XMUT315 Control Systems Engineering

Note 12b: Analysis and Design with Nyquist Diagram

Topic:

- Construction of Nyquist diagram.
- Poles on the complex axis.
- Gain and phase margins in Nyquist diagram.
- Stability in Nyquist diagram.
- Compensators in Nyquist diagram.
- Analysis with Nichols chart.
- Design procedures of control systems with Nyquist diagram and Nichols chart.

1. Introduction to Analysis with Nyquist Diagram

Most of the analysis performed using Nyquist diagram is for stability analysis of the control systems.

1.1. Construction of a Nyquist Diagram

We can draw a Nyquist diagram directly, without needing to draw a Bode plot first. We just consider the system response as we move around the required contour.

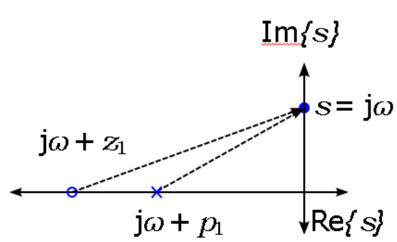


Figure 1: Graph of pole and zero vectors in the s-plane

Recall the gain and phase of frequency response.

Gain:

$$|G(s)| = \frac{|K||s+z_1||s+z_2| \dots |s+z_k|}{|s+p_1||s+p_2| \dots |s+p_k|}$$

Phase:

$$\angle G(s) = \angle(s+z_1) + \angle(s+z_2) \dots + \angle(s+z_k)$$

$$-\angle(s+p_1) - \angle(s+p_2) \dots - \angle(s+p_k)$$

For example, consider the contribution of a single LHP pole at pole location of $-a$.

$$G(s) = \frac{1}{(s+a)}$$

As we move from zero to infinite frequency, the phase will move from zero to −90°.

$$\angle \mathrm{G}(j\omega) \;= \tan^{-1}\left(\frac{j\omega}{a}\right) = \theta°$$

At the same time, the gain will drop.

$$|G(j\omega)| = \frac{1}{\sqrt{(j\omega)^2 + (a)^2}}$$

We can therefore sketch the Nyquist diagram that results.

1.2. Examples of Nyquist Diagram Construction

We will see two examples of construction of Nyquist diagram in this section e.g. first-order and second-order systems.

Example for Tutorial 1: Nyquist Diagram of First-Order System

Consider a first-order system with the transfer function:

$$G(s) = \frac{1}{s\; + \;0.2}$$

a. Determine the equations for calculating gain and phase of the frequency response of the system. [4 marks]

b. Calculate the gain and phase of the frequency response of the system for ω = 0, 0.2, and 1 rad/s. [6 marks]

c. Sketch the Nyquist diagram based on the results obtained in part (b). [4 marks]

Answer

a. The equations for calculating gain and phase of the frequency response of the system are:

Gain:

$$|G(s)| = \frac{1}{\sqrt{(j\omega)^2 + (0.2)^2}}$$

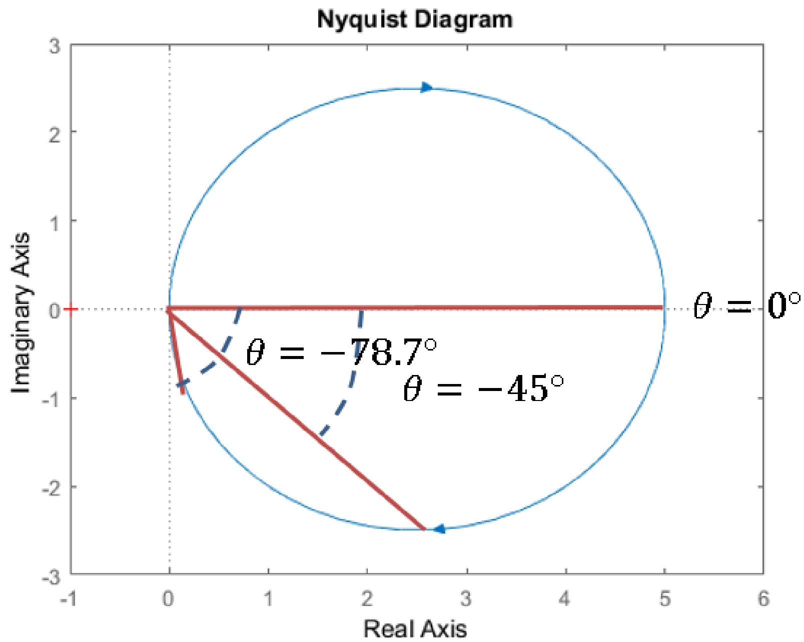
Phase:

$$\angle G(s) = -\tan^{-1}\left(\frac{j\omega}{0.2}\right)$$

b. The gain and phase values of the above given system is shown in the following table.

$\boldsymbol{\omega}$	$\boldsymbol{\lvert G(s)\rvert}$	$\boldsymbol{\angle G(s)}$
0	$\frac{1}{\sqrt{(0)^2 + (0.2)^2}} = 5$	$\tan^{-1}\left(\frac{0}{0.2}\right) = 0°$
0.2	$\frac{1}{\sqrt{(0.2)^2 + (0.2)^2}} = 3.5$	$\tan^{-1}\left(\frac{0.2}{0.2}\right) = -45°$
1	$\frac{1}{\sqrt{(1)^2 + (0.2)^2}} = 0.98$	$\tan^{-1}\left(\frac{1}{0.2}\right) = -78.7°$

c. The Nyquist diagram of the first-order system with transfer function $G(s) = 1/(s + 0.2)$.



Example for Tutorial 2: Nyquist Diagram of Second-Order System

We can use the same procedure for drawing more complex system examples. See the following second-order system with transfer function:

$$G(s) = \frac{1}{(s + j + 2)(s - j + 2)}$$

a. Determine the equations for calculating gain and phase of the frequency response of the system. [4 marks]

b. Calculate the gain and phase of the frequency response of the system for ω = 0, 1, and 10 rad/s. [6 marks]

c. Sketch the Nyquist diagram based on the results obtained in part (b). [4 marks]

Answer

a. Simply start with a pole-zero map and consider what happens as we follow the Nyquist contour.

