XMUT315 Control Systems Engineering

Note 7: Time Response Analysis

Topic

- First-order responses analysis (e.g. time constant, rise time, and settling time).
- Second-order responses analysis (e.g. damping ratio, rise time, settling time, time-to-peak, percentage overshoot, and steady-state error).
- Damping of the systems.
- Second-order system responses.
- Trends in second-order system responses.

1. Introduction to Time Response

It is the time response of a system to an input that sets the criteria for our control systems. Many quantitative criteria have been defined to characterise the time response of a system.

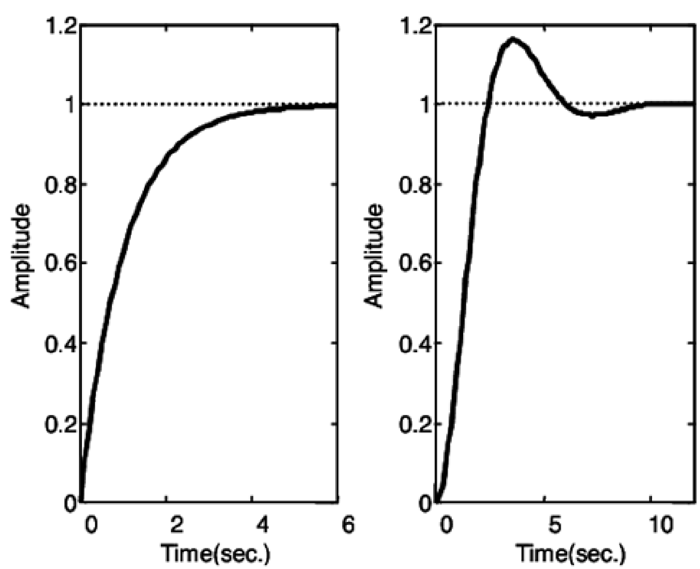


Figure 1: Time responses of systems (first-order system (left) and second-order system (right))

Time response of higher order system could be approximated from first and second order systems.

The term transient response is often used to describe the initial time response of the system that is occurring at the beginning of the response as opposed to the steady-state response of the system that typically is happening at the end of the response.

2. Time Response of First-Order System

The time constant and system gain of a first-order system are useful in its analysis, but other criteria describe the time response more accurately to an engineer.

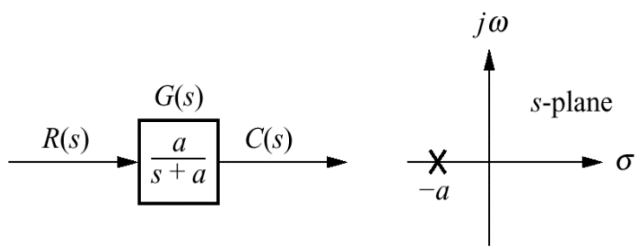


Figure 2: Block diagram and s-plane diagram of first-order system

A first-order system may be written as:

$$\frac{C(s)}{R(s)} = \frac{K}{sT+1} \qquad \text{or} \qquad \frac{C(s)}{R(s)} = \frac{a}{s+a}$$

Where: $K = 1$ and $T = 1/a$

Rearrange the equation given above:

$$C(s) = R(s)\left(\frac{a}{s+a}\right)$$

For a unit step e.g., $R(s) = 1/s$:

$$C(s) = R(s)\left(\frac{a}{s+a}\right) = \frac{1}{s}\left(\frac{a}{s+a}\right)$$

Apply partial fraction:

$$C(s) = \frac{1}{s} - \frac{1}{s+a}$$

By using Laplace table, we obtain the standard response (note: f = forced response and n = natural response).

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

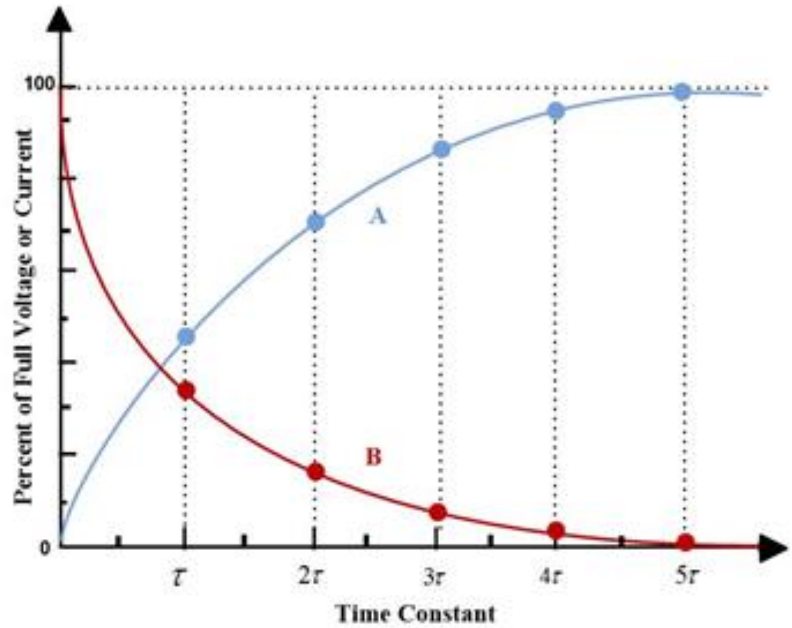


Figure 3: Exponential time responses of first-order system (increasing (A) and decreasing (B))

The time response for a first-order system depends on the gain and time constant of the system. Generally, the time response of a first-order system is exponential. Changing the gain or constant only changes the steady state value and time. Typical parameters are:

- Time constant (τ).
- Rise time (T_r).
- Settling time (T_s).

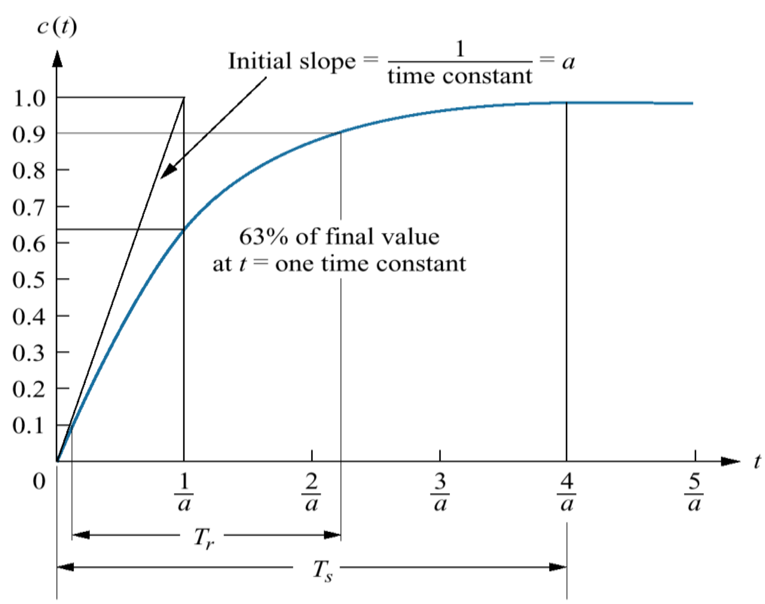


Figure 4: Time response of first-order system

2.1. Time Constant of First-Order System

For a first-order system response to a unit step, time constant (T) is defined as the time for the step response to rise to 63% of its final value.

$$\tau = 1/a$$

It is derived from the time for e^{-at} to decay to 37% of its final value.

$$e^{-at}|_{t=1/a} = e^{-1} = 0.37$$

For the given first-order system, the time domain equation of the system is:

$$c(t)|_{t=1/a} = 1 - e^{-at}|_{t=1/a}$$
$$= 1 - 0.37 = 0.63$$

2.2. Rise Time of First-Order System

For a first-order system response to a unit step, rise time (T_r) is defined as the time for the response to go from 0.1 to 0.9 of its final value:

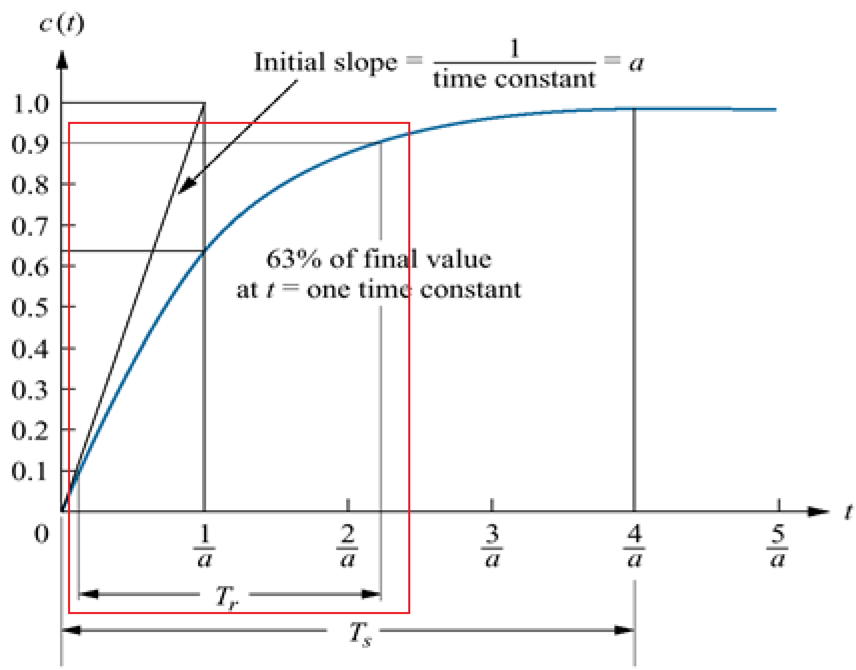


Figure 5: The rise time of first-order system

For $c(t) = 0.1$:

$$c(t) = \left(1 - e^{-aT_{(0.1)}}\right) = 0.1$$

Taking the natural log on both sides:

$$-aT_{(0.1)} = \ln(1 - 0.1)$$

Rearrange the equation above:

$$T_{(0.1)} = 0.11/a$$

For $c(t)$ = 0.9:

$$c(t) = \left(1 - e^{-aT_{(0.9)}}\right) = 0.9$$

Taking natural log on both sides:

$$-aT_{(0.9)} = \ln\ (1 - 0.9)$$

Rearrange the equation above:

$$T_{(0.9)} = 2.31/a$$

So, by subtracting the second equation with the first equation, this will yield the rise time (T_r) of first-order system that is:

$$T_r = T_{(0.9)} - T_{(0.1)}$$
$$= \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

2.3. Settling Time of First-Order System

For a first-order system response to a unit step, settling time (T_s) is calculated as 4 times the time constant (τ) of the system.

$$T_s = 4\tau = \frac{4}{a}$$

The settling time could be also determined as the time taken by the system to stay within 2 % of its final value (typically this 2 % is standard).

