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Analysis with Root Locus

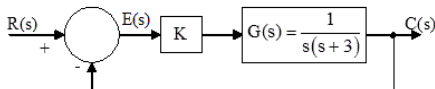
XMUT315 Control System Engineering

Topics

- Sketching root locus.
- Rules in root locus analysis.
- Examples of root locus analysis.
- Refining root locus analysis.
- Break away and break in.
- Imaginary axes crossing.
- Angle of departure and angle of arrival.

Sketching the Root Locus

- Direct solution (e.g. analytical method) for the closed-loop pole locations as a function of K becomes burdensome as the system order increases.

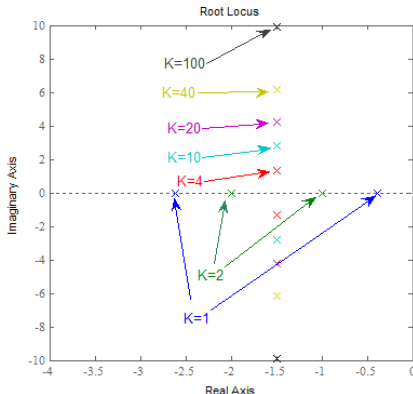


- Open-loop system:

$$G(s) = \frac{1}{s(s+3)}$$

- Closed-loop system:

$$T(s) = \frac{K}{s^2 + 3s + K}$$



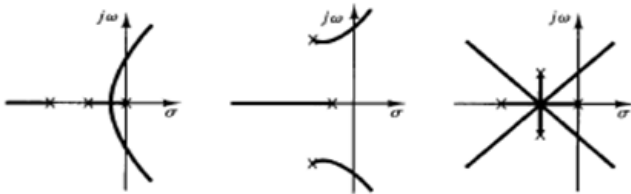
Sketching the Root Locus

- Walter Evans devised a set of rules that allows sketching of the root locus without brute force calculation.
- The basic root locus diagram is based on five rules -> e.g. Evans rule for sketching root locus.
- These rules are generally sufficient to give a good sense of the shape of the root locus, and consequently its “story” as the gain changes.
- There are additional rules that can be used to refine the shape of the locus when necessary.

Rule #1 for Sketching a Root Locus

The number of branches of the root locus is equal to the number of closed-loop poles.

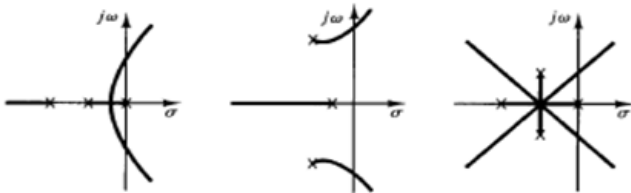
- A “branch” is a segment of the root locus traversed (or moved) by a single pole as the gain is varied.
- This rule implies that there is one and only one branch originating at each pole.



Rule #2 for Sketching a Root Locus

The root locus is symmetric about the real axis.

- Contours of the plots are symmetric about the real axis.
- Any complex roots (poles or zeros) must occur in complex conjugate pairs.
- Therefore, the root locus must be symmetric.



Rule #3 for Sketching a Root Locus

A branch of the root locus will only be on the real axis to the left of an odd number of finite open-loop roots.

- Recall that roots means both poles and zeros!
- As discussed earlier, the solution of the characteristic equation implies that $\angle CG = -180^\circ$ at all points on the root locus.
- Now, all roots on the further left than a test point will contribute no phase shift.
- However, both zeros and poles to the right of a test point contribute 180° .
- We therefore only satisfy the characteristic equation if there is an odd number of roots on the real axis that are further to the right than our test point.

Rule #4 for Sketching a Root Locus

The root locus begins at the finite and infinite poles of $C(s)G(s)$ and ends at the finite and infinite zeros of $C(s)G(s)$.

- Let's calculate what happens to $T = KCG/(1 + KCG)$ for very small and very large values of K .
- Notice the characteristic equations of the closed-loop system is:

$$1 + KC(s)G(s) = 0$$

- Factor $G(s)$ and rewrite the polynomial equation as:

$$1 + KC(s)G(s) = 1 + K \frac{N_C(s)N_G(s)}{D_C(s)D_G(s)} = 1 + K \frac{\prod_{i=1}^m (s - z_i)}{\prod_{i=1}^m (s - p_i)}$$

- Therefore, the characteristic equation becomes:

$$\prod_{i=1}^m (s - p_i) + K \prod_{i=1}^m (s - z_i) = 0$$

Rule #4 for Sketching a Root Locus

- Note that we have allowed for more interesting compensators here, by expanding the previous C into KC , where C now contains any compensator roots.

$$\lim_{k \rightarrow 0} \left[\prod_{i=1}^m (s - p_i) + K \prod_{i=1}^m (s - z_i) \right]$$

- Therefore, the poles of T for small K are the poles of CG . That is, the poles of T start at the open-loop poles of the plant and the open-loop poles of the compensator.

$$\lim_{k \rightarrow \infty} \left[\prod_{i=1}^m (s - p_i) + K \prod_{i=1}^m (s - z_i) \right]$$

- Therefore, the poles of T for large K are the roots of CG . These are the zeros of CG , which is just the combination of the zeros of C and the zeros of G .

