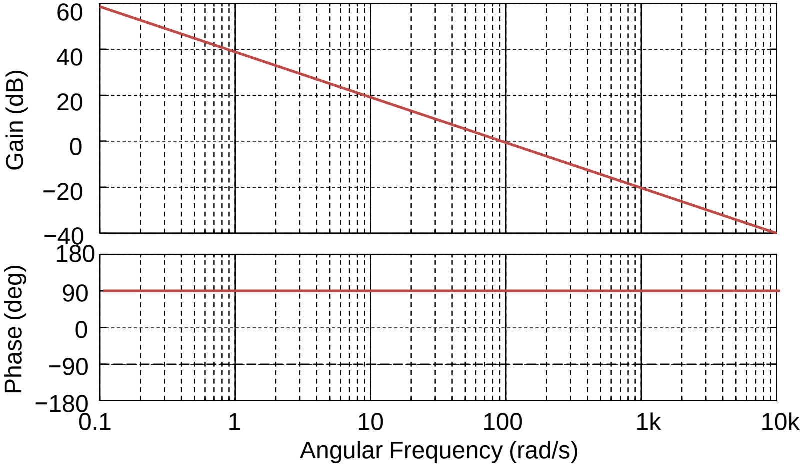
- The phase shift of the system is:

$$\angle(G(j\omega) = 100/j\omega) = -90°$$

Example of Negative Gain and Pole

- Sketch Bode plots of system with gain -100 and pole at origin:

$$G(s) = -100/s$$



Example of Negative Gain and Pole

- Notice the difference between positive gain and negative gain the Bode plot.
- It is difficult to distinguish which gain is positive and which one is negative.

$$|G(j\omega) = -100/j\omega| = 20\log(100) - 20\log(\omega)$$

$$= 40 - 20\log(\omega)\ \text{dB/dec}$$

- To cope with $K < 0$, you just need to account for the extra 180° phase shift associated with the negative gain.

$$\angle(G(j\omega) = -100/j\omega) = +90^\circ$$

Magnitude Response for a TF of Form $(s/a + 1)^{-1}$

- Consider a transfer function of a real pole form with $a > 0$.

$$G(s) = \frac{1}{(s/a) + 1}$$

- This is a system with a single pole at $s = -a$ (a low pass filter).
  - At low frequencies ($\omega \ll a$), the $|G(j\omega)| = 1$ (or 0 dB).
  - At high frequencies ($\omega \gg a$), the $|G(j\omega)| = \frac{1}{(\omega/a)} = \frac{a}{\omega}$
- The response in these two frequency regimes form a low frequency and a high frequency asymptote.

Magnitude Response for a TF of Form $(s/a+1)^{-1}$

- The low frequency asymptote is a straight line with zero slope and unity gain.
- On a dB scale, the high frequency asymptote is given by:

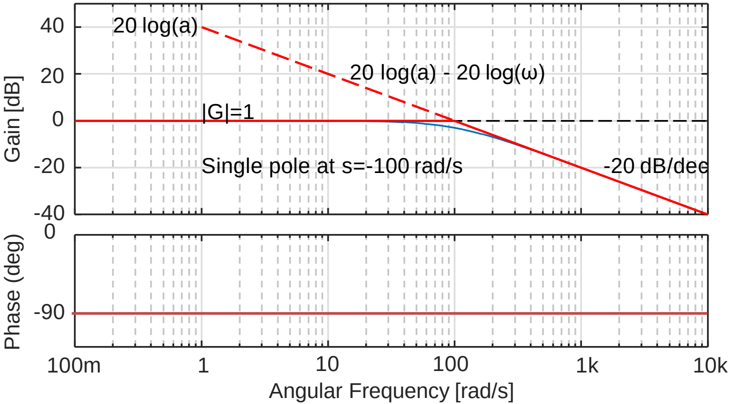
$$|G(j\omega)| = 20\log a - 20\log\omega$$

- This is therefore a straight line with a slope of −20 dB/decade.