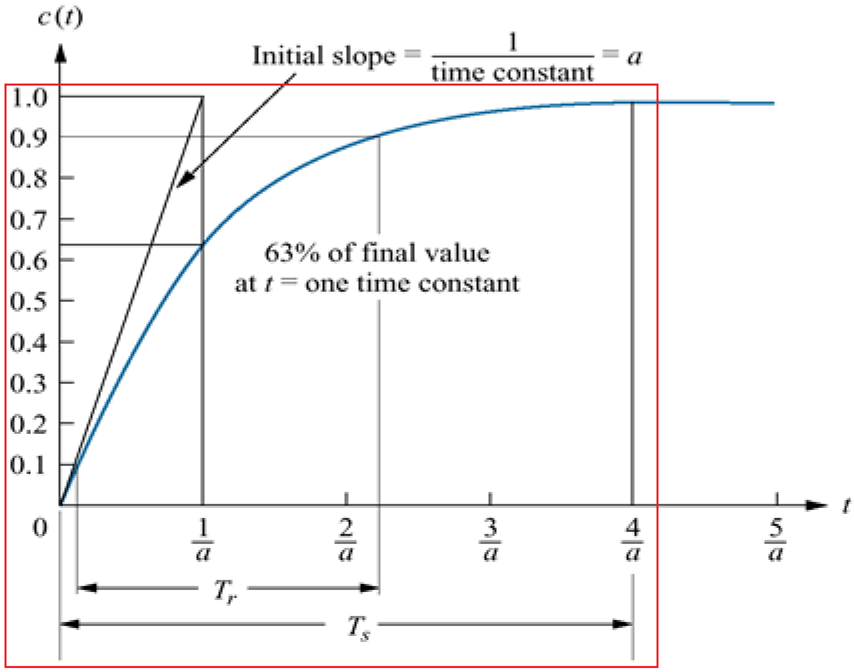


Figure 6: The settling time of first-order system

For a first-order system, since $\tau = 1/a$, the response of the system is calculated from:

$$c(t) = 1 - e^{at} = 1 - e^{-t/\tau}$$

Rearrange the equation above:

$$e^{-t/\tau} = 1 - c(t)$$

To calculate the 2% settling time e.g. 0.98 of final value:

$$e^{-T_s/\tau} = 1 - 0.98 = 0.02$$

Thus, the settling time (T_s) of the first order system is:

$$T_s = -\tau \ln 0.02 = 3.9\tau \approx 4\tau$$

Example for Tutorial 1: Time-Response Analysis of First-Order System

For a first order given as the transfer function given below, calculate the following time-domain parameters of the system.

$$\frac{C(s)}{R(s)} = \frac{2.5}{s+3}$$

a. The time constant (τ).

b. The rise time (T_r).
c. The settling time (T_s).

Answer

a. The time constant (τ) is calculated from:

$$\tau = \frac{1}{a} = \frac{1}{3} = 0.33 \text{ s}$$

b. The rise time (T_r) is calculated from:

$$T_r = \frac{2.2}{a} = \frac{2.2}{3} = 0.733 \text{ s}$$

c. The settling time (T_s) is calculated from:

$$T_s = 4\tau = 4(0.33) = 1.32 \text{ s}$$

3. Time Response of Second-Order System

The time response of a second order system depends on the characteristics of the system, notably the natural frequency (ω_n) and damping ratio (ζ) of the system.

So, it is important that we discuss the details of these two parameters of the second-order system before we start covering the other time domain parameters of the second-order system.

Consider a second-order system with the following transfer function equation:

$$\frac{C(s)}{R(s)} = \frac{k}{a's^2 + b's + c'}$$

To work out the natural frequency and damping characteristics, convert the transfer function equation to a monic polynomial form with unity in front of the leading coefficient (s^2 term) and k such that:

$$\frac{C(s)}{R(s)} = \frac{b}{s^2 + as + b}$$

Note that the constants a', b', and c' are not equivalent to a and b.

As seen above in the equation, given that there is the transfer function equation of the second-order system, we can determine its natural frequency and damping ratio.

3.1. Natural Frequency of Second-Order System

Natural frequency is when there is no damping in the system (a = 0 in this case).

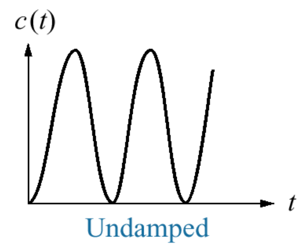


Figure 7: Transient response of undamped system

We obtain:

$$\frac{C(s)}{R(s)} = \frac{b}{s^2 + b}$$

With poles:

$$s_{1,2} = \pm j\sqrt{b}$$

We know that: $\omega = \sqrt{b} = \omega_n$ as a complex number. So, the frequency of oscillation is $+j\omega$, which is termed the natural frequency.

3.2. Exponential Decay Frequency

Exponential decay frequency is when the exponential function shapes up the sinusoidal oscillation function of the system response.

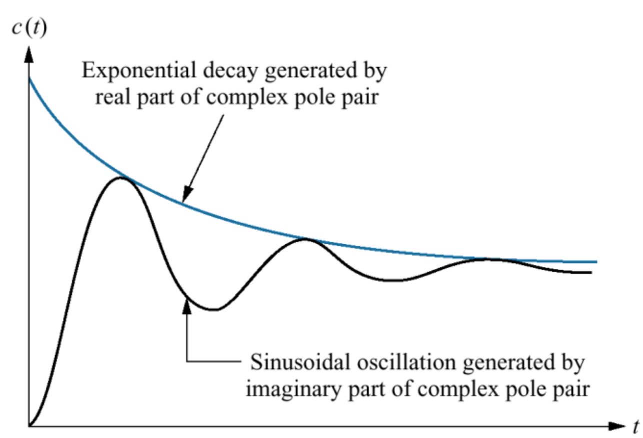


Figure 8: Exponential decay response of a system

Considering an underdamped system:

$$\frac{C(s)}{R(s)} = \frac{b}{s^2 + as + b}$$

The real part of $s = \sigma + j\omega$, is $\sigma = -a/2$, where $|\sigma|$ is termed the exponential decay frequency.

3.3. Damping Ratio

Damping ratio (ζ) is defined as measure describing how oscillations in a system decay after a disturbance. It is equated as the ratio of the exponential decay frequency with the natural frequency.

$$\zeta = \frac{|\sigma|}{\omega_n} = \frac{\left(\frac{a}{2}\right)}{\omega_n}$$

Where:

- $|\sigma|$ is the exponential decay frequency.
- ω_n is the natural frequency.

3.4. Natural Frequency and Damping Ratio

Consider a second order system with the following transfer function equation:

$$\frac{C(s)}{R(s)} = k\left(\frac{c}{as^2 + bs + c}\right)$$

The equation above can be written as a standardized equation for second order system in terms of damping ratio and natural frequency:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Consider the roots of the characteristic equation:

$$s_{1,2} = \zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Depending on the root i.e. technically the natural frequency of the system (ω_n), or the damping ratio (ζ), you will end up with various types of time response of the second order system.

The following figure shows damping ratios (ζ), roots (ω_n), the location of poles in the s-plane, and the step response of various second-order systems based on their damping ratios.

Notice that the time response is based on the step input that is very commonly used for analysing and testing of the second order system. The other types of input are ramp and parabolic inputs.

