Analysis with Bode Plots

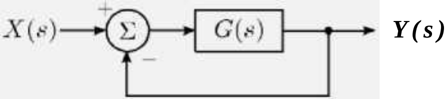
XMUT315 Control System Engineering

Topics

- Closed-loop stability.
- Gain and phase margins.
- Determining gain and phase margins in Bode plots.
- Damping and phase margin.
- Transient response parameters from Bode plots.
- System types.
- Steady-state errors.
- System errors and inputs.
- Determining steady-state errors in Bode plots.

Closed-Loop Stability

- Imagine a situation where we have a system described by a transfer function $G(s)$. We now enclose the system in a unity gain feedback loop.



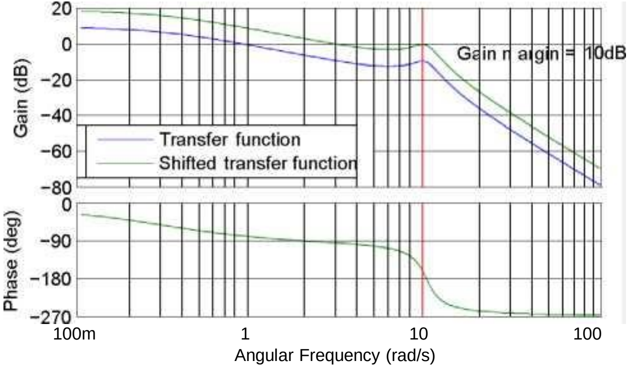
- We know that negative feedback is useful in stabilising a system. However, instability results when the feedback is positive.
- The feedback in the system shown becomes positive when the plant transfer function $G(s)$ contributes 180° of phase shift to the overall system.

Closed-Loop Stability

- System stability is one of the basic concerns when designing a control system.
- We would like to be able to meaningfully talk about how close a system is to instability, not just whether it is stable or not.
- For many systems, we can assess the stability by finding the frequency at which the phase curve crosses -180° and reading the gain at that point.
  - If the gain > 1, then the system will be unstable.
  - If the gain < 1 at the frequency where the phase crosses -180°, then not enough gain to sustain the oscillations.
- This approach leads to a metric known as the gain margin.

Gain Margin

- The gain margin is the amount by which we can increase the gain of a stable system before it becomes unstable.



- To determine the gain margin of a system, read the gain at the frequency where the phase curve crosses 180°.

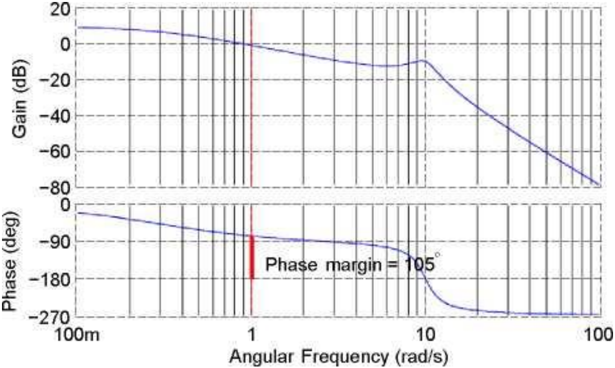
- The gain margin must be positive for the system to be stable!

Unity Gain

- In control applications, we often use the Unity Gain Frequency, which is the frequency at which the system's gain has dropped to one (0 dB).
- We can use the Bode plot to simply read off the frequency where the gain plot crosses the 0 dB line.
- Note that some systems have multiple unity gain frequencies because their gain curves cross and recross the 0 dB line.
- In crude terms, the unity gain frequency of a control system is the highest frequency at which the control is doing anything useful.
- Beyond this point, the gain is too small to improve the system.

Phase Margin

- The phase margin is the amount by which we can decrease the phase of a stable system before it become unstable.
- To determine the phase margin of a system, find the unity gain frequency and read the system phase at that point.

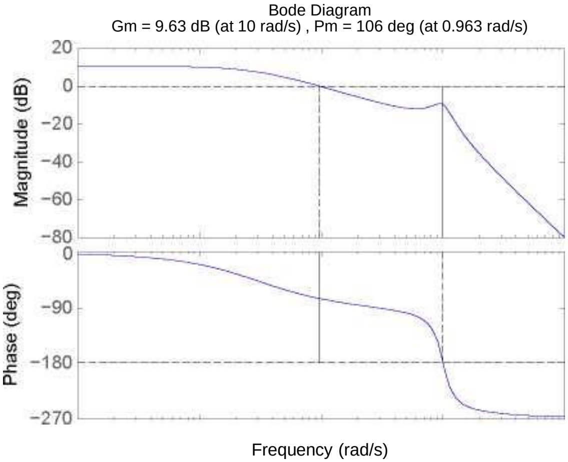


- This reveals how much extra phase lag we could tolerate before instability sets in.

Gain and Phase Margins with MATLAB

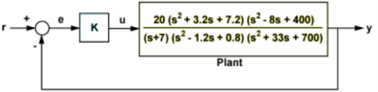
- The `margin` MATLAB command will tell you the gain and phase margins and the frequencies at which they occur.

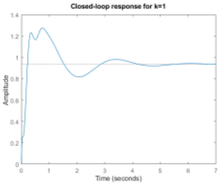
- If you call it without any return arguments, it will draw a plot displaying the same information

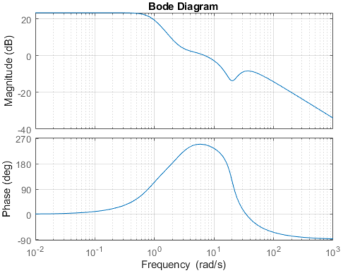


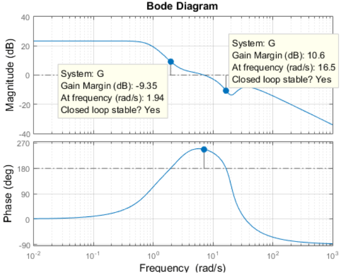
Phase Margin with Multiple Unity Gain Crosses

- The following Bode plot shows a higher order system that has multiple crossing of the 0 dB gain curve with unity gain line.









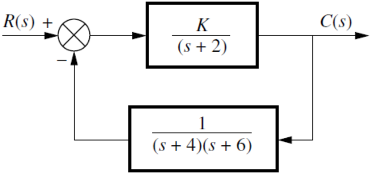
- There are two 180° phase crossings with corresponding gain margins of -9.35 dB and +10.6 dB.

Phase Margin with Multiple Unity Gain Crosses

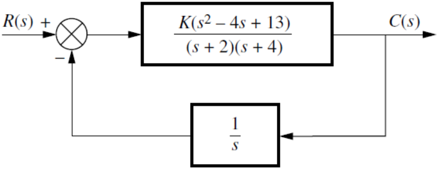
- For some systems that cross the 0 dB gain curve more than once, in general, there will be a different phase margin associated with each of these crossings.
- It is possible to define the system phase margin as the worst (smallest) of the individual phase margins.
- However, this is dangerous as there are some systems like this appear to be stable, but they are not.
- When you see a system with multiple crossings of the 0 dB line, you should double check the system stability with another method, such as a root locus diagram or (more traditionally) a Nyquist plot.