For the given second-order system:

$$G(s) = \frac{1}{(s + j + 2)(s - j + 2)} = \frac{1}{s^2 + 4s + 5}$$

Gain:

$$|G(s)| = 1/\sqrt{(5 - (j\omega)^2)^2 + (4j\omega)^2}$$

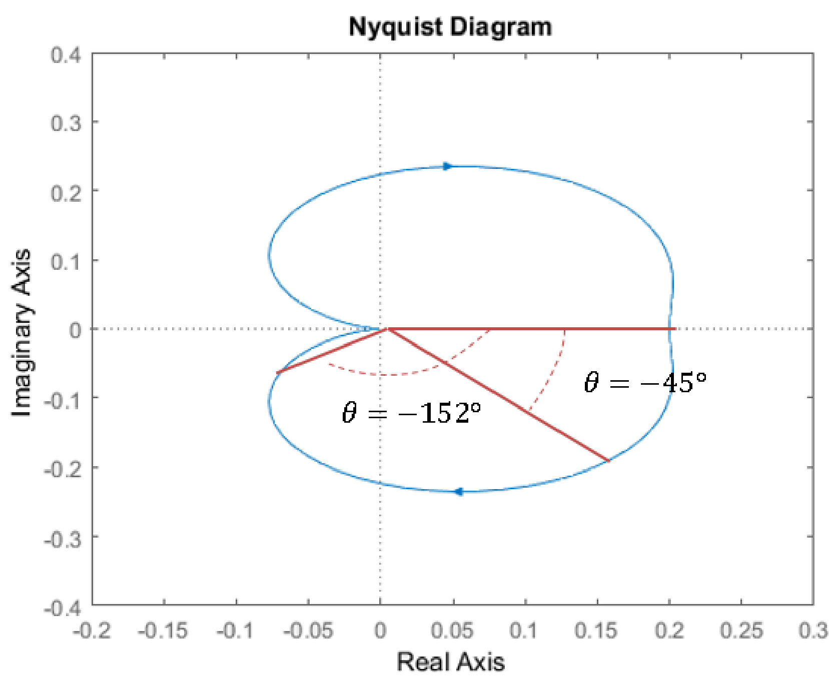
Phase:

$$\angle G(s) = -\tan^{-1}\left[\frac{4j\omega}{(5-(j\omega)^2)}\right]$$

b. The gain and phase values of the above given system are listed in the following table.

$\boldsymbol{\omega}$	$\lvert \boldsymbol{G(s)} \rvert$	$\angle \boldsymbol{G(s)}$
0	$\frac{1}{\sqrt{(5-0^2)^2+(0)^2}} = 0.2$	$-\tan^{-1}\left(\frac{0}{5}\right) = 0°$
1	$\frac{1}{\sqrt{(5-1^2)^2+(4)^2}} = 0.17$	$-\tan^{-1}\left(\frac{4}{4}\right) = -45°$
10	$\frac{1}{\sqrt{(5-10^2)^2+(40)^2}} = 0.097$	$-\tan^{-1}\left(\frac{40}{75}\right) = -152°$

c. The Nyquist diagram of the second-order system with transfer function: $G(s) = \frac{1}{(s+j+2)(s-j+2)}$



2. Poles on the Complex Axis

Following the contour we discussed before does not work if we have poles (or zeros) that would lie on the contour. Thus, if we have poles of zeros on the imaginary axis, we modify the contour so that it takes an infinitesimally small "detour" around the imaginary root. We then proceed as normal.

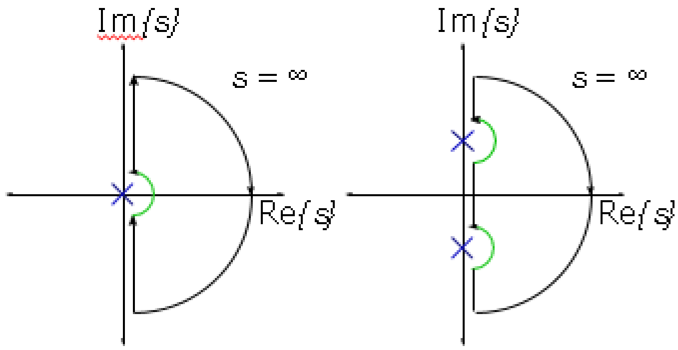


Figure 5: Poles in the complex axis in the Nyquist diagram

Example for Tutorial 3: Poles on the Complex Axis

For example, consider a second-order system with transfer function:

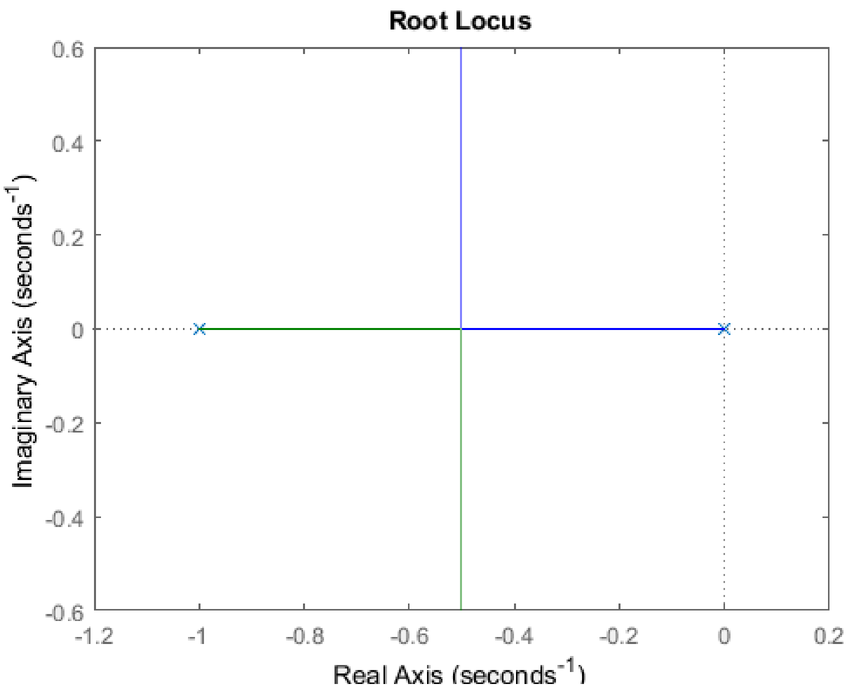
$$G(s) = \frac{4}{s(s+1)}$$

a. Simulate the Nyquist diagram of the system in MATLAB. [4 marks]

