XMUT315 Control Systems Engineering

Note 11b: Analysis with Root Locus

Topic

- Sketching root locus.
- Rules in root locus analysis.
- Examples of root locus analysis.
- Refining root locus analysis.
- Break away and break in.
- Imaginary axes crossing.
- Angle of departure and angle of arrival.

1. Introduction to Analysis with Root Locus

To perform analysis of the control systems using root locus, we need to know how to construct the root locus diagram.

1.1. Sketching the Root Locus

Direct solution for the closed loop pole locations as a function of K becomes burdensome as the system order increases.

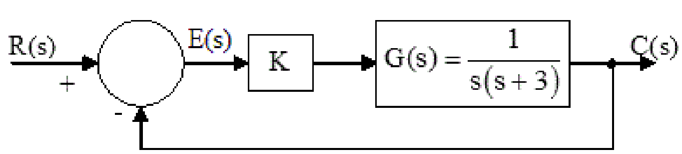


Figure 1: Feedback control system

As shown in the figure above, for the given control system, the transfer function of the open-loop system is:

$$G(s) = \frac{1}{s(s+3)}$$

Applying the feedback equation, the transfer function of the closed loop system is given below:

$$T(s) = \frac{K}{s^2 + 3s + K}$$

Notice that when we increase the gain of the controller (K) e.g.: K = 1, 2, 4, 10, 20, 40, and 100, the location of the pole is shifted as shown in the figure below.

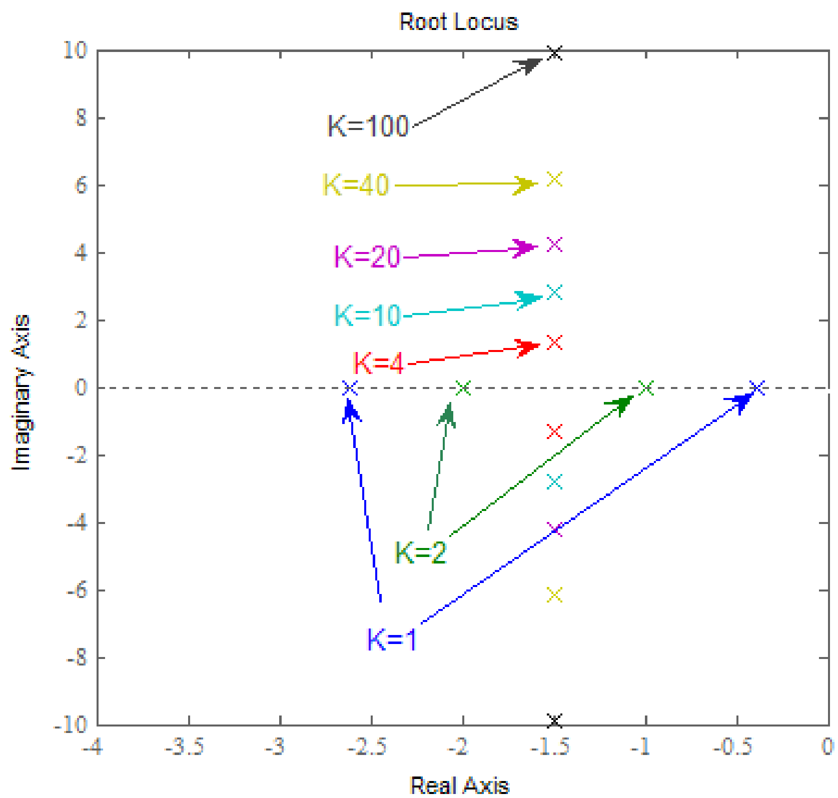


Figure 2: Root locus diagram of a closed loop system with gain varied

1.2. Rules for Sketching Root Locus

Walter Evans devised a set of rules that allows sketching of the root locus without brute force calculation. The basic root locus diagram is based on five rules i.e. Evans rule for sketching root locus.

These rules are generally sufficient to give a good sense of the shape of the root locus, and consequently its “story” as the gain changes. There are additional rules that can be used to refine the shape of the locus when necessary.

1.2.1. Rule #1 for Sketching a Root Locus

The number of branches of the root locus is equal to the number of closed loop poles. A “branch” is a segment of the root locus traversed (or moved) by a single pole as the gain is varied. This rule implies that there is one and only one branch originating at each pole.

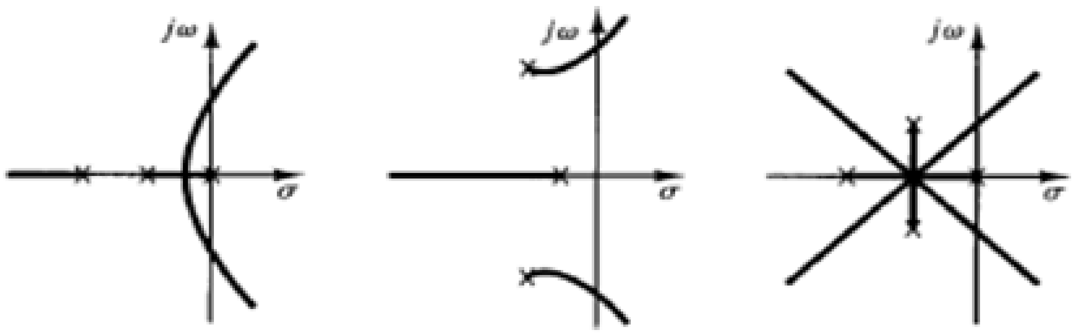


Figure 3: Symmetries in the root locus diagram

1.2.2. Rule #2 for Sketching a Root Locus

The root locus is symmetric about the real axis. Contours of the plots are symmetric about the real axis. Any complex roots (poles or zeros) must occur in complex conjugate pairs. Therefore, the root locus must be symmetric.

1.2.3. Rule #3 for Sketching a Root Locus

A branch of the root locus will only be on the real axis to the left of an odd number of finite open loop roots. Recall that roots means both poles and zeros! As discussed earlier, the solution of the characteristic equation implies that $\angle CG = -180°$ at all points on the root locus.

Now, all roots on the further left than a test point will contribute no phase shift. However, both zeros and poles to the right of a test point contribute 180°. We therefore only satisfy the characteristic equation if there is an odd number of roots on the real axis that are further to the right than our test point.

1.2.4. Rule #4 for Sketching a Root Locus

The root locus begins at the finite and infinite poles of $C(s)G(s)$ and ends at the finite and infinite zeros of $C(s)G(s)$. Let us calculate what happens to $T = KCG/(1 + KCG)$ for very small and very large values of K. Notice the characteristic equations of the closed-loop system is:

$$1 + KC(s)G(s) = 0$$

Factor $G(s)$ and rewrite the polynomial equation as:

$$1 + KC(s)G(s) = 1 + K\frac{N_C(s)N_G(s)}{D_C(s)D_G(s)} = 1 + K\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{m}(s - p_i)}$$

Therefore, the characteristic equation becomes:

$$\prod_{i=1}^{m}(s - p_i) + K\prod_{i=1}^{m}(s - z_i) = 0$$

Note that we have allowed for more interesting compensators here, by expanding the previous C into KC, where C now contains any compensator roots.

$$\lim_{k\to 0}\left[\prod_{i=1}^{m}(s - p_i) + K\prod_{i=1}^{m}(s - z_i)\right]$$

Therefore, the poles of T for small K are the poles of $CG.$ That is, the poles of T start at the open loop poles of the plant and the open loop poles of the compensator.

$$\lim_{k\to \infty}\left[\prod_{i=1}^{m}(s - p_i) + K\prod_{i=1}^{m}(s - z_i)]\right]$$

Therefore, the poles of T for large K are the roots of $CG.$ These are the zeros of $CG,$ which is just the combination of the zeros of C and the zeros of G.

1.2.5. Rule #5 for Sketching a Root Locus

The root locus approaches straight line asymptotes as the gain approaches infinity. The real-axis intercept, σ_a, and angles, θ_a, are given by:

$$\sigma_a = \frac{\sum_i p_i - \sum_i z_i}{P - Z} \qquad \text{and} \qquad \theta_a = \frac{(2k+1)\pi}{P - Z}$$

Where: P is the number of finite poles, Z is the number of finite zeros, p_i is the i-th pole, z_i is the i-th zero and $k \in Z$.

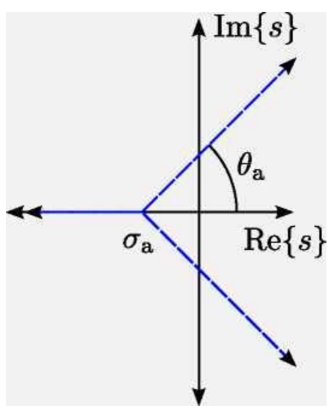


Figure 4: Real axis intercept and angles of asymptote in root locus diagram

In general, a root locus will have $P - Z$ asymptotes, so you will need to substitute $P - Z$ consecutive values for k into the θ_a equation.

