



Stability Analysis

XMUT315 Control Systems Engineering

Topics

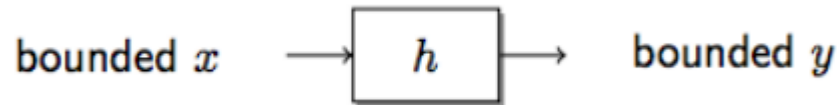
- Stability and system responses.
- Stability of the systems and stability analysis.
- Methods of stability analysis.
- Routh-Hurwitz criterion.
- Construction of the criterion.
- Special cases of Routh-Hurwitz criterion.
 - Zero in a single column.
 - Zeros in a row.

Stability

- A linear system is where the principles of superposition do apply (e.g. no saturation or hysteresis effects)
- A time-invariant system is where its characteristics do not vary with respect to time (e.g. no ageing)
- In LTI systems, we often approximate systems over a specific range or time-period.
- We want to build up a relationship between the response of the system and stability:
 - if input is bounded and output ($c(t)$) does not approach ∞ as t approaches ∞ (e.g. natural response is not approaching ∞).
 - if input is unbounded, we cannot conclude stability.

Stability

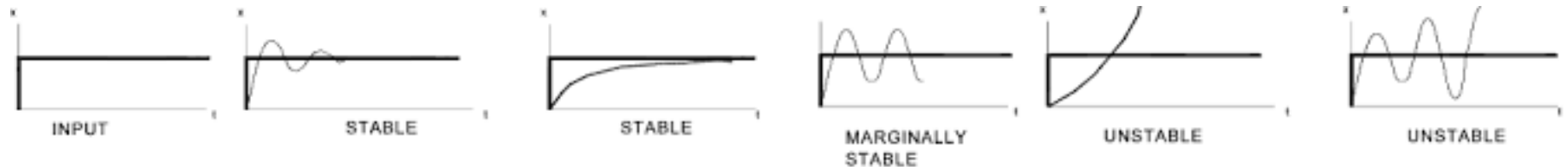
- A system is stable if every bounded input yields a bounded output or it is bounded-input bounded-output (BIBO).



- We want to build up a relationship between the total response and instability:
 - if the input is bounded, but the output ($c(t)$) is unbounded, the system is unstable.
 - if input is unbounded, we can not conclude instability.
- A system is unstable if any bounded input yields an unbounded output.

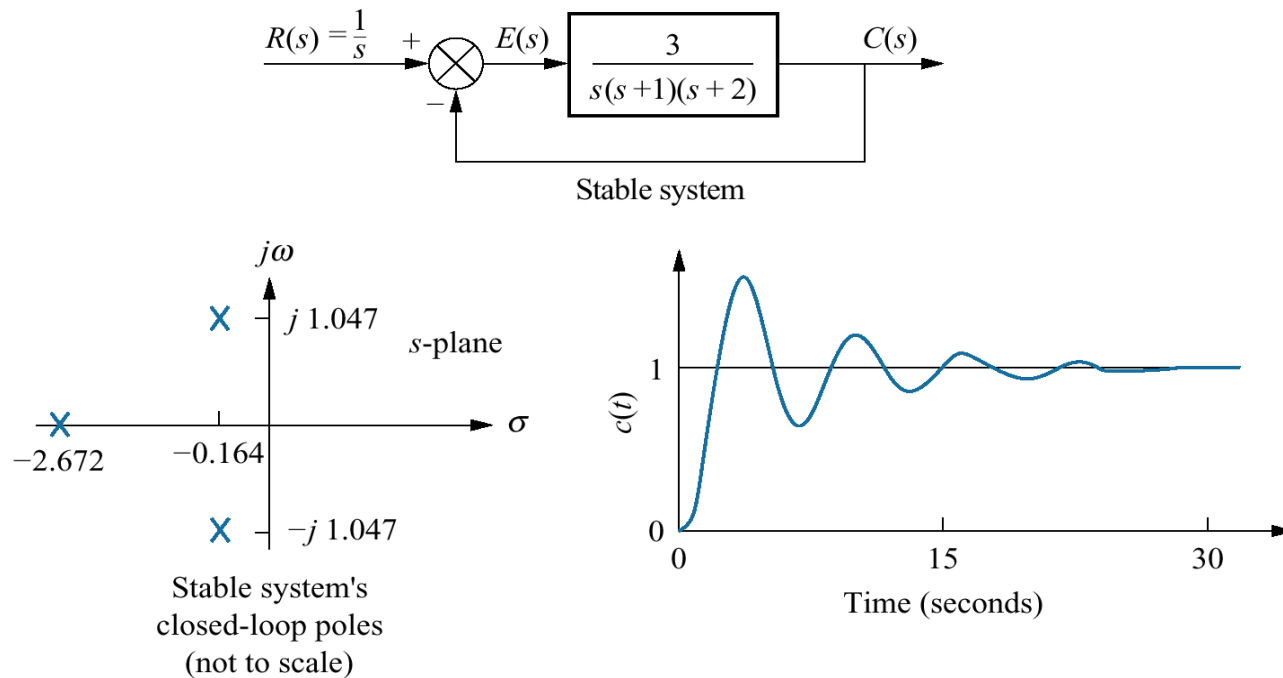
Stability and System Response

- A linear, time-invariant system is *stable* if: the natural response approaches zero as time approaches infinity.
- A linear, time-invariant system is *unstable* if: the natural response grows without bound as time approaches infinity.
- A linear, time-invariant system is *marginally stable* if: the natural response neither decays nor grows but remains constant or oscillates as time approaches infinity.



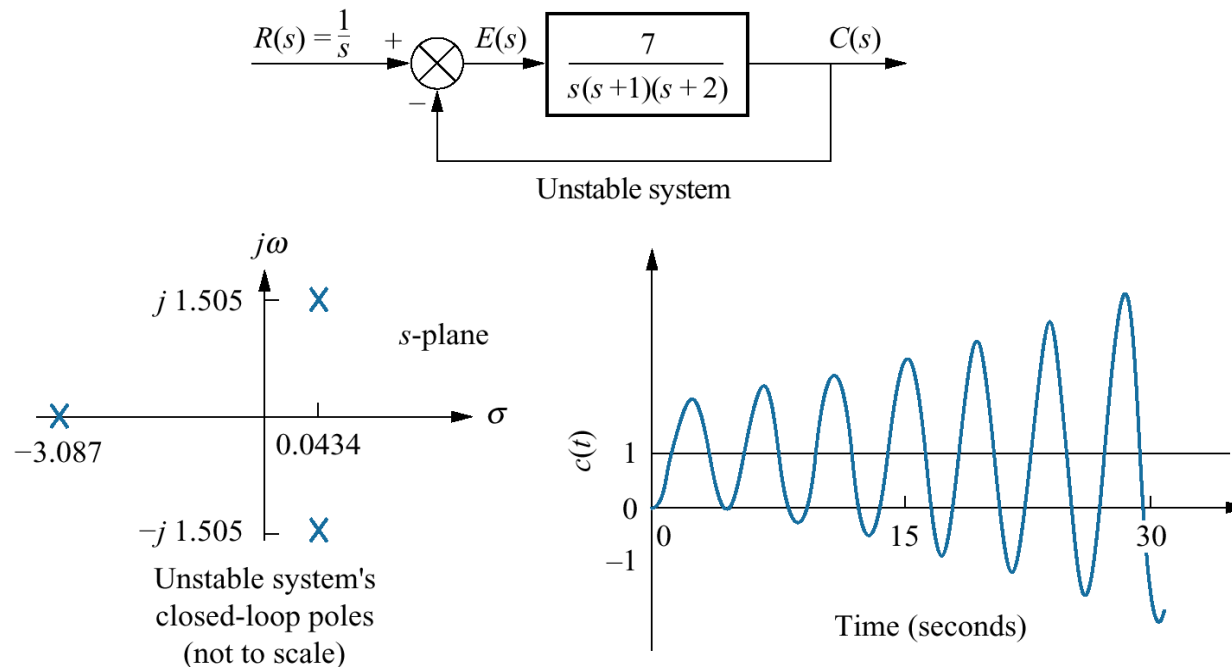
Stability and System Response

- **Stable**, a linear, time-invariant system is stable if the natural response approaches zero as time approaches infinity



Stability and System Response

- **Unstable**, if the natural response grows without bound as time approaches infinity.