Stability Analysis with Bode Plots

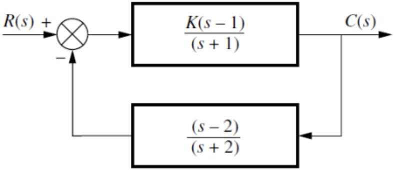
For each system given below, find the gain margin and phase margin if the value of gain K is 1, 100, 1000, and 0.1. Write a summary on the stability of each system. [30 marks]



System 1



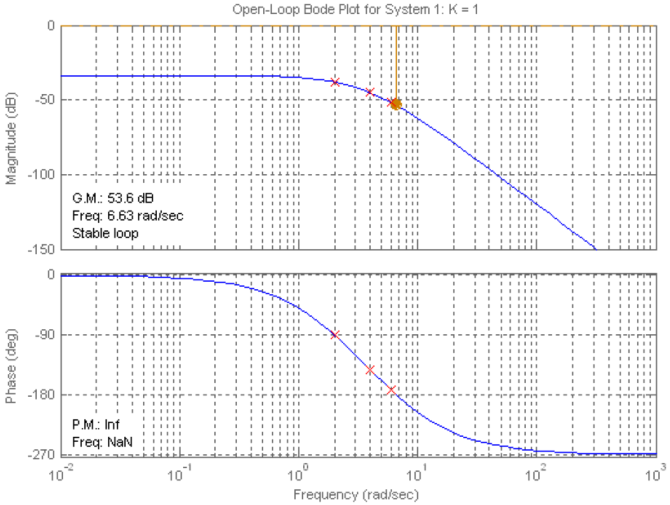
System 2



System 3

Stability Analysis with Bode Plots

a. System 1: Plotting for $K = 1$ yields the following Bode plots.



Stability Analysis with Bode Plots

$K = 1$:

For K = 1, when the phase response is 180° at ω = 6.63 rad/s, the gain margin is 53.6 dB. Phase margin is $+\infty$ at any frequency.

K = 100:

- For K = 100, gain curve is raised by 40 dB yielding -13.6 dB at 6.63 rad/s. Thus, the gain margin is 13.6 dB.
- Phase margin: Raising the gain curve by 40 dB yields 0 dB at 2.54 rad/s, where the phase curve is 107.3°. Hence, the phase margin is 180° - 107.3° = 72.7°.

Stability Analysis with Bode Plots

K = 1000:

- For K = 1000, gain curve is raised by 60 dB yielding +6.4 dB at 6.63 rad/s. Thus, the gain margin is -6.4 dB.
- Phase margin: Raising the gain curve by 60 dB yields 0 dB at 9.07 rad/s, where the phase curve is 200.3°. Hence, the phase margin is 180° - 200.3° = -20.3°.

K = 0.1:

- For K = 1, when phase response is 180° at ω = 6.63 rad/s, the gain margin is increased to 53.6 dB at this frequency.

Stability Analysis with Bode Plots

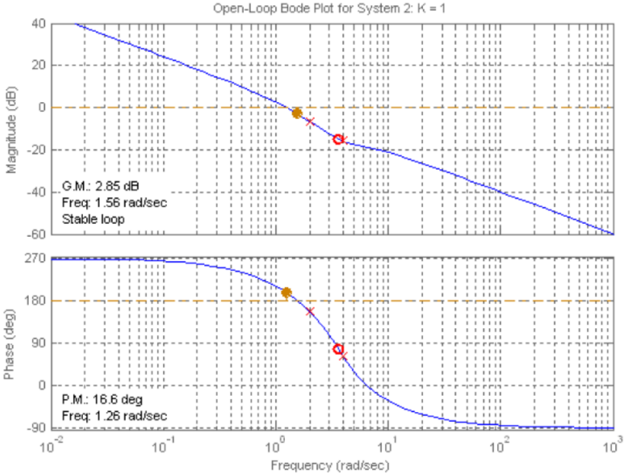
- For K = 0.1, gain curve is lowered by 20 dB yielding -73.6 dB at 6.63 rad/s. Thus, the gain margin is increased to 73.6 dB.

Stability Summary of System 1:

- When $K = 1$, considering positive gain margin (GM = 53.6 dB at 6.63 rad/s) and phase margin (PM = ∞ at any frequency), the system is found to be stable.
- Any increase of the gain might reduce the gain margin of the system. If the increase is excessive, the system could be unstable.
- If the gain is lowered, the system stays stable with the margins are increased.

Stability Analysis with Bode Plots

b. System 2: Plotting for $K = 1$ yields the following Bode plots.



Stability Analysis with Bode Plots

$K = 1$:

For K = 1, when the phase response is 180° at ω = 1.56 rad/s, the gain margin is -2.85 dB and phase margin is -18.6° at 1.26 rad/s.

K = 100:

- For K = 100, gain curve is raised by 40 dB yielding +37.15 dB at 1.56 rad/s. Thus, the gain margin is -37.15 dB.
- Phase margin: Raising the gain curve by 40 dB yields 0 dB at 99.8 rad/s, where the phase curve is -84.3°. Hence, the phase margin is 180° - 84.3° = 95.7°.

Stability Analysis with Bode Plots

$K = 1000$:

- For $K = 1000$, gain curve is raised by 60 dB yielding +57.15 dB at 1.56 rad/s. Thus, the gain margin is -57.15 dB.
- Phase margin: Raising the gain curve by 54 dB yields 0 dB at 500 rad/s, where the phase curve is -91.03°. Hence, the phase margin is 180° - 91.03° = 88.97°.

$K = 0.1$:

- For $K = 0.1$, gain curve is lowered by 20 dB yielding -22.85 dB at 1.56 rad/s. Thus, the gain margin is -22.85 dB.

Stability Analysis with Bode Plots

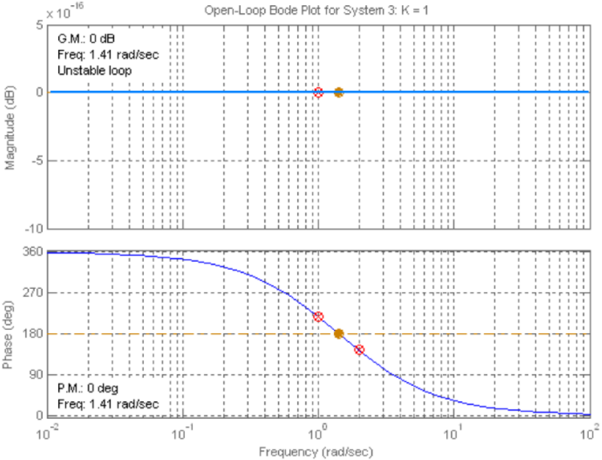
- Phase margin: Lowering the gain curve by 20 dB yields 0 dB at 0.162 rad/s, where the phase curve is -99.8°. Hence, the phase margin is 180° - 99.86° = 80.2°.