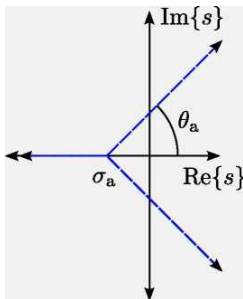
Rule #5 for Sketching a Root Locus

The root locus approaches straight-line asymptotes as the gain approaches infinity. The real-axis intercept, σ_a , and angles, θ_a , are given by:

$$\sigma_a = \frac{\sum_i p_i - \sum_i z_i}{P - Z} \quad \text{and} \quad \theta_a = \frac{(2k + 1)\pi}{P - Z}$$

Where: P is the number of finite poles, Z is the number of finite zeros, p_i is the i -th pole, z_i is the i -th zero and $k \in \mathbb{Z}$.

- In general, a root locus will have $P-Z$ asymptotes, so you will need to substitute $P-Z$ consecutive values for k into the θ_a equation.



Example for Root Locus Construction

Sketch the root locus of the transfer function:

$$T(s) = \frac{s + 3}{s(s + 1)(s + 2)(s + 4)}$$

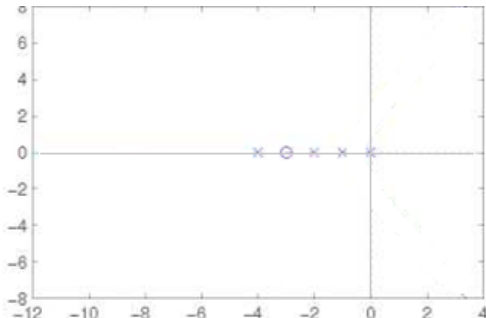
- Work out and determine the locations of the poles and zeros on the system in the s-plane diagram. [2 marks]
- Calculate the departure point and indicate this on the s-plane diagram. [3 marks]
- Calculate the angle of asymptotes and sketch these angles on the s-plane diagram. [4 marks]
- Determine the branches of the root locus and sketch them on the s-plane diagram. [4 marks]
- Evaluate the construction of root locus diagram of the system using the rules for sketching a root locus. [10 marks]
- Simulate the root locus diagram in MATLAB. [5 marks]

Example for Root Locus Construction

- Knowing the transfer function of the system:

$$T(s) = \frac{s + 3}{s(s + 1)(s + 2)(s + 4)}$$

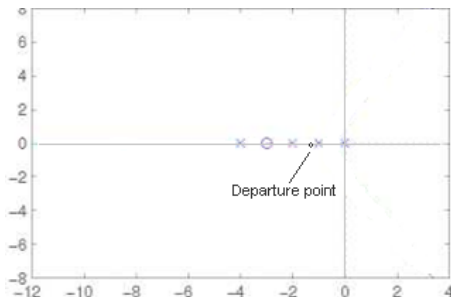
- For the given open-loop system, we have poles at $s = 0, -1, -2$ and -4 and a zero at $s = -3$.



Example for Root Locus Construction

- Let's first determine the parameters of the asymptotes. First, the departure point:

$$\begin{aligned}\sigma_a &= \frac{\sum_i p_i - \sum_i z_i}{P - Z} \\ &= \frac{(0 - 1 - 2 - 4) - (-3)}{4 - 1} \\ &= \frac{-7 + 3}{3} = -\frac{4}{3}\end{aligned}$$



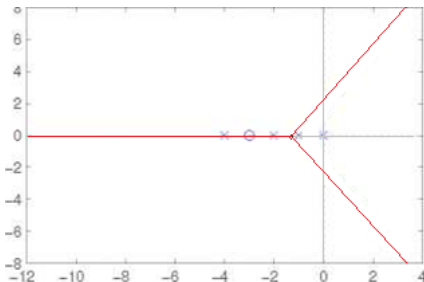
- We can therefore indicate this point in the root locus.

Example for Root Locus Construction

- Work on the angle of the asymptotes:

$$\begin{aligned}\theta_a &= \frac{(2k+1)\pi}{P-Z} \\ &= \frac{(2k+1)\pi}{4-1}\end{aligned}$$

$$\theta_a = \begin{cases} \frac{\pi}{3} & \text{for } k = 0 \\ \pi & \text{for } k = 1 \\ -\frac{\pi}{3} & \text{for } k = -1 \end{cases}$$



- From the departure point there are three asymptotes: at $\pi/3$ (60°), $-\pi/3$ (-60°), and π (-180°).
- We can therefore sketch in the asymptotes.

Example for Root Locus Construction

- With the asymptotes in place we can sketch in the various branches of the root locus.

- Asymptotes are:

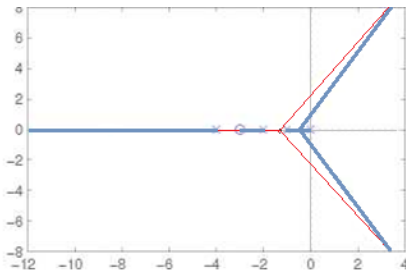
Asymptote 1: Pole $(0,0) \rightarrow +\infty$

Asymptote 2: Pole $(-1,0) \rightarrow -\infty$

Asymptote 3: Pole $(-2,0) \rightarrow$

Zero $(-3,0)$

Asymptote 4: Pole $(-4,0) \rightarrow -\infty$



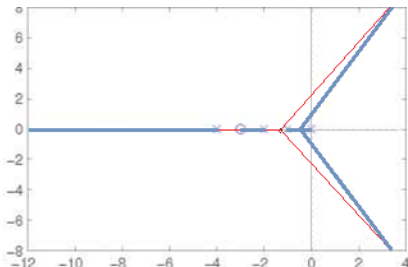
Example for Root Locus Construction

- R1 (*The number of branches of the root locus is equal to the number of closed-loop poles*) :

We expect four branches to the locus.