System Damping (Indicator of Stability)

- Consider a control system described as:

$$as^2 + bs + c$$

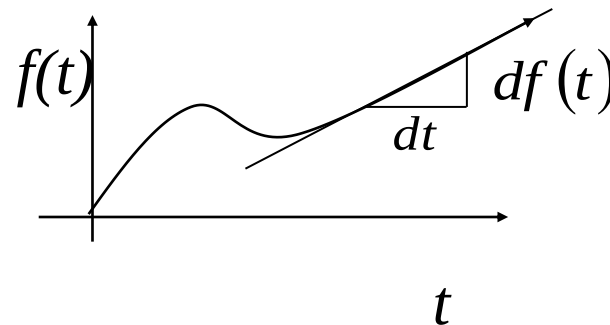
- Find roots of a quadratic equation:

$$\text{Root}_{1,2} = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

Equation	Roots	Transient Response
$b^2 - 4ac > 0$	Real, different	Overdamped
$b^2 - 4ac = 0$	Real, same	Critically damped
$b^2 - 4ac < 0$	Complex, different	Underdamped
$b = 0$	Complex, same	Undamped

's' Variable

- 's' Laplace operator or Laplace transform variable.
- 's' can be considered as a differentiator ($df(t)/dt$).
- 's' can be considered as a gradient.



- A variable at an instance is a number:
 - Numbers can be real or imaginary.

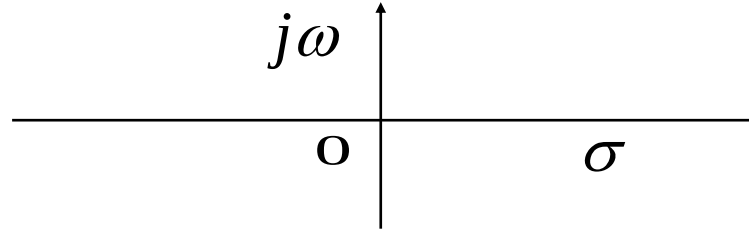
$$s = a \quad \text{or} \quad s = a + bj$$

- Usually given in control systems as:

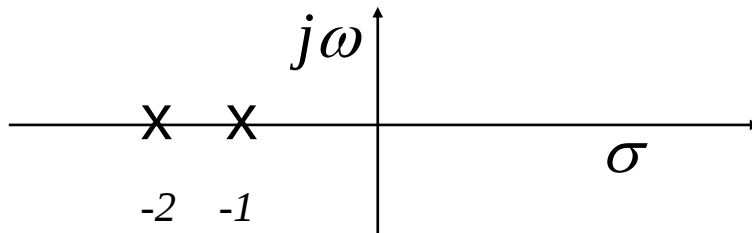
$$s = \sigma \quad \text{or} \quad s = \sigma + j\omega$$

's' Domain

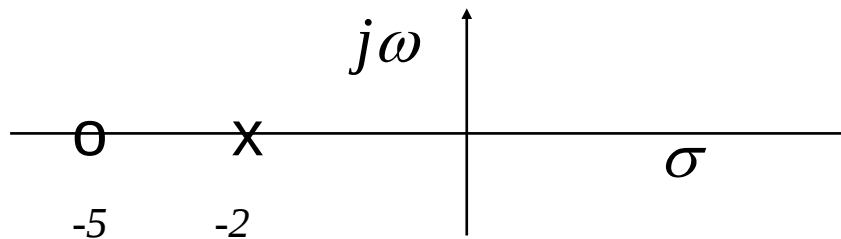
- We can plot the s variable on a s -domain diagram:



- Consider the systems given below:



Poles (x) cause system to be infinity in the s -domain



Zeros (o) cause system to be zero in the s -domain

System Response

Transfer Function

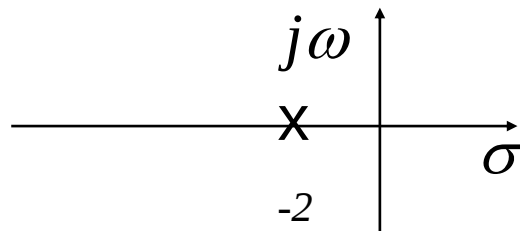
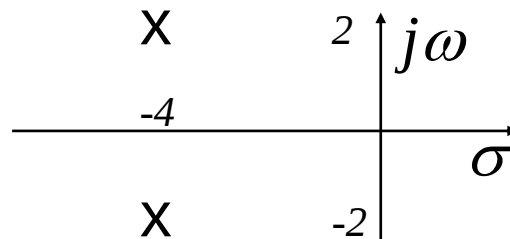
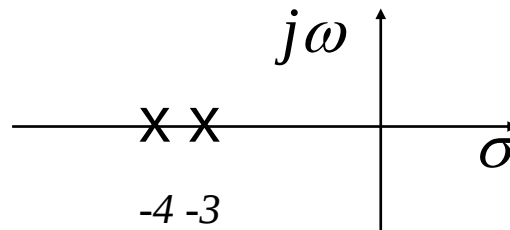
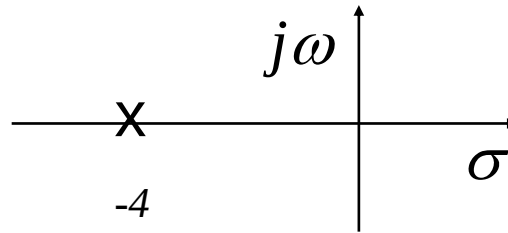
$$G_1(s) = \frac{10}{1 + 0.25s}$$

$$G_2(s) = \frac{3}{s^2 + 7s + 12}$$

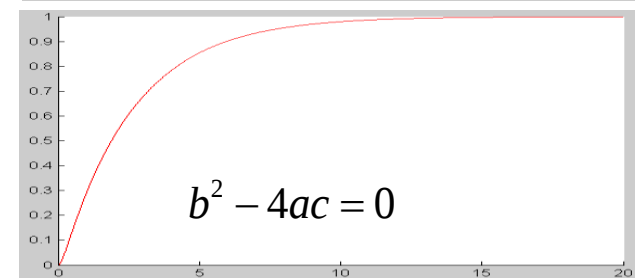
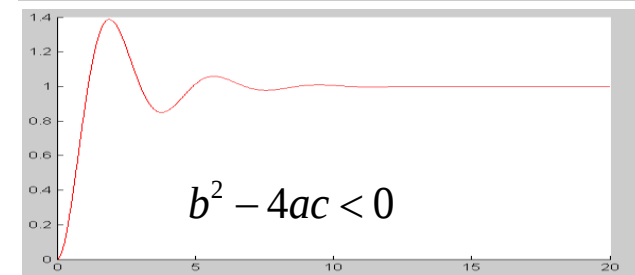
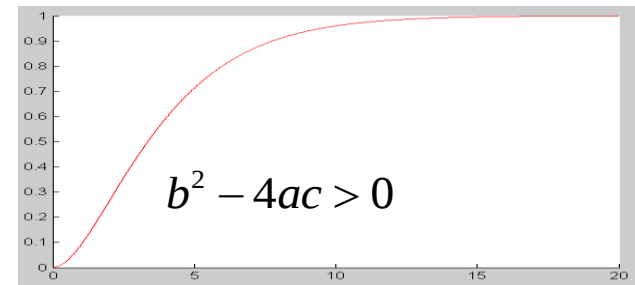
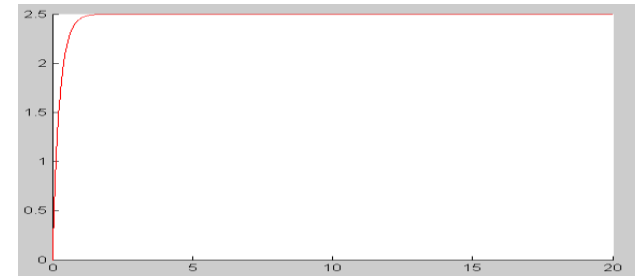
$$G_3(s) = \frac{3}{s^2 + 8s + 20}$$

$$G_4(s) = \frac{1}{s^2 + 4s + 4}$$

s-Domain



Time Response



Example of Stability of Systems

Consider the following two control systems:

i. System 1:

$$G_1(s) = \frac{1}{(s + 1)(s + 2)}$$

ii. System 2:

$$G_2(s) = \frac{s + 5}{s + 2}$$

What happens when:

- a. Root $s = -1$? [4 marks]
- b. Root $s = -2$? [4 marks]
- c. Root $s = -5$? [4 marks]

Zeros (0) and Poles (∞)

What happens when the following conditions existed:

a. When $s = -1$?

$$G_1(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{(1-1)(2-1)} = \infty$$

$$G_2(s) = \frac{s+5}{s+2} = \frac{5-1}{2-1} = \frac{4}{1} = 1$$

Notice that the output of system 1, $G_1(s)$ is ∞ when $s = -1$ that indicates its response is a growing. The system is unstable.