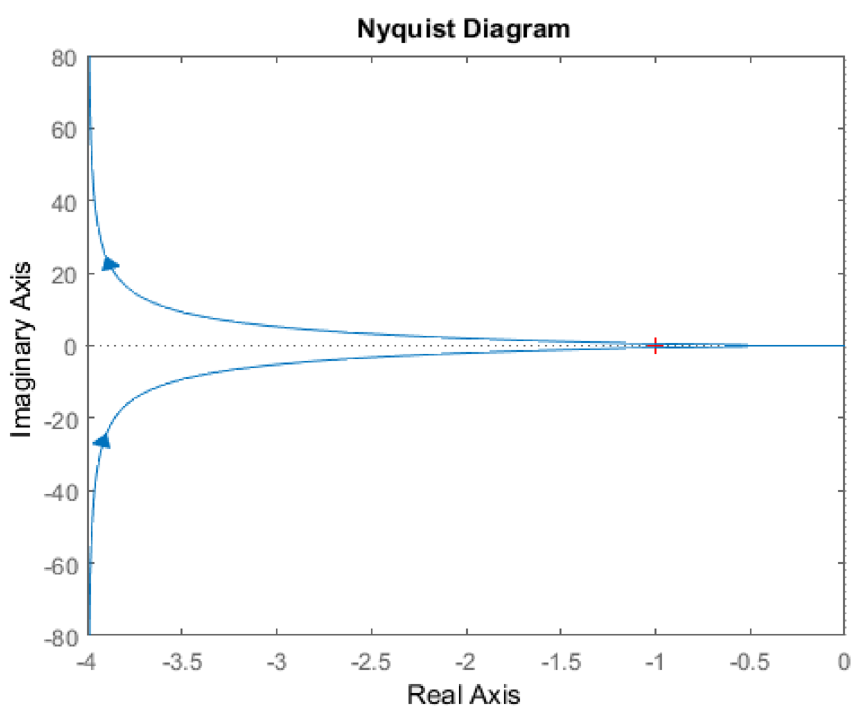
b. Determine the stability of the system by evaluating the encirclement at the test point (-1, 0). [4 marks]

Answer

a. The Nyquist diagram of the above given system is shown in the figure below.



b. If we blow up the region near the test point (1,0), the Nyquist diagram of the given system is shown in the figure below.



3. Margins on the Nyquist Diagram

We can examine the Nyquist diagram to determine the gain and phase margins. The phase margin can simply be read as the difference between the phase = −180° line and the point where the curve crosses the unit circle. The gain margin is the inverse of the distance to the point where the curve crosses the negative real axis.

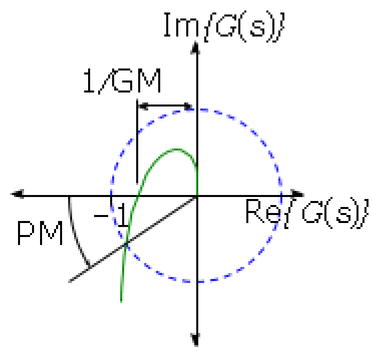


Figure 8: Gain and phase margins in Nyquist diagram

Note: that we can again have multiple gain and phase margins if the curve crosses the negative x-axis multiple times.

3.2. Stability from the Nyquist Diagram

Nyquist showed mathematically that the number of poles in the right-half plane of the closed-loop transfer function can be determined by examining the Nyquist diagram of the open-loop transfer function.

> Number of closed-loop poles in right-half of s-plane = Number of open-loop poles in right-half of s-plane + Number of clockwise encirclements of −1 + 0j.

Remember that for the system to be stable we must have no closed-loop poles.

If we have the transfer function of the system, we can easily determine the number of open-loop poles. So, if we use the Nyquist diagram to count the clockwise encirclements, we will be able to determine the closed-loop stability.

3.3. Counting Encirclements

To count the clockwise encirclements:

- Draw a straight line from −1 + 0j to ∞ in any direction.
- Count how many times the Nyquist diagram crosses from left to right over your chosen line. For each such crossing add one to your count of the number of encirclements.

- Count how many times the Nyquist diagram crosses from right to left over your chosen line. For each such crossing subtract one from your count of total encirclements.

If you have drawn a correctly constructed Nyquist diagram, then the final number that you get will be the same regardless of which orientation you chose for your original line (so choose a direction that makes counting easy).

Example for Tutorial 4: Nyquist Diagram of Third-Order System

Consider a third-order system as given in the transfer function equation below:

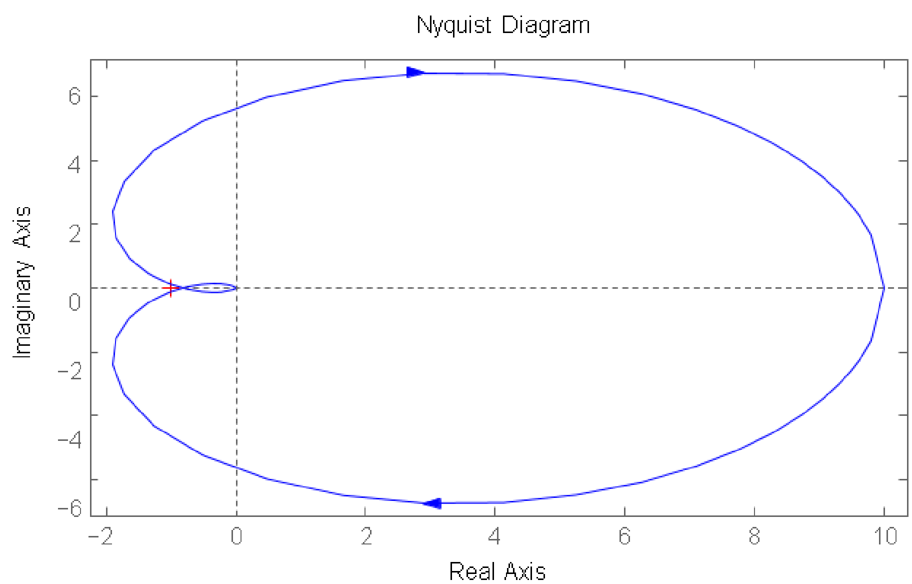
$$G(s) = \frac{100}{(s+1)(s+2)(s+5)}$$

a. Simulate the Nyquist diagram of the system in MATLAB. [4 marks]

b. Determine the stability of the system. [2 marks]

Answer

a. The Nyquist diagram of the above system is shown in the figure below.



b. This system will be closed loop stable, but if we were to increase the gain, then it would eventually encircle *s* = −1 and become unstable.