Example of Pole

- Sketch Bode plots of system with pole at -100:

$$G(s) = \frac{1}{(s/100) + 1}$$



Example of Pole

- Notice that the high and low frequency asymptotes form a reasonable approximation to the real response.
- At low frequency, it has a phase of 0° in this region:

$$G(s) = 1$$

- At high frequency, the asymptote has a fixed value of −90°:

$$G(s) \approx \frac{a}{j\omega} = -j\left(\frac{a}{\omega}\right)$$

- At the breakpoint ($\omega = a$) we have a phase of −45°:

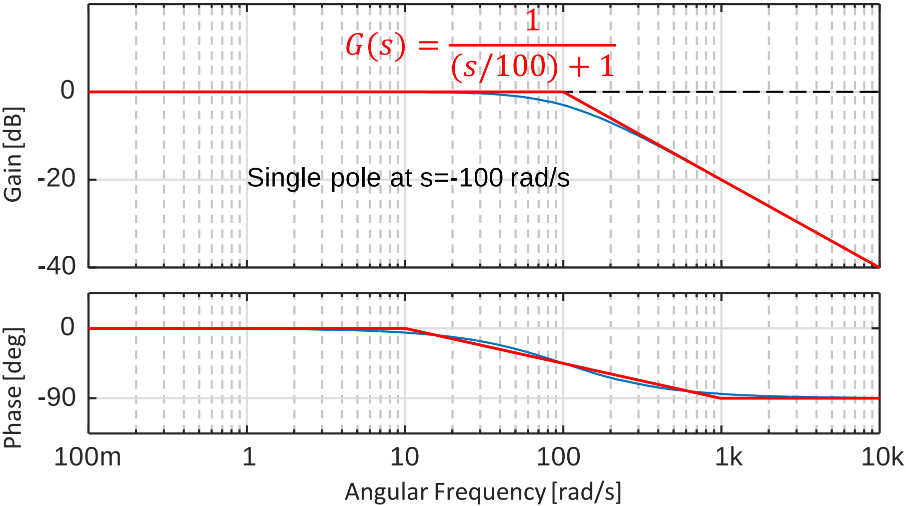
$$G(s) = \frac{1}{1+j} = \frac{1-j}{2}$$

Example of Pole

- The normal approximation for the phase response is to draw a straight line at 0° up to a frequency a factor of ten below the break point,
- a straight line with phase of −90° beyond ten times the breakpoint and
- then join the two asymptotes with a straight line.

Example of Pole

- Again, for the example of a system with pole:



Example of Pole

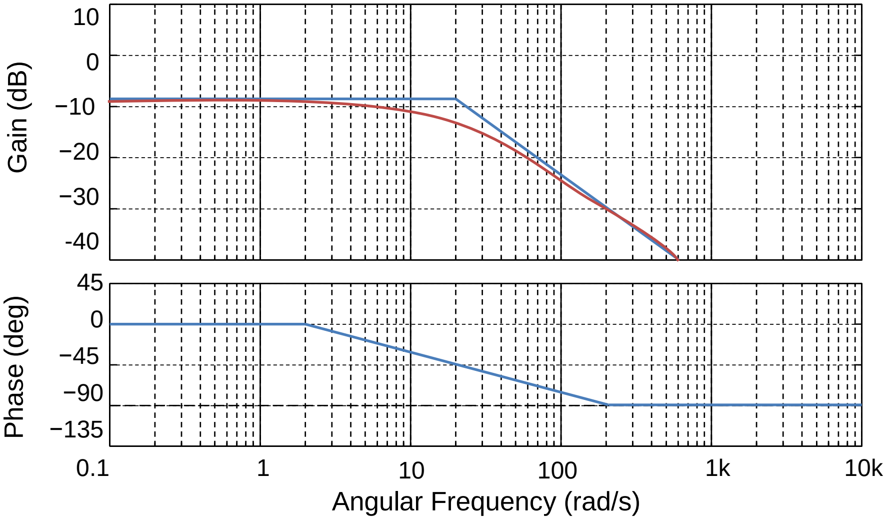
- Let us begin by putting the transfer function into a form suitable for Bode plotting.

$$G(s) = \frac{7}{s+20} = \frac{7}{20\left(\frac{s}{20}+1\right)} = \frac{7}{20}\left(\frac{1}{1+\frac{s}{20}}\right)$$