On the other hand, the output of system 2, $G_2(s)$ is 1 when $s = -1$ that indicates its response settles at 1. The system is stable.

Zeros (0) and Poles (∞)

b. When $s = -2$?

$$G_1(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{(-1)(0)} = \infty$$

$$G_2(s) = \frac{s+5}{s+2} = \frac{3}{0} = \infty$$

Both systems 1 and 2 are growing. Both systems are unstable.

c. When $s = -5$?

$$G_1(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{(-4)(-3)} = \frac{1}{12}$$

$$G_2(s) = \frac{s+5}{s+2} = \frac{0}{-3} = 0$$

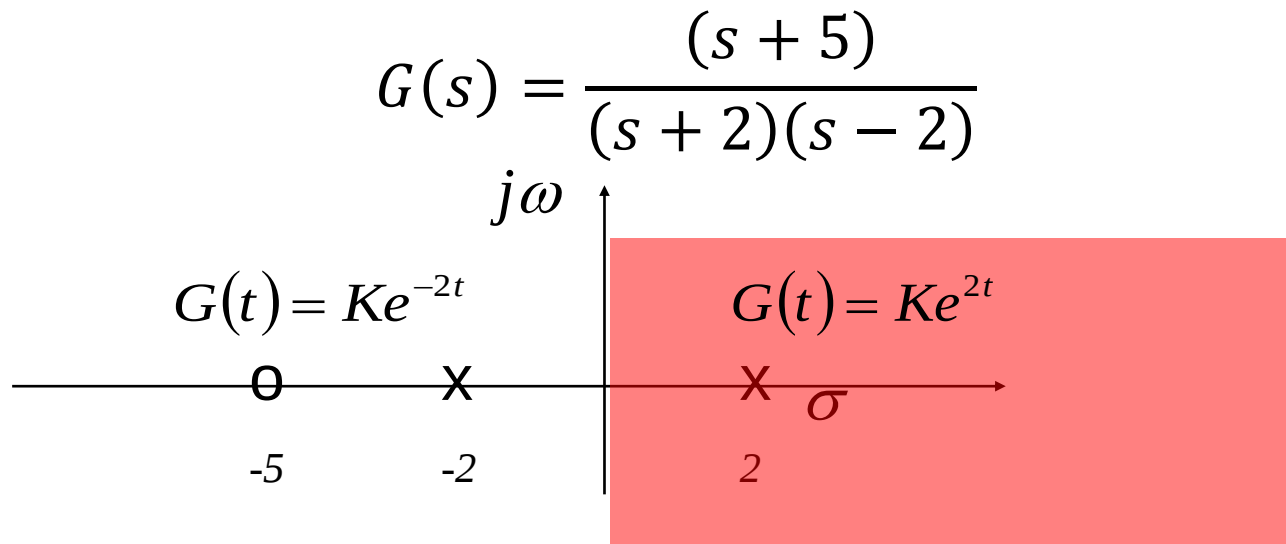
Both systems settle to $1/12$ and 0 . Both systems are stable.

Time Domain

- We are worried when system is infinite or 0 in TIME domain.
- System response often contains an exponential component:

$$G(t) = ke^{at}$$

- When $a > 0$, the output will reach infinity! We MUST avoid right half of plane poles.



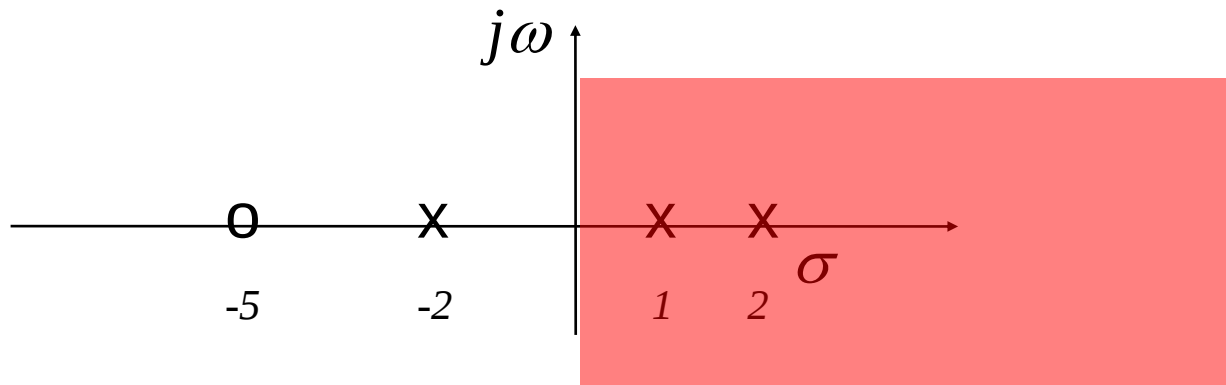
All poles must be in left half plane for stability

Any poles in right half plane will cause the system to be UNSTABLE

Stability

- If the poles of a system transfer function all lie in the left half of the s -plane, then that system is stable.

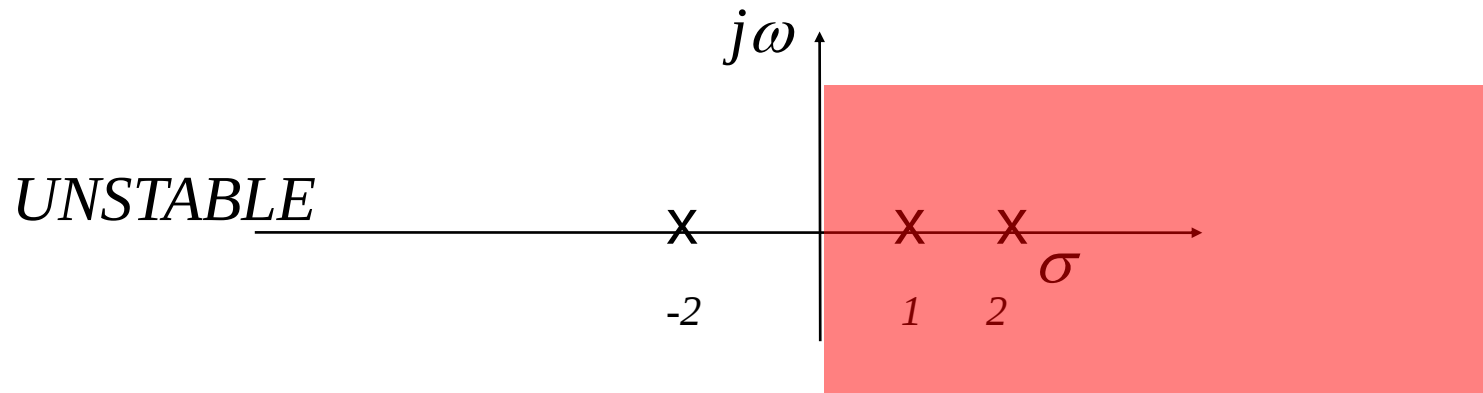
$$G(s) = \frac{(s + 5)}{(s + 2)(s - 1)(s - 2)}$$



Stability

- It is only the poles of a system transfer function which are important as far as stability is concerned (non-cancelling zeros can be ignored)

$$G(s) = \frac{1}{(s + 2)(s - 1)(s - 2)}$$



Stability

- The poles of a system are the polynomial roots obtained when the system denominator is equated with 0.

$$G(s) = \frac{(s + 5)}{(s + 2)(s - 1)(s - 2)}$$

- The system denominator is known as the *characteristic polynomial*.

$$G(s) = \frac{(s + 5)}{(s + 2)(s - 1)(s - 2)}$$

- The system denominator equated to zero is the *characteristic equation*.

$$(s + 2)(s - 1)(s - 2)$$

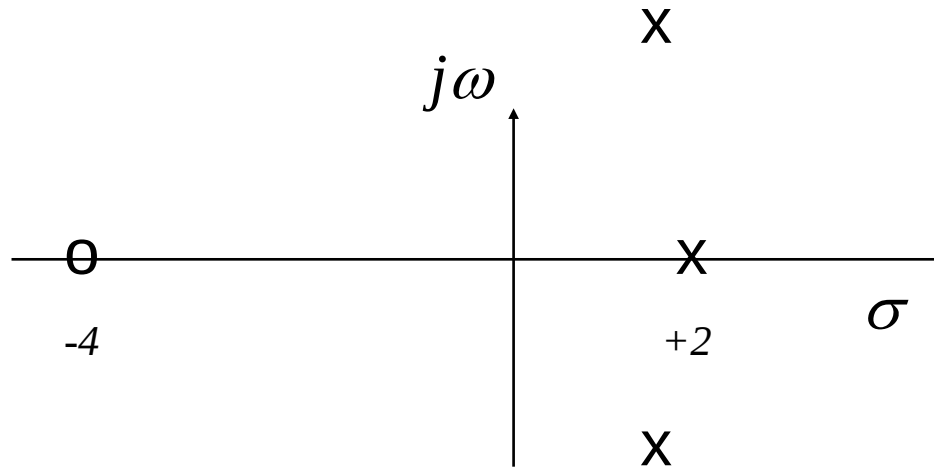
- Open-loop stable may/may not be closed-loop stable
- Open-loop unstable, may/may not be closed-loop unstable!