Example for Tutorial 5: Conditionally Stable System

Consider a complex system given as the following transfer function equation:

$$G(s) = \frac{s^2 + 2s + 4}{s(s + 4)(s + 6)(s^2 + 1.4s + 1)}$$

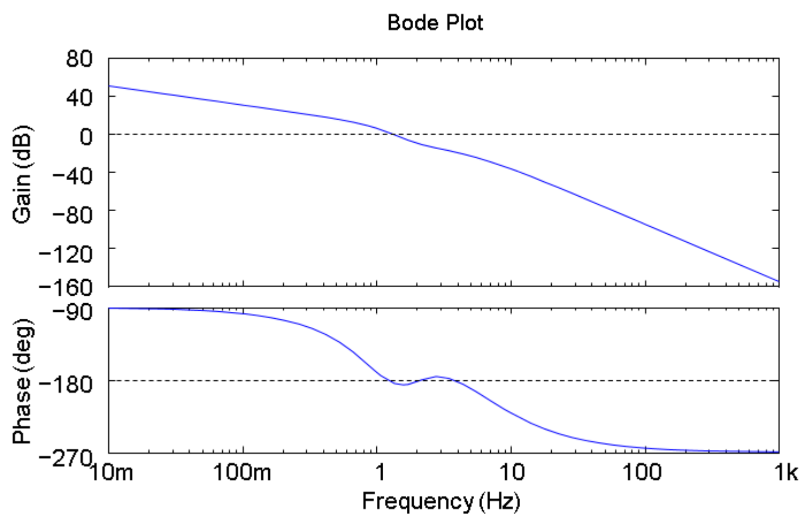
a. Simulate the Bode plots of the system in MATLAB. [4 marks]

b. Determine the stability of the system from results obtained in part (a). [2 marks]

c. Simulate the Nyquist diagram of the system in MATLAB. Comment on the difference using this method compared with Bode plots. [6 marks]

Answer

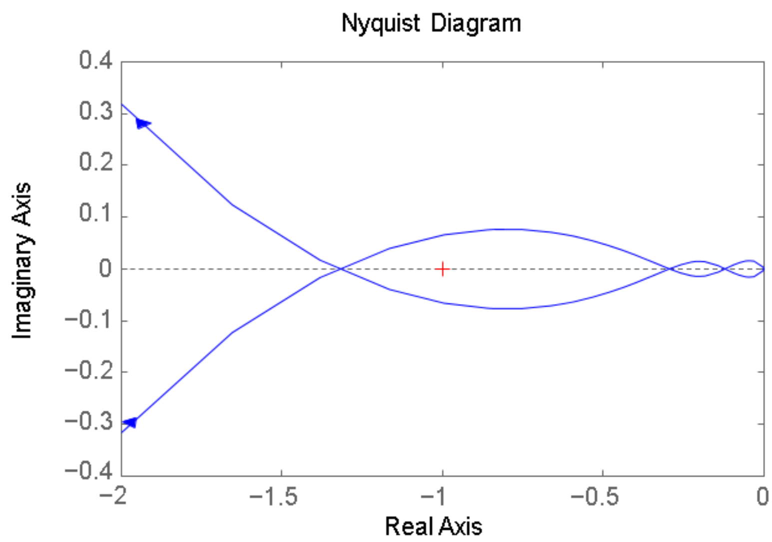
a. The Bode plot of the system given above is shown in the following figure.



b. It looks like that the Bode plots show a conditionally stable system. We need to determine the gain margin (GM) and phase margin (PM) of the system in the Bode plots to find out if it is stable or not.

c. Using Bode plots, for some gains, this will be stable, for others unstable. You cannot easily determine this from a Bode plot.

The Nyquist diagram of the system is as shown in the figure below. If we blow up the region near the test point (1, 0), we are able to determine that the system is actually unstable as there is encirclement around the test point.



4. Compensators in Nyquist Diagram

If they are implemented in the control systems, the compensators will influence the profiles of the graph of Nyquist diagram. We will evaluate their affects in this section.

4.1. Proportional Compensators

A proportional compensator simply scales the Nyquist contour. Consider a third-order system as shown in the transfer function equation below:

$$G(s) = \frac{50}{(s+1)(s+2)(s+3)}$$

The Nyquist diagram that shows the system given above is as shown in the figure below.

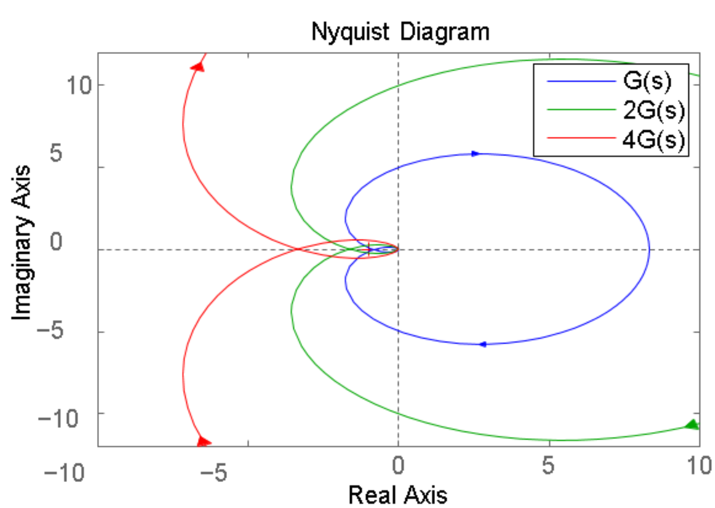


Figure 12: Effect of proportional compensator in Nyquist diagram

Even if the system does not become unstable, the phase margin will typically still be reduced by increasing the gain. This example below shows a second-order system, for which the phase margin is reduced when a proportional compensator with a gain of 4 is used.

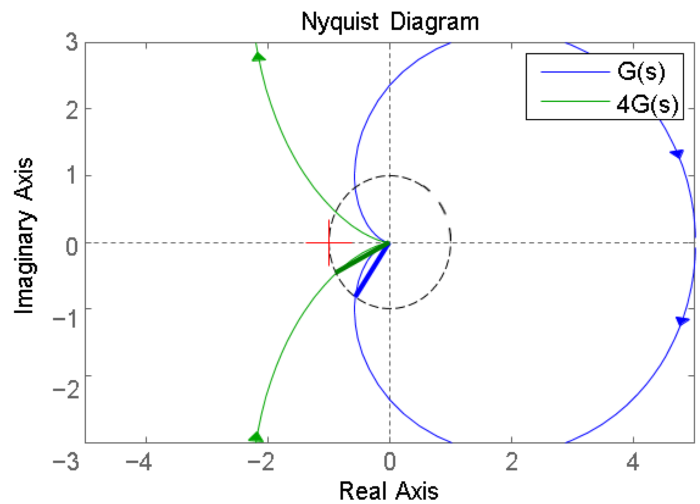


Figure 13: Effect of proportional compensator in Nyquist diagram

4.2. Lag Compensators

A lag compensated system starts with a higher gain than the uncompensated system, but approaches the Nyquist contour of the uncompensated system before it is near −1 + j0. Consider a lag compensator with transfer function:

$$C(s) = \frac{s + 0.4}{s + 0.1}$$

Where: $\omega_b = 0.1$ and $\alpha = 4$

The Nyquist diagram that shows the system given above is as shown in the figure below.