Stability Summary of System 2:

- Both of the gain and phase margins of the system are negative i.e. -2.85 dB at 1.56 rad/s and -18.6° at 1.26 rad/s respectively. The system is unstable due to these negative margins.
- Increasing the gain reduces the gain margin further making the system to become more unstable.
- Decreasing the gain of the system might increase the margin and might turn the system to become stable.

Stability Analysis with Bode Plots

c. System 3: Plotting for K = 1 yields the following Bode plots.



Stability Analysis with Bode Plots

$K = 1$:

- For $K = 1$, when the phase response is 180° at $\omega = 1.41$ rad/s, the gain margin is 0 dB. Phase margin is 0° at 1.41 rad/s.

$K = 100$:

- For $K = 100$, gain curve is raised by 40 dB yielding 40 dB at 1.41 rad/s. Thus, the gain margin is - 40 dB.
- Phase margin: Raising the gain curve by 40 dB yields no frequency where the gain curve is 0 dB. Hence, the phase margin is infinite.

Stability Analysis with Bode Plots

$K = 1000$:

- For $K = 1000$, gain curve is raised by 60 dB yielding 60 dB at 1.41 rad/s. Thus, the gain margin is - 60 dB.
- Phase margin: Raising the gain curve by 60 dB yields no frequency where the gain curve is 0 dB. Hence, the phase margin is infinite.

$K = 0.1$:

- For $K = 0.1$, gain curve is lowered by 20 dB yielding -20 dB at 1.41 rad/s. Thus, the gain margin is 20 dB.

Stability Analysis with Bode Plots

- Phase margin: Lowering the gain curve by 20 dB yields no frequency where the gain curve is 0 dB. Hence, the phase margin is infinite.

Stability Summary of System 3:

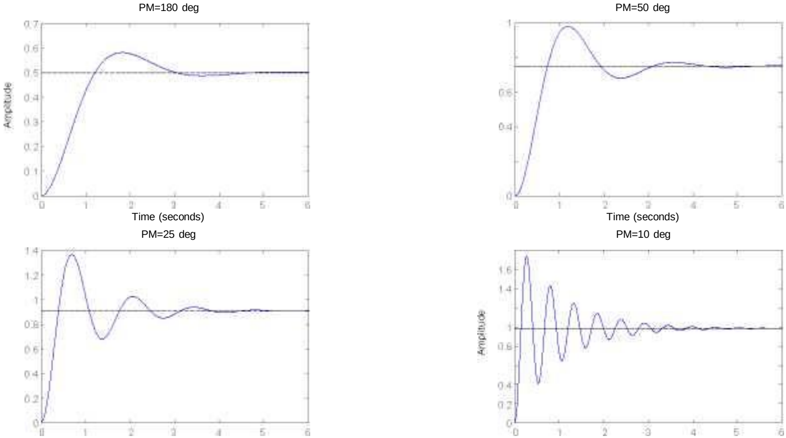
- Both of the gain and phase margins are zero at 1.41 rad/s and 1.41 rad/s respectively. The system is critically stable.
- Increasing the gain might turn the system to become unstable.
- Reducing the gain increases the margins and these make the system to become stable.

Transient Response in Bode Plots

- From the given Bode plots, we can determine a variety of transient response parameters:
    - Damping ratio.
    - Settling time.
    - Peak time.
- For transient response analysis, we should know the values of these parameters to work out the transient response parameters:
    - Phase margin -> damping ratio
    - Closed loop bandwidth (+ damping ratio) -> settling time and peak time.

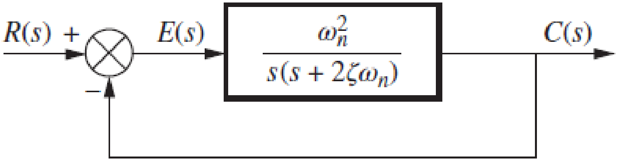
Phase Margin and Damping Ratio

- Phase margin is useful because there is a direct link between a system's phase margin and its damping in the closed-loop case.
- The smaller the phase margin, the badly the system will ring.



Phase Margin and Damping Ratio

- It can be shown that there is a relationship between the phase margin and the damping ratio of the closed-loop response.



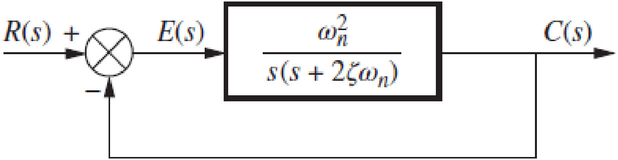
- For a standardised second order equation as shown in the figure below, the open-loop transfer function of the plant is:

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

Phase Margin and Damping Ratio

- The closed-loop transfer function of the system is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



- To evaluate the phase margin, find the frequency for which $|G(j\omega)| = 1$.

$$|G(j\omega)| = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega} = 1$$

Phase Margin and Damping Ratio

- The frequency, ω_1, that satisfies the equation above is:

$$\omega_1 = \omega_n \sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}$$

- The phase angle of $G(j\omega)$ at this frequency is:

$$\angle G(j\omega) = -90° - \tan^{-1}\left(\frac{\omega_1}{2\zeta\omega_n}\right)$$

- Substitute the equation for ω_1 into the equation above.

$$\angle G(j\omega) = -90° - \tan^{-1}\left(\frac{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}{2\zeta}\right)$$

Phase Margin and Damping Ratio

- The difference between the angle of the equation above and -180° is the phase margin, ϕ_m.

$$\phi_m = 90° - \tan^{-1}\left(\frac{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}{2\zeta}\right)$$

- The accurate relation of damping ratio with the phase margin of the system over the full range is:

$$\phi_m = \tan^{-1}\frac{2\zeta}{\sqrt{2\zeta^2 + \sqrt{1 + 4\zeta^4}}}$$

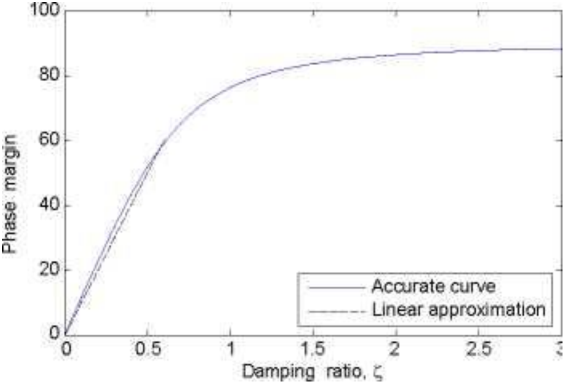
- To keep the damping reasonable, we generally try to preserve a phase margin of about 60°.

Phase Margin and Damping Ratio

- Rearrange the equation given above, the damping ratio is:

$$\zeta = \sqrt[4]{\frac{1}{\left(\frac{4}{\tan^2 \phi_m} + 2\right)^2 - 4}}$$

- The relationship between phase margin and damping ratio is as shown in the graph below.



- For damping ratios less than 0.65, use the approximate relation $\phi_m = 100\zeta$ as shown in the graph above.