Unstable System Response

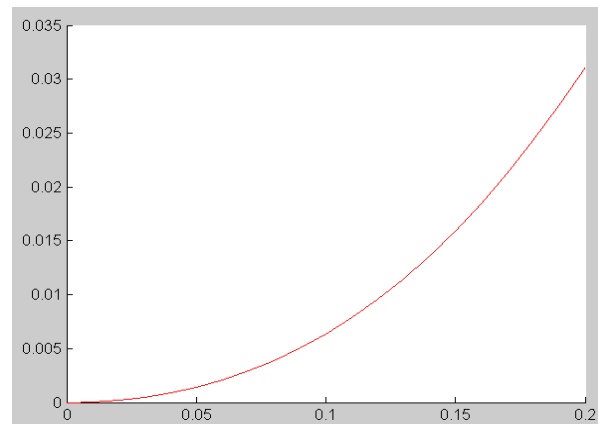
- Transfer function:

$$G(s) = \frac{(s + 4)}{s^3 - 4s^2 + 21s - 34}$$

- s-Domain

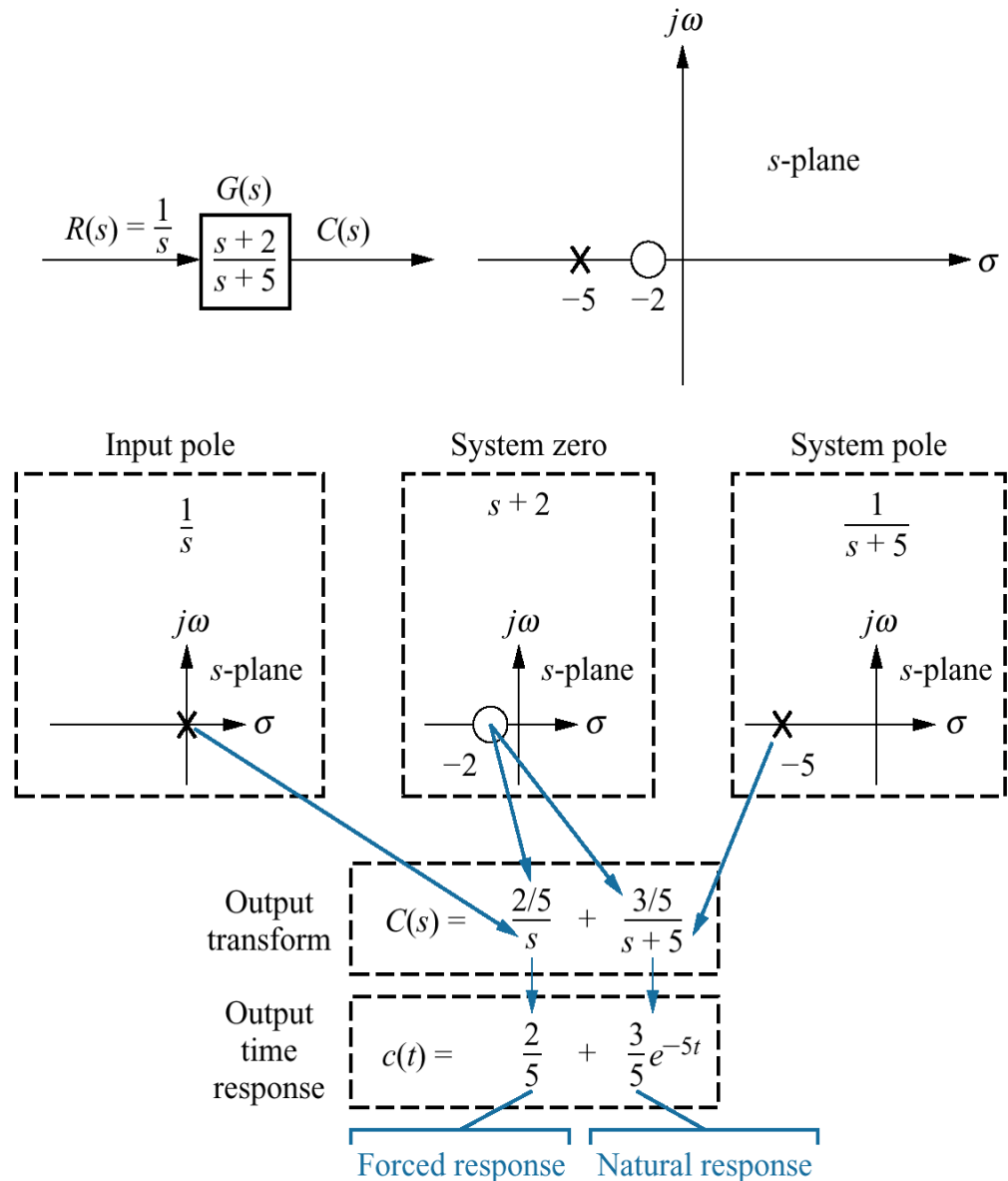


- Time response

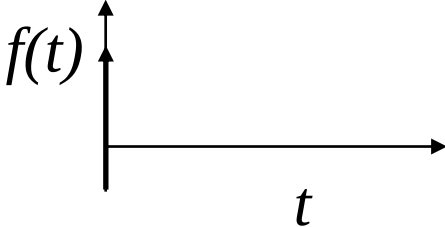
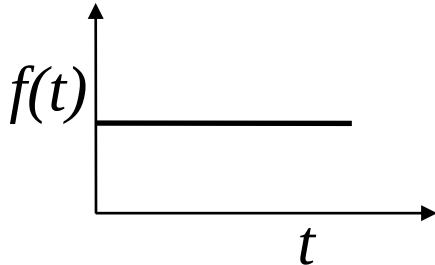
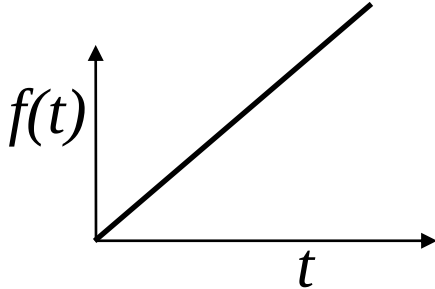
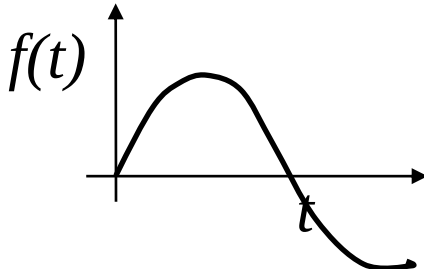


Forced and Natural Responses

- Natural response: output due to the response of the system itself, without external input.
- Forced response: output due to intentional input, external to the system.
- Stability of the system mainly depends on the natural response of the system.