Damping (ζ)	**Roots (ω_n)**	**Poles in the S-plane**	**Step Response of System**

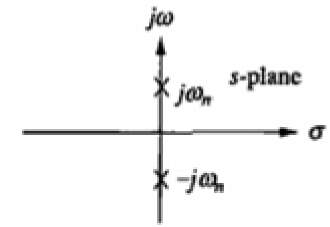
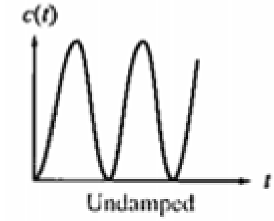
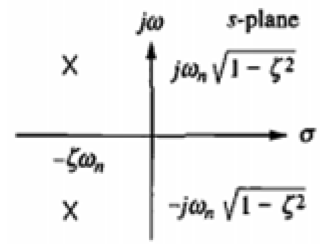
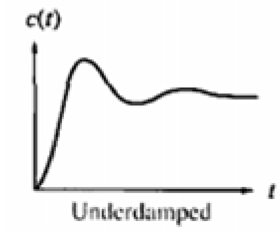
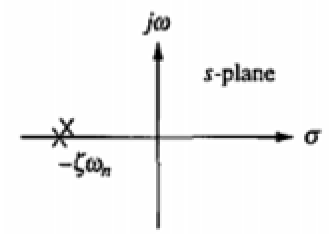
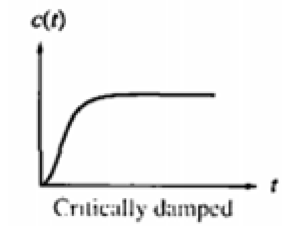
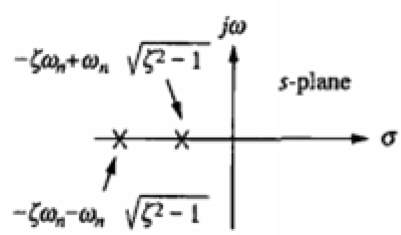
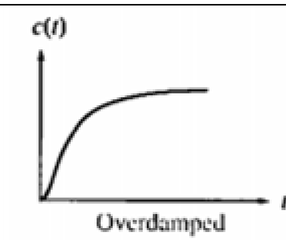
$\zeta = 0$	$\pm j\omega_n$		
$0 < \zeta < 1$	$-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$		
$\zeta = 1$	$\zeta\omega_n$		
$\zeta > 1$	$-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$		

Table 1: Damping ratios, their root locations in s-plane, and their transient responses

3.4. Determining Natural Frequency and Damping Ratio

Consider a second order system with the transfer function equation given below:

$$\frac{C(s)}{R(s)} = \frac{k}{xs^2 + ys + z} \qquad (Eq.1)$$

Rearranging the equation so this system is with unity in front of the s^2 term and k such that:

$$\frac{C(s)}{R(s)} = \frac{b}{s^2 + as + b} \qquad (Eq.2)$$

Equating the transfer function equation of the system with the standardized equation for second order system.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n\zeta s + \omega_n^2} \qquad (Eq.3)$$

After equating equation (3) with equation (1), the equations for determining the natural frequency and damping ratio of the second-order system are determined as follow.

First, derive the natural frequency is:

$$\omega_n^2 = b \qquad (Eq.4)$$

Thus, the natural frequency is:

$$\omega_n = \sqrt{b} \qquad (Eq.5)$$

The damping ratio of the second order system is calculated from:

$$2\omega_n\zeta = a \qquad (Eq.6)$$

Substituting equation (5) into equation (6), knowing that $\omega_n = \sqrt{b}$, the damping ratio (ζ) of the second order system is:

$$\zeta = \frac{a}{2\sqrt{b}}$$

Example for Tutorial 2: Time-Response Analysis of Second-Order System

For the following second-order system, determine the following time-domain parameters of the system:

$$G(s) = \frac{4}{3s^2 + 6s + 9}$$

a. Poles and zeros of the system.

b. Natural frequency (ω_n).

c. Damping ratio (ζ).

Answer

a. Convert the transfer function equation of the system into a monic polynomial first.

$$G(s) = \frac{4}{3s^2 + 15s + 9} = \frac{4}{(3)(3)}\left(\frac{3}{s^2 + 5s + 3}\right)$$

Notice that $4/[(3)(3)] = 4/9$ term is becoming the gain of the system.
To determine the poles and zeros of the system, we use the standard equation for determining the roots of the second order equation.

$$s_{1,2} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Poles and zeros of the system are determined from:

$$s_{1,2} = -\frac{(5)}{2(1)} \pm \frac{\sqrt{(5)^2 - 4(1)(3)}}{2(1)} = -2.5 \pm \frac{\sqrt{13}}{2}$$

The poles and zeros of the system are $-2.5 + \sqrt{13}/2$ and $-2.5 - \sqrt{13}/2$.

b. To calculate the natural frequency of the system, we use the standardised equation for the second order system.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{3}{s^2 + 5s + 3}$$

Natural frequency (ω_n) of the system is calculated from:

$$\omega_n^2 = 3$$

Thus, the natural frequency of the system is $\sqrt{3}$.

c. Knowing the natural frequency of the system (ω_n) from part (b), the damping ratio (ζ) of the system is calculated from:

$$2\zeta\omega_n = 5$$

Thus

$$\zeta = \frac{5}{2\omega_n} = \frac{5}{2\sqrt{3}} = \frac{5}{6}\sqrt{3}$$

As a result, the damping ratio of the system is $(5/6)\sqrt{3}$.

4. Time Response of Second-Order System

The time response of the overdamped second-order system is very similar to the time response of first-order system.

But, for the underdamped second-order system, its time response is very different from overdamped system, and hence from first-order system as well.

As a result, most of the parameters of the second-order system are derived following the characteristics and behaviour of underdamped system.

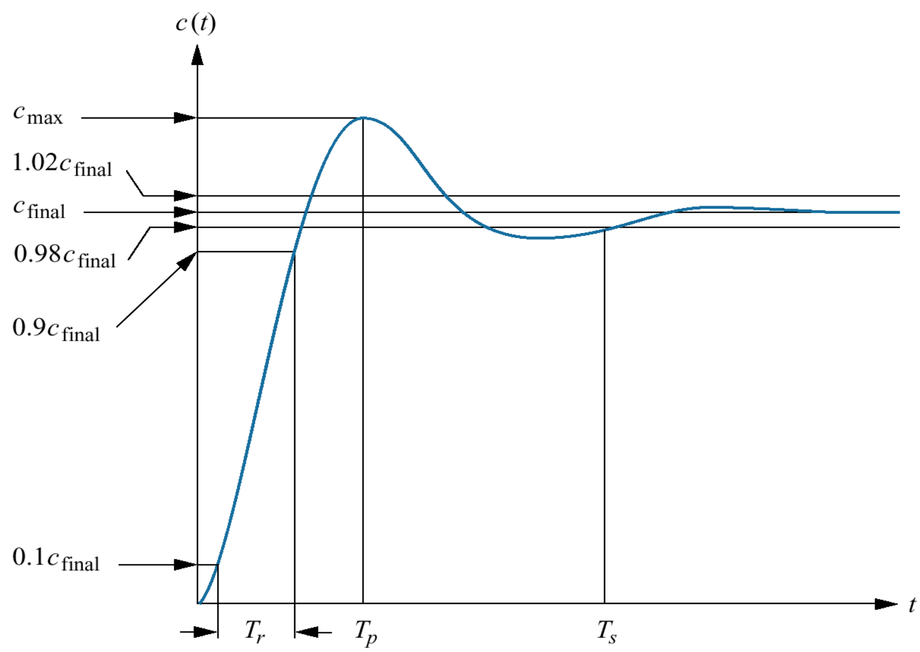


Figure 9: Time response of second-order system

We cannot use equations for determining the characteristics and behavior of the first-order system as these are not accurate for second-order system. The following sections will derive these parameters for the second-order systems.

Consider the parameters of time response that are common for the first-order systems and second-order systems such as:

- Rise time (T_r).
- Settling time (T_s).

But we also have parameters that are specific for second-order system:

- Time-to-peak (T_p).
- Percentage overshoot ($\%OS$).
- Steady-state error ($e(\infty)$).

4.1. Rise Time of Second-Order System

The rise time (T_r) is defined as the time for the response to go from 0.1 to 0.9 of its final value. We could use the rise time of the first order response – but this is not very accurate. To simplify the mathematics required, we consider the rise time of the second-order system as the time response from zero to its final value.

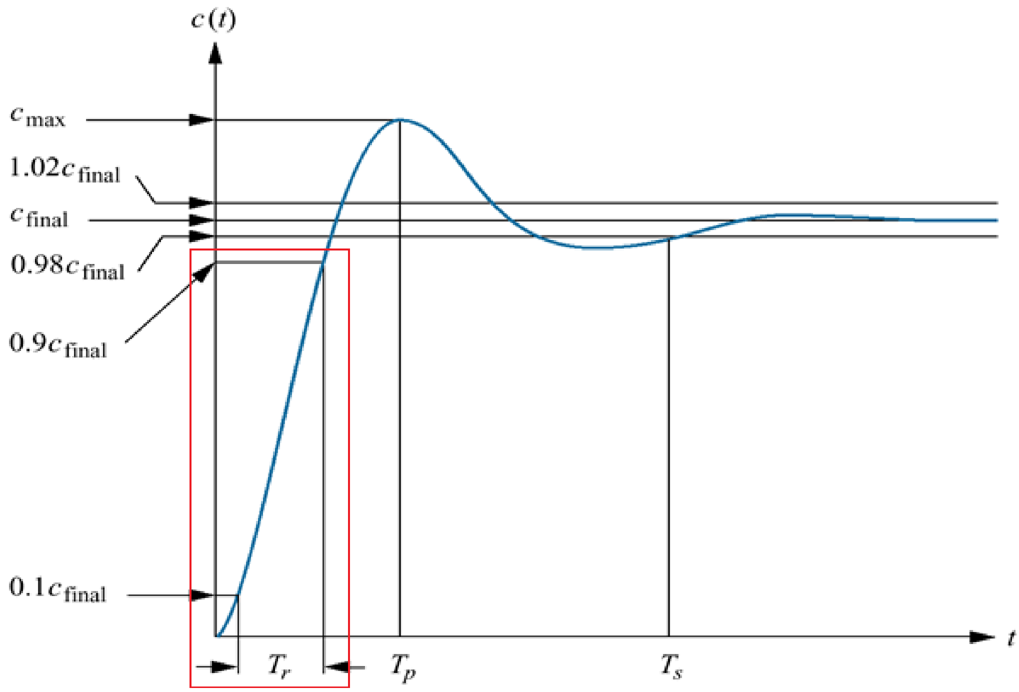


Figure 10: The rise time of second-order system

For an underdamped second-order system, its time response:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_d t + \phi)$$

Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

The magnitude of the output signal at rise time is approximated to 1 for easy calculation e.g., $c(t) = 1$.

$$1 - \left(\frac{e^{-\zeta\omega_n T_r}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_d T_r + \phi) = 1$$

Thus, equating both sides:

$$\left(\frac{e^{-\zeta\omega_n T_r}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_d T_r + \phi) = 0$$

Knowing that $\sin^{-1}(0) = n\pi$, then the above given equation becomes:

$$\omega_d T_r + \phi = n\pi$$

The rise time is calculated from:

$$T_r = \frac{\pi - \phi}{\omega_d}$$

Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$ $\quad$ (i.e. convert to rad/s)

As $\omega_d = \omega_n\sqrt{1-\zeta^2}$, the formulae for calculating rise time (T_r) of the second order system is typically a function of damping ratio (ζ) and natural frequency (ω_n):

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$$

Where: $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

We have also an alternative formula for calculating rise time (T_r) of the second-order system. This formula is derived from approximating the experiment results of a second-order system with curve fitting techniques.

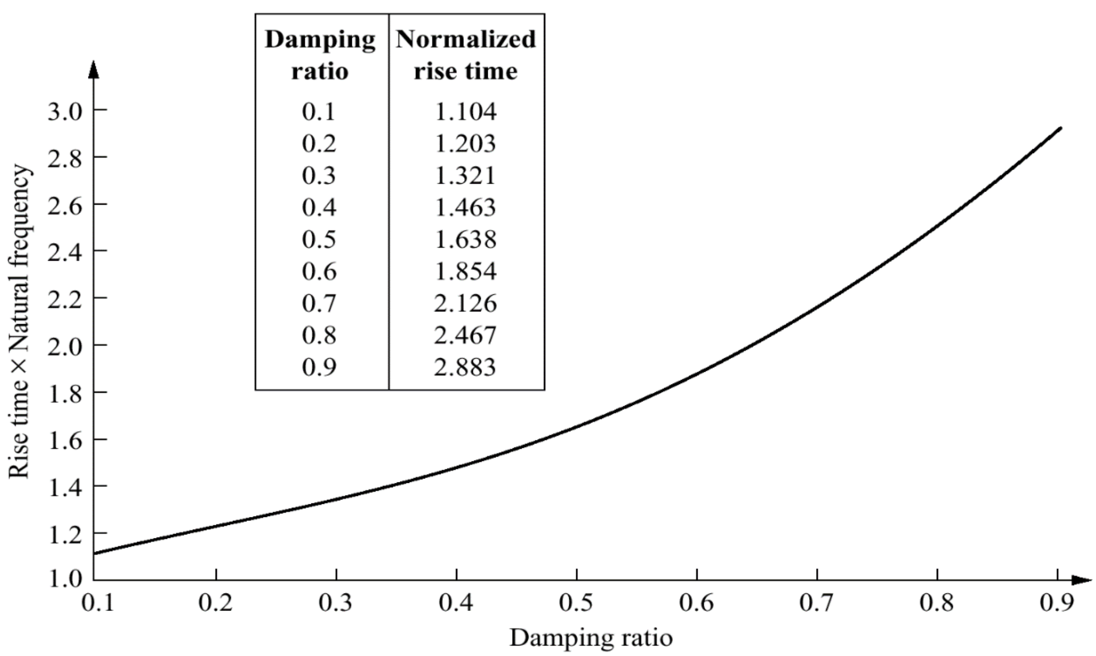
$$T_r = \frac{(1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)}{\omega_n}$$

If analytical method using equation and math do not appeal you, we can also use the graphical method to determine the rise time of second-order system.

To find damping ratio, we can use also the normalized rise time vs. damping ratio for a second-order underdamped response (e.g. equation of the curve given in the figure below):

$$t_r\omega_0 = 2.230\zeta^2 - 0.078\zeta + 1.12$$

This approach is quicker to do and without implementing mathematical processes to analyse.



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Figure 11: Graph of normalised rise time vs. damping ratio

4.2. Settling Time of Second-Order System

The settling time (T_s) is defined as the time for the response to reach and stay within its final steady-state value. There are several settling time standards that exist in control system engineering e.g. 0.1%, 0.5%, 1%, 2%, 5%, etc.

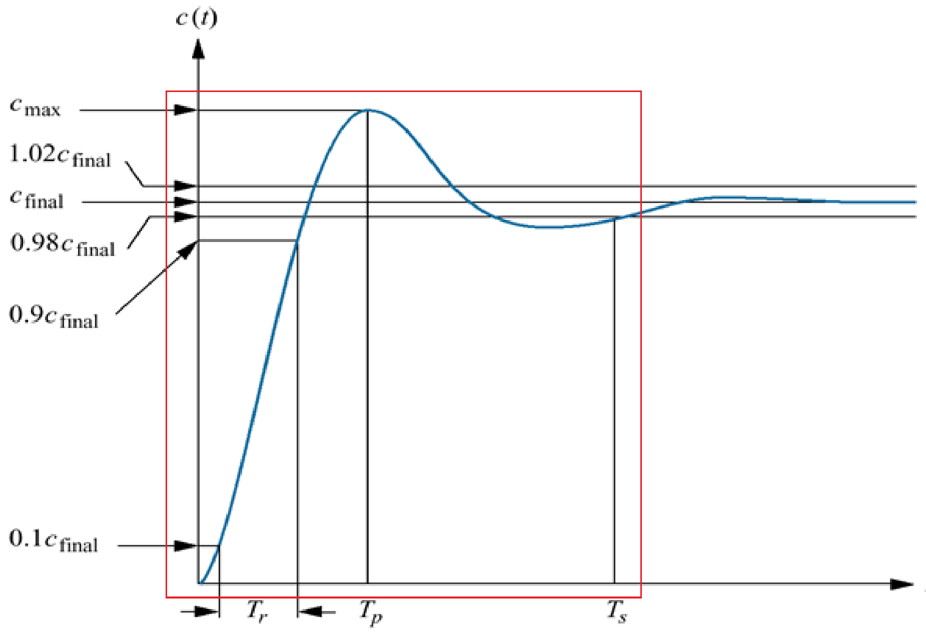


Figure 12: The settling time of second-order system

For an underdamped second-order system, its time response:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_d t + \phi)$$

Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

Thus, to find the settling time, $\sin(\omega_d t + \phi) = 1$:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n T_s}}{\sqrt{1-\zeta^2}}\right)$$

Rearrange the equation above:

$$\left(\frac{e^{-\zeta\omega_n T_s}}{\sqrt{1-\zeta^2}}\right) = 1 - c(t)$$

For 2% settling time standard, therefore $1 - c(t) = 0.02$:

$$\left(\frac{e^{-\zeta\omega_n T_s}}{\sqrt{1-\zeta^2}}\right) = 0.02$$

For underdamped second order system, ζ lies between 0 and 1. As a result, neglect the denominator for easy calculation.

$$e^{-\zeta\omega_n T_s} = 0.02$$

Taking natural log on both sides:

$$-\zeta\omega_n T_s = \ln 0.02$$

So, the settling time of the system for 2% standard is:

$$T_s = \frac{3.9}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n}$$

For 5% settling time standard, therefore $1 - c(t) = 0.05$:

$$\left(\frac{e^{-\zeta\omega_n T_s}}{\sqrt{1-\zeta^2}}\right) = 0.05$$

For underdamped second-order system, thus:

$$e^{-\zeta\omega_n T_s} = 0.05$$

Taking natural log on both sides:

$$-\zeta\omega_n T_s = \ln 0.05$$

So, the settling time for 5% standard is:

$$T_s = \frac{2.9957}{\zeta\omega_n} \approx \frac{3}{\zeta\omega_n}$$

4.3. Time-To-Peak of Second-Order System

The time-to-peak (T_p), it is the time required to reach the first peak or maximum peak. For an underdamped second-order system, its time response:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_d t + \phi)$$

Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

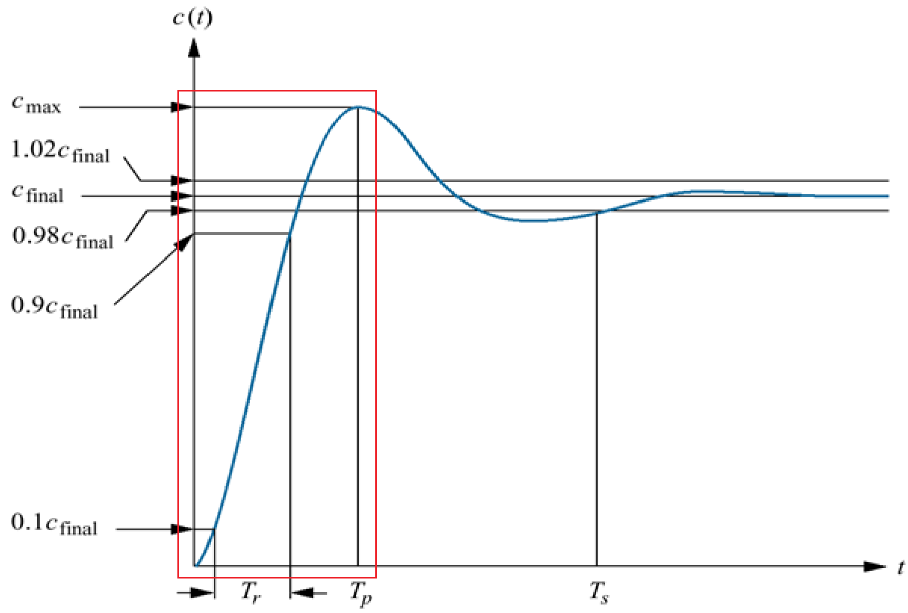


Figure 13: The time-to-peak of second-order system

As per definition at the peak time, the response curve reaches to its maximum value. Hence at that point,

$$\frac{dc(t)}{dt} = 0$$

Now, substitute $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and perform partial differentiation of the equation above (i.e. $uv' + u'v$):

$$\frac{dc(t)}{dt} = \left(\frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\right)\omega_n\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}T_p + \phi\right) + \left[\frac{(-\zeta\omega_n)e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\right]\sin\left(\omega_n\sqrt{1-\zeta^2}T_p + \phi\right)$$