Bandwidth of Control Systems

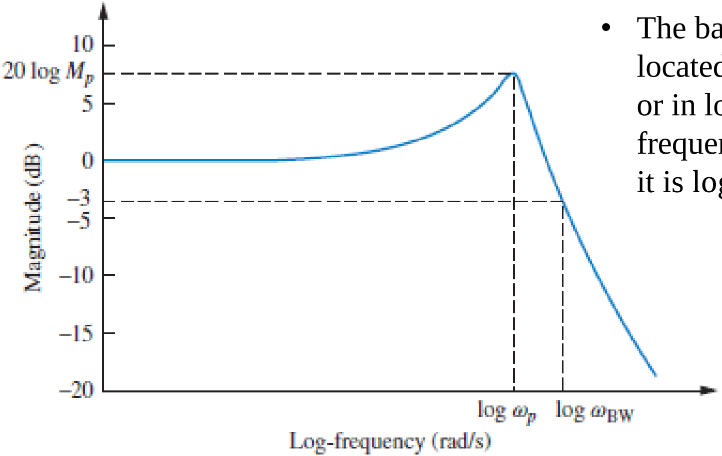
- The magnitude or gain of the frequency response of the given control system is:

$$|T(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

- To determine the transient response of the control system, we need to find the closed-loop bandwidth from the Bode plots.
- For open-loop system, the bandwidth of the control systems (ω_{BW}) is the width of frequency of gain of the system from DC (0 rad/s) to the half-power point (i.e. -3 dB).

Bandwidth of Control Systems

- For a typical second-order system, the magnitude plot of the equation given above is shown in the figure below.



- The bandwidth is located at ω_{BW}, or in log frequency scale, it is $\log \omega_{BW}$.

Closed-Loop Bandwidth

- The bandwidth of the standardised control systems (ω_{BW}) is determined by finding the frequency for which $|T(j\omega)| = 1/\sqrt{2}$ (i.e. that is -3 dB).

$$|T(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

- Equate the equation above to be equal to $1/\sqrt{2}$ that happens when $\omega = \omega_{BW}$:

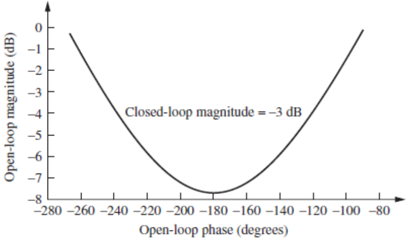
$$\frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega_{BW}^2)^2 + 4\zeta^2\omega_n^2\omega_{BW}^2}} = \frac{1}{\sqrt{2}}$$

Closed-Loop Bandwidth

- Rearranging the equation above, the bandwidth of the control system is:

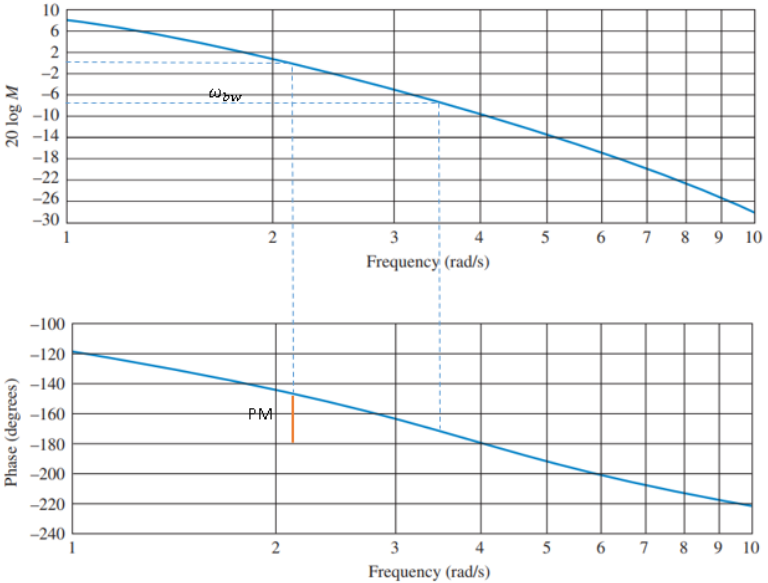
$$\omega_{BW} = \omega_n \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

- The closed-loop bandwidth, ω_{BW} is the frequency at which the closed-loop magnitude response is -3 dB.



- It equals the frequency at which the open-loop magnitude response is between -6 and -7.5 dB (i.e. if the open-loop phase response is between -135° and -225°).

Closed-Loop Bandwidth in Bode Plots



Closed-Loop Bandwidth in Bode Plots

- Given the Bode plots of a control system as shown in the figure, we can determine the phase margin and gain margin, and bandwidth of the system.
- From the plots, phase margin, PM is 180° - 150° = 30°
- Also, we found that gain margin, GM is 10 dB at 4 rad/s.
- Considering the open-loop system and looking at the frequency when the gain of the system is -7.5 dB, the bandwidth, ω_{BW} is approximately 3.5 rad/s.

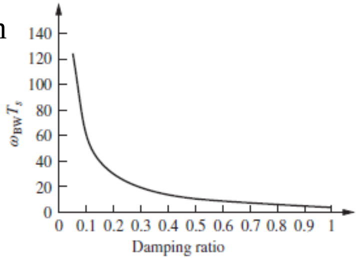
Closed Loop Bandwidth in Bode Plots

We could determine the transient response parameters of the system from the graphs for:

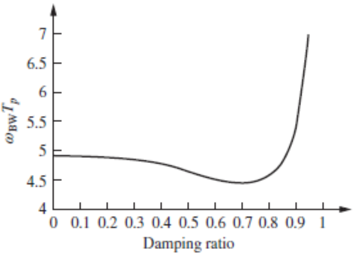
a. Settling time.

b. Peak time.

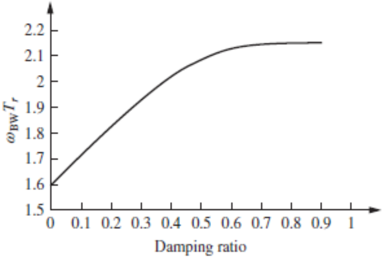
c. Rise time.



(a)



(b)



(c)

Settling Time in Bode Plots

- Knowing that for 2% settling time standard:

$$T_s = \frac{4}{\omega_n \zeta} \qquad \text{thus} \qquad \omega_n = \frac{4}{T_s \zeta}$$

- The bandwidth of the closed loop control system (ω_{BW}) vs. settling time (T_s).

$$\omega_{BW} = \frac{4}{T_s \zeta}\sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

Where: ζ is the damping ratio.

- Hence, for 2% settling time standard, the settling time is:

$$T_s = \frac{4}{\omega_{BW} \zeta}\sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

Peak Time in Bode Plots

- Like the settling time, we can determine also the time-to-peak (T_p) from the Bode plots through the bandwidth of the closed loop system (ω_{BW}). Since

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad \text{thus} \qquad \omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}}$$

- The previous equation becomes:

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Where: ζ is the damping ratio.

