Introduction to Root Locus

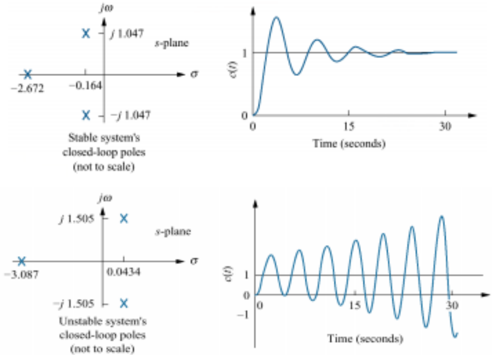
XMUT315 Control System Engineering

Topics

- Introduction to Pole- Zero Diagrams
- Poles and Zeros of Systems
- System Performance
- Closed-Loop Poles and Zeros
- Examples of Root Locus Diagram
- Design using Root Locus Diagram
- Formalisations of Root Locus Diagram
- Examples of Plots in Root Locus.

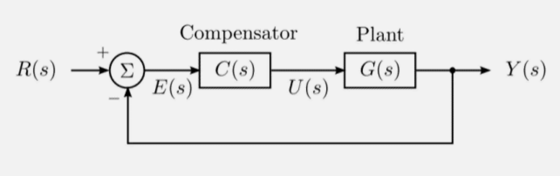
Introduction

- We know that the location of the system poles and zeros determine the response of a LTI system.
  - Poles on the y-axis -> undamped response.
  - Complex poles -> underdamped response.
  - Single double poles on the x-axis -> critical damped response.
  - Poles on the x-axis -> overdamped response.



Introduction

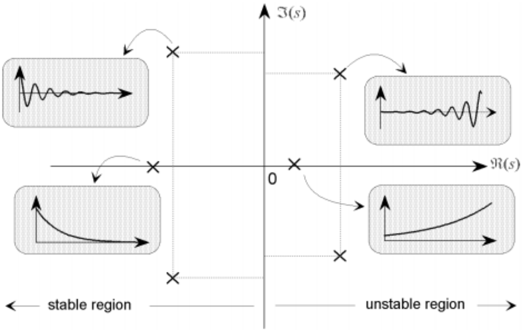
- To change the system response, we therefore need to change these root (pole and zero) location(s).
- We will do this by applying feedback and including a compensator (or controller) into the system.



- Application of feedback enables us to manage the system.
- Compensator (or controller) enable us to determine the specific response of the system.

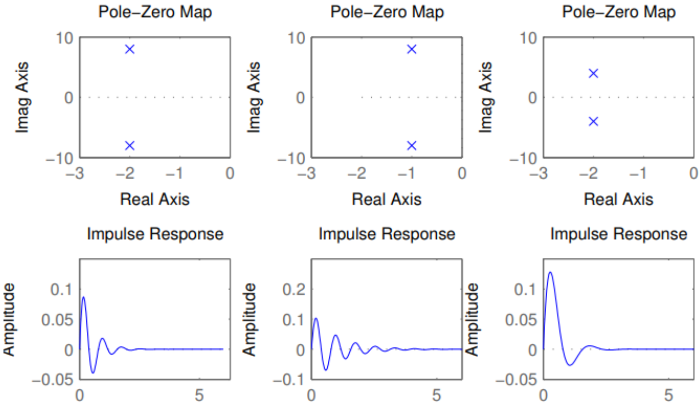
Introduction

- The root locus is a visual presentation of the way the roots of a system change as we vary some parameters of our compensator.
- We will usually be interested in how altering the gain affects the system response, but we can use the root locus for other parameters too.



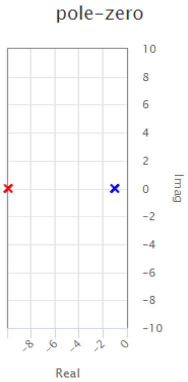
Poles on the Pole-Zero Map

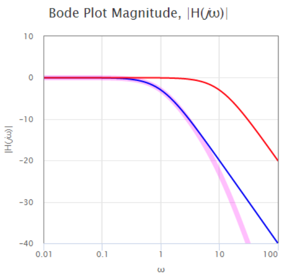
- We know that pole location determines which modes will be present in a time response.
- The real part of a pole location determines the damping of a mode and the imaginary part determines the natural frequency of the mode.