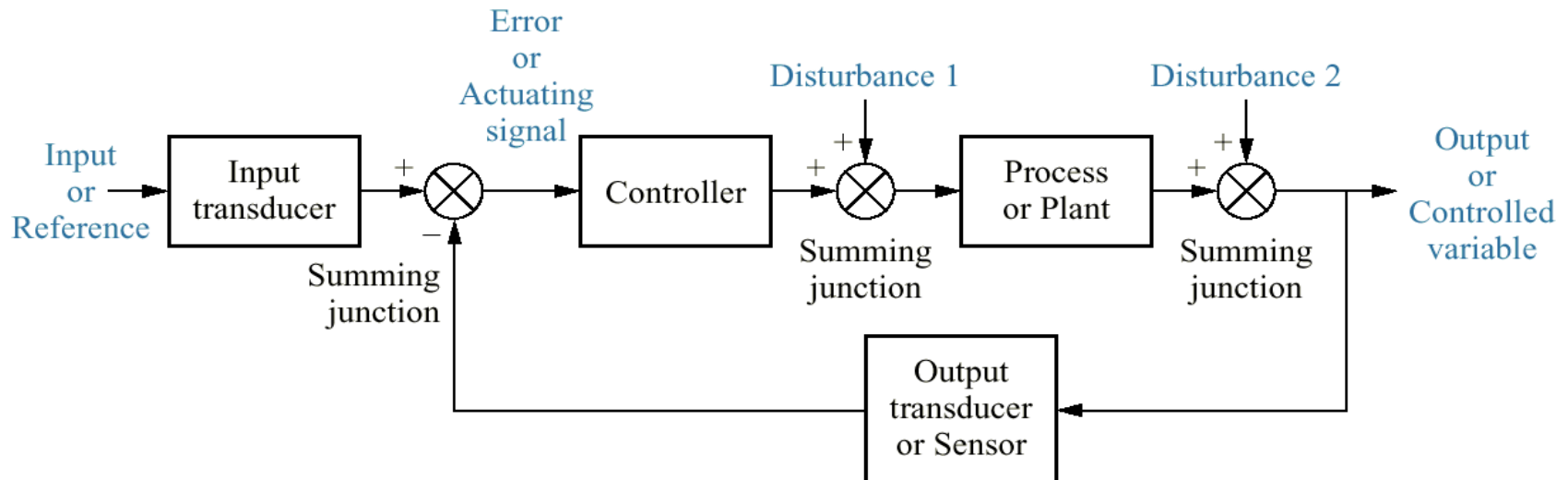


Forced Response

Input	Function	Sketch	s-domain
Impulse	$\delta(t)$		1
Step	$u(t)$		$\frac{1}{s}$
Ramp	$tu(t)$		$\frac{1}{s^2}$
Sinusoid	$\sin t$		$\frac{\omega}{s^2 + \omega^2}$

Improvements to Stability of System

- Negative feedback can reduce effects of disturbances and changes to input parameters -> stability of the system.



- Negative feedback is when:

$$|closed\ loop\ gain| < |open\ loop\ gain|$$

- This reduces the steady-state error by making the output closer to the input.
- To improve the stability of the system further, we could add a controller or compensator.

Improvements to Stability of System

- Controllers or compensators change the natural response of the system.
- They adjust the poles of the system.
- They help achieve the desired output from a given input.
- Controllers or compensators can be mechanical, natural or electrical (used in industry).
- Three main types of controller or compensator:
 - Gain (proportional).
 - Lead or lag (lead, lag, and lead-lag).
 - PID (proportional, integral, derivative, or any of their combinations).

Stability Analysis

- Various type of stability analysis in control systems:
 - Analytical (i.e. requiring model of the physical system and maths to solve differential equations – mostly approximation for complex system).
 - Experimentation (i.e. running a set of trials and observation or measurement).
- Approaches used in the course:
 - Routh-Hurwitz criterion – mathematical process.
 - Nyquist plot – graphical tool.
 - Nichols chart – graphical tool.
 - Bode plot – graphical tool.
 - Root locus diagram – graphical tool.

Routh-Hurwitz Criterion

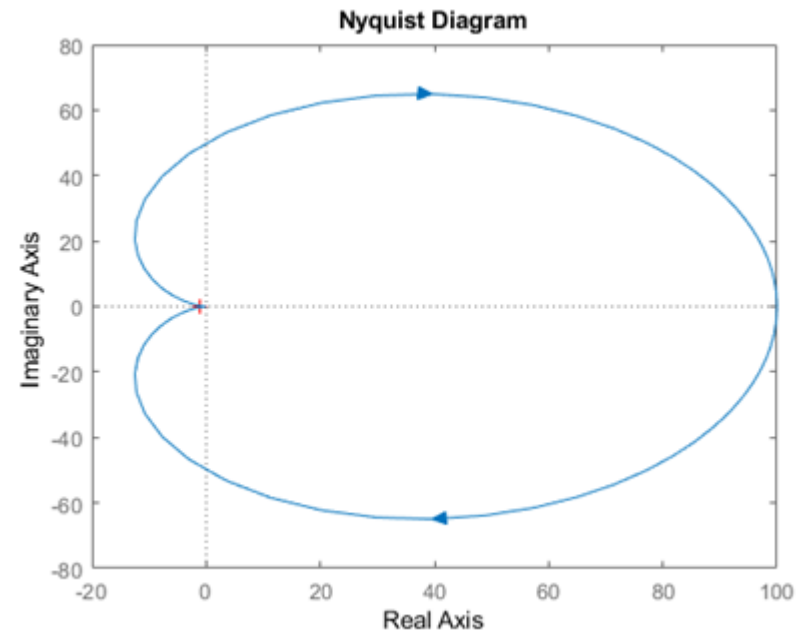
- Analyse stability of a system through mathematical analysis of characteristic equation.

$$G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s^1 + a_0s^0}$$

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

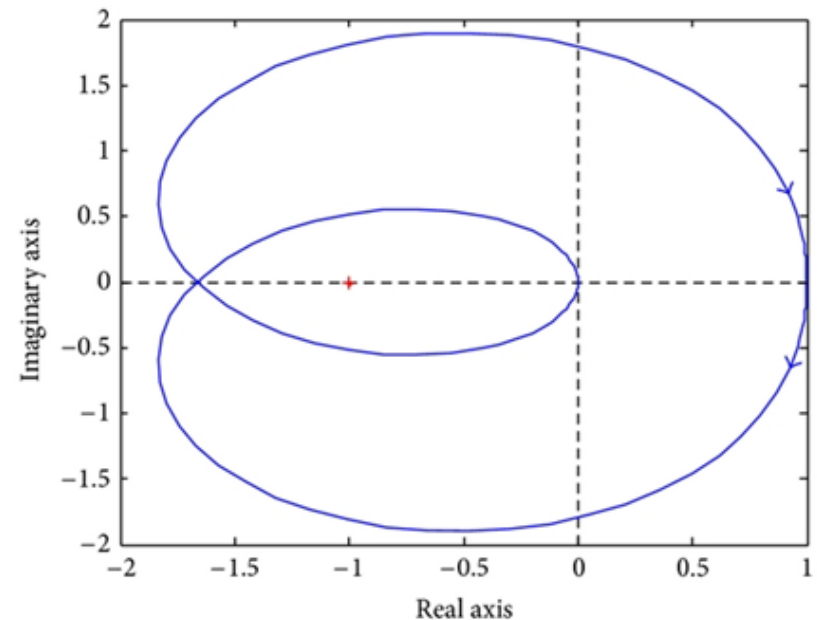
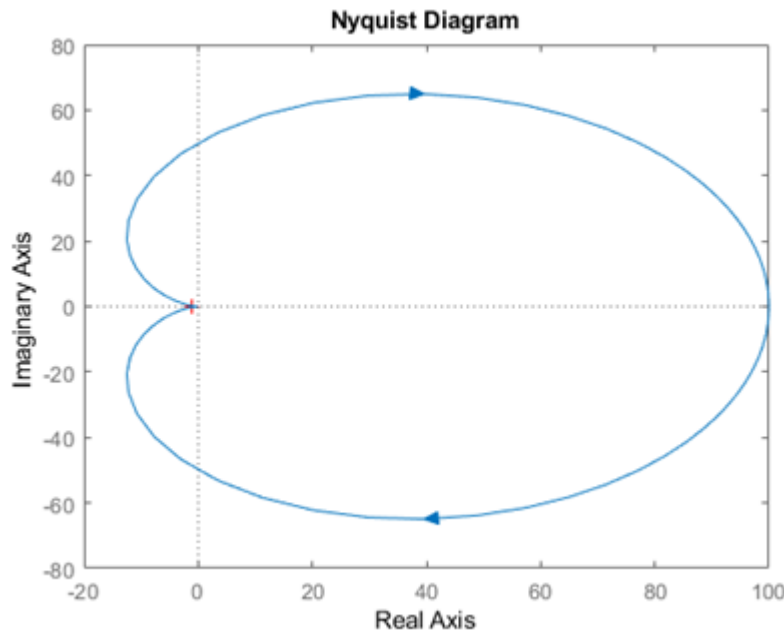
Nyquist Plot

- Analyse stability of a system through use of polar plot of system equation.
- In Cartesian coordinates, the real part of the transfer function is plotted on the x-axis.
- The imaginary part is plotted on the y-axis. The frequency is swept as a parameter, resulting in a plot per frequency.
- Alternatively, in polar coordinates, the gain of the transfer function is plotted as the radial coordinate, while the phase of the transfer function is plotted as the angular coordinate.