- Hence, the peak time is:

$$T_p = \frac{\pi}{\omega_{BW}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Rise Time in Bode Plots

- To relate the bandwidth to rise time (T_r), knowing the desired ζ, we can calculate it from:

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$$

Where:

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

- Rearranging the equation above

$$\omega_n = \frac{\pi - \phi}{T_r\sqrt{1-\zeta^2}}$$

Rise Time in Bode Plots

- Then, substituting ω_n into the bandwidth equation

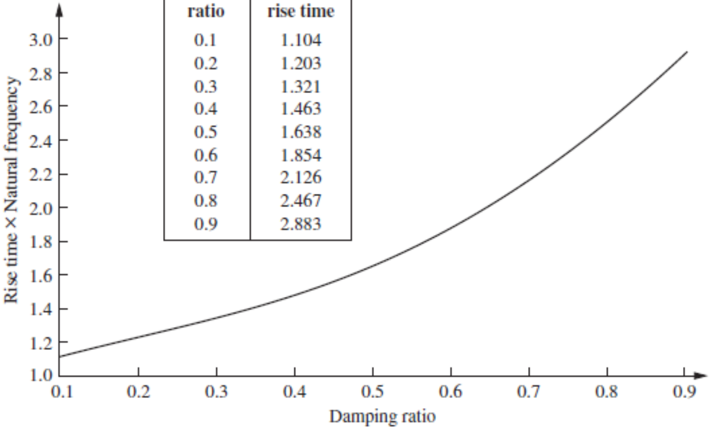
$$\omega_{BW} = \frac{\pi - \phi}{T_r\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

- As a result, the rise time is:

$$T_r = \frac{\pi - \phi}{\omega_{BW}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Rise Time in Bode Plots

- Alternatively, to relate the bandwidth to rise time, T_r, we use the graph given below, knowing the desired ζ and T_r.



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Rise Time in Bode Plots

- For example, assume $\zeta = 0.4$ and $T_r = 0.2$ second.
- Using the graph given above, for $\zeta = 0.4$, the ordinate $T_r\omega_n$ = 1.463, from which ω_n = 1.463/T_r = 1.463/0.2 = 7.315 rad/s.

$$\omega_{BW} = \omega_n\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

$$= 7.315\sqrt{(1-2(0.4)^2)+\sqrt{4(0.4)^4-4(0.4)^2+2}}$$

$$= 10.05 \text{ rad/s}$$

- Using the above given equation, the bandwidth ω_{BW} is 10.05 rad/s.

Transient Response Analysis with Bode Plots

The Bode plots for a plant, $G(s)$, used in a unity feedback system are shown in the figure below. Do the following:

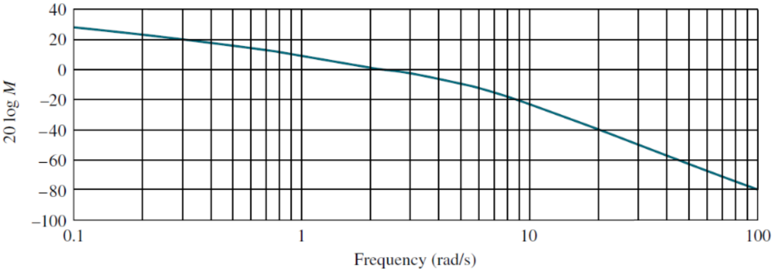
a. Find the gain margin, phase margin, 0 dB frequency (unity gain), 180° frequency, and the closed-loop bandwidth.

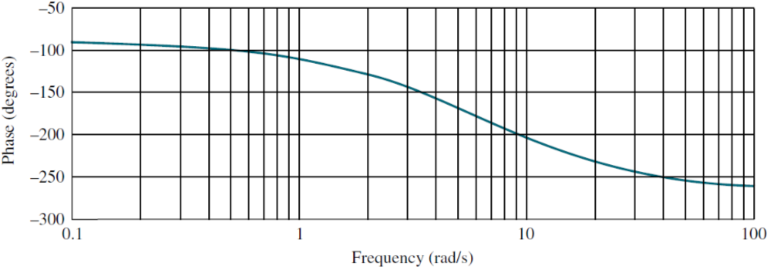
[10 marks]

b. Use your results in part (a) to estimate the damping ratio, percent overshoot, settling time, and peak time.

[10 marks]

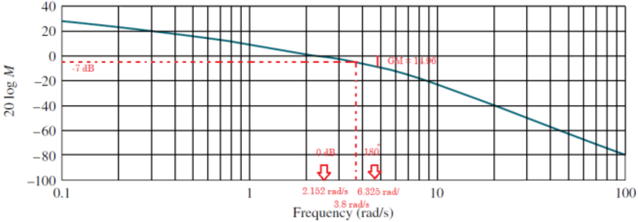
Transient Response Analysis with Bode Plots

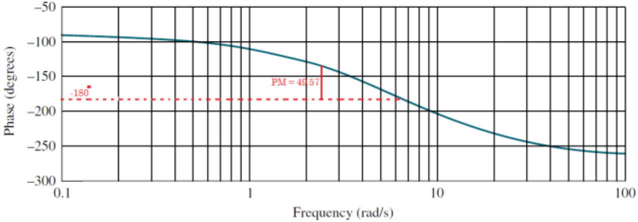




Transient Response Analysis with Bode Plots

- From the Bode plots given below, the gain margin, phase margin, 0 dB frequency (unity gain), 180° frequency, and the closed-loop bandwidth are determined from the plots.





Transient Response Analysis with Bode Plots

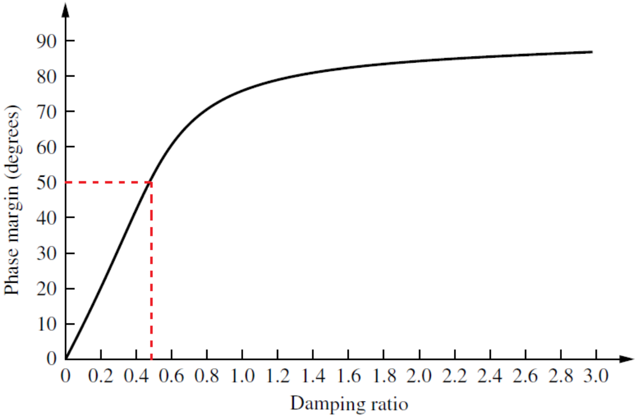
- The results estimated from the graphs given above:
  - Gain margin = 14.96 dB.
  - Phase margin = 49.57°.
  - Unity (0 dB) frequency = 2.152 rad/s.
  - 180° frequency = 6.325 rad/s.
  - Bandwidth (@-7 dB point) = 3.8 rad/s.

$$\zeta = \sqrt[4]{\frac{1}{\left(\frac{4}{\tan^2 \phi_m} + 2\right)^2 - 4}} = \sqrt[4]{\frac{1}{\left(\frac{4}{\tan^2 49.57} + 2\right)^2 - 4}} = 0.48$$

- The damping ratio of the system, ζ is 0.48.

Transient Response Analysis with Bode Plots

- Or, from the graph given below, the damping ratio of the system, ζ is estimated to be 0.5.