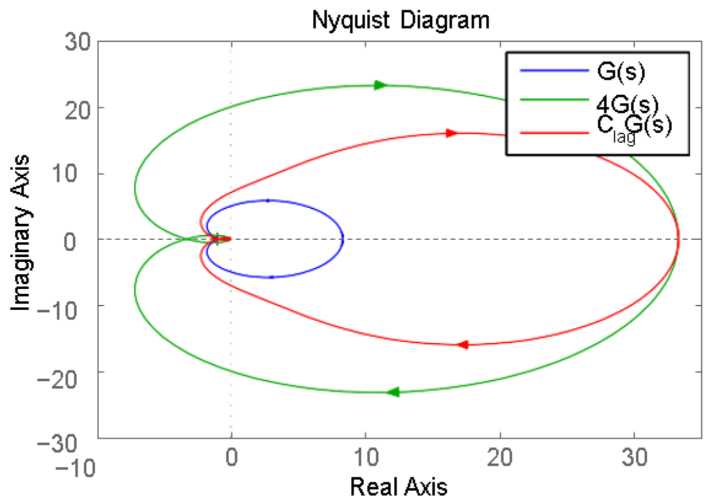


Figure 14: Effect of lag compensator in Nyquist diagram

4.3. Lag Compensators - Effect of $\boldsymbol{\omega_b}$

If ω_b is reduced, then the compensated system approaches the uncompensated more "quickly".

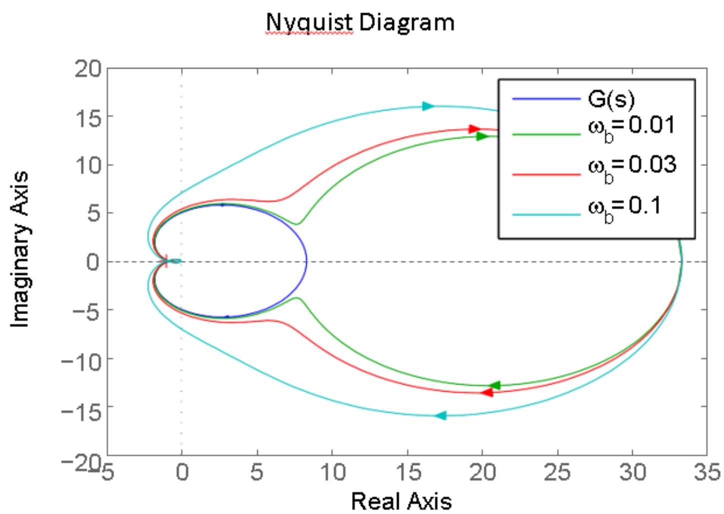


Figure 15: Effect of ω_b of lag Compensators in Nyquist diagram

4.4. PI Compensators - Effect of $\boldsymbol{\omega_b}$

A PI compensator changes the shape of the Nyquist diagram at low frequencies. Again, ω_b determines how quickly the system approaches the uncompensated system.

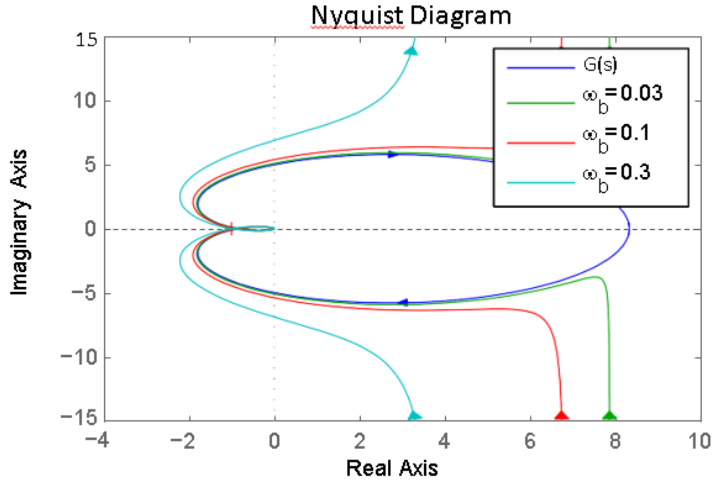


Figure 16: Effect of ω_b of PI Compensators in Nyquist diagram

4.5. Lead Compensators

The lead compensator "rotates" the Nyquist contour away from −1.

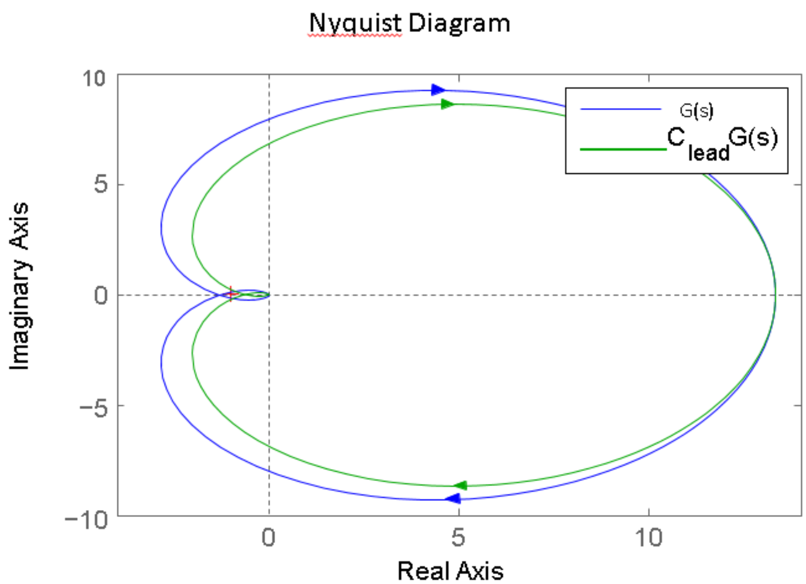
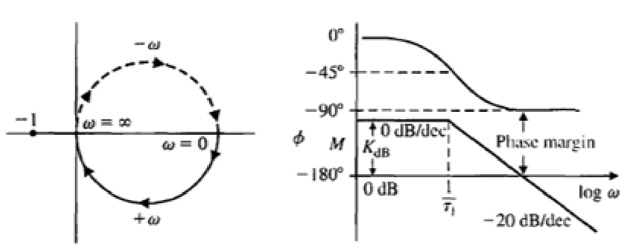
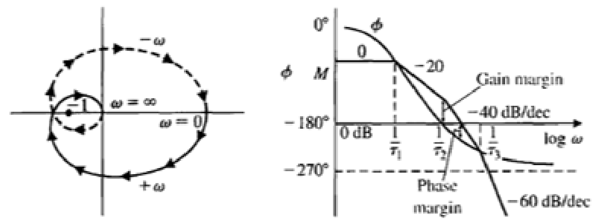