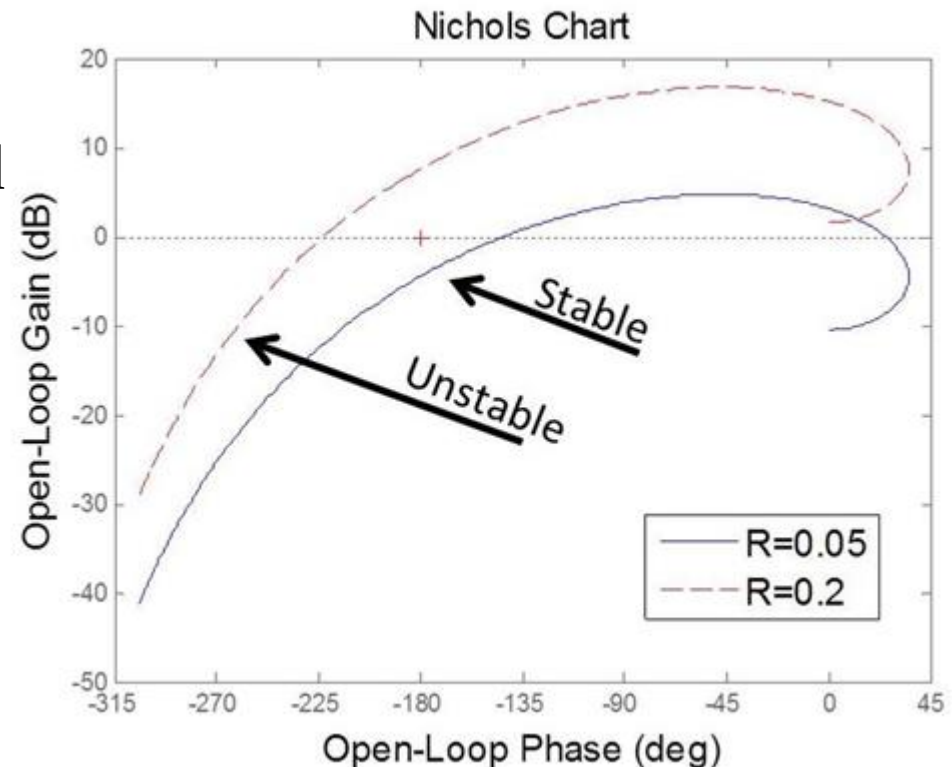
Nyquist Plot

- System is stable if the plot does not encircle the unity gain point $(-1, 0)$ in the Nyquist plot.
- The system below on the left is deemed to be stable as the plot is not encircling the Nyquist stability node $(-1, 0)$.
- The system on the right is not stable as the plot is encircling the Nyquist stability node $(-1, 0)$.



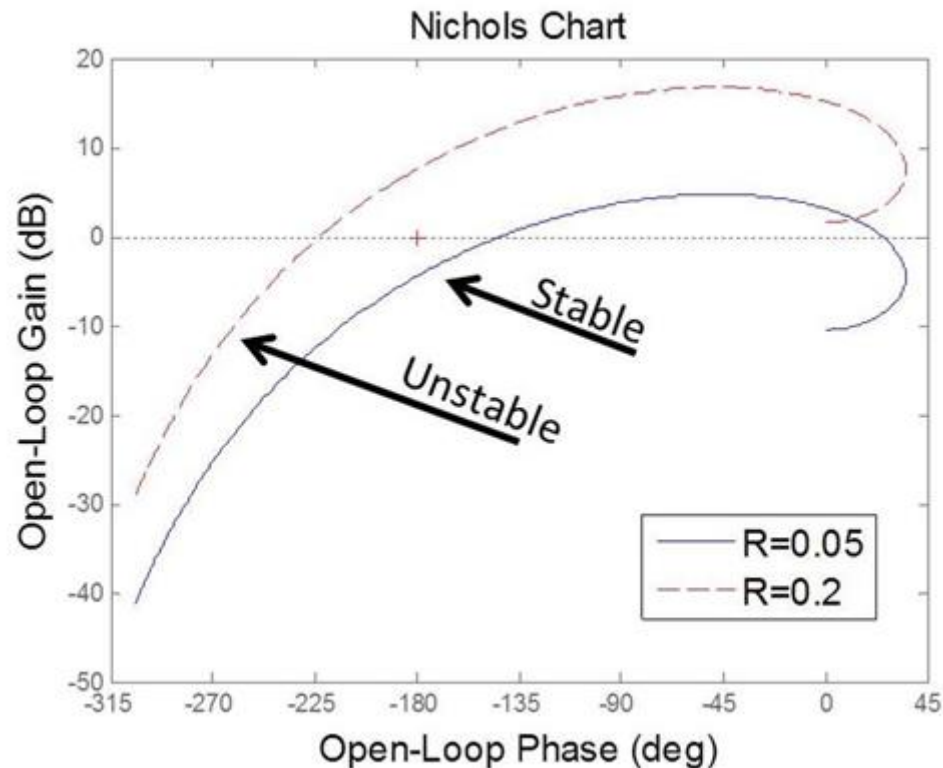
Nichols Chart

- Analyse stability of a system through use of gain and phase plot of the system.
- A Nichols chart displays the magnitude (in dB) plotted against the phase (in degrees) of the system response.
- It is used further to analyse stability of a system beyond the Nyquist plot.
- System is considered to be stable if the value of gain at 180° in the Nichols chart is positive.



Nichols Chart

- The system below is deemed to be stable as the curve of the plot is at the gain > 0 dB whenever phase = 180° in the Nichols chart.
- Contrast with the unstable system when its gain < 0 dB at 180° .



Routh-Hurwitz Criterion

This stability analysis technique is based on the works of Edward Routh with his algorithm proposed in 1876 and Adolf Hurwitz who independently proposed contribution in 1895.



Edward Routh



Adolf Hurwitz

Routh-Hurwitz Criterion

- Any system that has closed-loop poles in the right-half of the s -plane will be unstable.
- The Routh Hurwitz table is a simple method that allows us to determine the number of poles in the left- and right-halves of the s -plane and the number of poles on the imaginary axis.
- It tells us nothing about *where* those poles are, so it is not generally sufficient to design a control system.
- It is just a specialised tool to tell us about system stability.

Routh-Hurwitz Criterion

- Using this method, we can tell how many closed-loop system poles are in the left half-plane, in the right half-plane, and on the $j\omega$ -axis.
- We can not tell where, but only how many are in each plane determining the system's stability.
- The method requires two steps:
 - Generate a data table called a Routh table.
 - Interpret the Routh table to tell how many closed-loop system poles are in the left half-plane, the right half-plane, and on the $j\omega$ -axis.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Construction of the Routh Array

- Consider a *closed loop* transfer function having form:

$$G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s^1 + a_0s^0}$$

- When considering stability, we are interested only in the poles of $G(s)$, therefore we examine the denominator polynomial of $G(s)$.
- To form the Routh array, start by writing powers of s down the left-hand side and then fill in the coefficients of the denominator polynomial.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Construction of the Routh Array

- Note that in the first row, we start with the coefficient of the highest power of s and then write every second coefficient.
- In the second row, we write in the coefficients that were not included in the top row.
- Include a zero at the end of row two if necessary to fill out the array.
- To complete the array, we fill out successive rows as a function of the two rows above.
- See the table for the required pattern.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Filling in the Array

- The Routh table:

$$G(s) = \frac{N(s)}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s^1 + a_0s^0}$$

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Filling in the Array

- The Routh criterion states that the number of closed loop poles in the right half of the s-plane is equal to the number of sign changes in the first column of the Routh table.
- For stability, we therefore require zero sign changes in the first column.
- Any change of sign indicates that the system is unstable.

s^4	a_4	a_2	a_0	+/-
s^3	a_3	a_1	0	+/-
s^2	b_1	b_2	0	+/-
s^1	c_1	0	0	+/-
s^0	d_1	0	0	+/-

Interpreting a Routh Table

- If the closed-loop transfer function has all poles in the left-half of the s-plane, the system is stable.
- Thus, a system is stable if there are no sign changes in the first column of the Routh table.