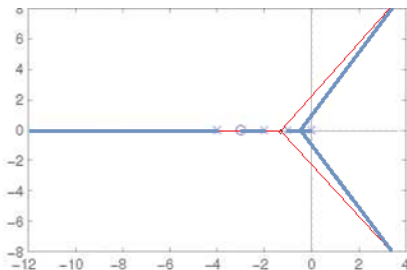
- R2 (*The root locus is symmetric about the real axis*) :

The sketch is symmetric about the real axis.



Example for Root Locus Construction

- R3 (A branch of the root locus will only be on the real axis to the left of an odd number of finite open loop roots) :
- The locus will pass along the real axis between the slowest two poles and between the third slowest pole and the zero.
- There is a branch on the real axis beyond the fastest pole.



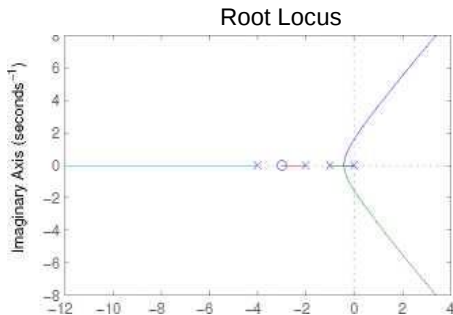
Example for Root Locus Construction

- R4 (The root locus begins at the finite and infinite poles of $C(s)G(s)$ and ends at the finite and infinite zeros of $C(s)G(s)$) :*

The locus are from poles to infinities and from a pole at -2,0 to zero at -3,0.

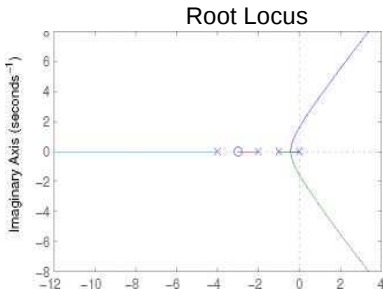
- R5 (The root locus approaches straight line asymptotes as the gain approaches infinity at the real-axis intercept, σ_a , and angles, θ_a) :*

There are three asymptotes from (-1.33,0) at $\pi/3$, $-\pi/3$, and π .

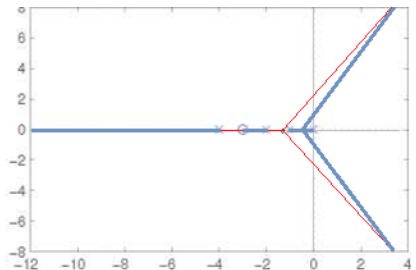


Example for Root Locus Construction

- Root locus diagram of the given system from MATLAB simulation.
- The result shows a similar likeness with the result of the manual sketch.



MATLAB simulation



Sketch

Example Exercises of Root Locus Construction

For each of the system with the following transfer function equations:

$$G_1(s) = \frac{s^2 - 4s + 24}{(s + 2)(s + 4)}$$

$$G_2(s) = \frac{1}{s[(s + 4)^2 + 16]}$$

$$G_3(s) = \frac{1 + s}{s^2}$$

$$G_4(s) = \frac{1}{s(s + 2)[(s + 1)^2 + 4]}$$

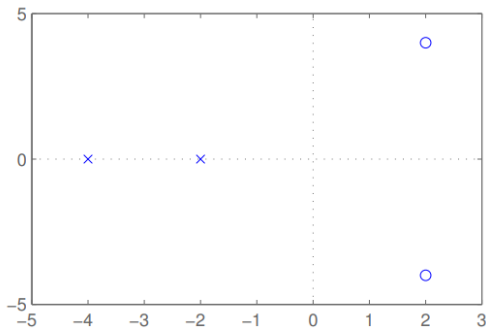
Example Exercises of Root Locus Construction

- a. Determine the poles and zeros of the system. [2 marks]
- b. Sketch the poles and zeros of the system in the s-plane diagram. [4 marks]
- c. Determine the asymptotes of the root locus. [2 marks]
- d. Sketch the asymptotes on the s-plane diagram. [4 marks]
- e. Simulate the root locus diagram of the system in MATLAB. [5 marks]

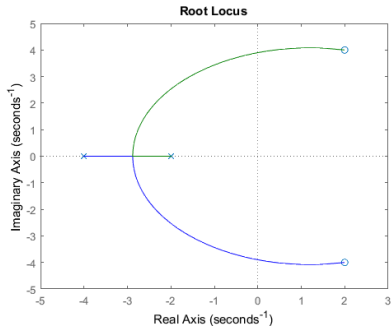
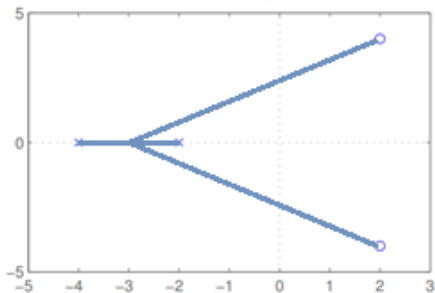
Root Locus Construction Exercise #1

$$G_1(s) = \frac{s^2 - 4s + 24}{(s + 2)(s + 4)}$$

- Pair of complex zero at $s_{1,2} = -2 \pm j\sqrt{20}$.
- Real pole at $s = -2$.
- Real pole at $s = -4$.