Assign the $dc(t)/dt$ to zero, thus:

$$\left(\frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\right)\omega_n\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}T_p + \phi\right) + \left[\frac{(-\zeta\omega_n)e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\right]\sin\left(\omega_n\sqrt{1-\zeta^2}T_p + \phi\right) = 0$$

Rearranging and equating both sides:

$$\tan\left(\omega_n\sqrt{1-\zeta^2}T_p + \phi\right) = \frac{\sqrt{1-\zeta^2}}{\zeta} = \tan\phi$$

The equation above becomes:

$$\left(\omega_n\sqrt{1-\zeta^2}\right)T_p = n\pi$$

The time-to-peak of the second-order system is:

$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Where: n = n-th peak (n = 1, first peak; n = 2, second peak, …,n = n-th peak)

Note: the maximum overshoot of the given second–order system occurs at n = 1.

4.4. Percentage Overshoot of Second-Order System

The percentage overshoot ($\%OS$), is the amount that the waveform overshoots the steady-state or final value compared with value at the peak time. It is typically expressed as a percentage of the steady-state value.

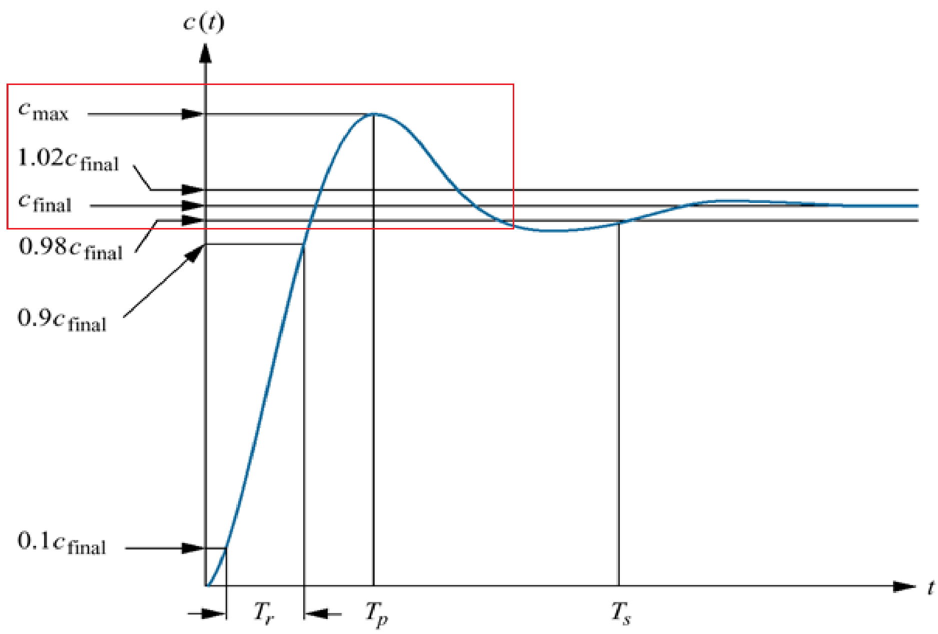


Figure 14: The percentage overshoot of second-order system

For an underdamped second-order system, its time response:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_d t + \phi)$$

Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

To determine the percentage overshoot, substitute $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

$$c(t)_{max} = 1 - \left(\frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_n\sqrt{1-\zeta^2}T_p + \phi)$$

Put the expression of peak time in the expression of output response $c(t)$.

$$c(t)_{max} = 1 - \left[\frac{e^{-\zeta\omega_n\left(\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\right)}}{\sqrt{1-\zeta^2}}\right]\sin(\omega_n\sqrt{1-\zeta^2}\left(\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\right) + \phi)$$

Since $\sin(\pi + \phi) = -\sin\phi$, thus:

$$c(t)_{max} = 1 - \left(\frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\right)\sin(\pi + \phi) = 1 - \left(\frac{e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}}{\sqrt{1-\zeta^2}}\right)(-\sin\phi)$$

We know that $\sin\phi = \sqrt{1-\zeta^2}$.

$$1 + \left(\frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\right)\sin\phi = 1 + \left(\frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\right)\sqrt{1-\zeta^2} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Thus, maximum overshoot is:

$$M_p = c(t)_{max} - 1 = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} - 1 = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

The percentage overshoot of the second-order system is:

$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

4.5. Percentage Overshoot and Damping Ratio

By rearranging the equation for percentage overshoot, we could find the damping ratio of the system from the percentage overshoot.

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100\%$$

Equation both sides with natural log.

$$-\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = \ln(\%OS/100)$$

Rearrange the equation.

$$\zeta\pi = -\sqrt{1-\zeta^2}\ln(\%OS/100)$$

Squaring both sides.

$$(\zeta\pi)^2 = 1 - \zeta^2\,[\ln(\%OS/100)]^2$$

Rearrange the equation.

$$\zeta^2 = \frac{[\ln(\%OS/100)]^2}{(\pi)^2 + [\ln(\%OS/100)]^2}$$

The relationship between percentage overshoot ($\%OS$) and damping ratio (ζ).

$$\zeta = -\frac{-\ln(\%O\,S/100)}{\sqrt{\pi^2 + [\ln(\%O\,S/100)]^2}}$$

Selection of the damping ratio is a tradeoff between maximum percentage overshoot ($\%OS$) and time where the peak overshoot occurs (time-to-peak). Smaller damping ratio decreases time-to-peak (desirable), but it increases $\%OS$ (undesirable).

4.6. Steady-State Error of Second-Order System

The steady-state error ($e(\infty)$), is the difference between the input ($r(t)$) and output ($c(t)$) for a prescribed test input at steady-state period ($t \to \infty$).

$$e(\infty) = \lim_{t\to\infty}[r(t) - c(t)]$$

We will look more closely the steady-state response and steady-state error of the system in the subsequent topic in the course.

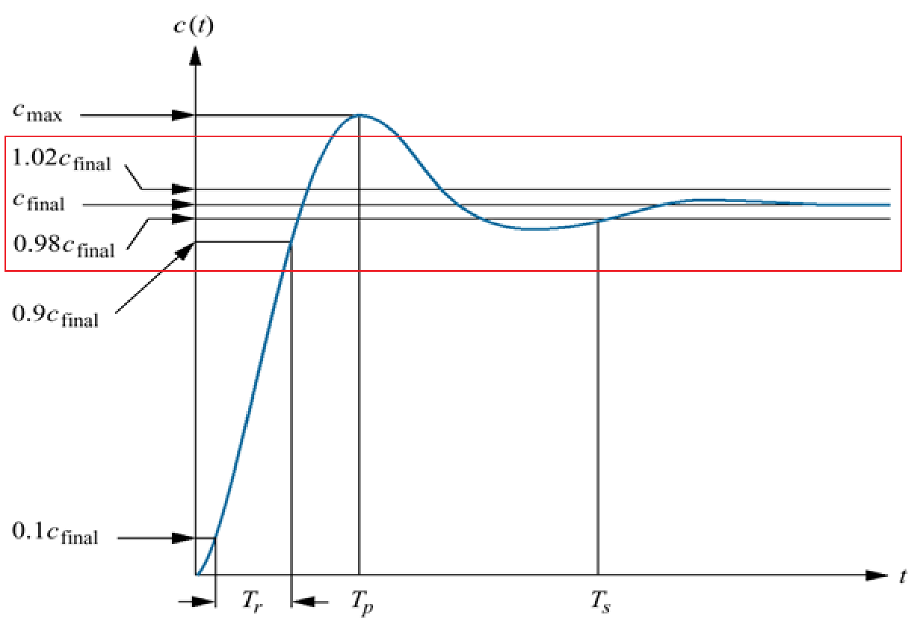


Figure 15: The steady-state error of second-order system

Example for Tutorial 3: Further Time-Response Analysis of Second-Order System

For the following second-order system, determine the following time-domain parameters of the system:

$$G(s) = \frac{81}{s^2 + 15s + 81}$$

a. Natural frequency (ω_n). [2 marks]

b. Damping ratio (ζ). [2 marks]

c. Rise time (T_r). [2 marks]

d. Settling time (T_s). [2 marks]

e. Time-to-peak (T_p). [2 marks]

f. Percentage overshoot ($\%OS$). [2 marks]

g. Perform transient response simulation in MATLAB and determine natural frequency (ω_n), damping ratio (ζ), rise time (T_r), settling time (T_s), time-to-peak (T_p), and percentage overshoot ($\%OS$) of the given system. Simulate the transient response of the system and comment on the result. [12 marks]