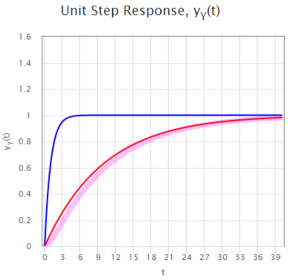


Dominant Poles

- The pole with the slowest response will dominate the response of any system.



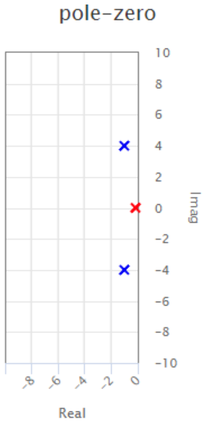


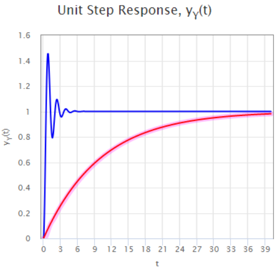


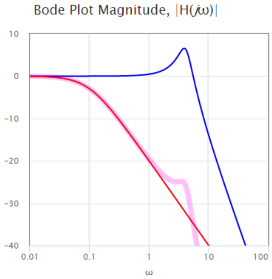
- The pole near the origin (blue – with the slowest response) compared with the pole further from the origin (red).

Dominant Poles

- The pole with the slowest response will dominate the response of any system.







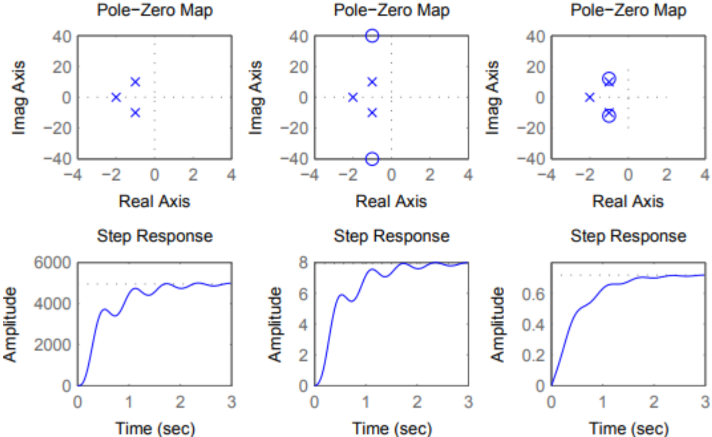
- The pole at the origin (red – with the slowest response) compared with complex poles (blue) further from origin.

Dominant Poles

- We can therefore often approximate the response of a system as either a first order, or a second order system.
- We can then use the location of the dominant poles to determine the system's settling time, overshoot, damping, natural frequency etc.
- So, if the dominant pole (pair) is sufficiently far away from any other poles, then we can ignore the other poles when designing our control system.
- This approximation is normally safe if other poles are at least a factor of three further to the left of the *s*-plane.

Zeros on the Pole-Zero Map

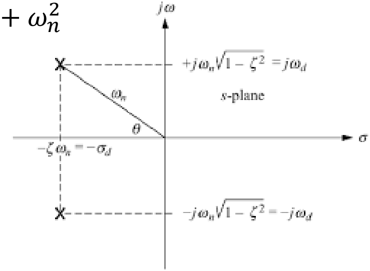
- Zeros near a pole suppress the mode corresponding to the nearby pole.
- In the extreme case where the zeros and poles are co-located, we get complete cancellation of the corresponding mode.