Example for Tutorial 1: Root Locus Diagram 1 (Real Poles and Zeros)

Sketch the root locus of a system expressed as the transfer function shown below:

$$T(s) = \frac{s+3}{s(s+1)(s+2)(s+4)}$$

a. Work out and determine the locations of the poles and zeros on the system in the s-plane diagram. [2 marks]

b. Calculate the departure point and indicate this on the s-plane diagram. [3 marks]

c. Calculate the angle of asymptotes and sketch these angles on the s-plane diagram. [4 marks]

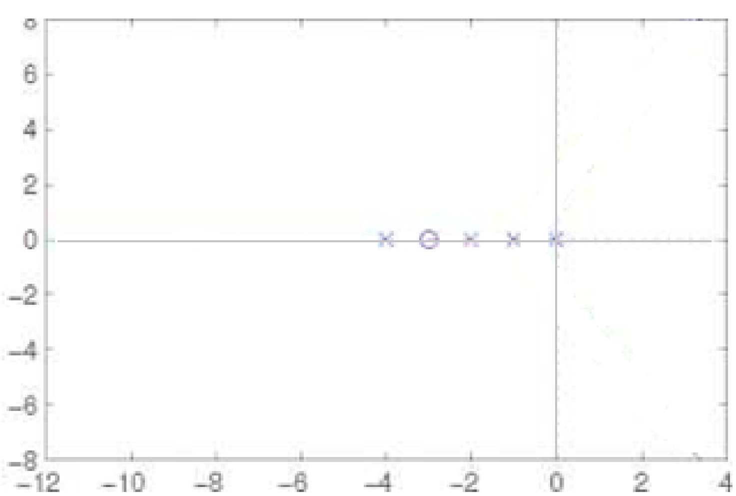
d. Determine the branches of the root locus and sketch them on the s-plane diagram. [4 marks]

e. Evaluate the construction of root locus diagram of the system using the rules for sketching a root locus. [10 marks]

f. Simulate the root locus diagram of the system in MATLAB. [5 marks]

Answer

a. For the given open loop system, we have poles at s = 0, -1, -2 and -4 and a zero at s = -3. The following figure shows poles and zeros locations of the system in the s-plane.

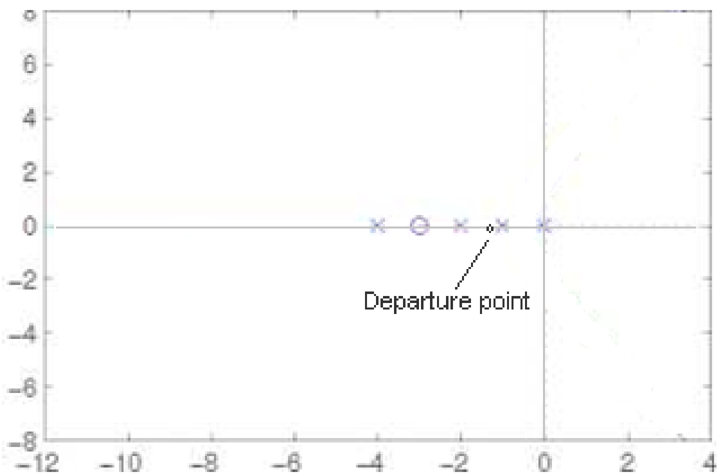


b. Let us first determine the parameters of the asymptotes. First, the departure point:

$$\sigma_a = \frac{\sum_i p_i - \sum_i z_i}{P - Z}$$

$$= \frac{(0-1-2-4)-(-3)}{4-1} = \frac{-7+3}{3} = -\frac{4}{3}$$

We can therefore indicate this point in the root locus. The following figure outlines the departure point of the system in the s-plane.



c. Work on the angle of the asymptotes that are determined from the following equation:

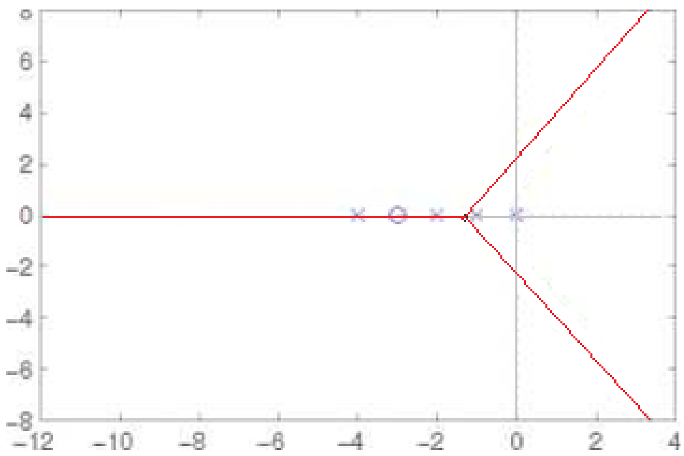
$$\theta_a = \frac{(2k+1)\pi}{P - Z}$$

$$= \frac{(2k+1)\pi}{4-1}$$

The angles of all these asymptotes are:

$$\theta_a = \begin{cases} \frac{\pi}{3} & \text{for } k = 0 \\ \pi & \text{for } k = 1 \\ -\frac{\pi}{3} & \text{for } k = -1 \end{cases}$$

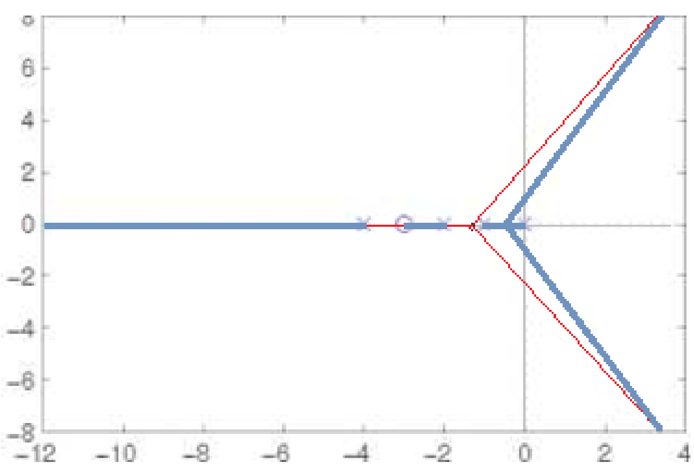
From the departure point, there are three asymptotes: at π/3 (60°), -π/3 (-60°), and π (-180°). We can therefore sketch in the asymptotes. The following diagram shows sketch of the asymptotes.



d. With the asymptotes in place, we can sketch in the various branches of the root locus. These asymptotes are:

- Asymptote 1: Pole (0,0) -> +∞
- Asymptote 2: Pole (-1,0) -> -∞
- Asymptote 3: Pole (-2,0) -> Zero (-3,0)
- Asymptote 4: Pole (-4,0) -> -∞

The following figure shows the sketch of these asymptotes of the system in the s-plane.



e. Based on the rules for constructing root locus diagram, we go through the construction of the root locus as listed below.

- R1 (*The number of branches of the root locus is equal to the number of closed loop poles)*:

  We expect four branches to the locus.

- R2 (*The root locus is symmetric about the real axis*):

The sketch is symmetric about the real axis

- R3 (*A branch of the root locus will only be on the real axis to the left of an odd number of finite open loop roots)*:

  The locus will pass along the real axis between the slowest two poles and between the third slowest pole and the zero. There is a branch on the real axis beyond the fastest pole.

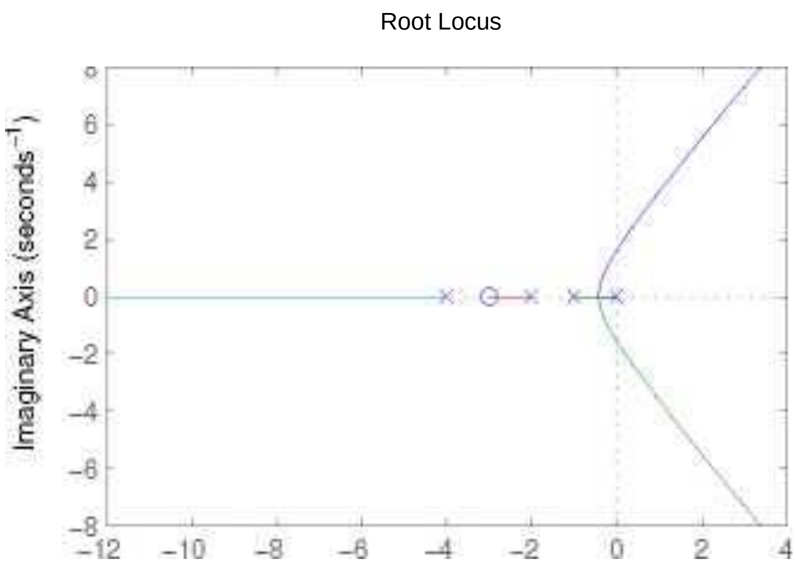
- R4 *(The root locus begins at the finite and infinite poles of $C(s)G(s)$ and ends at the finite and infinite zeros of $C(s)G(s)$)*:

  The loci are from poles to infinities and from a pole at (-2, 0) to zero at (-3, 0).