Answer

a. Equating the transfer function equation with the standardised equation for second order system, the natural frequency (ω_n) of the given system is calculated from:

$$G(s) = \frac{81}{s^2 + 15s + 81} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Thus, the natural frequency of the system is as:

$$\omega_n = \sqrt{81} = 9 \text{ rad/s}$$

b. Damping ratio (ζ) is calculated from:

$$\zeta = \frac{15}{2\omega_n} = \frac{15}{2(9)} = 0.833$$

c. Rise time (T_r) is calculated from:

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi - \pi(0.664)}{9\sqrt{1-(0.833)^2}} = 0.212 \text{ s}$$

$$\text{Where: } \phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \tan^{-1}\left(\frac{\sqrt{1-(0.833)^2}}{0.833}\right) = 0.664$$

Using alternative equation for rise time, it is:

$$T_r = \frac{(1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)}{\omega_n}$$
$$= \frac{(1.76(0.833)^3 - 0.417(0.833)^2 + 1.039(0.833) + 1)}{9}$$
$$= 0.288 \text{ s}$$

d. If the settling time standard is 2%, the settling time (T_s) is calculated from:

$$T_s = \frac{3.9}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n} = \frac{4}{(0.833)(9)} = 0.533 \text{ s}$$

e. For the first (max) peak (n = 1), the time-to-peak (T_p) is calculated from:

$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{9\sqrt{1-(0.833)^2}} = 0.63 \text{ s}$$

Where: n = n-th peak

f. Percentage overshoot ($\%OS$) is calculated from:

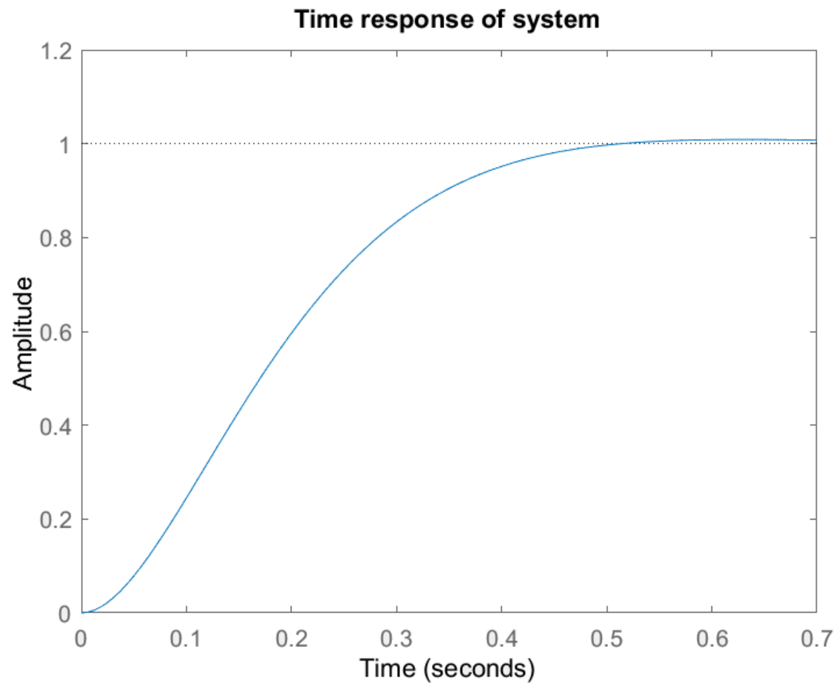
$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\% = e^{-\frac{(0.833)\pi}{\sqrt{1-(0.833)^2}}} \times 100\% = 0.88\%$$

g. The results of the simulation in MATLAB are listed below.

Workspace

Name	Value
den	[1,15,81]
num	81
omegan	9
percent	0.8773
T	1x1 tf
Tp	0.6315
Tr	0.2883
Ts	0.5333
zeta	0.8333

The transient simulation of the system is given in the figure below.



It seems that from the plots, the rise time and settling time found to be approximately 0.3 and 0.5 respectively. The transient response of the system is underdamped and it settles down to amplitude of 1 in the end.

5. Step Response of Second-Order System

For given second-order systems, there are various time response characteristics of these systems depending on their damping ratio values.

Step function is typically used for analysing and testing the response of the system.

If the second order systems are given a step input, their step responses typically are:

- $\zeta = 0 \rightarrow$ Undamped response.
- $0 < \zeta < 1$ (small ζ) $\rightarrow$ Underdamped response.
- $\zeta = 1 \rightarrow$ Critically damped.
- $\zeta > 1$ (large ζ) $\rightarrow$ Overdamped response.

The following figure outlines graphs of step responses of underdamped second-order system with various degrees of damping ratio i.e. from 0.1 to 0.8.

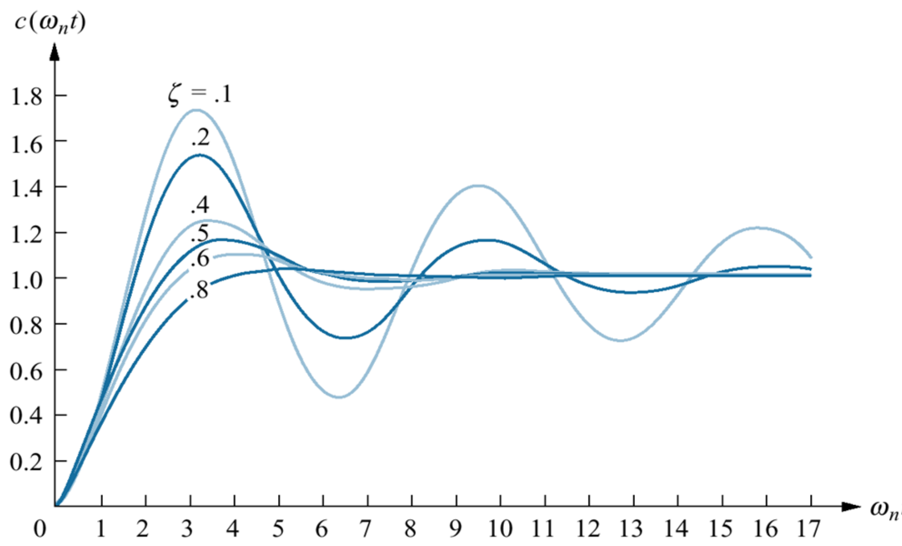


Figure 16: Step responses of systems with various damping ratios

5.1. Examples of Step Responses of Second Order System

There are several types of response of second-order systems. The response of the second order system depends on the characteristics of the system and the type of input applied to the system.

For a step input, the responses of the second order system are overdamped, critically damped, underdamped, and undamped.

5.1.1. Undamped Step Response of Second-Order System

Undamped second-order step responses are typically generated by complex poles that lie on the y-axis in the s-plane.

$$\frac{C(s)}{R(s)} = \frac{k}{(s+jb)(s-jb)}$$

For a step input ($R(s) = 1/s$), after implementing partial fraction expansion, the transfer function equation is:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+jb)} + \frac{K_3}{(s-jb)}$$

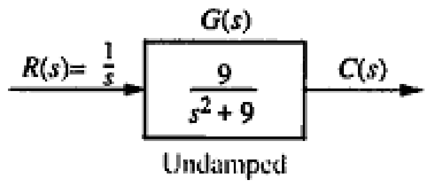
Taking the inverse Laplace transform, the pole at origin becomes a constant and a pair of complex roots become co-sinusoidal function. The time domain equation of the system is:

$$c(t) = K_1 - \cos bt$$

As a result, the step response of the system is a constant amplitude sinusoid.

Example for Tutorial 4: Step Response of Undamped System

For a given system described as transfer function given below, perform the following tasks:



a. Roots of the characteristic equation. [2 marks]

b. Derive the expression for the transient response of the system. [6 marks]

c. Illustrate its poles and zeros in the s-plane. [4 marks]

d. With a help of diagram, determine the step response of the system. [4 marks]

Answer

a. For the given system, factorise its transfer function equation:

$$\frac{C(s)}{R(s)} = \frac{9}{s^2+9}$$

$$= \frac{9}{(s+j3)(s-j3)}$$

Roots of the characteristic equation are:

$$s_1 = j3 \text{ and } s_2 = -j3$$

Thus

$$\frac{C(s)}{R(s)} = \frac{9}{s^2+9} = \frac{9}{(s+j3)(s-j3)}$$

b. Apply step function to the system, the transfer function equation becomes:

$$\frac{C(s)}{(1/s)} = \frac{9}{(s+j3)(s-j3)}$$

Implement partial fraction expansion of the transfer function equation.

$$C(s) = \frac{9}{s(s+j3)(s-j3)} = \frac{K_1}{s} + \frac{K_2}{(s+j3)} + \frac{K_3}{(s-j3)}$$