System Performance - Settling Time

- The further a pole is to the left of the s-plane, the faster the corresponding mode decays.
- Therefore, if we are given a specification for system response time, we can convert this to a requirement on pole position.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



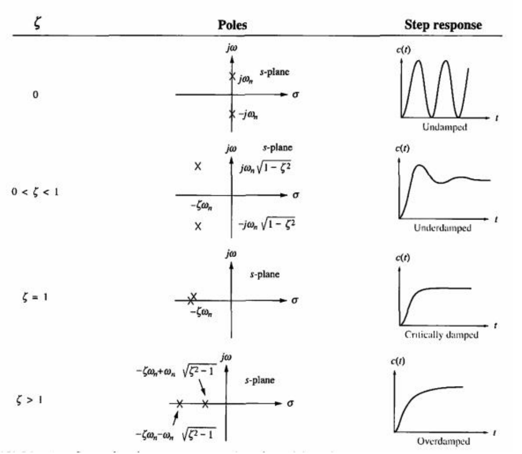
- Roots of the characteristic equation:

$$s = -\frac{2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\sigma \pm \omega_d$$

System Performance - Settling Time

- For the given roots of characteristic equation:
  - The two roots are imaginary when $\zeta = 0$.
  - The two roots are complex conjugate when $0 < \zeta < 1$.
  - The two roots are real and equal when $\zeta = 1$.
  - The two roots are real but not equal when $\zeta > 1$.



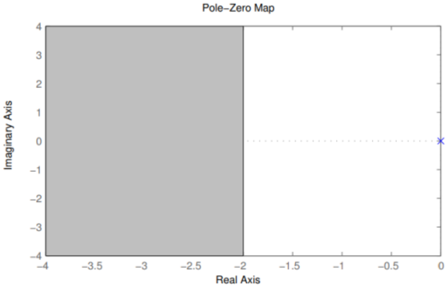
System Performance - Settling Time

- The settling time of a second order system (T_s) is approximately (e.g. for the 2% steady-state settling time standard):

$$T_s = \frac{4}{\zeta \omega_n} = \frac{4}{\sigma}$$

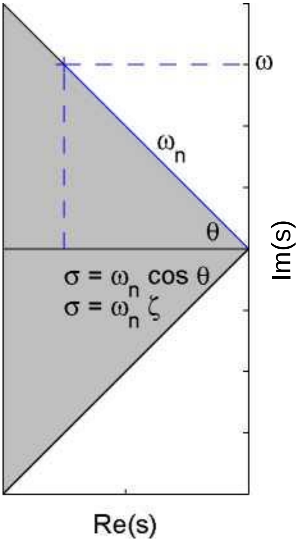
Where: σ is the real part of the pole pair.

- So, for example if we have a requirement that settling time, T_s < 2 second, then we must have the dominant pole further left than s = - 2.



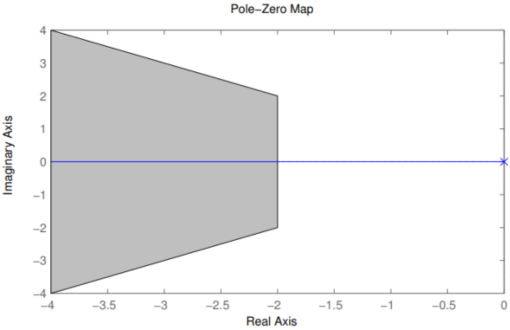
System Performance - Damping

- We are often also given a requirement on damping ratio.
- Recall that a damping ratio ζ occurs on the pole-zero diagram as a straight line with an angle of $\theta = \cos^{-1}\zeta$ to the negative real axis.
- We can thus similarly construct an allowed region for the poles if we are given a damping specification.
- For example, if we require ζ > 0.707, then the poles must lie less than $\theta = \cos^{-1} 0.707 \approx 45°$ from the negative real axis.



System Performance - Allowed Pole Region

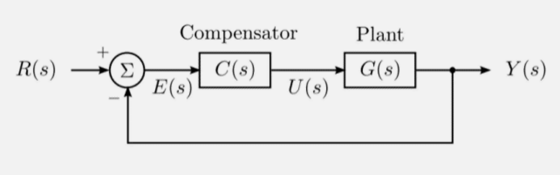
- We are often given specifications on both settling time and damping, so we must combine the constraints imposed by the two.
- We therefore obtain a composite region in which we can place the poles.