Root Locus Construction Exercise #1



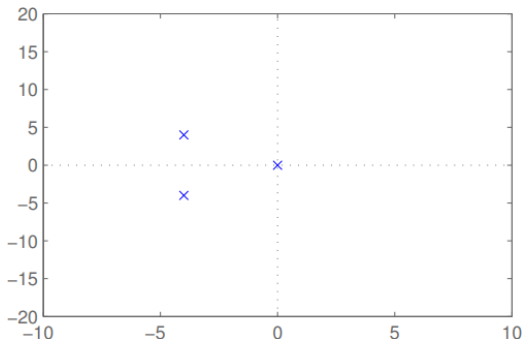
Asymptotes:

- Pole at $s = -2 \rightarrow$ zero at $2 + j\sqrt{20}$
- Pole at $s = -4 \rightarrow$ zero at $2 - j\sqrt{20}$

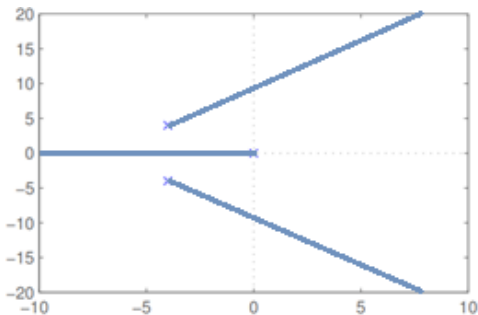
Root Locus Construction Exercise #2

$$G_2(s) = \frac{1}{s[(s+4)^2 + 16]}$$

- No zero.
- A pole at origin.
- Pair of complex poles at $s_{1,2} = -4 \pm j4$.

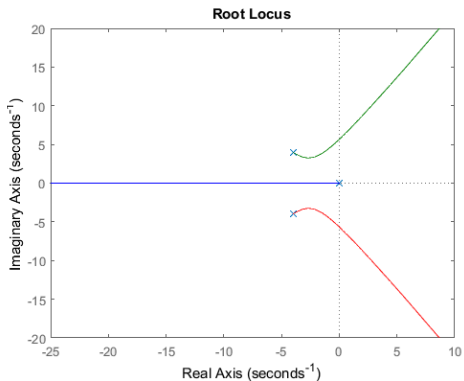


Root Locus Construction Exercise #2



Asymptotes:

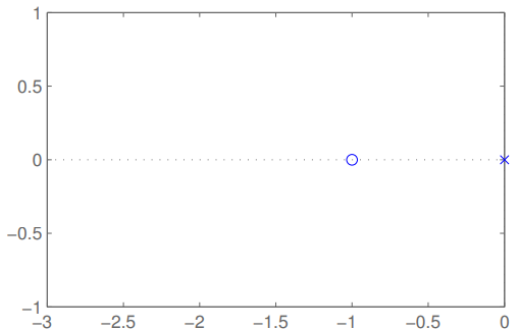
- Pole at origin $\rightarrow -\infty$
- Pole at $s = -4 + j4 \rightarrow +\infty (+j\infty)$
- Pole at $s = -4 - j4 \rightarrow -\infty (-j\infty)$



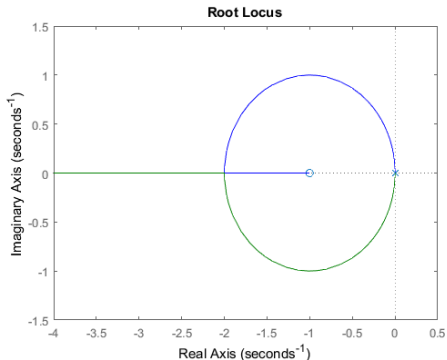
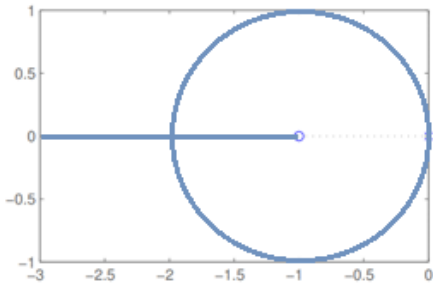
Root Locus Construction Exercise #3

$$G_3(s) = \frac{1+s}{s^2}$$

- A real zero at $s = -1$.
- Double poles at origin.



Root Locus Construction Exercise #3



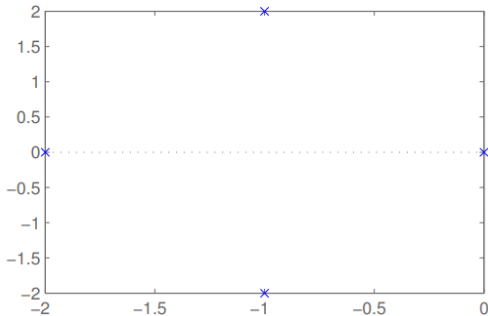
Asymptotes:

- Pole at origin $\rightarrow$ zero at $s = -1$.
- Pole at origin $\rightarrow -\infty$.