- Note that:

$$20\log\ (7/10) = -9.1\ \mathrm{dB} \approx -10\ \mathrm{dB}$$

Example of Pole



Example of Pole

- We can check our work using MATLAB.
- Be careful using `zpk()` function – check that you have the right dc gain and put the pole at −20 rad/s not +20 rad/s.

```
>> G = zpk([],[-20],7)

>> bode(G)
```

- You can also use `tf()` function instead. The following will both work:

```
>> G = tf(7,[1,20])

>> G = tf(7/20,[1/20,1])
```

Example of Pole

- Though MATLAB is very useful for control design, it can be error prone.
- One important reason to understand how to draw a Bode plot by hand is that it allows you to recognize errors when using computer-based tools.
- Most of the errors are due to algorithm used in MATLAB and accuracy of the simulation results.
- It is also possible errors are due to extreme points e.g. infinite results obtained in the simulation.

Transfer Function of Form $(s/a + 1)$

- Now, consider the case of a single real zero at $s = -a$, where $a > 0$.
- The low frequency asymptote arising from a zero is the same as that for a pole (a straight line at 0 dB).
- However, for a zero, the high frequency asymptote is given by:

$$|G(j\omega)| = -20 \log a + 20 \log \omega$$

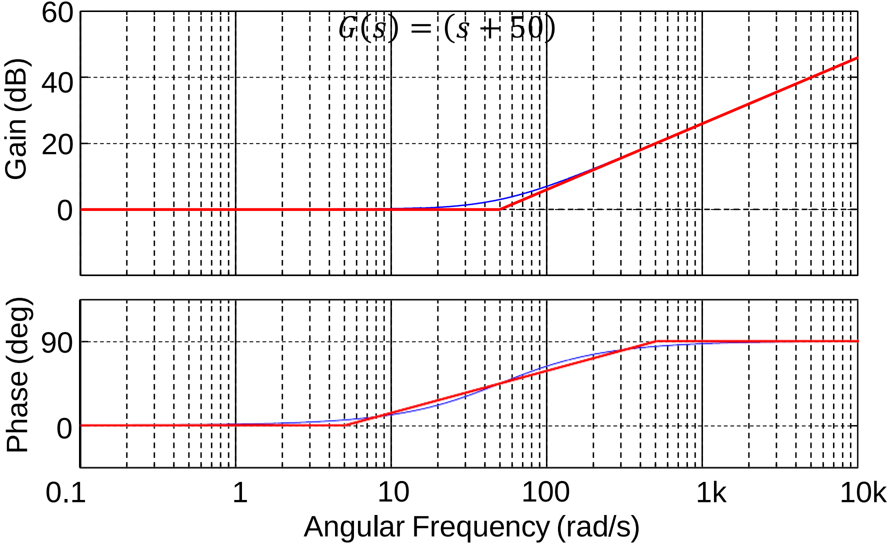
- The high frequency asymptote is therefore a straight line with slope of +20 dB/decade.

Transfer Function of Form $(s/a + 1)$

- The phase response is also the opposite of that produced by a pole.
- At high frequencies, we have $G(j\omega) \approx j\omega$, leading to a phase shift of +90°.
- As we might expect, the phase is +45° at the breakpoint.

Example of Zero

- Sketch Bode plots of system with a zero at −50:



Repeated Roots

- As a transition to complex pairs of poles/zeros, consider the case of a transfer function with a double pole:

$$G(s) = \frac{1}{\left(1+\frac{s}{a}\right)^2} = \left(\frac{1}{1+\frac{s}{a}}\right)\left(\frac{1}{1+\frac{s}{a}}\right) \qquad \text{for} \quad a > 0$$

- We know that the magnitude and phase responses are the sum of the two parts.

- So, we will have a response that falls off at −40 dB/decade beyond the breakpoint and moves from 0° to −180° in phase (over the same frequency range that a single pole TF would take to move 90°).