- So, if the closed-loop system poles lie within the allowed region then we have satisfied the design specification.

Closed-Loop Poles

- If we have a plant described by an open-loop transfer function $G(s)$, it will have a certain set of open-loop poles and zeros.



- We now enclose the plant within a unity-gain feedback loop including a compensator with a transfer function $C(s)$, which results in a closed-loop transfer function:

$$T(s) = \frac{Y(s)}{R(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

Closed-Loop Poles

- Notice that the closed-loop transfer function will not have the same pole locations as the open-loop transfer function.
- There is no reason that the denominator of $G(s)$ and the denominator of $T(s)$ would have the same roots.
- Closing the loop has moved the poles, but we do not yet know to where it is heading.

- Open-loop transfer function:

$$G(s) = \frac{1}{(s+10)}$$

- Root of characteristic equation:

$$s = -10$$

- Closed-loop transfer function:

$$T(s) = \frac{\frac{1}{(s+10)}}{1 + \frac{1}{(s+10)}} = \frac{1}{s+11}$$

- Root of characteristic equation:

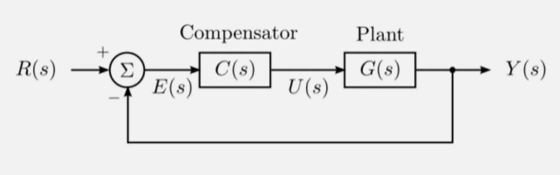
$$s = -11$$

Closed-Loop Poles

- We will find is that the location of the closed-loop poles will depend on the dc gain K or the system. In this case:

$$K = C(s)G(s)\Big|_{s=0}$$

- As we can change $C(s)$, we can use K as a tuning parameter to move the closed-loop poles to a desired location.
- The desired location for the poles will, in turn, be determined by the performance specification for our control system.



Closed-Loop Zeros

- Let's consider the effect that feedback has on zeros.
- Again, consider a plant with a transfer function $G(s)$ that we have controlled by adding a series compensator $C(s)$.
- Again, the closed-loop transfer function is:

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)}$$

- The only way that $T(s)$ can be zero is if either $C(s)$ or $G(s)$ is zero (or both).
- Thus, the set of zeros of the closed-loop system is the combination of the zeros of the plant and the compensator.

It is not possible to move the zeros of a plant using feedback.

Example #1

- Consider a system that has a single open-loop pole at s = -1.

$$G(s) = \frac{1}{s+1}$$

- Let us use a proportional compensator $C(s) = K_p$ and see what effect this has on the pole.

$$T(s) = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{\left(\frac{K_p}{s+1}\right)}{1 + \left(\frac{K_p}{s+1}\right)} = \frac{K_p}{s + \left(1 + K_p\right)}$$

- Thus, the closed-loop pole is at $s = -1 - K_p$, compared to the open-loop pole at s = -1.

- We can choose the closed-loop pole position by choosing K_p, though only along a constrained path (e.g. a locus).

Example #1

- Depending on the value of proportional compensator, K, the locations of the closed-loop poles are along a constraint path (e.g. a locus).

- Closed loop system:

$$T(s) = \frac{K_p}{s + (1 + K_p)}$$

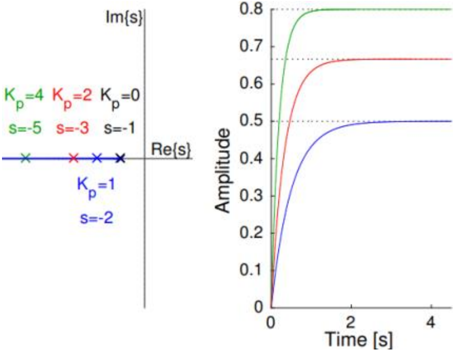
- These locus are:

$$K_p = 0 \quad \rightarrow \quad s = -1$$

$$K_p = 1 \quad \rightarrow \quad s = -2$$

$$K_p = 2 \quad \rightarrow \quad s = -3$$

$$K_p = 4 \quad \rightarrow \quad s = -5$$



Example #2

- Consider a DC motor with a transfer function $G(s) = 1/s(s+1)$.
- Let us add a proportional controller $C(s) = K_p$ to this plant.