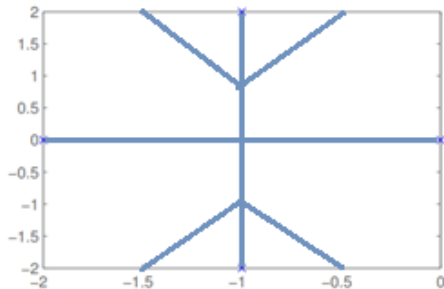
Root Locus Construction Exercise #4

$$G_4(s) = \frac{1}{s(s+2)[(s+1)^2+4]}$$

- No zero.
- A pole at origin.
- A real pole at $s = -2$.
- A pair of complex poles $s = -1 \pm j2$.

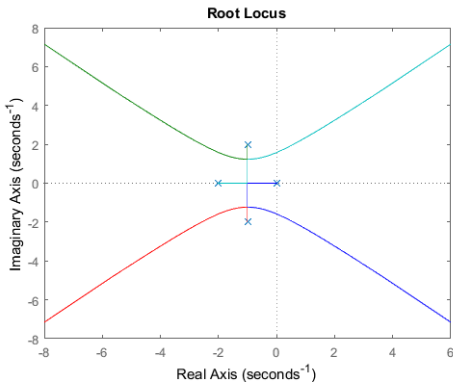


Root Locus Construction Exercise #4



Asymptotes:

- Pole at origin $\rightarrow +\infty$ ($-j\infty$).
- Pole at $s = -2 \rightarrow +\infty$ ($+j\infty$).
- Complex pole at $s = -1 + j2 \rightarrow -\infty$ ($+j\infty$).
- Complex pole at $s = -1 - j2 \rightarrow -\infty$ ($-j\infty$).



Refining the Root Locus

- There are many features of the root locus as yet undetermined.
 - Where do pole pairs break away / break into the real axis?
 - Where do pole pairs cross the imaginary axis?
 - At what angles do pole pairs depart from the open-loop poles and enter open-loop zeros?
- We have two options for determining these factors:
 - Use MATLAB (or some other tools).
 - Apply some more rules.

Pole Break-away / Break-in Angles

- We often encounter systems where poles break away from the real axis into the complex plane, or conversely merge back into the plane.
- The angle at which the poles leave the plane is given by:

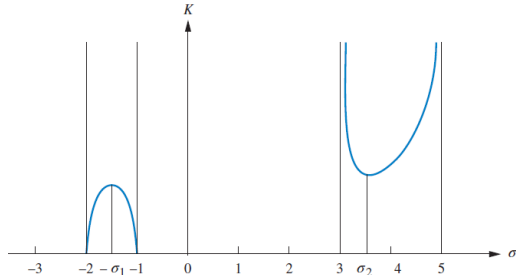
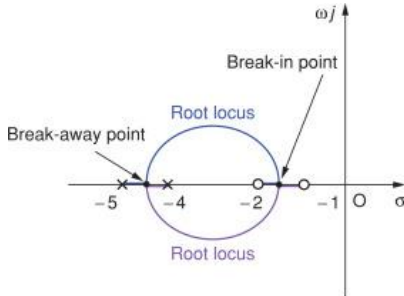
$$\angle\theta = \frac{180^\circ}{n}$$

Where: n is the number of poles breaking away/in.

- Typically, $n = 2$, so the angle of departure/arrival from the real axis is 90° .

Pole Break-away Locations

- We know that a break-away occurs as gain is increasing.
- Therefore, the break-away point has the highest gain of any point where the locus is on the real axis.
- In cases where there are multiple segments on the real axis, it might not be the highest gain globally, but it will be a local maximum.



Pole Break-in Locations

- Similarly, around a break-in point, the locus first touches the real axis at some value of gain.
- The locus then continues on the real axis as the gain increases. The break-in point will therefore correspond to a local minimum of gain.
- We know that the root locus satisfies the characteristic equation:

$$1 + KG = 0$$

- Thus, everywhere along the root locus:

$$K = -\frac{1}{G(s)}$$

- Along the real axis, we know that s is real, so we can write $s = \sigma$.
- Hence, $K = -G(\sigma)$ on the real axis.

Pole Break-in Locations

- For a given system, the open loop transfer function equation is:

$$G(s) = \frac{(s + z_1)(s + z_2) \dots (s + z_k)}{(s + p_1)(s + p_2) \dots (s + p_k)} = \frac{n_1 s^k + n_2 s^{k-1} \dots + n_k}{d_1 s^m + d_2 s^{m-1} \dots + d_m}$$

- Thus

$$K = -\frac{1}{G(s)} = \frac{d_1 s^m + d_2 s^{m-1} \dots + d_m}{n_1 s^k + n_2 s^{k-1} \dots + n_k} = \frac{P(s)}{Z(s)}$$

- Knowing the form of equation is $u/v = (u'v - uv')/v^2$, the first derivative of the equation above is:

$$\frac{dK}{ds} = \frac{\frac{dP(s)}{ds} Z(s) - P(s) \frac{dZ(s)}{ds}}{[D(s)]^2}$$

Pole Break-away / Break-in without Differentiation

- There is also an easier method without using differentiation, though it is a less obvious technique.
- Using this method, the *breakaway and break-in points*, $s = \sigma_{1,2}$, satisfy the relationship:

$$\sum_{i=1}^Z \frac{1}{\sigma_b - z_i} = \sum_{j=1}^P \frac{1}{\sigma_b - p_j}$$

Where: p_i and z_i are the pole and zero values of CG , where we have Z total zeros and P total poles.