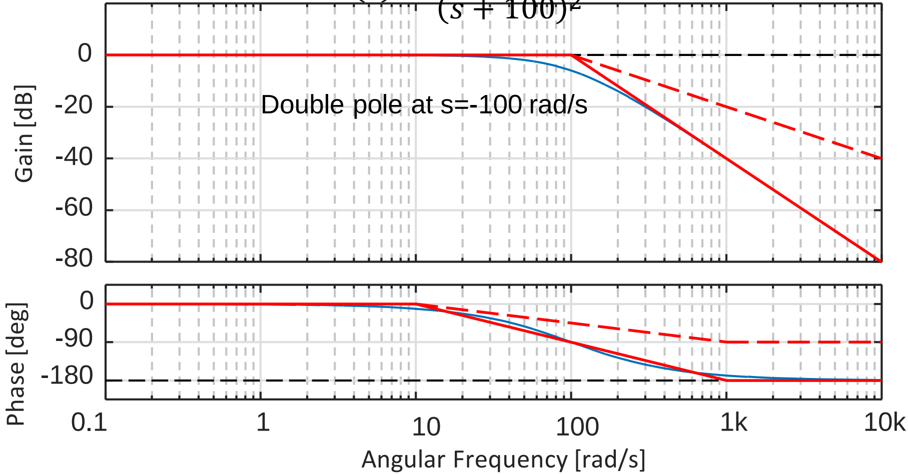
Repeated Roots

- Notice that the presence of two poles means that the gain at $s = a$ is 6 dB down from the dc value.
- The plots for a repeated zero are opposite, with a slope of 40 dB/decade and a phase that moves from 0° to 180°.

Example of Repeated Roots

- Sketch Bode plot of a system with a double pole at -100:

$$G(s) = \frac{1}{(s+100)^2}$$



Transfer Functions of Form $\left[\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1\right]^{\pm 1}$

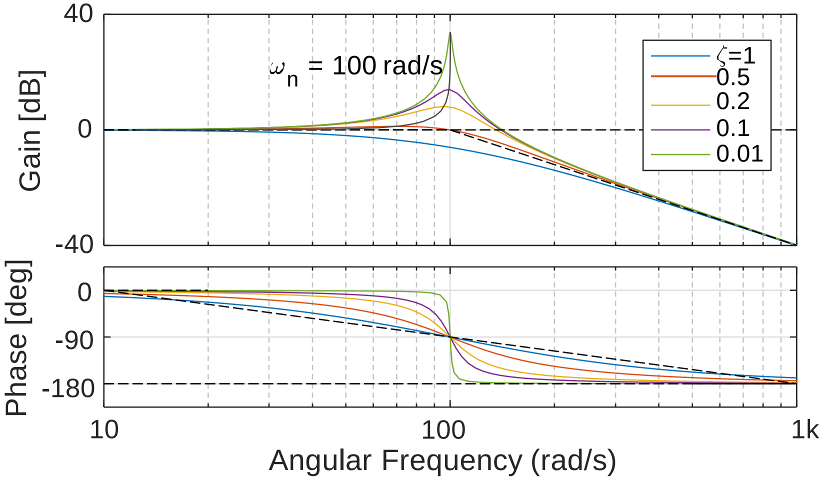
- We might expect that the transfer function produced by a pair of complex poles would look something like that produced by a double pole.
- Away from the breakpoint this is true; the gain rolls off at -40 dB/decade at high frequencies and the phase moves from $0°$ at low frequencies to $-180°$ at high frequencies.
- However, when the transfer function is underdamped it leads to some significant deviations in the region of the breakpoint.

Transfer Functions of Form $\left[\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1\right]^{\pm 1}$

- The smaller the damping the larger the effect. As damping decreases we get:
  - increasing peak in the magnitude response.
  - sharper transition in the phase response.
- These effects are shown in the figure.

Plot of a TF of Form $\left[\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1\right]^{-1}$

- Frequency response plot of a pair of complex poles.



Plot of a TF of Form $\left[\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1\right]^{-1}$

- We represent this family of curves with a straight-line approximation identical to the repeated real pole example above.
- Note though that the corner point is at ω_n for the resonance, not at the real part of the pole pair.
- If the damping is very low ($\zeta < 0.01$ say), you might prefer to approximate the phase response as a step at the natural frequency.