- R5 *(The root locus approaches straight line asymptotes as the gain approaches infinity at the real-axis intercept, σ_a, and angles, θ_a)*:

  There are three asymptotes from -1.33, 0 at $\pi/3$, $-\pi/3$, and π. Root locus diagram of the given system from the MATLAB simulation. The result shows a similar likeness with the result of the manual sketch.

f. The following figure shows the result of root locus simulation of the system in MATLAB.



Example for Tutorial 2: Root Locus Diagram 2 (Complex Zeros)

For a system with the following transfer function equation:

$$G(s) = \frac{s^2 - s + 20}{(s+2)(s+4)}$$

a. Determine the poles and zeros of the system. [2 marks]

b. Sketch the poles and zeros of the system in the s-plane diagram. [4 marks]

c. Determine the asymptotes of the root locus. [2 marks]

d. Sketch the asymptotes on the s-plane diagram. [4 marks]

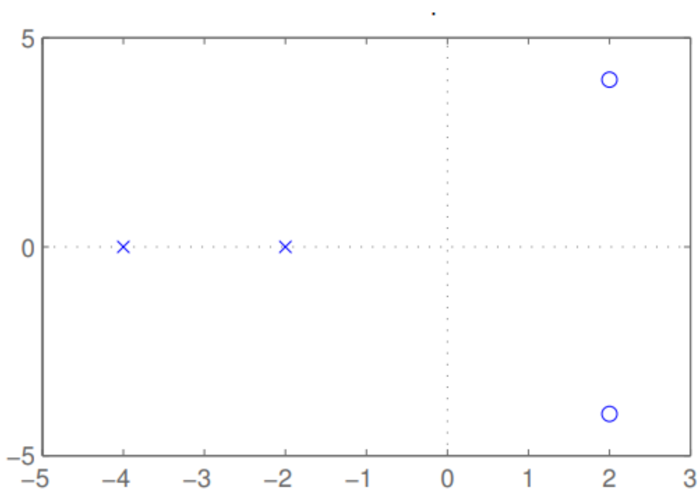
e. Simulate the root locus diagram of the system in MATLAB. [5 marks]

Answer

a. The poles and zeros of the system are:

- Pair of complex zero at -2 $\pm j\sqrt{20}$.
- Real pole at s = -2.
- Real pole at s = -4.

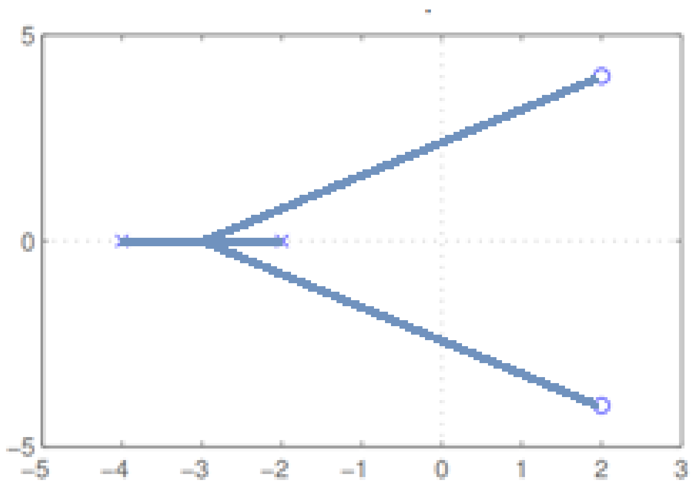
b. The sketch of the poles and zeros of the system in the s-plane is as shown in the figure below.



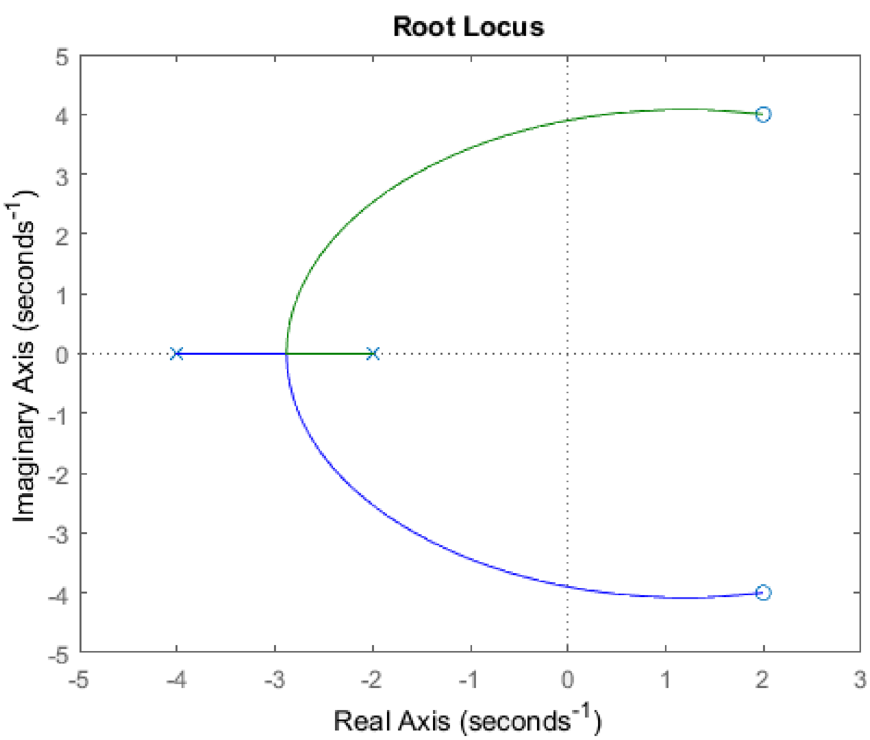
c. Asymptotes of the root locus are:

- Pole at s = -2 -> zero at 2 + $j\sqrt{20}$.
- Pole at s = -4 -> zero at 2 - $j\sqrt{20}$.

d. The sketch of the asymptotes in the s-plane is shown in the following diagram.



e. The result of root locus simulation of the system using MATLAB is shown in the figure below.



Example for Tutorial 3: Root Locus Diagram 3 (Complex Poles)

For a system with the following transfer function equation:

$$G(s) = \frac{1}{s[(s+4)^2 + 16]}$$

a. Determine the poles and zeros of the system. [2 marks]

b. Sketch the poles and zeros of the system in the s-plane diagram. [4 marks]

c. Determine the asymptotes of the root locus. [2 marks]

d. Sketch the asymptotes on the s-plane diagram. [4 marks]

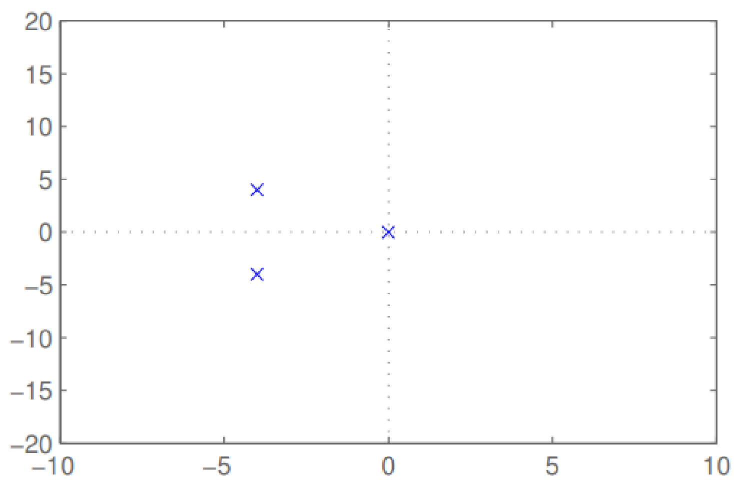
e. Simulate the root locus diagram of the system in MATLAB. [5 marks]

Answer

a. The poles and zeros of the system are:

- No zero.
- A pole at origin.
- Pair of complex poles at s = -4 ± j4.