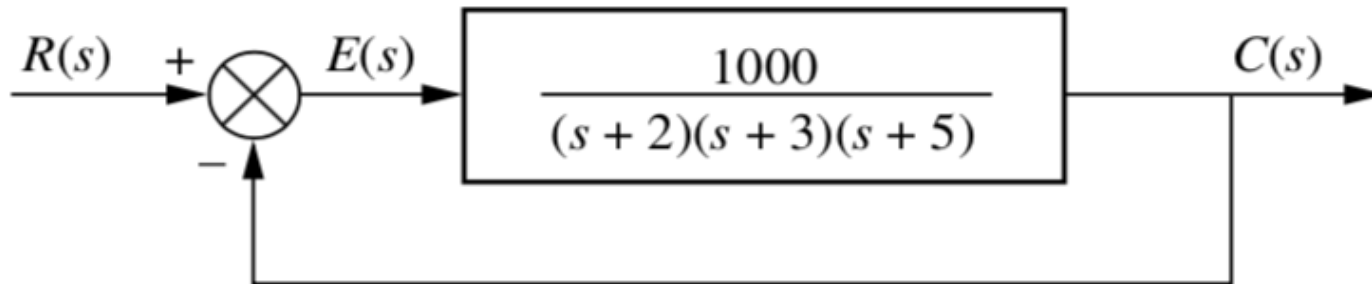
Sign change	Interpretation
0	No right-hand side pole
1	1 right-hand side pole
2	2 right-hand side poles
...	
N	Nth right-hand side poles

Example of Routh Array Construction

- Consider a system with a transfer function of:

$$G(s) = \frac{1000}{(s + 2)(s + 3)(s + 5)}$$

- Determine whether this system is stable when closed in a unity gain feedback loop. [14 marks]

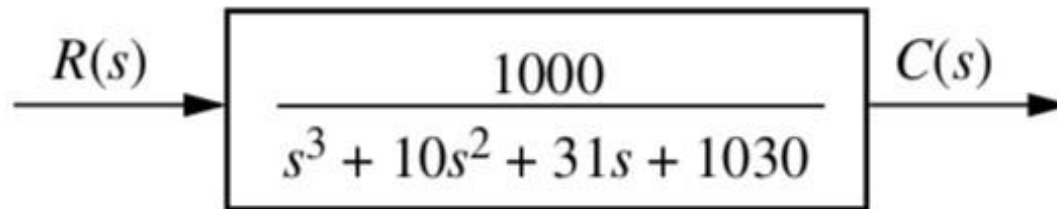


Find Closed Loop Transfer Function

- We first need to find the closed loop transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{1000}{s^3 + 10s^2 + 31s + 1030}$$

- Notice that we have completely expanded the denominator rather than leaving it as the product of poles as we normally do.



- The characteristic equation of the system is:

$$s^3 + 10s^2 + 31s + 1030$$

Scaling in the Rows in the Table

- Now write the powers of s down the left side and fill in the coefficients from the denominator polynomial of the closed loop transfer function.

s^3	1	31
s^2	10	1030
s^1		
s^0		

- We are allowed to divide any row of the array through by a *positive* number if it will simplify the calculation.
- We will divide the second row though by ten.

s^3	1	31
s^2	$10/10 = 1$	$1030/10 = 103$
s^1		
s^0		

Filling in the Array

- We see that the first column changes from positive to negative after the second row and then from negative to positive after the third row.
- We therefore have two sign changes.
- The Routh criterion tells us that the transfer function has two poles in the right half of the s-plane.

s^3	1	31	0
s^2	1	103	0
s^1	$\frac{- \begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\frac{- \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	$\frac{- \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\frac{- \begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\frac{- \begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\frac{- \begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

Interpreting a Routh Table

- Simply stated, the Routh-Hurwitz criterion declares that the number of roots of the polynomial that are in the right hand-plane is equal to the number of sign changes in the first column.

s^3	1	31	0	+
s^2	1	103	0	+
s^1	$-\frac{\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$-\frac{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	$-\frac{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	-
s^0	$-\frac{\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$-\frac{\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$-\frac{\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	+

Special Cases of the Routh Array

- A couple of special situations can arise when constructing a Routh array.
- While constructing the array we might find that we end up with:
 - A zero in the first column.
 - A row of zeros.
- Either of these occurrences will “break” the procedure outlines above.
- However, there are simple tricks to deal with such situations
 - consult any control text for further information.

Special Cases of the Routh Array

- A zero in the first column.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 = (0)$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

- A row of zeros.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 = (0)$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 = (0)$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = (0)$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Routh-Hurwitz Special Cases

Zero in the first row/column:

- In this case, the zero is replaced with epsilon (ϵ).

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 = (0)$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

- Evaluate the sign change in the first column. Interpret stability of the system.

s^4	a_4	a_2	a_0	+/-
s^3	a_3	a_1	0	+/-
s^2	$b_1 = (\epsilon)$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$	+/-
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	+/-
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	+/-

Example of Routh-Hurwitz Special Cases 1

Given the following transfer function of a control system, we perform stability analysis of the system using Routh-Hurwitz method. [6 marks]

$$T(s) = \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$$

- The characteristic equation of the system is:

$$2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1$$

- Create Routh table based on the equation given above.

Example of Routh-Hurwitz Special Cases 1

- This example shows an example of zero in the first row or column.
- In this case, zero is replaced with epsilon (ϵ) and will tend to zero.

s^5	2	2	2
s^4	3	3	1
s^3	$\frac{-\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}}{3} = 0$	$\frac{-\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}}{3} = 4/3$	$\frac{-\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}}{3} = 4/3$
s^2	$\frac{-\begin{vmatrix} 3 & 3 \\ 0 & 4/3 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} 2 & 1 \\ 0 & 4/3 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} 2 & 1 \\ 0 & 4/3 \end{vmatrix}}{0} = \infty$
s^1	$\frac{-\begin{vmatrix} 0 & 4/3 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$	$\frac{-\begin{vmatrix} 0 & 4/3 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$	$\frac{-\begin{vmatrix} 0 & 4/3 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$
s^0	$\frac{-\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$

Example of Routh-Hurwitz Special Cases 1

- There are two sign changes in the first row of the Routh table.
- System has two unstable poles located on the right-hand side of the s-plane.

s^5	2	2	2	+
s^4	3	3	1	+
s^3	(ϵ)	$4/3$	0	+
s^2	$\frac{3\epsilon - 4}{\epsilon}$	1	0	-
s^1	$\frac{12\epsilon - 16 - 3\epsilon^2}{9\epsilon - 12}$	0	0	+
s^0	1	0	0	+

Routh-Hurwitz Special Cases

Entire row of zeros:

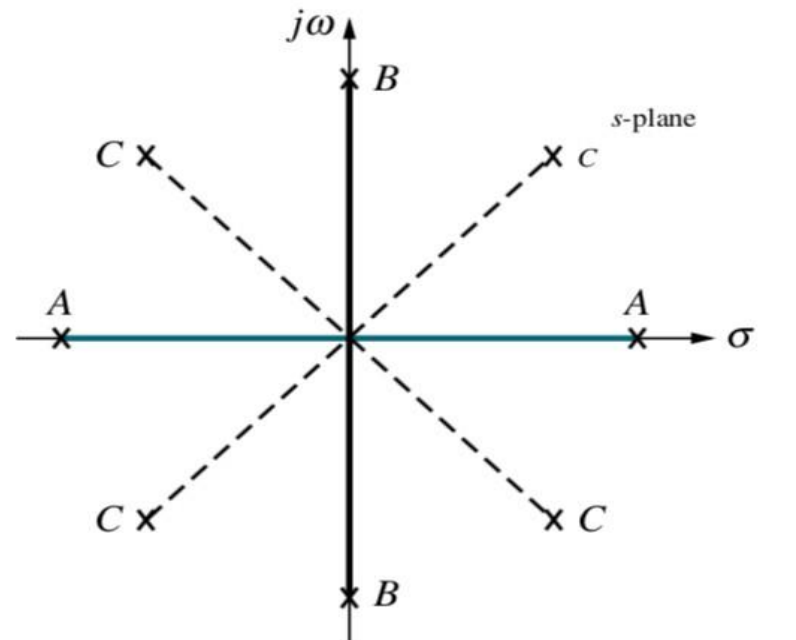
s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 = (0)$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 = (0)$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = (0)$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

- There is an even polynomial that is a factor of the original polynomial.
- This case have to be handled differently from the case with a zero in the first column of a row.

Routh-Hurwitz Special Cases

Entire row of zeros in the s-plane:

- Real and symmetrical about the origin (A).
- Imaginary and symmetrical about the origin (B).
- Quadrantal and symmetrical about the origin (C).



- A: Real and symmetrical about the origin ———
- B: Imaginary and symmetrical about the origin ———
- C: Quadrantal and symmetrical about the origin - - - -

Routh-Hurwitz Special Cases

- The row of zeros tells us of the existence of an even polynomial whose roots are symmetric about the origin.
- Some of these roots could be on the $j\omega$ -axis.
- On the other hand, since $j\omega$ roots are symmetric about the origin, if we do not have a row of zeros, we cannot possibly have $j\omega$ roots.
- Another characteristic of the Routh table for the case in question is that the row previous to the row of zeros contains the even polynomial that is a factor of the original polynomial.
- Finally, everything from the row containing the even polynomial down to the end of the Routh table is a test of only the even polynomial.