When $s = 0$, the value of coefficient K_1 is:

$$K_1 = \frac{9}{(s+j3)(s-j3)} = \frac{9}{(j3)(-j3)} = 1$$

When $s = j3$, the value of coefficient K_2 is:

$$K_2 = \frac{9}{s(s+j3)} = \frac{9}{(j3)(j3+j3)} = -\frac{9}{18}$$

When $s = -j3$, the value of coefficient K_3 is:

$$K_3 = \frac{9}{s(s-j3)} = \frac{9}{(-j3)(-j3-j3)} = \frac{9}{18}$$

So

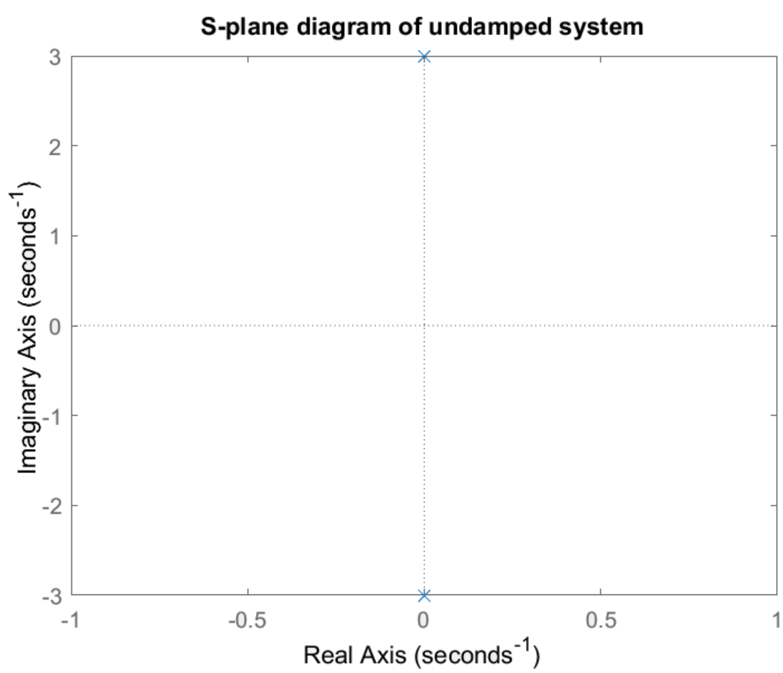
$$\frac{C(s)}{R(s)} = \frac{9}{(s+j3)(s-j3)} = \frac{1}{s} - \frac{1}{2(s+j3)} + \frac{1}{2(s-j3)}$$

Taking the inverse Laplace transform, the time domain equation of the system is:

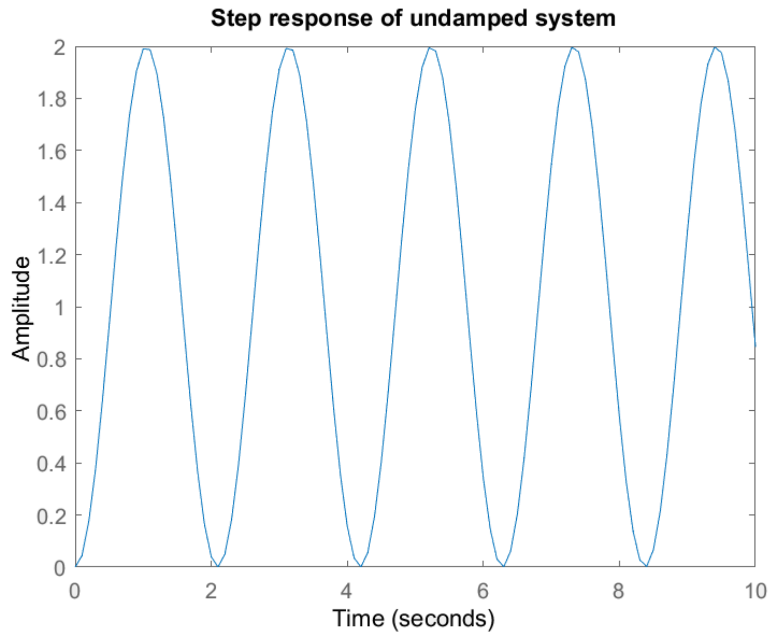
$$c(t) = 1 - \cos 3t$$

As a result, the step response of the system is a constant amplitude sinusoid.

c. Using MATLAB for simulation, the location of the poles in the s-plane is illustrated in the following figure. Notice a pair of complex poles on the x-axis in the graph.



d. Simulating the system in MATLAB, the step response of the system is shown in the following figure. The result of simulation confirms the constant amplitude sinusoid obtained in part (b).



5.1.2. Underdamped Step Response of Second-Order System

Underdamped second-order step responses are typically generated by complex poles. Given a second order system with a pair of complex poles as shown below.

$$\frac{C(s)}{R(s)} = \frac{k}{(s + a + jb)(s + a - jb)}$$

For a step input ($R(s) = 1/s$), after implementing partial fraction expansion, the transfer function equation is:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s + a + jb)} + \frac{K_3}{(s + a - jb)}$$

Taking inverse Laplace transform, the complex pair of roots become the co-sinusoidal function.

$$\frac{c(t)}{r(t)} = Ae^{-t}\cos(bt - \phi°)$$

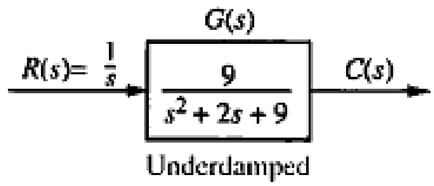
Where:

$$\phi = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}$$

As a result, the time response of the under damped second order system is an exponentially decaying sinusoidal oscillation.

Example for Tutorial 5: Step Response of Underdamped System

For a given system described as transfer function given below, perform the following tasks:



a.	Roots of the characteristic equation.	[2 marks]
b.	Derive the expression for the transient response of the system.	[6 marks]
c.	Illustrate its poles and zeros in the s-plane.	[4 marks]
d.	With a help of diagram, determine the step response of the system.	[4 marks]

Answer

a. For the given system, factorise its transfer function equation:

$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 2s + 9}$$

Considering the characteristic equation of the system, apply equation for solving roots of second order equation.

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(9)}}{2(1)}$$

This gives roots of the characteristic equation:

$$s_{1,2} = -1 \pm \sqrt{-8} = -1 \pm j\sqrt{8}$$

Thus

$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 2s + 9} = \frac{9}{\left(s + 1 + j\sqrt{8}\right)\left(s + 1 - j\sqrt{8}\right)}$$

b. Apply step response to the system.

$$\frac{C(s)}{(1/s)} = \frac{9}{\left(s + 1 + j\sqrt{8}\right)\left(s + 1 - j\sqrt{8}\right)}$$

Implement partial fraction expansion of the transfer function equation.

$$C(s) = \frac{9}{s\left(s + 1 + j\sqrt{8}\right)\left(s + 1 - j\sqrt{8}\right)} = \frac{K_1}{s} + \frac{K_2}{\left(s + 1 + j\sqrt{8}\right)} + \frac{K_3}{\left(s + 1 - j\sqrt{8}\right)}$$

When $s = 0$, the value of K_1 coefficient is:

$$K_1 = \frac{9}{(s+1+j\sqrt{8})(s+1-j\sqrt{8})} = \frac{9}{(1+j\sqrt{8})(1-j\sqrt{8})} = 1$$

When $s = -1 - j\sqrt{8}$, the value of K_2 coefficient is:

$$K_2 = \frac{9}{s(s+1-j\sqrt{8})} = \frac{9}{(-1-j\sqrt{8})(-1-j\sqrt{8}+1-j\sqrt{8})} = \frac{9}{2j\sqrt{8}-16}$$

When $s = -1 + j\sqrt{8}$, the value of K_3 coefficient is:

$$K_3 = \frac{9}{s(s+1+j\sqrt{8})} = \frac{9}{(-1+j\sqrt{8})(-1+j\sqrt{8}+1+j\sqrt{8})} = \frac{9}{-2j\sqrt{8}-16}$$

Thus:

$$C(s) = \frac{1}{s} + \frac{\left(\frac{9}{2j\sqrt{8}-16}\right)}{(s+1+j\sqrt{8})} + \frac{\left(\frac{9}{-2j\sqrt{8}-16}\right)}{(s+1-j\sqrt{8})}$$

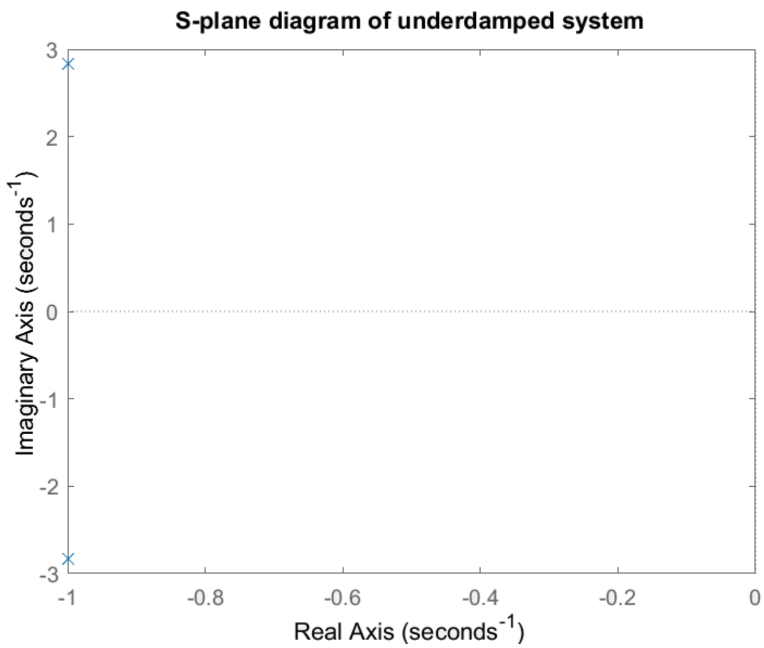
Taking inverse Laplace transform, the real part generates: e^{-t}. The complex part generates: $k\cos(\sqrt{8}t - \phi)$.

$$c(t) = 1 - e^{-t}\left(\cos\sqrt{8}t + \frac{\sqrt{8}}{8}\sin\sqrt{8}t\right)$$