Example of Pole Break-away / Break-in Location

Given a control system with the following transfer function equation, it is connected in series with a proportional controller with a gain of K :

$$G(s) = \frac{(s - 3)(s - 5)}{(s + 1)(s + 2)}$$

- Determine the expression for the closed-loop system for determining value of K in terms of value of σ . [4 marks]
- Simulate in MATLAB the break-away and break-in points of the system. [5 marks]
- Determine the break-away and break-in points of the system using differentiation method. [4 marks]
- Determine the break-away and break-in points of the system without using differentiation method. Comment on the difference using this method. [4 marks]

Example of Pole Break-away / Break-in Location

Answer

a. Along the root locus, $KG = -1$.

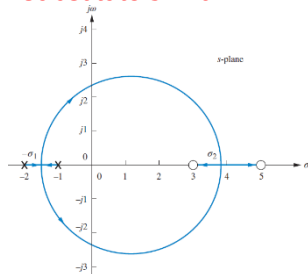
$$\frac{K(s-3)(s-5)}{(s+1)(s+2)} = -1$$

Considering gains only on the real axis $\rightarrow$ substitute $s = \sigma$.

$$\frac{K(\sigma-3)(\sigma-5)}{(\sigma+1)(\sigma+2)} = -1$$

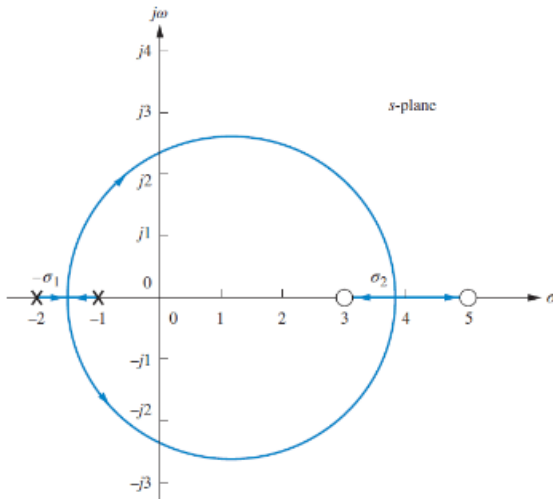
We can solve for K and obtain:

$$K = -\frac{\sigma^2 + 3\sigma + 2}{\sigma^2 - 8\sigma + 15}$$



Example of Pole Break-away / Break-in Location

- b. The following figure shows the results of MATLAB simulation of break-away and break-in points of system.



Example of Pole Break-away / Break-in Location

- c. Find local extrema in the K value by finding the derivative of the expression for K and setting it to zero.

$$K = -\frac{\sigma^2 + 3\sigma + 2}{\sigma^2 - 8\sigma + 15}$$

Differentiate the equation above ($d(u/v) = (u'v - uv')/v^2$):

$$\begin{aligned}\frac{dK}{d\sigma} &= \frac{(2\sigma + 3)(\sigma^2 - 8\sigma + 15) - (\sigma^2 + 3\sigma + 2)(2\sigma - 8)}{(\sigma^2 - 8\sigma + 15)^2} \\ &= -\frac{11\sigma^2 + 26\sigma + 61}{(\sigma^2 - 8\sigma + 15)^2}\end{aligned}$$

Solve $11\sigma^2 + 26\sigma + 61 = 0$ using the quadratic equation to find the critical values of σ .

Knowing $\sigma_{1,2} = -1.45, 3.82$. Thus, the breakaway point is at $s = -1.45$ and the break-in point is at $s = 3.82$.

Example of Pole Break-away / Break-in Location

- d. Without the differentiation method, the *breakaway and break-in points*, $s = \sigma_{1,2}$, satisfy the relationship:

$$\sum_{i=1}^Z \frac{1}{\sigma_b - z_i} = \sum_{j=1}^P \frac{1}{\sigma_b - p_j}$$

Where: p_i and z_i are the pole and zero values of CG , where we have Z total zeros and P total poles.

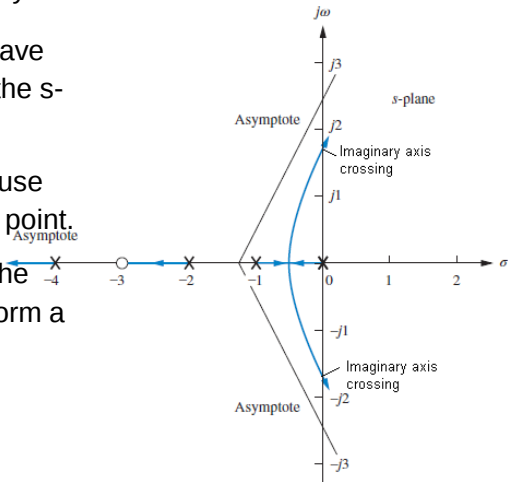
For our example, we get:

$$\frac{1}{\sigma_b - 3} + \frac{1}{\sigma_b - 5} = \frac{1}{\sigma_b + 1} + \frac{1}{\sigma_b + 2}$$

After some algebra, we get $11\sigma^2 + 26\sigma + 61 = 0$ as before. We can complete the problem as earlier.