Routh-Hurwitz Special Cases

- For entire row of zeros, in the Routh-Hurwitz table, go up a row above the row with all zero coefficients.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 = (0)$	$-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 = (0)$	$-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = (0)$
s^1	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 = \infty$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$
s^0	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

- Create an equation that is made up of the coefficients of the given row.

$$P = a_3 s^3 + a_1 s^1 + 0$$

Routh-Hurwitz Special Cases

- Differentiate the respected equation with regard to the ‘s’ function.

$$\frac{dP}{ds} = 3a_3s^2 + a_1$$

- Replace the row with all zero coefficients with the coefficients of the resulting function after the differentiation.

s^4	a_4	a_2	a_0	+/-
s^3	a_3	a_1	0	+/-
s^2	$b_1 = (3a_3s^2)$	$b_2 = (a_1)$		+/-
s^1	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	+/-
s^0	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	+/-

- Evaluate the sign change in the first column of the table. Interpret the stability of the system.

Example of Routh-Hurwitz Special Cases 2

For the transfer function equation of a fifth order system, perform stability analysis using Routh-Hurwitz method.

[10 marks]

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

The characteristic equation of the system is:

$$s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56$$

Create Routh table based on the equation above.

Example of Routh-Hurwitz Special Cases 2

- The Routh-Hurwitz table of the control system given above is shown in the figure below.
- Row s^4 is multiply through with $1/7$ for convenience.
- Notice that row with s^3 is all zero.
- The Routh table is summarised as shown below.

s^5	1	6	8
s^4	$\left(\frac{7}{7}\right) = 1$	$\left(\frac{42}{7}\right) = 6$	$\left(\frac{56}{7}\right) = 8$
s^3	$-\frac{\begin{vmatrix} 1 & 6 \\ 1 & 6 \end{vmatrix}}{1} = 0$	$-\frac{\begin{vmatrix} 1 & 8 \\ 1 & 8 \end{vmatrix}}{1} = 0$	$-\frac{\begin{vmatrix} 1 & 8 \\ 1 & 8 \end{vmatrix}}{1} = 0$
s^2	$-\frac{\begin{vmatrix} 1 & 6 \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$-\frac{\begin{vmatrix} 1 & 8 \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$-\frac{\begin{vmatrix} 1 & 8 \\ 0 & 0 \end{vmatrix}}{0} = \infty$
s^1	$-\frac{\begin{vmatrix} 0 & 0 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$	$-\frac{\begin{vmatrix} 0 & 0 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$	$-\frac{\begin{vmatrix} 0 & 0 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$
s^0	$-\frac{\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$-\frac{\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$-\frac{\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$

s^5	1	6	8
s^4	1	6	8
s^3	(0)	(0)	(0)
s^2	∞	∞	∞
s^1	0	0	0
s^0	∞	∞	∞

Example of Routh-Hurwitz Special Cases 2

- In the Routh-Hurwitz table, go up a row above the row with all zero coefficients.
- Create an equation that is made up of the coefficients of the given row.
- At the row of s^4 , the equation below is constructed from the coefficients in this row.

$$P = s^4 + 6s^2 + 8$$

- Differentiate the respected equation with regard to the ‘ s ’ function.
- The first derivate of the equation above is calculated as follow:

$$\frac{dP}{ds} = 4s^3 + 12s + 0$$

Example of Routh-Hurwitz Special Cases 2

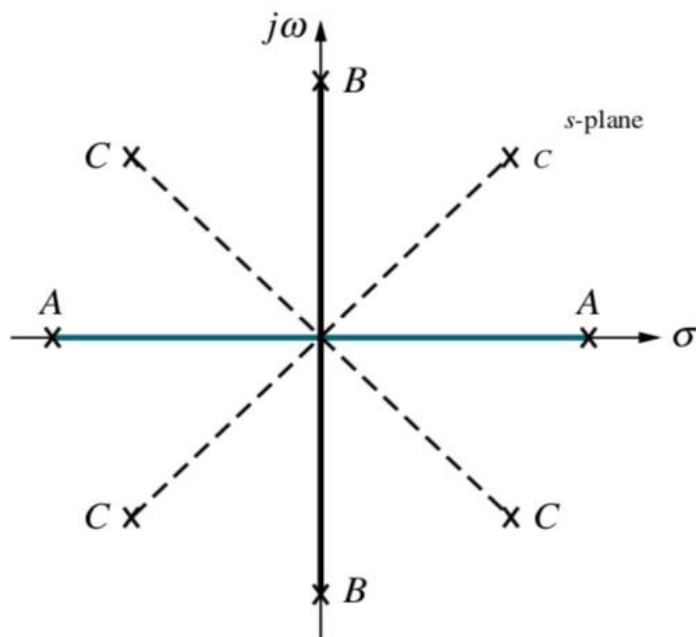
- Replace the row with all zero coefficients with the coefficients of the resulting function after the differentiation.
- As a result, the Routh table becomes as shown in the table below.

s^5	1	6	8	+
s^4	1	6	8	+
s^3	(4)	(12)	(0)	+
s^2	3	8	0	+
s^1	1/3	0	0	+
s^0	8	0	0	+

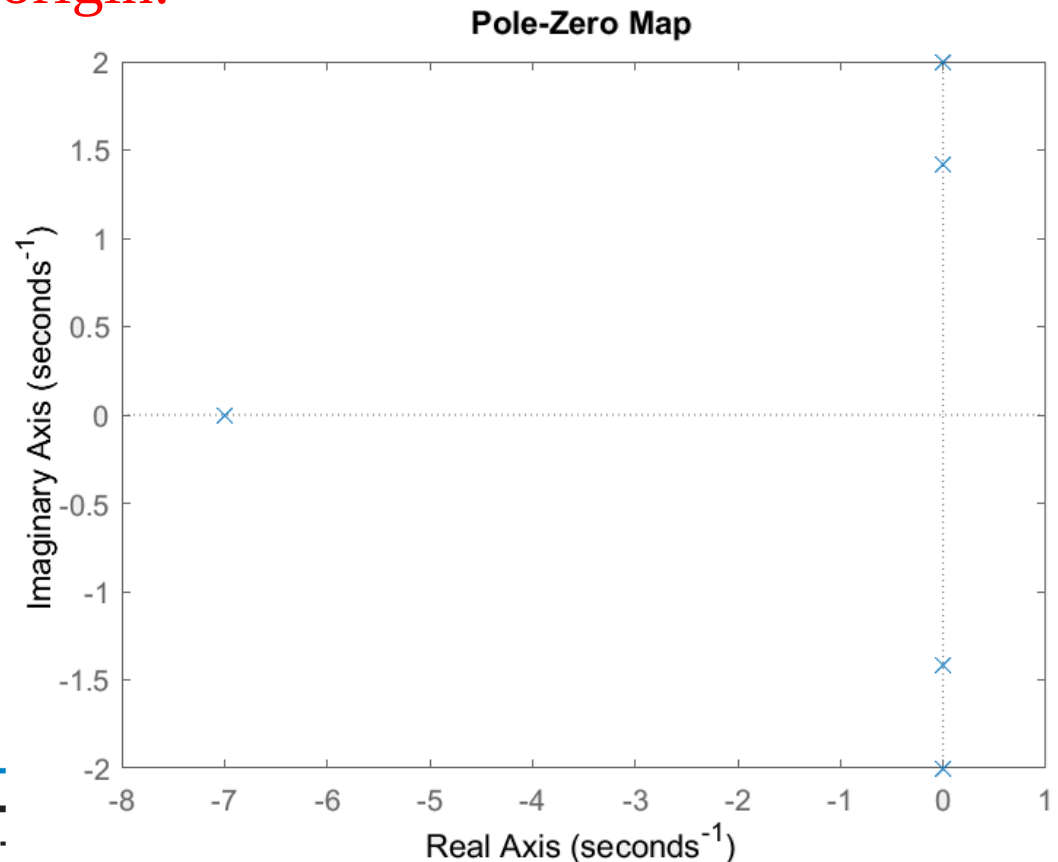
- From the result of the analysis, with all zeros in a given row, it seems that there is no change of sign in the first column of the Routh table.
- As a result, the given system has real and symmetrical poles about the origin.

Example of Routh-Hurwitz Special Cases 2

- As indicted in the s-plane diagram of the system as shown in the figure below, the poles for this system are real and symmetrical about the origin.



- A: Real and symmetrical about the origin
B: Imaginary and symmetrical about the origin
C: Quadrantal and symmetrical about the origin



Example of Routh-Hurwitz Special Cases 2

- Evaluating the even polynomial $P = s^4 + 6s^2 + 8$, no change of sign in the first column from row with s^4 to the last row. So, the four poles of the polynomial are located on the y-axis.
- No change of sign from row with s^5 to row with s^4 , so the remaining fifth pole is located on the left half-side of the s-plane.