Figure 17: Effect of lead compensator in Nyquist diagram

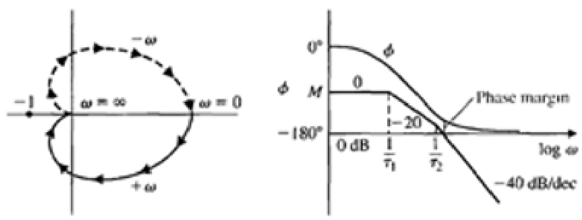
5. Other Examples of Nyquist Diagram

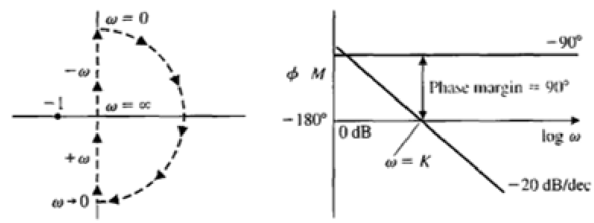
We will overview of other examples of Nyquist diagram in this section.



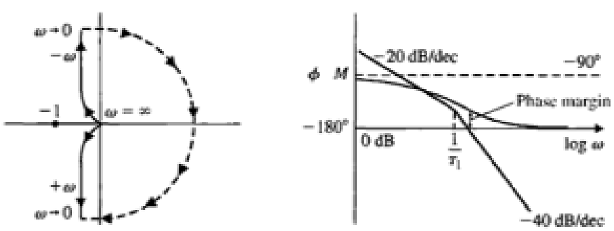


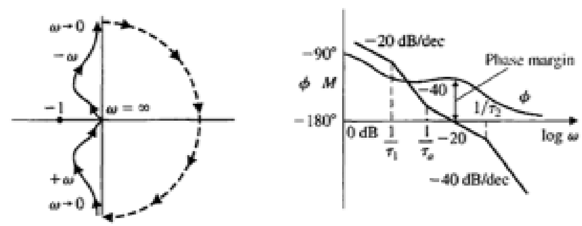
$$G1(s) = \frac{K}{s\tau_1 + 1} \qquad G3(s) = \frac{K}{(s\tau_1 + 1)(s\tau_2 + 1)(s\tau_3 + 1)}$$



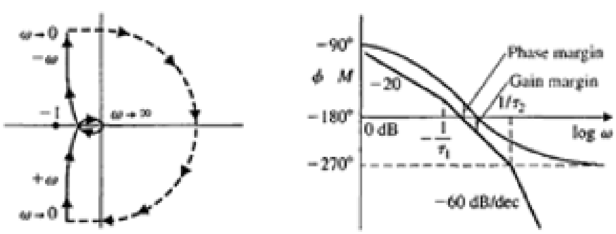


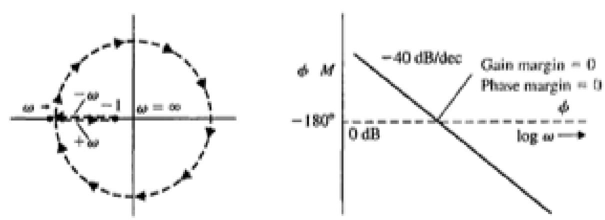
$$G2(s) = \frac{K}{(s\tau_1 + 1)(s\tau_2 + 1)} \qquad G4(s) = \frac{K}{s}$$



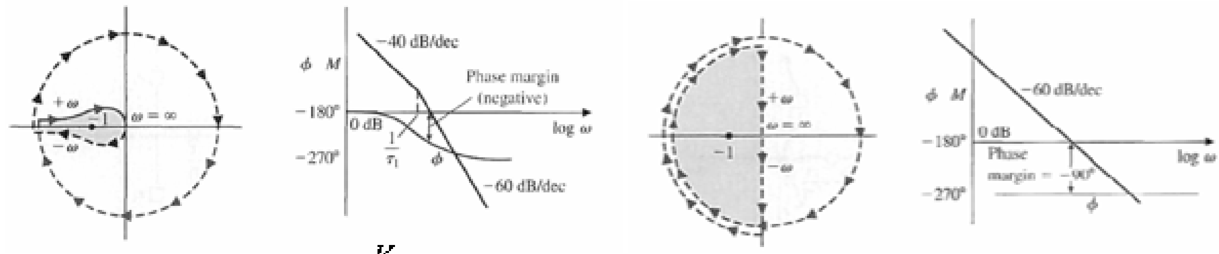


$$G5(s) = \frac{K}{s(s\tau_1 + 1)} \qquad G7(s) = \frac{K(s\tau_a + 1)}{(s\tau_1 + 1)(s\tau_2 + 1)}$$

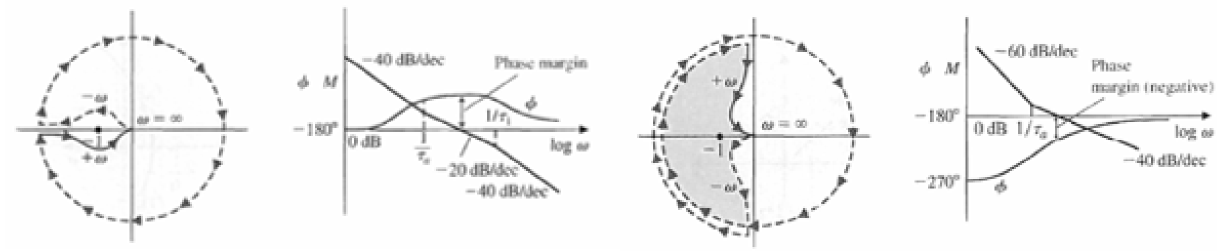




$$G6(s) = \frac{K}{s(s\tau_1 + 1)(s\tau_2 + 1)} \qquad G8(s) = \frac{K}{s^2}$$



$$G9(s) = \frac{K}{s^2(s\tau_1 + 1)} \qquad G11(s) = \frac{K}{s^3}$$



$$G10(s) = \frac{K(s\tau_a + 1)}{s^2(s\tau_1 + 1)} \quad (\tau_a > \tau_1) \qquad G12(s) = \frac{K(s\tau_a + 1)}{s^3}$$

Figure 18: Nyquist diagram of various control systems

6. Analysis with Nichols chart

Similar to the Nyquist diagram, Nichols chart is also used to analyse the stability of control systems. Its graph is based on the gain vs. phase (in degrees) of the system's frequency response. So, the stability analysis of the system can be determined in terms of these parameters.