$$T(s) = \frac{\left[\frac{K_p}{s(s+1)}\right]}{1 + \left[\frac{K_p}{s(s+1)}\right]}$$

$$= \frac{K_p}{s^2 + s + K_p}$$

- Now, the closed-loop poles are located where the denominator is equal to zero.
- Thus, we find them by solving $s^2 + s + K_p = 0$ (or more generally), by solving the characteristic equation, $1 + CG = 0$.

Example #2

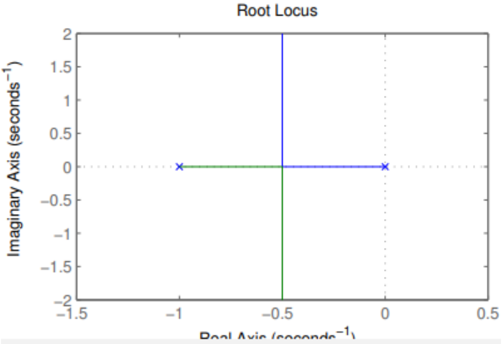
- So let us find the roots of $s^2 + s + K_p = 0$ using the quadratic equation.

$$s = \frac{-1 \pm \sqrt{1 - 4K_p}}{2}$$

- For K_p < 1/4, we get two real roots at:

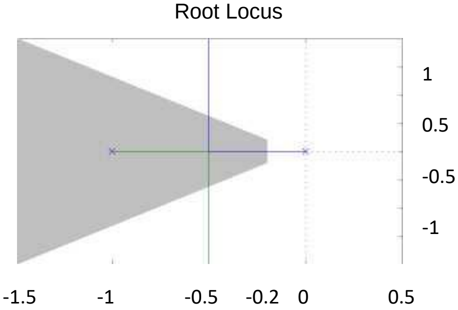
$$s = -\frac{1}{2} \pm \frac{\sqrt{1 - 4K_p}}{2}$$

- For K_p > 1/4 we change from a first order to a second order (oscillatory) response as the poles become complex.



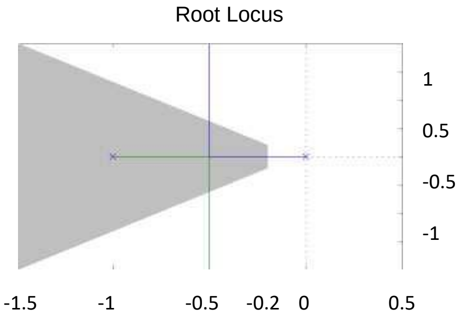
Design using the Root Locus

- Let us imagine that we have been asked to design a controller for the dc motor so that it has a settling time of less than 20 seconds and a damping ratio better than ζ = 0.707.
- As we need $T_s = 4/\sigma < 20$ the dominant pole must be further left than s = -0.2 (e.g. σ = 4/20 = 0.2).
- Damping ratio, ζ > 0.707 requires that the angle from the negative real axis be no greater than $\theta = \cos^{-1} 0.707 = 45°$.



Design using the Root Locus

- In this example, there are a range of possible K values that would satisfy the design specification.
- Let’s work out the minimum and maximum gains that would be acceptable.
- The dominant pole crosses into the allowed region at s = -0.2.