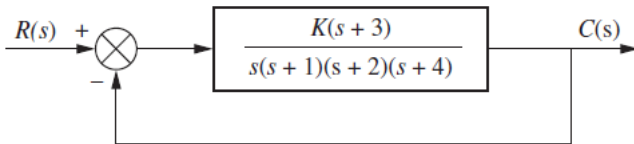
Imaginary Axis Crossings

- The gain K at which the root locus crosses the imaginary axis is the value at which the system becomes unstable.
- Stable systems cannot have poles in the right-half of the s -plane.
- There is no easy way to use the root locus to find this point.
- If you need to calculate the crossing point, then perform a Routh-Hurwitz test.



Example of Imaginary Axis Crossings

For the system given below, find the frequency and gain, K , for which the root locus crosses the imaginary axis. For what range of K is the system stable?



Example of Imaginary Axis Crossings

Answer

- The closed-loop transfer function for the system is:

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{K(s + 3)}{s^4 + 7s^3 + 14s^2 + (8 + K)s + 3K}$$

- The Routh table is as shown below.

s^4	1	14	$3K$
s^3	7	$8 + K$	
s^2	$90 - K$	$21K$	
s^1	$\frac{-K^2 - 65K + 720}{90 - K}$		
s^0	$21K$		

Example of Imaginary Axis Crossings

- A complete row of zeros yields the possibility for imaginary axis roots. For positive values of gain, those for which the root locus is plotted, only the s^1 row can yield a row of zeros. Thus,

$$-K^2 - 65K + 720 = 0$$

- From this equation K is evaluated as:

$$K = \frac{-65 \pm \sqrt{(-65)^2 - 4(-1)(720)}}{2(-1)} = 9.65$$

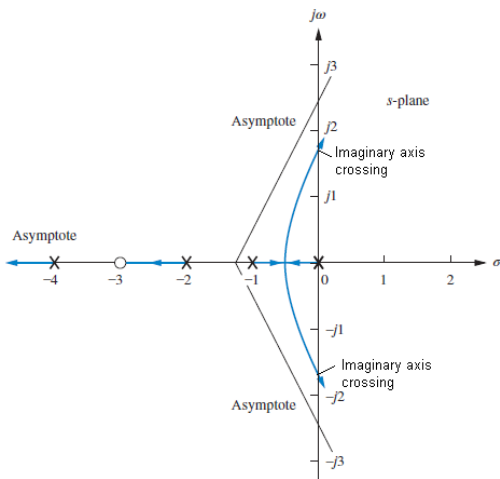
- Forming the even polynomial by using the s^2 row with $K = 9.65$, we obtain:

$$\begin{aligned}(90 - K)s^2 + 21K &= (90 - 9.65)s^2 + 21(9.65) \\ &= 80.35s^2 + 202.65\end{aligned}$$

- Solving the equation, the s is found to be equal to $\pm j1.59$.

Example of Imaginary Axis Crossings

- Thus, the root locus crosses the $j\omega$ -axis at $\pm j1.59$ at a gain K of 9.65.



- Also, we could conclude that the system is stable for $0 \leq K < 9.65$.

Angles of Departure/Arrival at Complex Roots

- The angles when locus leaving or entering the complex roots (e.g. it is NOT the same as the asymptote angle).

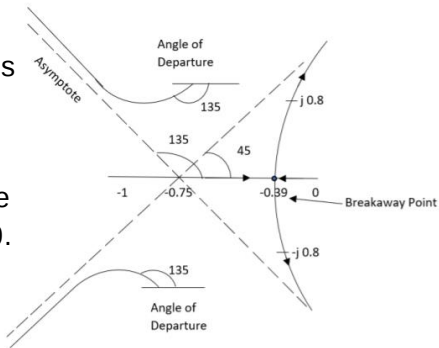
- All points on a root locus satisfy the characteristic equation, $1 + CG = 0$.

$$0 = 1 + CG \Rightarrow -1 = CG$$

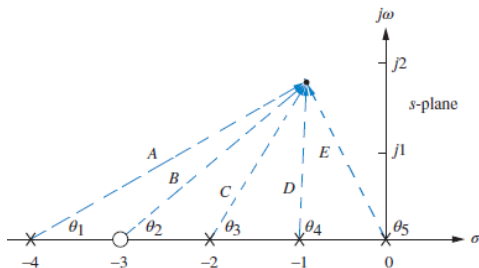
- Thus:

$$|CG| = 1 \quad \text{and} \quad \angle CG = -(2k + 1)180^\circ$$

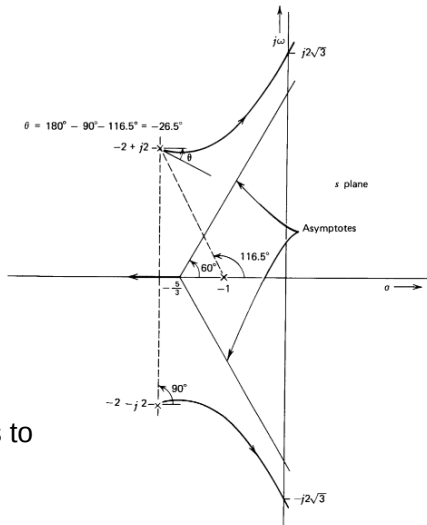
- Angle (phase) of CG is the sum of the phases of the poles and zeros that make up CG .
- To be on the root locus, the sum of the angles to the closed loop roots is always -180° .