Transient Response Analysis with Bode Plots

- From the equation given below, the percentage overshoot of the system can be calculated from:

$$\%OS = e^{\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\% = e^{\frac{\pi(0.48)}{\sqrt{1-(0.48)^2}}} \times 100\%$$

$$= 17.93\%$$

- The percentage overshoot of the system, $\%OS$ is 17.93%.

- From the equation given below, the settling time of the system (2% standard) can be calculated from:

$$T_s = \frac{4}{\omega_{BW}\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Transient Response Analysis with Bode Plots

- Thus

$$T_s = \frac{4\sqrt{(1-2(0.48)^2)+\sqrt{4(0.48)^4-4(0.48)^2+2}}}{(3.8)(0.48)} = 2.84 \text{ s}$$

- The settling time of the system, T_s is 2.84 s.
- From the equation given below, the time-to-peak (n = 1) can be calculated from:

$$T_p = \frac{\pi}{\omega_{BW}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Transient Response Analysis with Bode Plots

- Thus

$$T_p = \frac{\pi\sqrt{(1-2(0.48)^2)+\sqrt{4(0.48)^4-4(0.48)^2+2}}}{(3.8)\sqrt{1-(0.48)^2}}$$

$$= 1.22 \text{ s}$$

- The time-to-peak of the system, T_p is 1.22 s.

Steady-State Characteristics in Bode Plots

- From the given Bode plots, we can determine a variety of steady-state parameters.
- Steady-state parameters that can be derived and approximated from the Bode diagrams are:
    - System type.
    - Steady-state static error constants (K_p, K_v, and K_a).
    - Steady-state errors.

Steady-State Characteristics in Bode Plots

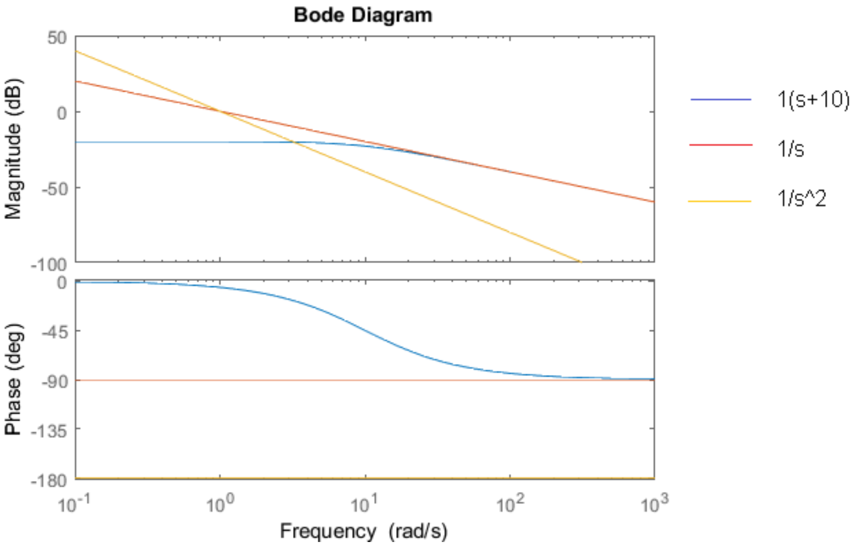
- With the Bode plots, we could determine the steady-state parameters and analyse the characteristics and behaviour of the control system at steady-state conditions:
  - Type of systems.
  - Static error constants (K_p, K_v, and K_a).
  - Steady-state errors.

		Type 0		Type 1		Type 2	
Input	**Steady-state error formula**	**Static error constant**	**Error**	**Static error constant**	**Error**	**Static error constant**	**Error**
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

System Type on a Bode Plot

- The type of a system is defined to be equal to the number of the integrators in the open loop transfer function.
- We can find the type of a system by examining its Bode plot.
  a. A type 0 system has a slope of 0 and a phase of 0° at low frequencies.
  b. A type 1 system has a slope of -20 dB/decade and a phase of -90° at low frequencies.
  c. A type 2 system has a slope of -40 dB/decade and a phase of -180° at low frequencies.
- “Low frequencies” in this context means in the frequency range below any of the system zeros or poles.
- Examination of an experimental frequency response lets you to determine the system type without needing a transfer function.

System Type on a Bode Plot



System Type on a Bode Plot

- As illustrated in the diagram below, the type of the system can be determined as follows:
  - For the first system 1/(s+10) with blue line, the gain at low frequency is 0 dB and the phase shift at low frequency is 0 degree -> type 0.
  - For the second system 1/s with orange line, the gain of at low frequency is a slope with -20 dB/decade and the phase shift at low frequency is -90 degree -> type 1 system.
  - For the second system $1/s^2$ with yellow line, the gain at low frequency is a -40 dB/decade slope and the phase shift at low frequency is -180 degree -> type 2 system.

Steady-State Error from a Bode Plot

- The system type is related to the error that a closed-loop system will exhibit when attempting to follow a reference signal.
- Reminder:

a. A type 0 system will have an error $1/(1+K_p)$ for a step input and infinite error for ramp and paraboloid.

b. A type 1 system will have zero error for a step, an error of $1/K_v$ for a ramp and an infinite error for an input paraboloid.

c. A type 2 system will have zero error when tracking input steps or ramps, but an error $1/K_a$ when tracking a command paraboloid.

Steady-State Error with a Step Input

- The error to a steady-state unity magnitude step is given by:

$$e(\infty) = \frac{1}{1 + K_p}$$

  Where: K_p is position-error constant.

- For a type 0 system, K_p is equal to the value of the open loop gain of the system.

- Thus, if the Bode plot indicates a type 0 system (zero slope at low frequency), we can directly read off the K_p value.

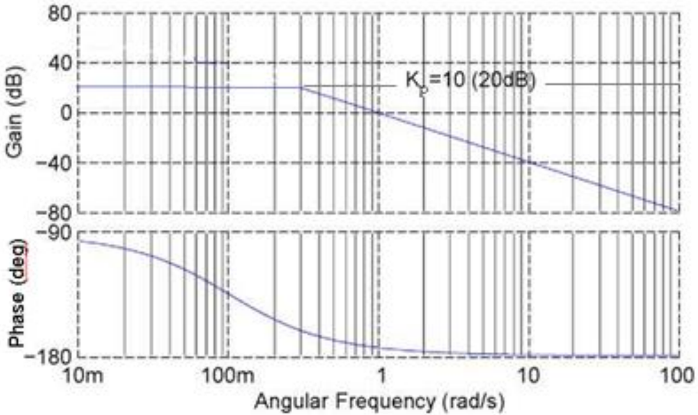
- For systems of higher type, the DC gain of the system is infinite, so the value of K_p is also infinite.

- This corresponds to zero static error to a step function for systems including one or more integrators in the forward path.

Steady-State Error with a Step Input

- This is actually the same as the value of the low-frequency gain of the system for type 0 system.