Design using the Root Locus

- We know that while the roots are real, they are located at:

$$s = -\frac{1}{2} \pm \frac{\sqrt{1-4K_p}}{2}$$

- So, in this case:

$$s = -0.5 \pm \frac{\sqrt{1-4K_p}}{2} = -0.2$$

- Solving the equation given above:

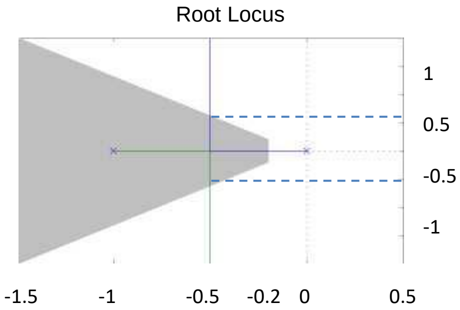
$$\frac{\sqrt{1-4K_p}}{2} = 0.3$$

- As a result, the proportional gain of the system is:

$$K_p = 0.16$$

Design using the Root Locus

- By inspection of the diagram, we can see that the poles leave the allowed region at $s = -0.5 \pm j0.5$.
- These points are the maximum values before that meet the design specification of the system.



Design using the Root Locus

- We previously calculated that the complex poles are located at:

$$s = -\frac{1}{2} \pm j\frac{\sqrt{4K_p - 1}}{2}$$

- So, σ = -0.5 and again we can equate to find the value of K_p on the boundary:

$$\frac{\sqrt{4K_p - 1}}{2} = 0.5$$

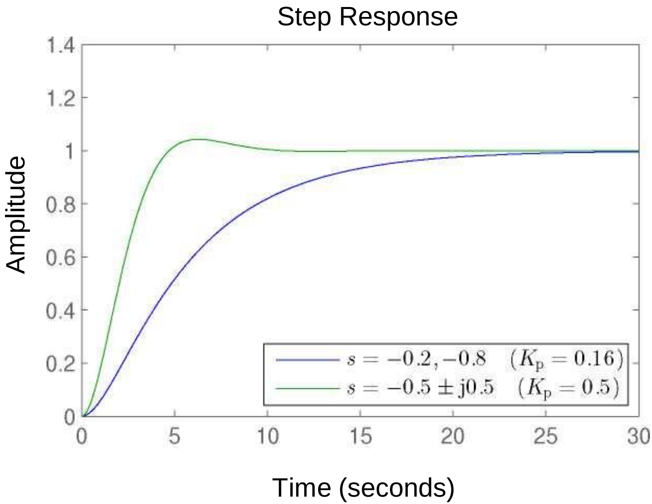
- So, the proportional gain of the system is:

$$K_p = 0.5$$

- So, we have found that the design specification will be met for 0.16 < K_p < 0.5.

Design using the Root Locus

- The graph shows the step responses of the system when $K_p = 0.16$ and $K_p = 0.5$.



The Evan's Root Locus

- We can always determine the root locus using mathematical analysis as in the examples above.
- However, this becomes tedious as the number of poles increases.
- The Evan's root locus is a graphical technique that automates the mathematics to provide a method to draw the locus directly.
- Examination of the root locus allows us to:
  a. Determine the stability of a system as gain changes.
  b. Choose an appropriate gain to produce a desired closed-loop response.
  c. Modify the form of $C(s)$ if an adequate closed-loop response cannot be achieved.

Root Locus Formalities

- The root locus is a pole zero diagram that shows the "tracks" taken by the system poles as some parameters (i.e. gain in our case) are varied.
- Formally, the root locus shows the locus traced out by the roots of the characteristic equation, $1 + C(s)G(s) = 0$, as the gain is varied.
- To find a root locus, we are therefore searching for values of s that satisfy the characteristic equation:

$$1 + C(s)G(s) = 0$$

- Rearrange the equation:

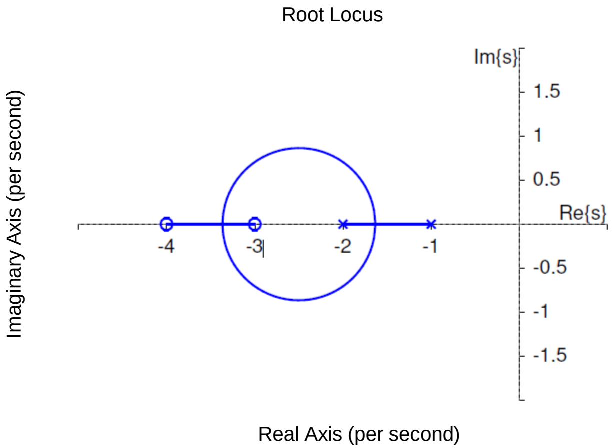
$$C(s)G(s) = -1$$

$$C(s)G(s) = 1\angle(2\mathrm{k}+1)180° \quad \text{for} \quad \mathrm{k} \in \mathbb{Z}$$