Angles of Departure/Arrival at Complex Roots



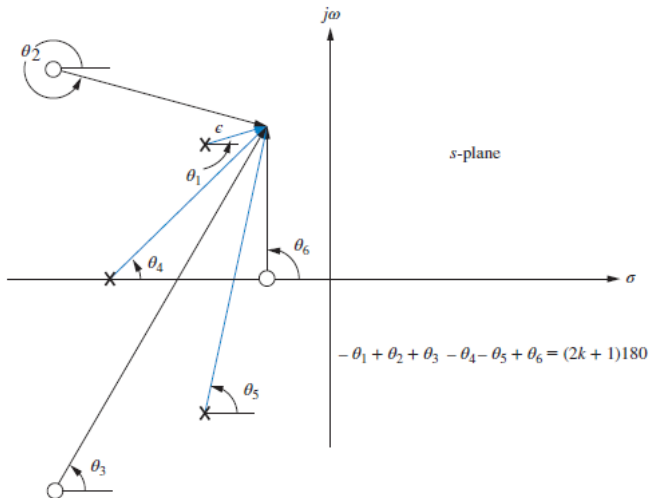
- Assume that we are on a point an arbitrarily small distance, ϵ , away from a system root.
- Angle from each of the other roots to our test point is the same as from that root to the root of interest.
- Calculate these other angles directly e.g. we can find the angle from ϵ to the root of departure or arrival.



Angles of Departure/Arrival at Complex Roots

- Angle of departure at complex roots:

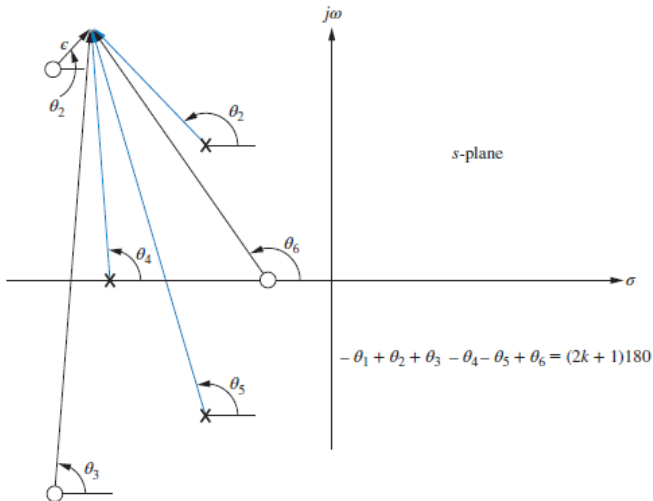
$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k + 1)180^\circ$$



Angles of Departure/Arrival at Complex Roots

- Angle of arrival at complex roots:

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k + 1)180^\circ$$



Example of Angles of Departure

Consider a transfer function:

$$G(s) = \frac{s + 2}{(s + 3)(s^2 + 2s + 2)}$$

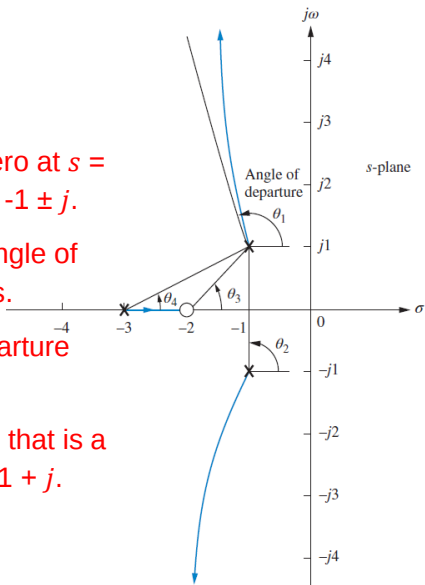
- a. Determine the poles and zeros of the open-loop system. [2 marks]
- b. Sketch the angle of departure of the system. [6 marks]
- c. Calculate the angle of departure from the pole at $s = -1 + j$. [6 marks]

Example of Angles of Departure

Answer

- This system has an open-loop zero at $s = -2$ and open-loop poles at $s = -3, -1 \pm j$.
- The following figure shows the angle of departure from the complex roots.
- Let us calculate the angle of departure from the pole at $s = -1 + j$.

Assume a point on the root locus that is a small distance ϵ away from $s = -1 + j$.



Example of Angles of Departure

- Equate all angles of the poles and zeros to -180° :

$$-180^\circ = -\theta_1 - \theta_2 + \theta_3 - \theta_4$$

$$= -\theta_1 - 90^\circ + 45^\circ - \tan^{-1}\left(\frac{1}{2}\right)$$

- Thus, rearrange the above equation:

$$\theta_1 = 180^\circ - 90^\circ + 45^\circ - 26^\circ = 109^\circ$$

- This is the required angle of departure from the open loop pole at $s = -1 + j$.
- Notice that zeros cause phase lead (so, we add their angle), while poles cause phase lag (so, we subtract their angle).