To determine the stability of the system in the Nichols chart, the system is stable if the curve is passing below the -180° mark and it is not stable if the curve is passing above this mark.

The following example shows how to construct and to perform the stability analysis of the control systems using Nichols chart.

Example for Tutorial 6: Construction of Nichols Chart

For the following control system with the transfer function given below, construct the Nichols chart of the system and determine the stability of the system.

$$G(s) = \frac{1}{(s + 2)}$$

a. Determine the equations for determining gain and phase shift of the frequency response. [4 marks]

b. Determine the gain and phase shift of the system for ω = 0, 1, 2, 5, and 10 rad/s. [10 marks]

c. Sketch the Nichols chart from the results obtained in part (b). [4 marks]

d. Simulate the Nichols chart in MATLAB. [5 marks]

e. Evaluate the frequency response of the system based on the results in part (d). [3 marks]

Answer

a. To construct the Nichols chart of the system, we calculate the gain and phase (in degrees) of the frequency response of the system. Their values are calculated from the equations given below.

Gain:

$$|G(s)| = \frac{1}{\sqrt{(2)^2 + \omega^2}}$$

Phase shift:

$$\angle\theta = -\tan^{-1}\left(\frac{j\omega}{2}\right)$$

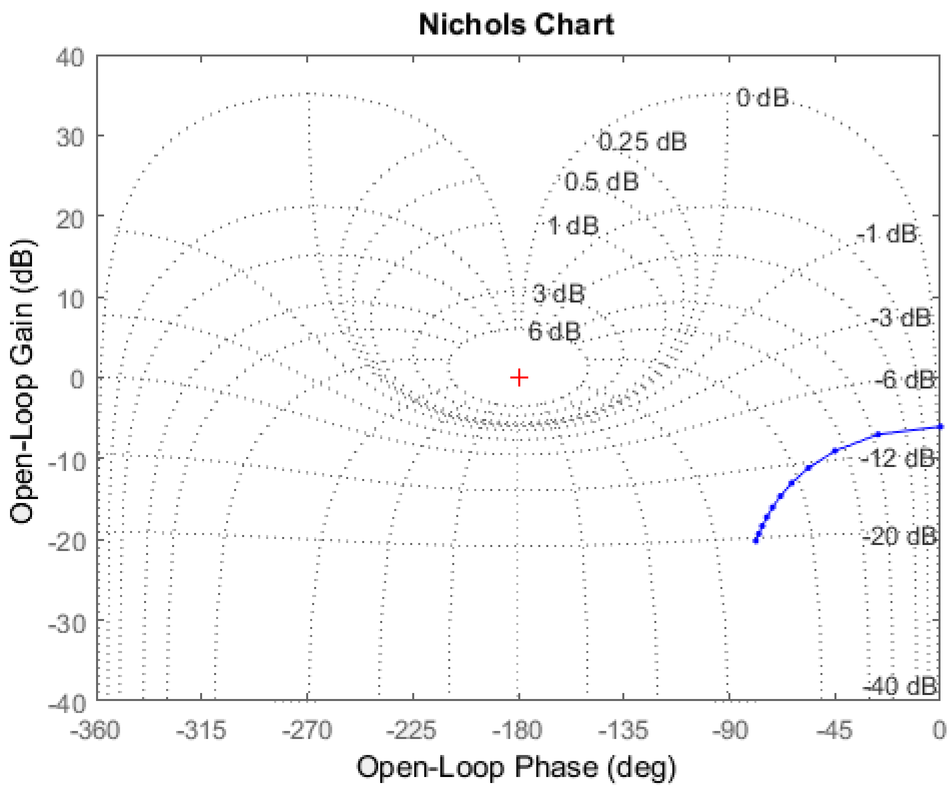
b. For given frequencies in the frequency response, the points used to plot the curve of the Nichols chart are calculated using the equations above.

Considering the frequency range of the points to be plotted are selected from some frequencies between 0 to 10 rad/s, the values of these points are tabulated in the table below.

Frequency (rad/s)	**Gain (dB)**	**Phase Shift (degree)**
0	$\frac{1}{\sqrt{(2)^2 - (0)^2}} = 0.5 = -6\text{ dB}$	$-\tan^{-1}\left(\frac{0}{2}\right) = 0°$
1	$\frac{1}{\sqrt{(2)^2 + (1)^2}} = 0.447 = -6.99\text{ dB}$	$-\tan^{-1}\left(\frac{1}{2}\right) = -26.56°$
2	$\frac{1}{\sqrt{(2)^2 + (2)^2}} = 0.353 = -9.04\text{ dB}$	$-\tan^{-1}\left(\frac{2}{2}\right) = -45°$
5	$\frac{1}{\sqrt{(2)^2 + (5)^2}} = 0.186 = -14.61\text{ dB}$	$-\tan^{-1}\left(\frac{5}{2}\right) = -68.2°$