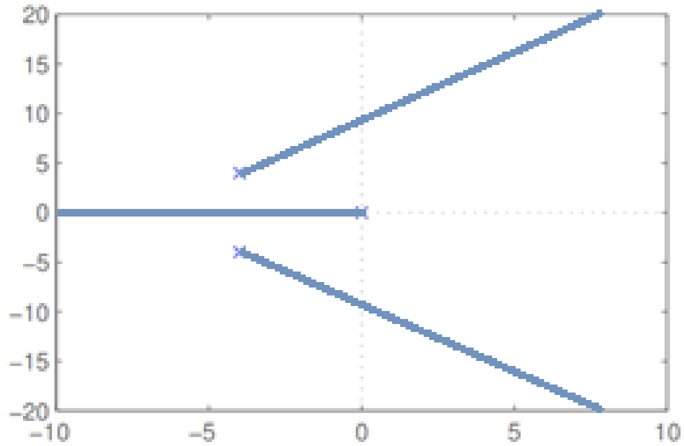
b. The sketch of the poles and zeros of the system in the s-plane is as shown in the figure below.



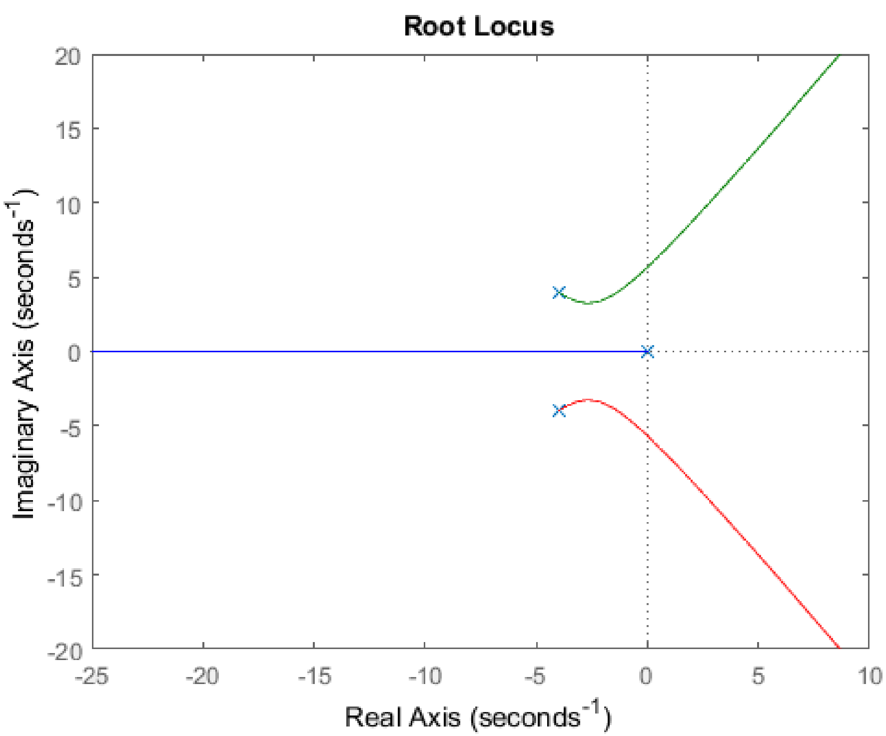
c. Asymptotes of the root locus are:

- Pole at origin -> asymptote (-∞, 0).
- Pole at s = -4 + j4 -> asymptote (+∞, +j∞).
- Pole at s = -4 - j4 -> asymptote (-∞, -j∞).

d. The sketch of the asymptotes in the s-plane is shown in the following diagram.



e. The result of root locus simulation of the system using MATLAB is shown in the figure below.



Example for Tutorial 4: Root Locus Diagram 4 (Poles at Origin)

The poles and zeros of the system in the s-plane is as shown in the figure below.

$$G(s) = \frac{1+s}{s^2}$$

a. Determine the poles and zeros of the system. [2 marks]

b. Sketch the poles and zeros of the system in the s-plane diagram. [4 marks]

c. Determine the asymptotes of the root locus. [2 marks]

d. Sketch the asymptotes on the s-plane diagram. [4 marks]

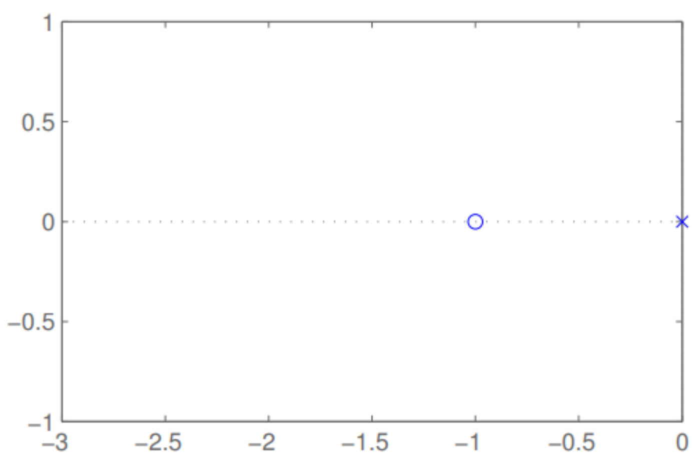
e. Simulate the root locus diagram of the system in MATLAB. [5 marks]

Answer

a. The poles and zeros of the system are:

- A real zero at s = -1.
- Double poles at origin.

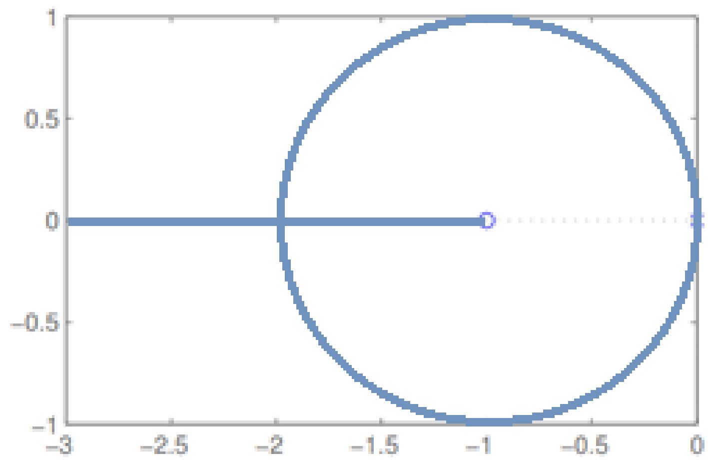
b. The sketch of the poles and zeros of the system in the s-plane is as shown in the figure below.



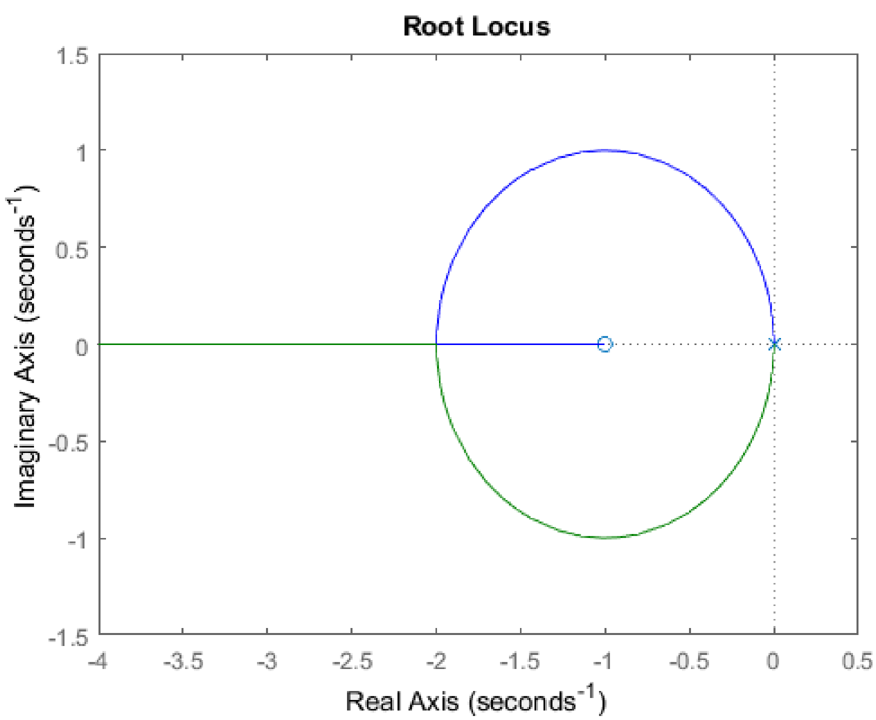
c. Asymptotes of the root locus are:

- Pole at origin -> zero at s = -1.
- Pole at origin -> asymptote (-∞, 0).

d. The sketch of the asymptotes in the s-plane is shown in the following diagram.



e. The result of root locus simulation of the system using MATLAB is shown in the figure below.