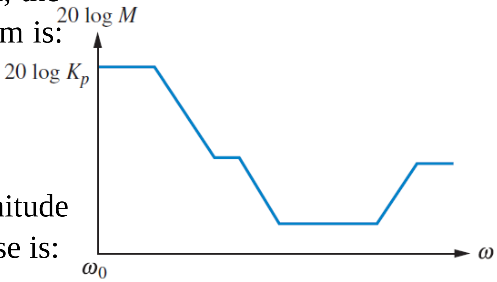
$$20\log K_p = |G(s)| \qquad \text{thus} \qquad K_p = \log^{-1}(20/20) = 10$$



Steady-State Error with a Step Input

- Or, for a given type 0 system, the transfer function of the system is:

$$G(s) = K\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)}$$



- The initial value of the magnitude plot of the frequency response is:

$$20\log|G(s)| = 20\log K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i}$$

- The value of position-error constant is:

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} K\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)} = K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i}$$

Steady-State Error with a Ramp Input

- The steady-state error in the presence of a unit ramp input is specified as:

$$e(\infty) = K_v$$

Where: K_v is velocity error constant.

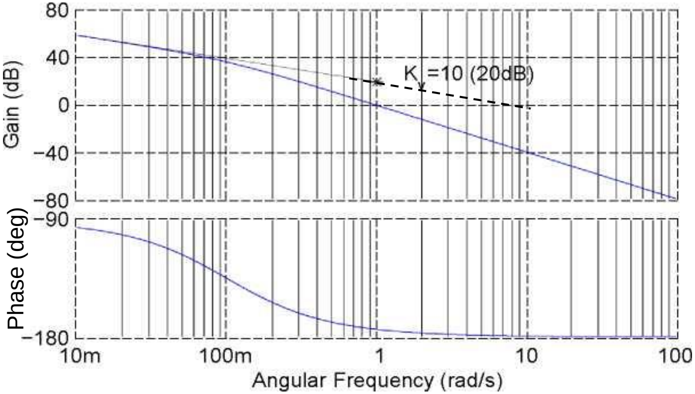
- Recall that $K_v = \lim_{s \to 0} sG(s)$.
- For a type 1 system, the multiplication by s would result in a level gain curve at low frequencies.
- If we were to plot a magnitude plot of $sG(s)$, then K_v would be the low frequency gain.
- Rather than plot this explicitly, we examine the gain that the 1/s part of the transfer function has at 1 rad/s.

Steady-State Error with a Ramp Input

- Similarly, the velocity error constant is found by determining the gain of the $1/s$ part of the transfer function if extended to $\omega = 1$.

- Velocity constant error is:

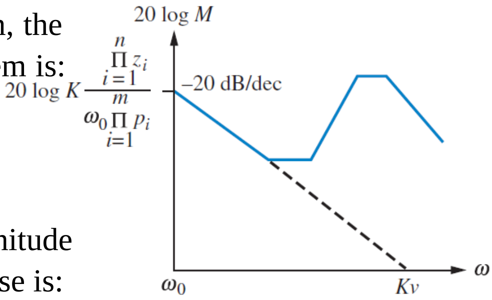
$$20 \log K_v = |G(s)| \qquad \text{thus} \qquad K_v = \log^{-1}(20/20) = 10$$



Steady-State Error with a Ramp Input

- Or, for a given type 1 system, the transfer function of the system is:

$$G(s) = K\frac{\prod_{i=1}^{n}(s+z_i)}{s\prod_{i=1}^{m}(s+p_i)}$$



- The initial value of the magnitude plot of the frequency response is:

$$20\log|G(s)| = 20\log K\frac{\prod_{i=1}^{n} z_i}{\omega_0\prod_{i=1}^{m} p_i}$$

- With a type 1 system, the -20 dB/decade slope of the frequency response can be considered as originated from a function:

$$G'(s) = K\frac{\prod_{i=1}^{n} z_i}{s\prod_{i=1}^{m} p_i}$$

Steady-State Error with a Ramp Input

- $G'(s)$ intersects the frequency axis when the frequency of the frequency response is:

$$\omega = K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i}$$

- Thus, the velocity-error constant of the system is:

$$K_v = \lim_{s\to 0} sG(s) = s\left(K\frac{\prod_{i=1}^{n} z_i}{s\prod_{i=1}^{m} p_i}\right) = K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i}$$

- Hence

$$K_v = \omega$$

- This is the same as the frequency-axis intercept.
- Extending the initial -20 dB/decade slope to the frequency axis will give you the velocity-error constant.

Steady-State Error with a Parabolic Input

- The error in the presence of a unit parabolic input is specified as:

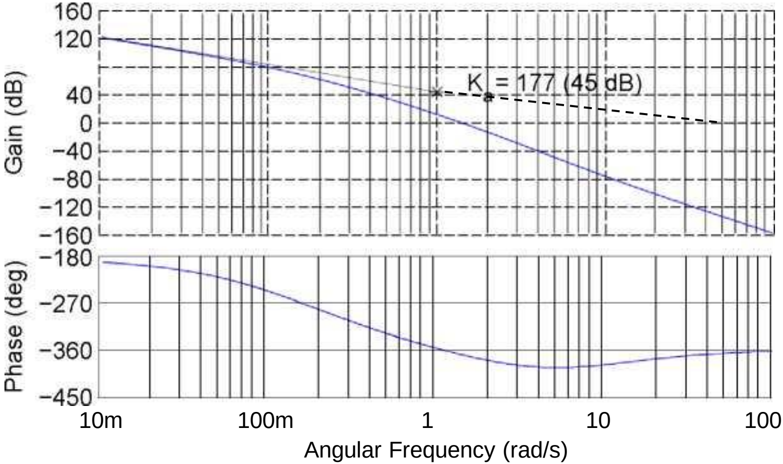
$$e(\infty) = K_a$$

Where: K_a is parabolic error constant.

- Recall that $K_a = \lim_{s \to 0} s^2 G(s)$.
- For a type 2 system, the multiplication by s^2 would result in a level gain curve at low frequencies.
- If we were to plot a magnitude plot of $s^2 G(s)$, then K_a would be the low frequency gain.
- Rather than plot this explicitly, we can instead examine the gain that the $1/s^2$ part of the transfer function has at 1 rad/s.

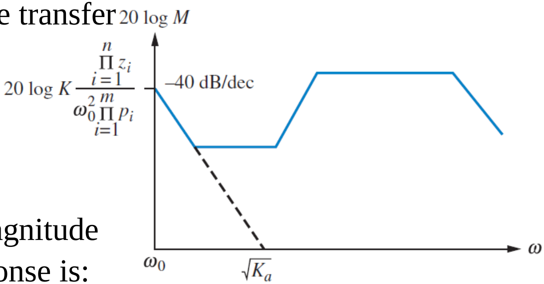
Steady-State Error with a Parabolic Input

- Thus, the acceleration error constant by determining the gain of the $1/s^2$ part of the transfer function if extended to $\omega = 1$.
- So, $20 \log K_a = |G(s)|$ thus $K_a = \log^{-1}(45/20) = 177$



Steady-State Error with a Parabolic Input

- Or, for a type 2 system, the transfer function of the system is:



$$G(s) = K \frac{\prod_{i=1}^{n}(s+z_i)}{s^2 \prod_{i=1}^{m}(s+p_i)}$$

- The initial value of the magnitude plot of the frequency response is:

$$20 \log|G(s)| = 20 \log K \frac{\prod_{i=1}^{n} z_i}{\omega_0^2 \prod_{i=1}^{m} p_i}$$