Corrections for Second Order Systems

- To draw an accurate frequency response for a second order system, it is necessary to make corrections by looking at at a previously plotted response.
- If you don't happen to have such a response handy, as a rough guide, the peak (or trough for zeros) in the gain response has a magnitude as follows at the breakpoint:

$$M_p = \frac{\sqrt{1}}{2\zeta\sqrt{1-\zeta^2}}$$

- For lightly damped systems, it is:

$$M_p \approx \frac{1}{2\zeta} = Q$$

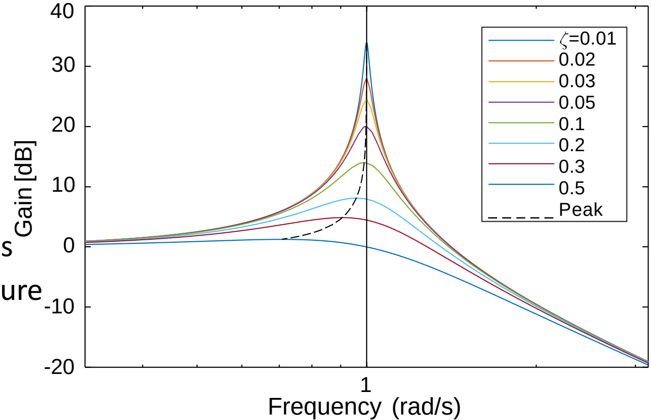
Resonance

This should be familiar, as it is just a description of resonance:

- The gain of the system becomes large in the vicinity of the resonant frequency.
- Highly resonant (lightly damped) systems have a more pronounced gain increase at resonance.
- All systems go through a 180° phase change in the vicinity of a resonance.

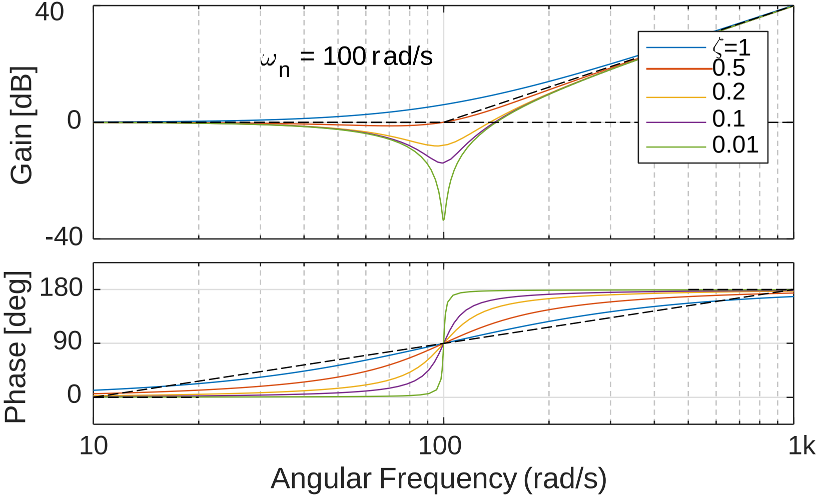
Damping and Resonant Peak

- For lightly damped systems, we can see that the resonant peak occurs at approximately ω_n.
- However, the peak in the magnitude response shifts downwards in frequency as damping increases.
- However, the passage of the phase response through $-90°$ always occurs at ω_n, which makes this a better feature to search for in experiments.



Plot of a TF with Form $\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1$

- As you might expect, the behaviour of a system with second-order zeros is opposite that with second order poles.