Example for Tutorial 5: Root Locus Diagram 5 (Complex Poles)

The poles and zeros of the system in the s-plane is as shown in the figure below.

$$G(s) = \frac{1}{s(s+2)[(s+1)^2+4]}$$

a. Determine the poles and zeros of the system. [2 marks]

b. Sketch the poles and zeros of the system in the s-plane diagram. [4 marks]

c. Determine the asymptotes of the root locus. [2 marks]

d. Sketch the asymptotes on the s-plane diagram. [4 marks]

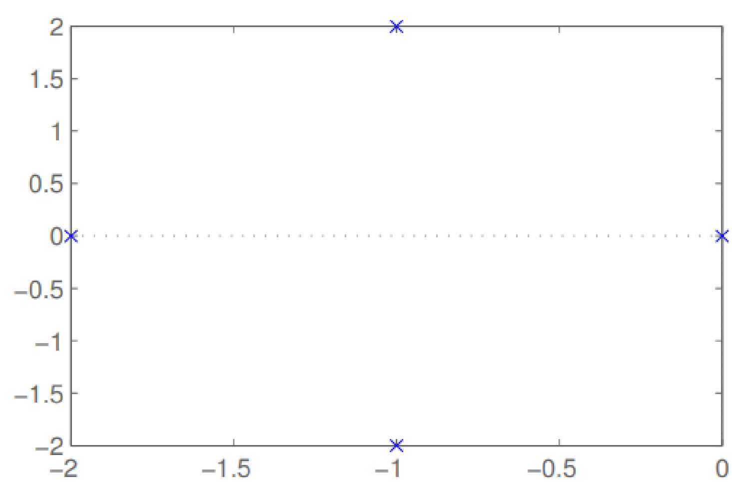
e. Simulate the root locus diagram of the system in MATLAB. [5 marks]

Answer

a. The poles and zeros of the system are:

- No zero.
- Pole at origin.
- A real pole at s = -2.
- A pair of complex poles s = -1 $\pm$ j2.

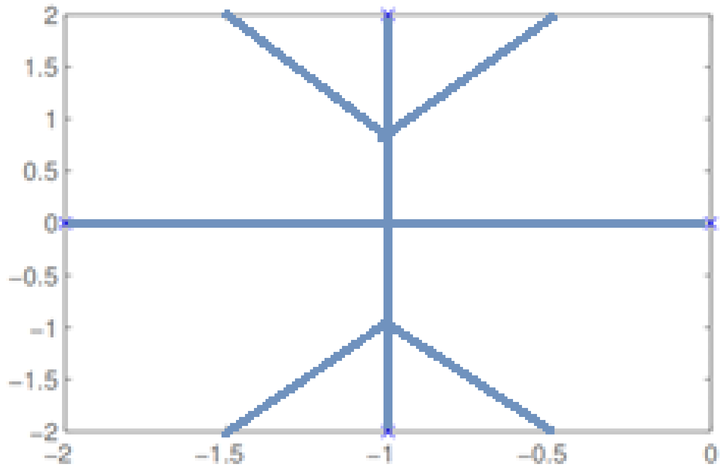
b. The sketch of the poles and zeros of the system in the s-plane is as shown in the figure below.



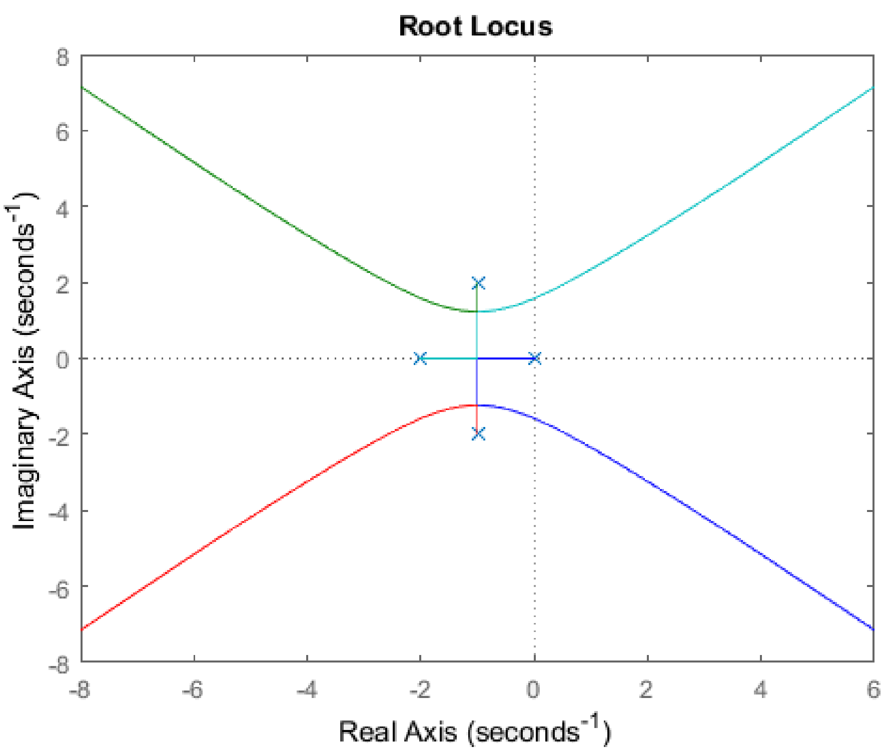
c. Asymptotes of the root locus are:

- Pole at origin -> asymptote (-∞, -j ∞).
- Pole at s = -2 -> asymptote (+∞, +j ∞).
- Complex pole at s = -1 + j2 -> asymptote (+∞, +j ∞).
- Complex pole at s = -1 - j2 -> asymptote (-∞, -j ∞).

d. The sketch of the asymptotes in the s-plane is shown in the following diagram.



e. The result of root locus simulation of the system using MATLAB is shown in the figure below.



3. Refining the Root Locus

There are many features of the root locus that are yet to be determined. Where do pole pairs break away or break into the real axis? Where do pole pairs cross the imaginary axis? At what angles do pole pairs depart from the open loop poles and enter open loop zeros?

- We have two options for determining these factors:

- Use MATLAB (or some other tool).
- Apply some more rules.

3.1. Pole Break-away / Break-in Angles

We often encounter systems where poles break away from the real axis into the complex plane, or conversely merge back into the plane. The angle at which the poles leave the plane is given by:

$$\angle\theta = \frac{180°}{n}$$

Where: n is the number of poles breaking away/in.

Typically, n = 2, so the angle of departure/arrival from the real axis is 90°.

3.1.1. Pole Break-away Locations

We know that a break-away occurs as gain is increasing. Therefore, the break-away point has the highest gain of any point where the locus is on the real axis. In cases where there are multiple segments on the real axis it might not be the highest gain globally, but it will be a local maximum.

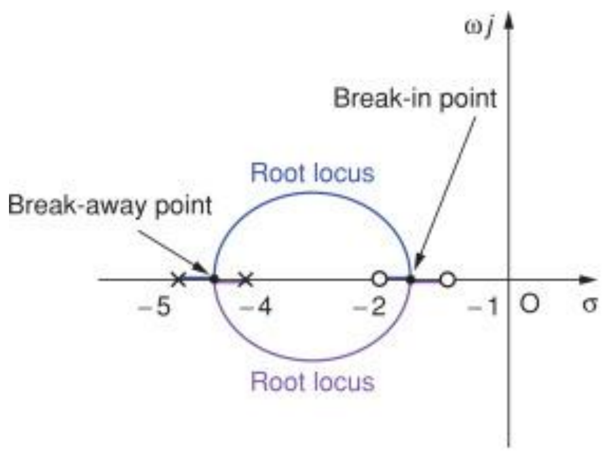


Figure 6: Break-away and break-in points of system in the root locus diagram

3.1.2. Pole Break-in Locations

Similarly, around a break-in point, the locus first touches the real axis at some value of gain. Then, the locus continues moving on the real axis as the gain increases. The break-in point will therefore correspond to a local minimum of gain. We know that the root locus satisfies the characteristic equation:

$$1 + KG = 0$$

Thus, everywhere along the root locus:

$$K = -\frac{1}{G(s)}$$

Along the real axis, we know that *s* is real, so we can write $s = \sigma$. Hence $K = -G(\sigma)$ on the real axis.