Example #3

$$G(s) = \frac{(s+3)(s+4)}{(s+1)(s+2)}$$

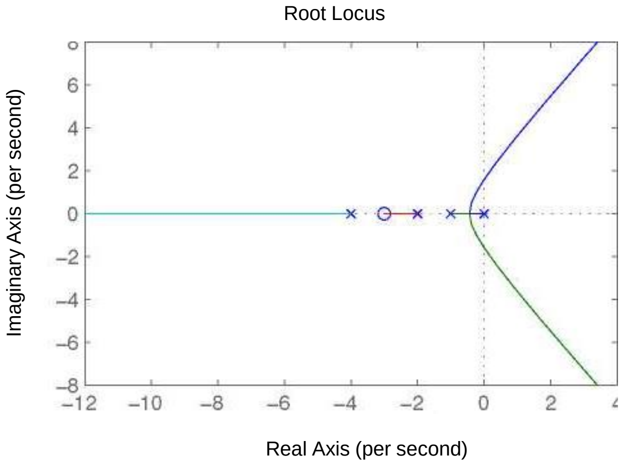
- Zeros: -3, -4.
- Poles: -1,-2.



Example #4

$$G(s) = \frac{s+3}{s(s+1)(s+2)(s+4)}$$

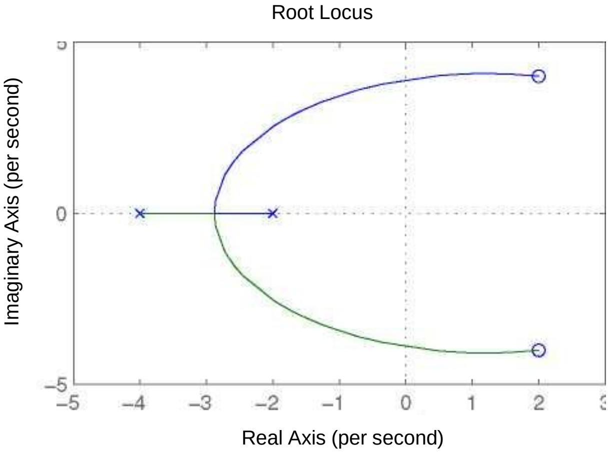
- Zeros: -3.
- Poles: 0,-1,-2,-4.



Example #5

$$G(s) = \frac{(s-2)^2 + 16}{(s+2)(s+4)}$$

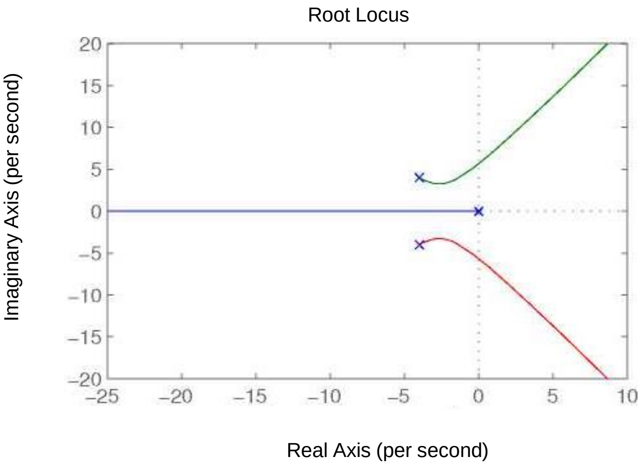
- Zeros: 2 ± j4.
- Poles: -2,-4.