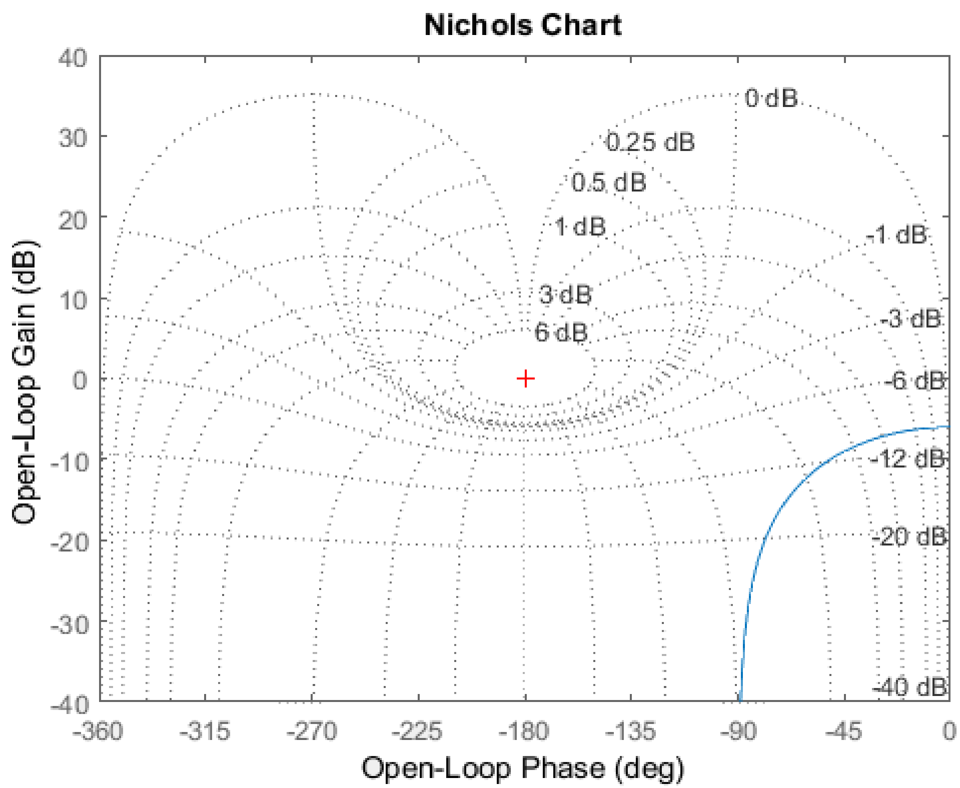
10	$\frac{1}{\sqrt{(2)^2+(10)^2}} = 0.098 = -19.82\text{ dB}$	$-\tan^{-1}\left(\frac{10}{2}\right) = -78.69°$

c. The Nichols chart as shown in the figure below is sketched based on the points given in the table.



d. The result of Nichols chart simulation in MATLAB is shown in the figure below. Notice the similarities between the result of the sketch with the simulation result.

In MATLAB, use `nichols()` function for plotting the Nichols chart and use of `ngrid()` function to activate the grids shown in the graph.



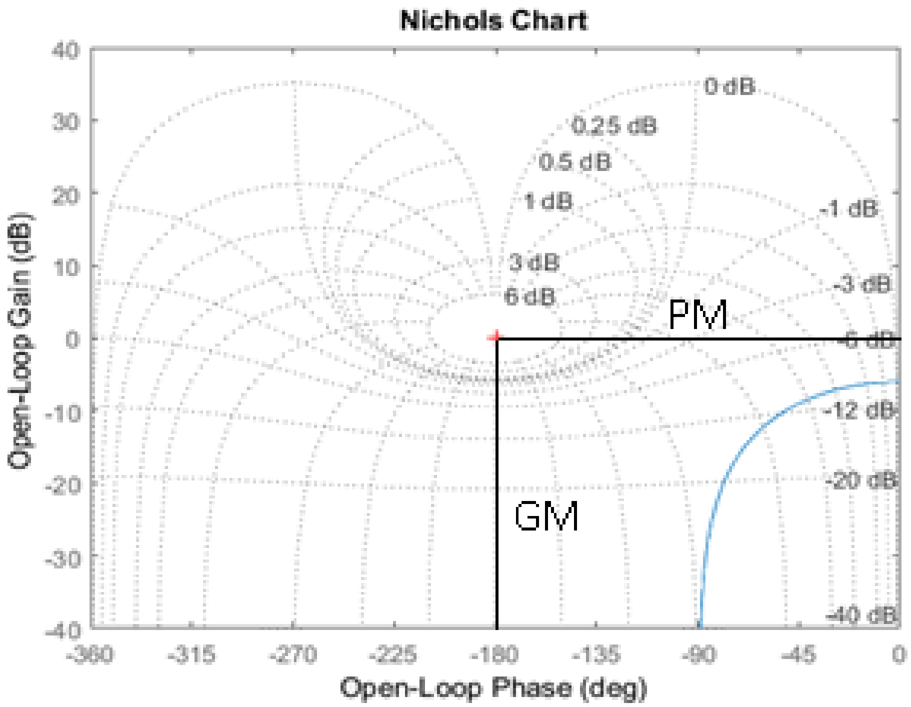
e. By referring to the plot of the Nichols chart of the system:

- At low frequency, the gain and phase of the system are around less than -6 dB and from 0 until less than -90° phase shift respectively.
- At high frequency, the gain and phase of the system are settling at negative small gain and -90° phase shift.
- The gain and phase shift are nowhere near the -180° mark, the gain and phase margins of the system are both positive values.

From the resulting graph of the Nichols chart, we could analyse the characteristics of the system.

- Gain margin: ∞ dB
- Phase margin: ∞ degree

Thus, the stability of the system is, since both of the margins are positive, then the given system is stable.



From the plot of the Nichols chart given in the construction of the Nichols chart, the system is found to be stable as its curve is never crossing the -180° mark and it always stays below this mark e.g. both gain and phase margins are positive values.

7. Design Procedures of Control Systems with Nyquist Diagram and Nichols Chart

In this section, the design procedures of control systems with Nyquist diagram and Nichols chart are outlined.

7.1. Design Procedures of Control Systems with Nyquist Diagram

Nyquist diagram is primarily used for analysing and designing the stability of the control systems.

Procedure for designing of control systems using Nyquist diagram.

i. Construct the Nyquist diagram for the given control system.

ii. Determine the stability of the system by evaluating the position of the curve relative to the critical test point of Nyquist diagram (-1, 0).

iii. Determine the gain and phase margins of the system.

iv. Increase or decrease the gain of the system to meet the required steady-state condition and transient response of the system.

v. If previous step is not successful, introduce compensator or controller to meet the required steady-state condition and transient response of the system.

vi. Readjust, if necessary, the gain of the system to meet the desired design specification.

7.2. Design Procedures of Control Systems with Nichols Chart

Like Nyquist diagram, Nichols chart could provide alternative for analysing and designing the stability of control systems.

Procedure for designing of control systems using Nichols chart:

i. Construct the Nichols chart for the given control system.

ii. Determine the stability of the system by evaluating the position of the curve relative to the critical test point of Nichols chart (-180°, 0).

iii. Determine the gain and phase margins of the system.

iv. Increase or decrease the gain of the system to meet the required steady-state condition and transient response of the system.

v. If previous step is not successful, introduce compensator or controller to meet the required steady-state condition and transient response of the system.

vi. Readjust, if necessary, the gain of the system to meet the desired design specification.