For a given system, the open loop transfer function equation is:

$$G(s) = \frac{(s + z_1)(s + z_2) \dots (s + z_k)}{(s + p_1)(s + p_2) \dots (s + p_k)} = \frac{n_1 s^k + n_2 s^{k-1} .. + n_k}{d_1 s^m + d_2 s^{m-1} .. + d_m}$$

Thus

$$K = -\frac{1}{G(s)} = \frac{d_1 s^m + d_2 s^{m-1} .. + d_m}{n_1 s^k + n_2 s^{k-1} .. + n_k} = \frac{P(s)}{Z(s)}$$

Knowing the form of equation is $u/v = (u'v - uv')/v^2$, the first derivative of the equation above is:

$$\frac{dK}{ds} = \frac{\frac{dP(s)}{ds} Z(s) - P(s) \frac{dZ(s)}{ds}}{[D(s)]^2}$$

3.1.3. Pole Break-away and Break-in Without Differentiation Method

There is a method without differentiation method that is easier, though it is less obvious technique. Using this method, the *Breakaway and Break-in points,* $s = \sigma_{1,2}$*, satisfy the relationship:*

$$\sum_{i=1}^{Z} \frac{1}{\sigma_b - z_i} = \sum_{j=1}^{P} \frac{1}{\sigma_b - p_j}$$

Where: p_i and z_i are the pole and zero values of CG*,* where we have Z total zeros and P total poles.

Example for Tutorial 6: Pole Break-away/Break-in Locations

Given a control system with the following transfer function equation, it is connected in series with a proportional controller with a gain of K:

$$G(s) = \frac{(s - 3)(s - 5)}{(s + 1)(s + 2)}$$

a. Determine the expression for the closed-loop system for determining value of K in terms of value of σ. [4 marks]

b. Simulate in MATLAB the break-away and break-in points of the system. [5 marks]

c. Determine the break-away and break-in points of the system using differentiation method. [4 marks]

d. Determine the break-away and break-in points of the system without using differentiation method. Comment on the difference using this method. [4 marks]

Answer

a. Along the root locus, $KG(s) = -1$

$$KG(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = -1$$

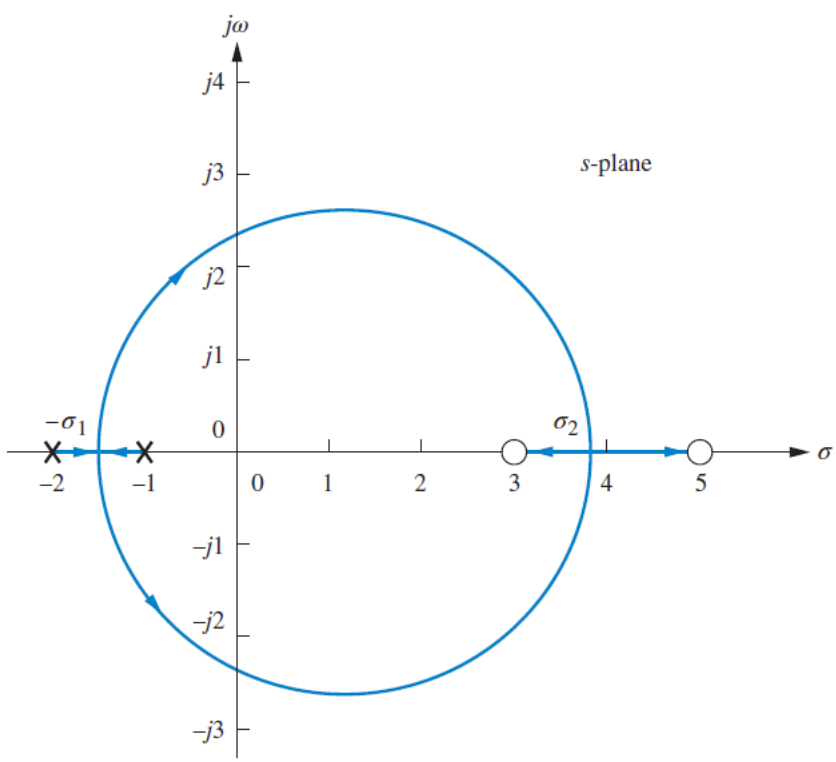
We are considering gains only on the real axis hence, we substitute $s = \sigma$.

$$\frac{K(\sigma-3)(\sigma-5)}{(\sigma+1)(\sigma+2)} = -1$$

We can solve for K and obtain the following equation:

$$K = -\frac{\sigma^2 + 3\sigma + 2}{\sigma^2 - 8\sigma + 15}$$

b. The following figure shows the results of MATLAB simulation of break-away and break-in points of system.



c. We want to find local extrema in the K value. We can find extrema by finding the derivative of the expression for K and setting it to zero.

$$K = -\frac{\sigma^2 + 3\sigma + 2}{\sigma^2 - 8\sigma + 15}$$

Differentiate the equation above ($d(u/v) = (u'v - uv')/v^2$):

$$\frac{dK}{d\sigma} = \frac{(2\sigma + 3)(\sigma^2 - 8\sigma + 15) - (\sigma^2 + 3\sigma + 2)(2\sigma - 8)}{(\sigma^2 - 8\sigma + 15)^2}$$

$$= \frac{(2\sigma^3 - 16\sigma^2 + 30\sigma + 3\sigma^2 - 24\sigma + 45) - (2\sigma^3 - 8\sigma^2 + 6\sigma^2 - 24\sigma + 4\sigma - 16)}{(\sigma^2 - 8\sigma + 15)^2}$$

$$= \frac{11\sigma^2 + 26\sigma + 61}{(\sigma^2 - 8\sigma + 15)^2}$$

Now, solve $a\sigma^2 + b\sigma + c = 11\sigma^2 + 26\sigma + 61 = 0$ using the quadratic equation to find the critical values of σ.

$$\sigma_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{(26)^2 - 4(11)(61)}}{2(11)}$$

We find that $\sigma_{1,2}$ = -1.45, 3.82. Thus, the breakaway point is at s = -1.45 and the break-in point is at s = 3.82.

d. This method is easier, though it is less obvious technique. *Breakaway and Break-in points, $s = \sigma_{1,2}$, satisfy the relationship:*

$$\sum_{i=1}^{Z} \frac{1}{\sigma_b - z_i} = \sum_{j=1}^{P} \frac{1}{\sigma_b - p_j}$$

Where: p_i and z_i are the pole and zero values of CG, where we have Z total zeros and P total poles.

For our example, we get:

$$\frac{1}{\sigma_b - 3} + \frac{1}{\sigma_b - 5} = \frac{1}{\sigma_b + 1} + \frac{1}{\sigma_b + 2}$$

After some algebra we get $11\sigma^2 + 26\sigma + 61 = 0$ as before. We can complete the problem as earlier.

3.3. Imaginary Axis Crossings

The gain K at which the root locus crosses the imaginary axis is the value at which the system becomes unstable. Stable systems cannot have poles in the right half of the s-plane.

There is no easy way to use the Root Locus to find this point. If you need to calculate the crossing point, then perform a Routh-Hurwitz test.

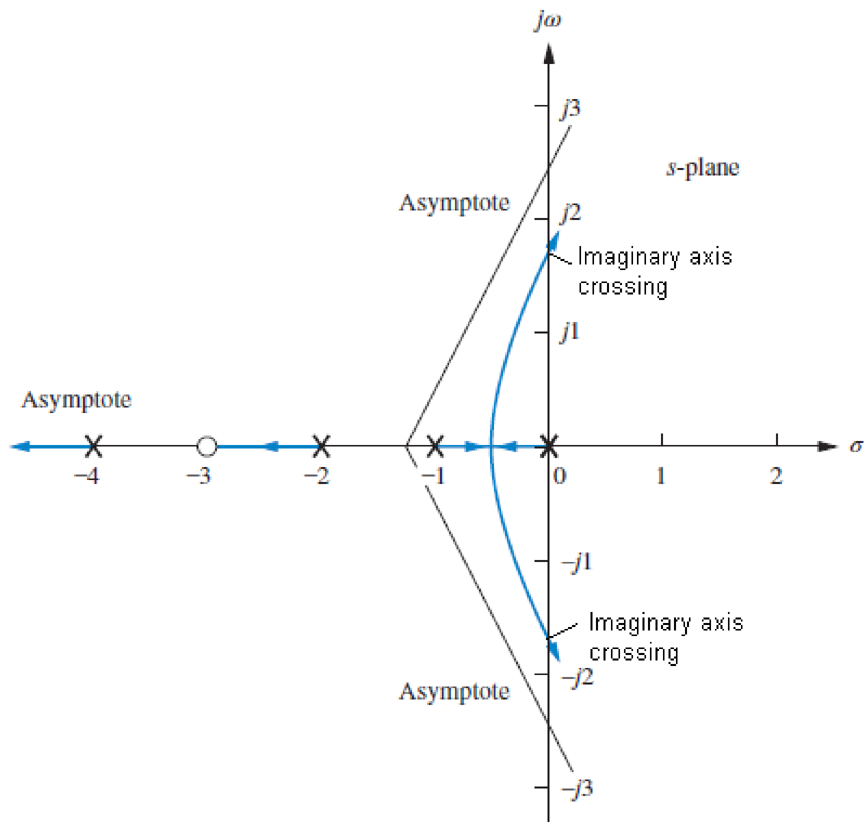
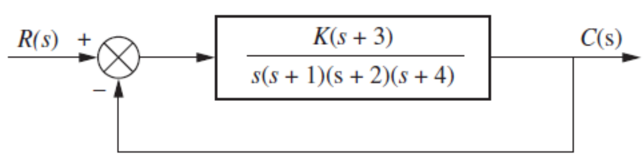