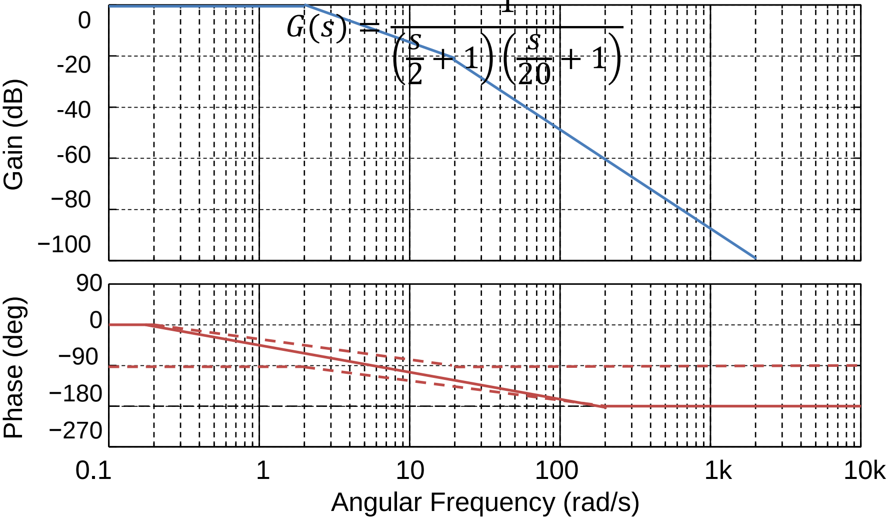
Building an Arbitrary Bode Plot

1. Arrange the transfer function into the convenient form.
2. Plot the straight line approximations for each term in the transfer function.
3. If required, make corrections to the approximations for complex pairs of poles.
4. Magnitude peaks are approximately.
5. Add the various curves graphically and draw in the final response curves.

Example of Poles

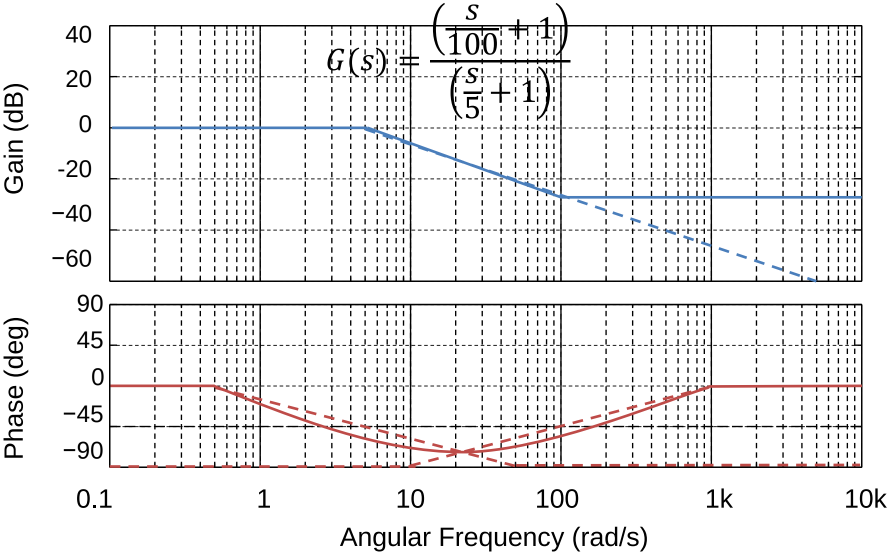
- Sketch a Bode plot of poles at -2 and -20:

$$G(s) = \frac{1}{\left(\frac{s}{2}+1\right)\left(\frac{s}{20}+1\right)}$$



Example of Pole and Zero

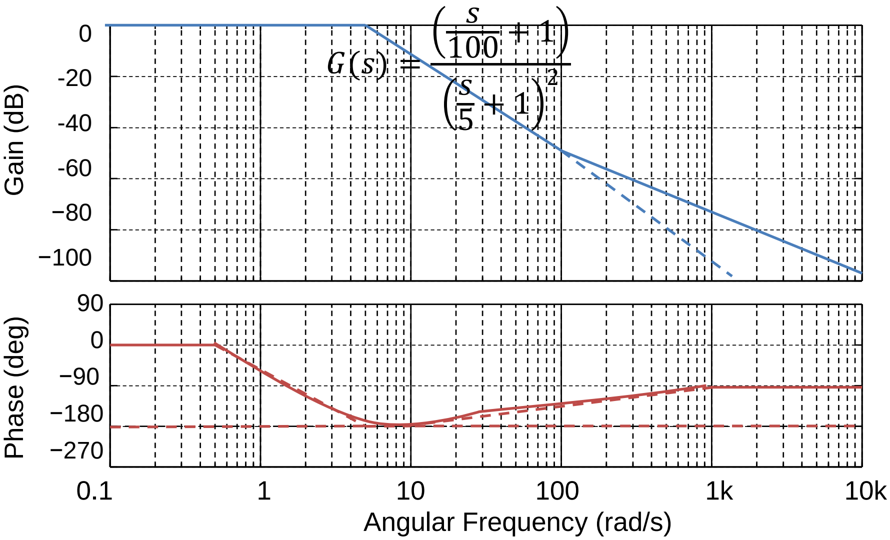
- Sketch Bode plots of pole at -5 and zero at -100:



Example of Double Poles and Zero

- Sketch Bode of double poles at -5 and zero at -100:

$$G(s) = \frac{\left(\frac{s}{100}+1\right)}{\left(\frac{s}{5}+1\right)^2}$$



Checking the Bode Plot

- You should always make sure that your final plot makes sense at both low and high frequencies.
- Low frequency:
  - At low frequency, the response is determined by only the differentiators/integrators in the system.
  - If the overall transfer function includes a factor of s^n, then the slope of the gain curve should be $20n$ dB/decade and the phase at low frequency should be $90n$ degrees.

Checking the Bode Plot

- High frequency:
  - The high-frequency behaviour is determined by the number of poles, P , and zeros, Z.
  - At high frequency, the slope of the gain should be $-20(P - Z)$ dB/decade and the phase should be at $-90(P - Z)$ degrees.

  Note: As we will see next, the phase checks only works when all the system poles and zeros are in the left half of the s-plane.

- The roots in the discussion have all been in the left half of the s-plane.

Bode Plots for Roots in the Right-Half Plane

- Let's first consider poles in the right-half plane. Consider the transfer functions as follow:

$$G_1(s) = \frac{1}{s-a} \quad \text{and} \quad G_2(s) = \frac{1}{s+a}$$

- The magnitude of these systems are:

$$|G_1(j\omega)| = \frac{1}{j\omega - a} = \frac{1}{\sqrt{a^2 + \omega^2}}$$

- And

$$|G_2(j\omega)| = \frac{1}{j\omega + a} = \frac{1}{\sqrt{a^2 + \omega^2}}$$

- The two transfer functions have identical magnitudes.

Phase Plot for Roots in the Right-Half Plane Roots

- Now, consider the phase responses of the two systems.

$$G_1(s) = \frac{1}{s-a} = \frac{-a - j\omega}{\omega^2 + a^2}$$

- And

$$G_2(s) = \frac{1}{s+a} = \frac{a - j\omega}{\omega^2 + a^2}$$

- The phase shifts of these systems are:

$$\angle G_1(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\{G_1(s)\}}{\mathrm{Re}\{G_2(s)\}}\right) = \tan^{-1}\left(\frac{\omega}{a}\right)$$

$$\angle G_2(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\{G_2(s)\}}{\mathrm{Re}\{G_2(s)\}}\right) = \tan^{-1}\left(\frac{-\omega}{a}\right) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

Phase Plot for Roots in the Right-Half Plane Roots

- The phase response is opposite to that we expect for a pole in the left-half side of the s-plane.

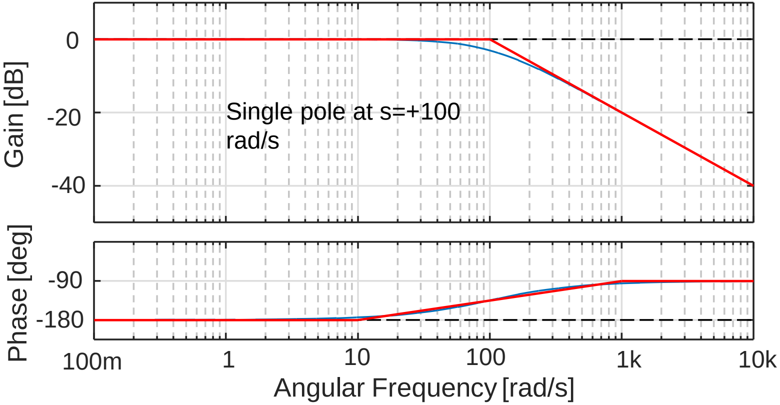
$$\angle G_1(j\omega) = -\angle G_2(j\omega)$$

- Having right-half plane poles will make the system to be unstable.
- The transient response of the system with right-hand plane poles is an increasing amplitude function.