Example #6

$$G(s) = \frac{1}{s[(s+4)^2 + 16]}$$

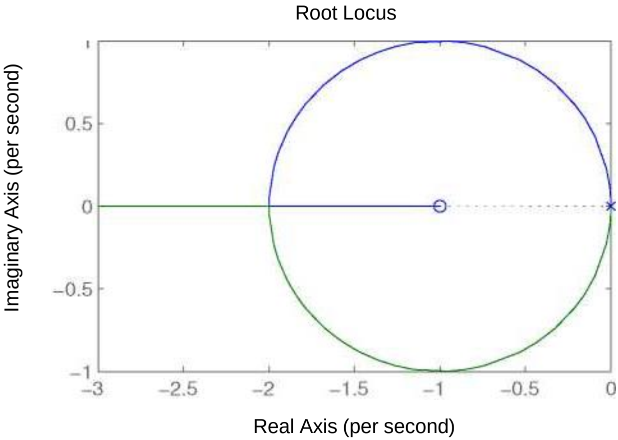
- Zeros:
- Poles: 0, -4 $\pm$ j4



Example #7

$$G(s) = \frac{1+s}{s^2}$$

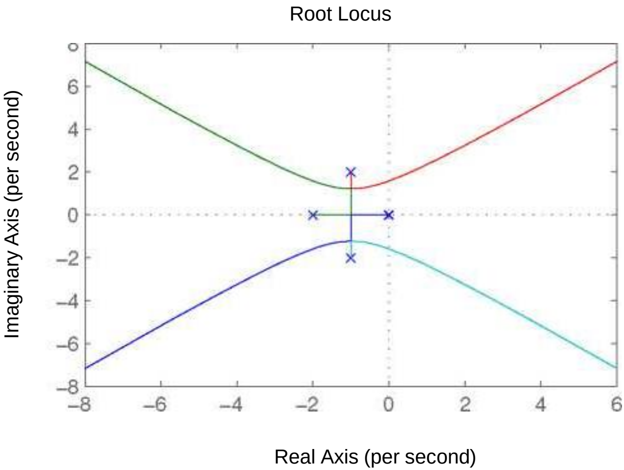
- Zeros: -1.
- Poles: -0 (double poles at origin).



Example #8

$$G(s) = \frac{1}{s(s+2)[(s+1)^2+4]}$$

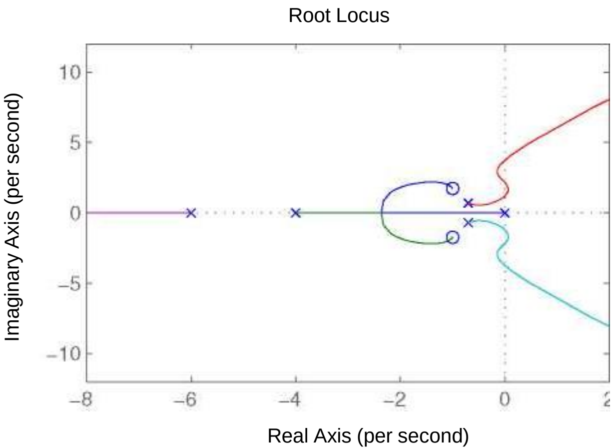
- Zeros:
- Poles: 0, -2, -1 $\pm$ j2



Example #9

$$G(s) = \frac{s^2 + 2s + 4}{s(s+4)(s+6)(s^2 + 1.4s + 1)}$$

- Zeros: -1 $\pm$ j$\sqrt{3}$.
- Poles: 0,-4,-6, -0.7 $\pm$ j$\sqrt{0.51}$



Things to Notice

These are the guidelines for construction of root locus diagram:

1. Each branch of the root locus begins at an open-loop pole.
2. Each branch of the root locus either terminates at a zero or goes to (complex) infinity.
3. One and only one branch leaves each pole.
4. One and only one branch enters each zero.
5. Like any pole-zero diagram, the root locus is always symmetric about the real axis (and complex poles always come in conjugate pairs).