- With a type 1 system, the -20 dB/decade slope of the frequency response can be considered as originated from a function:

$$G'(s) = K \frac{\prod_{i=1}^{n} z_i}{s^2 \prod_{i=1}^{m} p_i}$$

Steady-State Error with a Parabolic Input

- $G'(s)$ has an intersection with the frequency axis when the frequency of the frequency response is:

$$\omega = \sqrt{K \frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i}}$$

- But, since the acceleration-error constant of the system is:

$$K_a = \lim_{s \to 0} s^2 G(s) = s^2 \left[K \frac{\prod_{i=1}^{n} z_i}{s^2 \prod_{i=1}^{m} p_i} \right] = K \frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i}$$

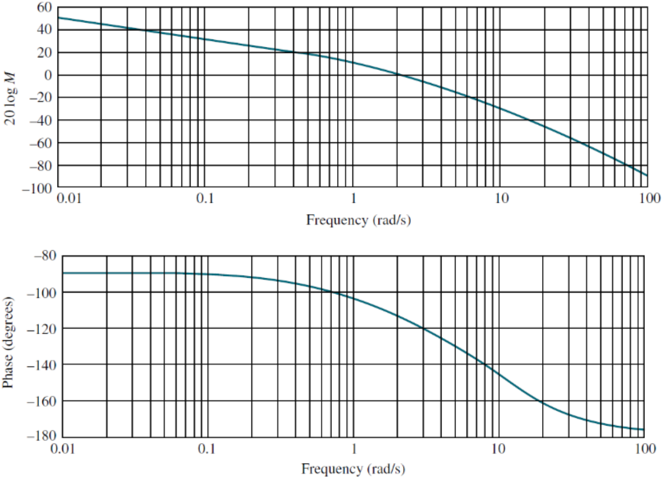
- Hence

$$\omega = \sqrt{K_a}$$

- Extending the initial -40 dB/decade slope to the frequency axis will give you the velocity-error constant at $\sqrt{K_a}$.

Steady-State Analysis with Bode Plots

The open-loop frequency response shown in the figure below was experimentally obtained from a unity feedback system. Estimate:



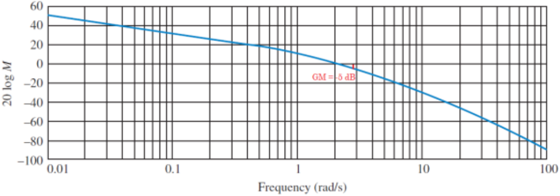
a. The percent overshoot of the closed-loop system.

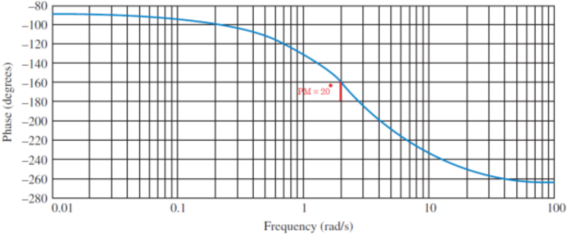
[20 marks]

a. The steady-state error of the closed-loop system.

[20 marks]

a. The phase margin of the closed-loop system is determined from following frequency response diagram.





From the given Bode plots, the phase margin of the given system is 20° and the gain margin is 5 dB.

Steady-State Analysis with Bode Plots

- Rearrange the equation given above, the damping ratio is:

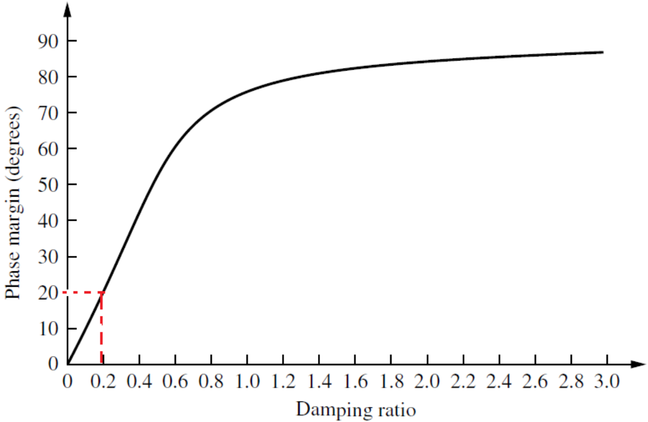
$$\zeta = \sqrt[4]{\frac{1}{\left(\frac{4}{\tan^2 \phi_m} + 2\right)^2 - 4}}$$

$$= \sqrt[4]{\frac{1}{\left(\frac{4}{\tan^2 20} + 2\right)^2 - 4}} = 0.176$$

- Or, using the phase margin vs. damping ration graph, the damping ratio can be estimated from the system's phase margin.

Steady-State Analysis with Bode Plots

- In graph given below, the damping ratio, ζ is approx. about 0.18.



Steady-State Analysis with Bode Plots

- The percentage overshoot of the system is calculated from the following equation:

$$\%OS = e^{\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\%$$

$$= e^{\frac{\pi(0.176)}{\sqrt{1-(0.176)^2}}} \times 100\% = 57\%$$

- The equation given above yields 57% overshoot.

b. The system is Type 1 since the initial slope is - 20 dB/dec and extending this slope intersection with the gain at 1 rad/s is 12 dB.

Continuing the low frequency slope down to the frequency axis (i.e. 0 dB line) yields 4 rad/s.

Steady-State Analysis with Bode Plots

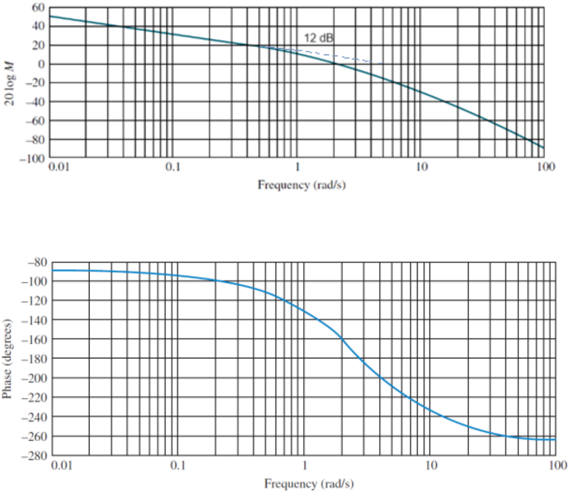
Knowing that we found the intersection of low frequency slope with 1 rad/s is 12 dB.

Thus, the velocity error constant of the system is:

$$K_v = \log^{-1}(12/20) = 4$$

Or, from the intersection with the frequency axis:

$$K_v = 4$$



Steady-State Analysis with Bode Plots

As a result, for $K_v = 4$ and given relevant inputs, the steady-state errors of the system are:

- For a unit step input, it is zero.
- For a unit ramp input, it is a finite value:

$$e(\infty)_{\text{ramp}} = \frac{1}{K_v} = 0.25$$

- For a parabolic input, it is infinite (∞).