Example of Right-Half Plane Pole

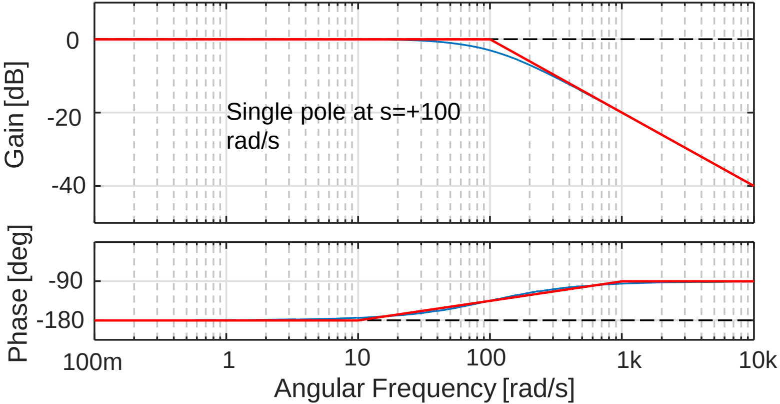
- Sketch Bode plots of system with right-half plane pole at 100:

$$G(s) = \frac{100}{s - 100}$$



Example of Right-Half Plane Pole

- A quick examination of a Bode plot is a good check whenever you enter a system in Matlab, as it is easy to put a root in the right-half plane unintentionally (particularly with `zpk`).



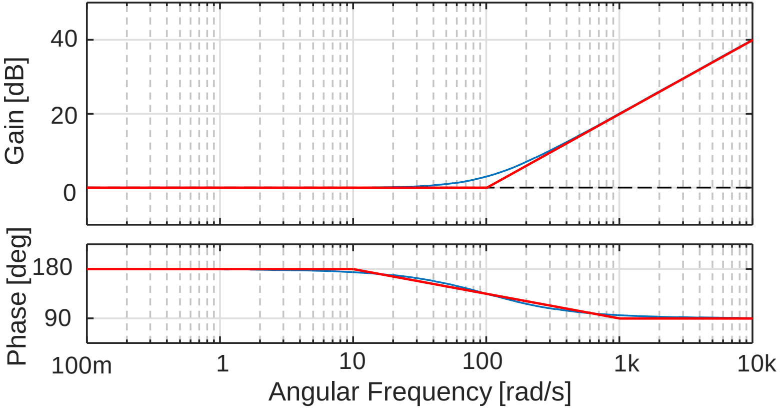
Non-Minimum Phase Systems

- The same analysis can be performed on systems having zeros in the right-half plane.
- Perhaps unsurprisingly, we find that these too have their magnitude response unchanged, but their phase response reversed from the left-half plane ones.
- Systems containing at least one right-half plane zero are called non-minimum phase systems.
- Non-minimum phase systems tend to be harder to control than minimum phase systems, but easier than open loop unstable systems (those with right-half plane poles).

Example of Non-Minimum Phase

- Sketch Bode of non-minimum phase:

$$G(s) = \frac{s - 100}{100}$$



Response of Non-Minimum Phase Systems

- Non-minimum phase systems are causal and stable systems whose inverses are causal, but unstable.
- Having a delay in a system or a zero on the right half of the s-plane may lead to a non-minimum phase system.
- Non-minimum phase systems are troublesome because their initial response is “the wrong way” when driven by an input.

Example Response of Non-Minimum Phase

- Compare the step responses of two systems having transfer functions:

$$G_1(s) = \frac{s+1}{s^2+4s+5}$$

And

$$G_2(s) = \frac{-(s-1)}{s^2+4s+5}$$

Example of Response of Non-Minimum Phase

- We can also compare their Bode plots.
- The greater change in the phase for G_2 is what leads to the name “non-minimum phase”.