Figure 7: Imaginary axis crossing of the root locus of the system

Example for Tutorial 7: Imaginary Crossing Axis

For the system given below, find the frequency and gain, K, for which the root locus crosses the imaginary axis. For what range of K is the system stable? [16 marks]



Answer

The closed-loop transfer function for the system is:

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{\frac{K(s+3)}{s(s+1)(s+2)(s+4)}}{1+\frac{K(s+3)}{s(s+1)(s+2)(s+4)}} = \frac{K(s+3)}{s^4+7s^3+14s^2+(8+K)s+3K}$$

Using the denominator and simplifying some of the entries by multiplying any row by a constant, we obtain the Routh table as shown below.

s^4	1	14	$3K$
s^3	7	$8+K$	
s^2	$90-K$	$21K$	
s^1	$\frac{-K^2-65K+720}{90-K}$		
s^0	$21K$		

A complete row of zeros yields the possibility for imaginary axis roots. For positive values of gain, those for which the root locus is plotted, only the s^1 row can yield a row of zeros. Thus,

$$-K^2-65K+720=0$$

From this equation K is evaluated as:

$$K = \frac{-65 \pm \sqrt{(-65)^2-4(-1)(720)}}{2(-1)} = 9.65$$

Forming the even polynomial by using the s^2 row with K = 9.65, we obtain:

$$(90-9.65)s^2+21(9.65) = 80.35s^2+202.65$$

and s is found to be equal to $\pm j1.59$. Thus, the root locus crosses the $j\omega$-axis at $\pm j1.59$ at a gain of 9.65. We conclude that the system is stable for $0 \le K < 9.65$.

3.4. Angles of Departure/Arrival at Complex Roots

These are the angles when locus leaving or entering the complex roots (e.g. it is NOT the same as the asymptote angle). Recall that at all points on a root locus we satisfy the characteristic equation, $1 + CG = 0$.

$$0 = 1 + CG$$

Thus

$$-1 = CG$$

As a result:

$$|CG| = 1$$

And

$$\angle CG = -(2k+1)180°$$

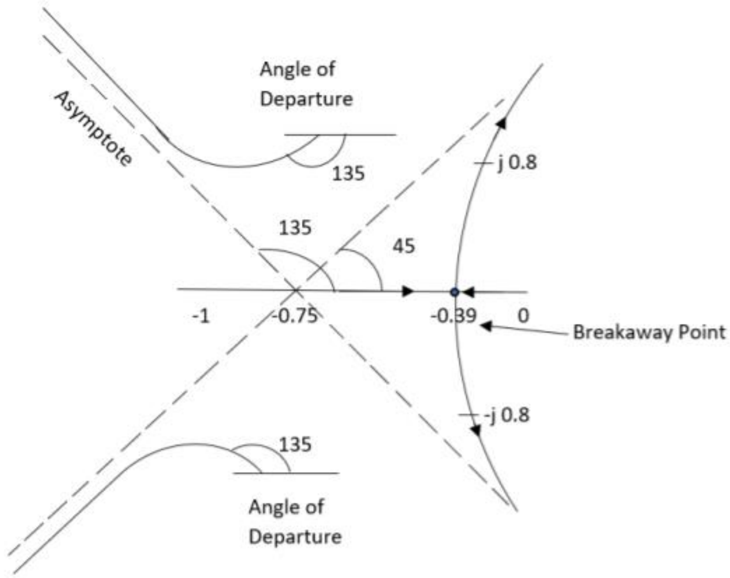


Figure 8: Angles of departure from complex root in the root locus diagram (Example 1)

Now, the angle (phase) of CG is the sum of the phases of the poles and zeros that make up $CG.$ That is, to be on the root locus, the sum of the angles to the closed loop roots is always -180°.

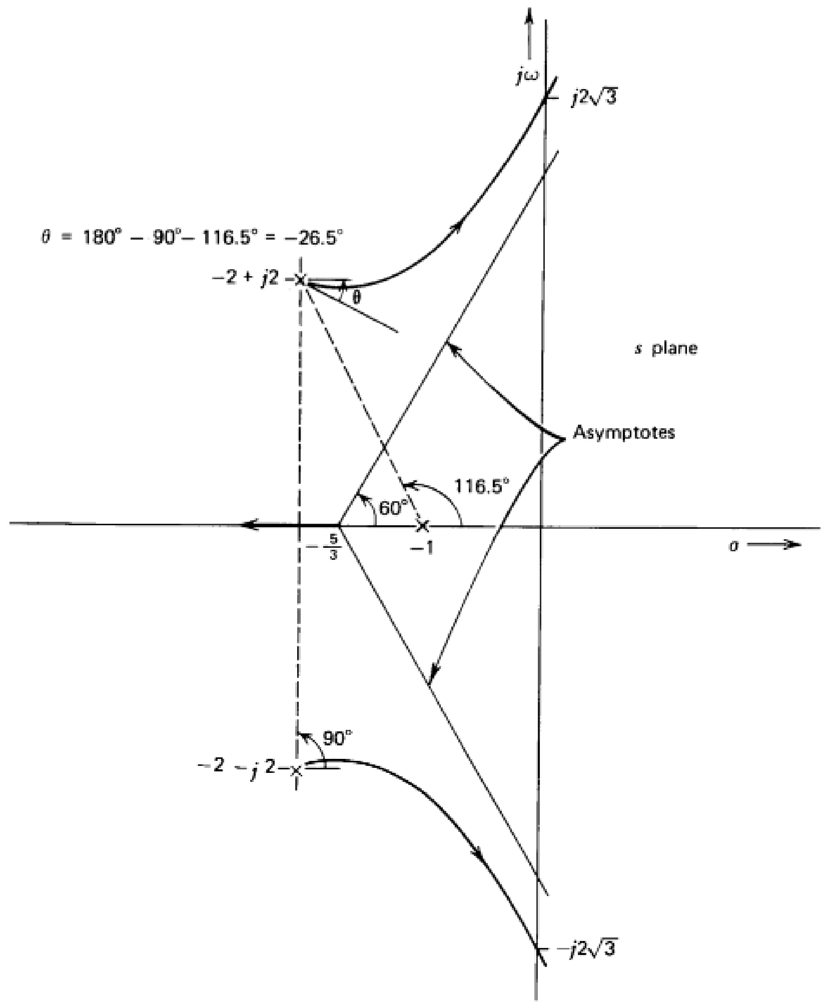


Figure 9: Angles of departure from complex root in the root locus diagram (Example 2)

Now, let us presume that we are on a point an arbitrarily small distance, ϵ, away from a system root. We can assume that the angle from each of the other roots to our test point is the same as from that root to the root of interest. We can calculate these other angles directly, enabling us to find the angle from ϵ to the root of departure or arrival.

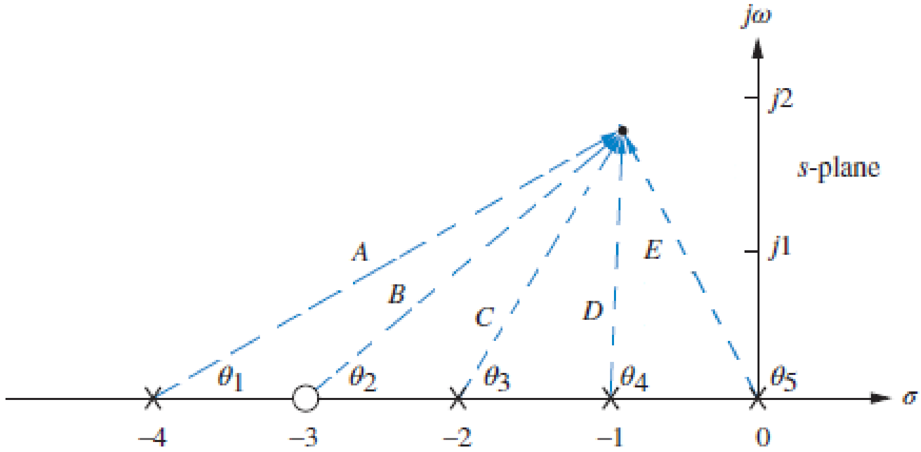


Figure 10: Point an arbitrarily small distance, ϵ, away from system roots

For the example root locus diagram of a given system, as shown in the figure above, the angle of departure at complex roots (θ_1) is determined from the following equation:

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)180°$$

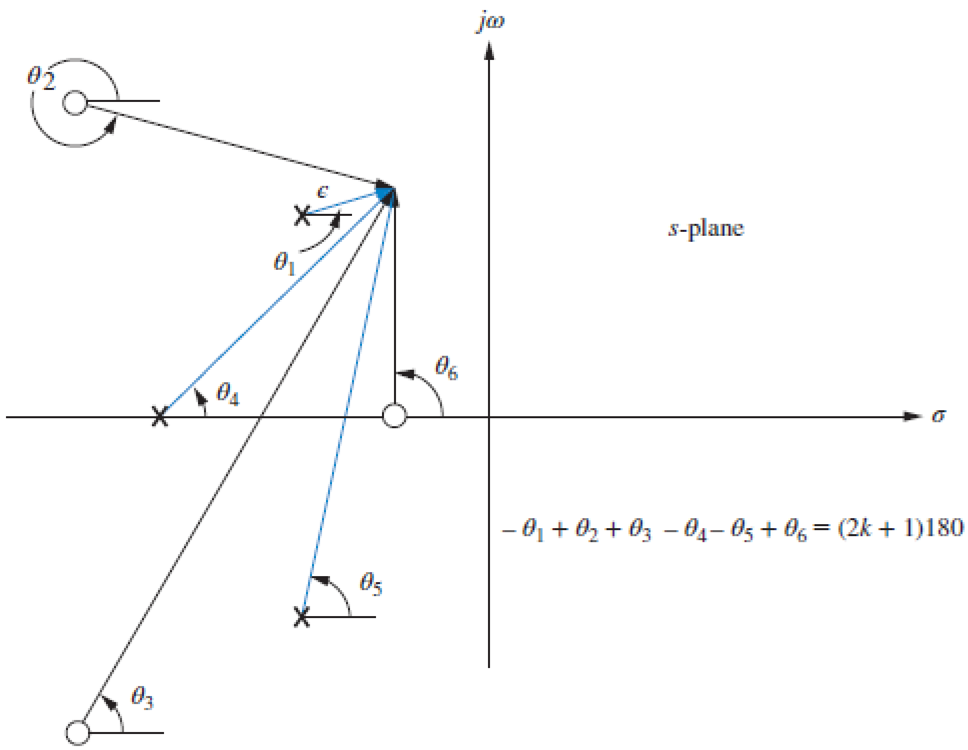


Figure 11: Angle of departure at complex roots

For the example root locus diagram of a given system, as shown in the figure below, the angle of arrival at complex roots (θ_2) is determined from the following equation:

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)180°$$

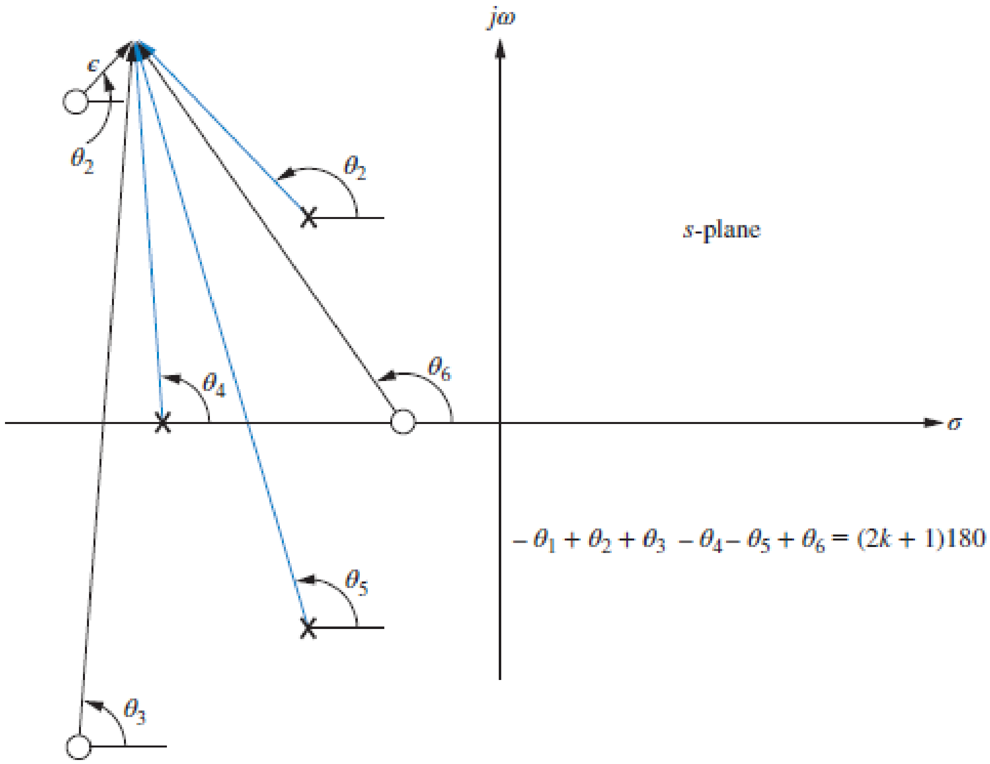


Figure 12: Angle of arrival at complex roots

Example for Tutorial 8: Angles of Departure

Consider a system given as the transfer function equation given below:

$$G(s) = \frac{s+2}{(s+3)(s^2+2s+2)}$$

a. Determine the poles and zeros of the open-loop system. [2 marks]

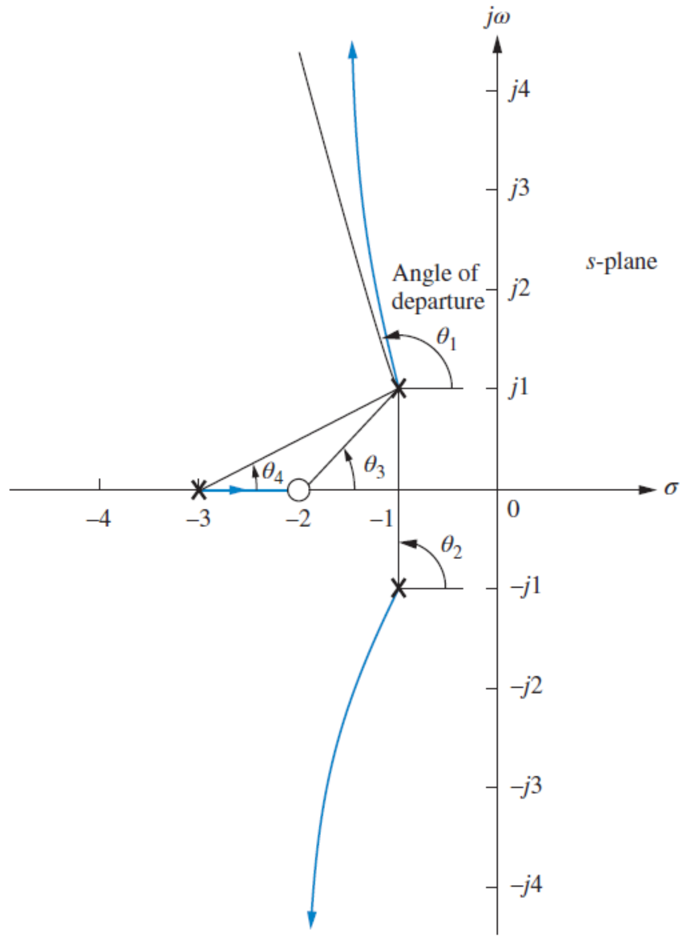
b. Sketch the angle of departure of the system. [6 marks]

c. Calculate the angle of departure from the pole at s = -1 + j. [6 marks]

Answer

a. This system has an open-loop zero at s = -2 and open-loop poles at s = -3, -1 ± j.

b. The following figure shows the angle of departure from the complex roots.



c. Let us calculate the angle of departure from the pole at s = -1 + j. Assume a point on the root locus that is a small distance ϵ away from s = -1 + j.

Equate all angles of the poles and zeros to -180°:

$$-180° = -\theta_1 - \theta_2 + \theta_3 - \theta_4$$

$$= -\theta_1 - 90° + 45° - \tan^{-1}\left(\frac{1}{2}\right)$$

Thus, rearrange the above equation:

$$\theta_1 = 180° - 90° + 45° - 26° = 109°$$

This is the required angle of departure from the open loop pole at s = -1 + j.

Notice that zeros cause phase lead (so we add their angle), while poles cause phase lag (so we subtract their angle).