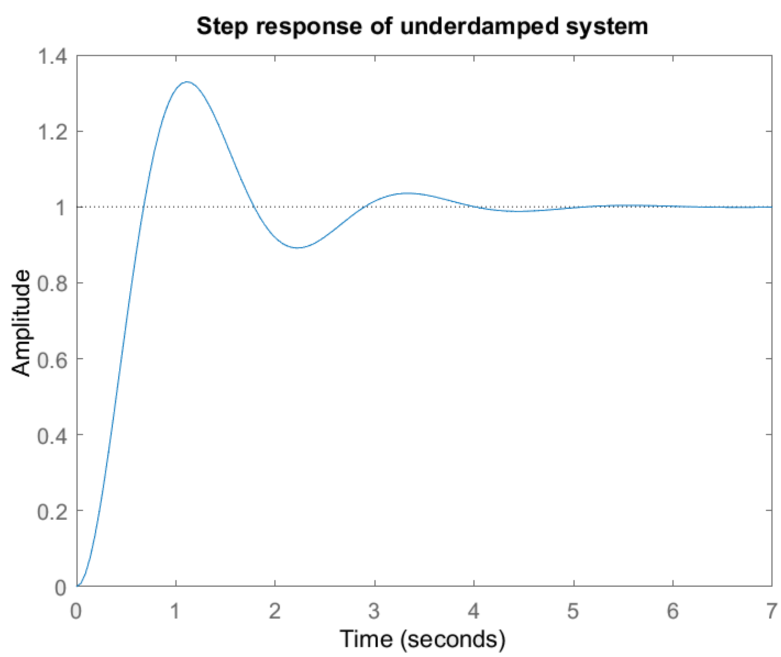
$$= 1 - 1.06e^{-t}\cos(\sqrt{8}t - 19.47°)$$

As a result, the step response of the system is a decaying sinusoidal oscillation.

c. Using MATLAB for simulation, the location of the poles in the s-plane is illustrated in the following figure. Notice the present of a pair of complex poles in the graph.



d. Simulating the system in MATLAB, the step response of the system is shown in the following figure. The result of simulation confirms the decaying sinusoidal oscillation obtained in part (b).



5.1.3. Overdamped Step Response of Second-Order System

Overdamped second-order step responses are typically generated by real poles. Consider a second order system with real poles as shown below.

$$\frac{C(s)}{R(s)} = \frac{k}{(s+a)(s+b)}$$

For a step input ($R(s) = 1/s$), after implementing partial fraction expansion, the transfer function equation is:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+a)} + \frac{K_3}{(s+b)}$$

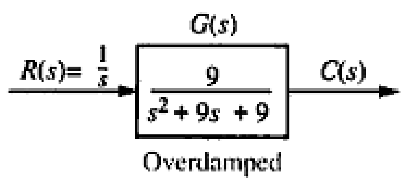
Taking inverse Laplace transform, the pole at origin becomes a constant and both of the real terms become exponential functions. The time domain equation of the system is:

$$c(t) = K_1 + K_2 e^{-at} + K_3 e^{-bt}$$

As a result, the step response of the system is an exponential increase.

Example for Tutorial 6: Step Response of Overdamped System

For a given system described as transfer function given below, perform the following tasks:



a. Roots of the characteristic equation. [2 marks]
b. Derive the expression for the transient response of the system. [6 marks]
c. Illustrate its poles and zeros in the s-plane. [4 marks]
d. With a help of diagram, determine the step response of the system. [4 marks]

Answer

a. For the given system, factorise its transfer function equation:

$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 9s + 9}$$

Considering the characteristic equation of the system, apply equation for solving roots of second order equation.

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(9) \pm \sqrt{(9)^2 - 4(1)(9)}}{2(1)} = -\frac{9}{2} \pm \frac{\sqrt{45}}{2}$$

The roots of the characteristic equation are:

$$s_1 = -7.854 \text{ and } s_2 = -1.146$$

Thus

$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 9s + 9} = \frac{9}{(s + 7.854)(s + 1.146)}$$

b. Apply step response ($1/s$) to the system.

$$\frac{C(s)}{(1/s)} = \frac{9}{(s + 7.854)(s + 1.146)}$$

Implement partial fraction expansion of the transfer function equation.

$$C(s) = \frac{9}{s(s + 7.854)(s + 1.146)} = \frac{K_1}{s} + \frac{K_2}{(s + 7.854)} + \frac{K_3}{(s + 1.146)}$$

When s = 0, the value of K_1 coefficient is:

$$K_1 = \frac{9}{(s + 7.854)(s + 1.146)} = \frac{9}{(7.854)(1.146)} = 1$$

When s = -1.146, the value of K_2 coefficient is:

$$K_2 = \frac{9}{s(s + 7.854)} = \frac{9}{(-1.146)(-1.146 + 7.854)} = 0.17$$

When s = -7.854, the value of K_3 coefficient is:

$$K_3 = \frac{9}{s(s + 1.146)} = \frac{9}{(-7.854)(-7.854 + 1.146)} = -1.17$$

Thus

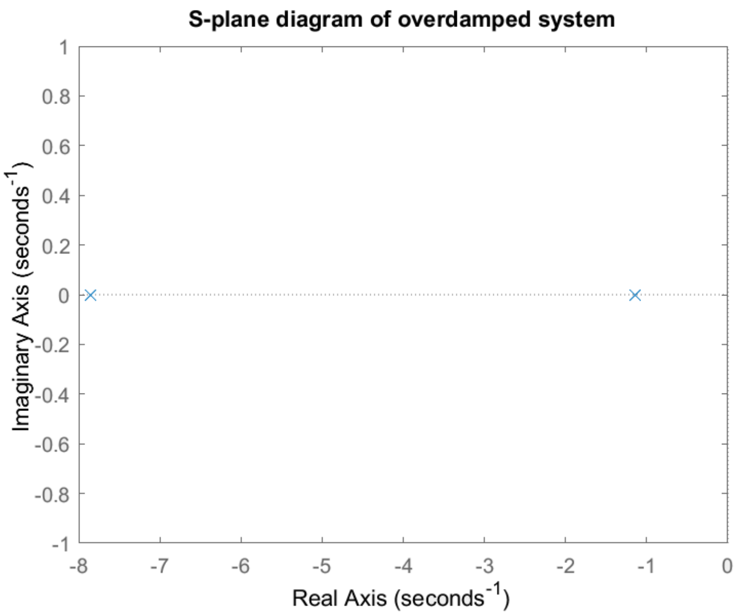
$$C(s) = \frac{1}{s} + \frac{0.17}{(s + 7.854)} + \frac{-1.17}{(s + 1.146)}$$

Taking the inverse Laplace transform, the time domain equation of the system is:

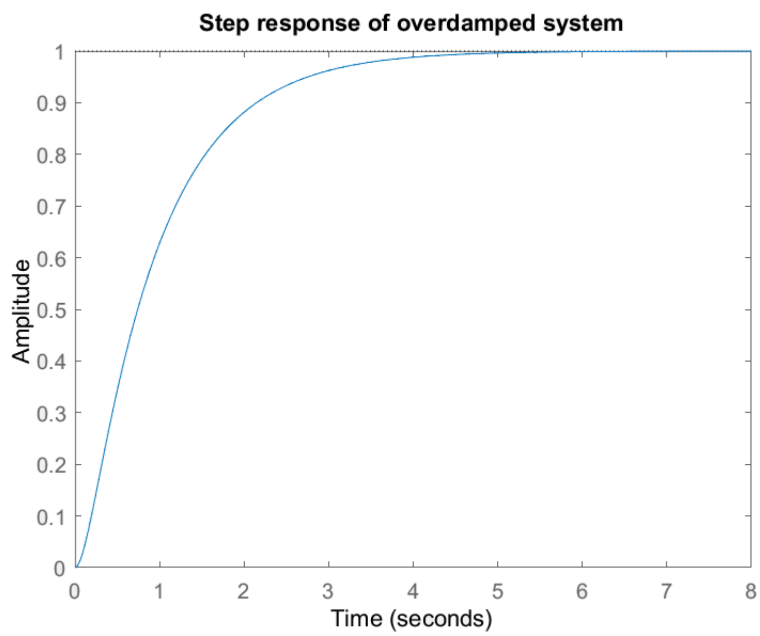
$$c(t) = 1 + 0.17e^{-7.854t} - 1.171e^{-1.146t}$$

As a result, the step response of the system is an exponential increase.

c. Using MATLAB for simulation, the location of the poles in the s-plane is illustrated in the following figure. Notice the two real poles in the graph.



d. Simulating the system in MATLAB, the step response of the system is shown in the following figure. The result of simulation confirms the exponential increase obtained in part (b).



5.2. Summary of Second Order Time Response

For a given second-order system, we might expect the following responses of the system:

Undamped responses:

- Poles: $s_{1,2} = \pm j\omega_0$

- Response: $c(t) = A\cos(\omega_0 t - \phi)$

Underdamped responses:

- Poles: $s_{1,2} = -\sigma_d \pm j\omega_d$
- Response: $c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$ where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Critically damped responses:

- Poles: $s_{1,2} = -\sigma_0$
- Response: $c(t) = K_1 e^{-\sigma_0 t} + K_2 t e^{-\sigma_0 t}$

Overdamped responses:

- Poles: $s_1 = -\sigma_1$, $s_2 = -\sigma_2$
- Response: $c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$

The time responses of the systems with various damping ratios are shown in the figure below.

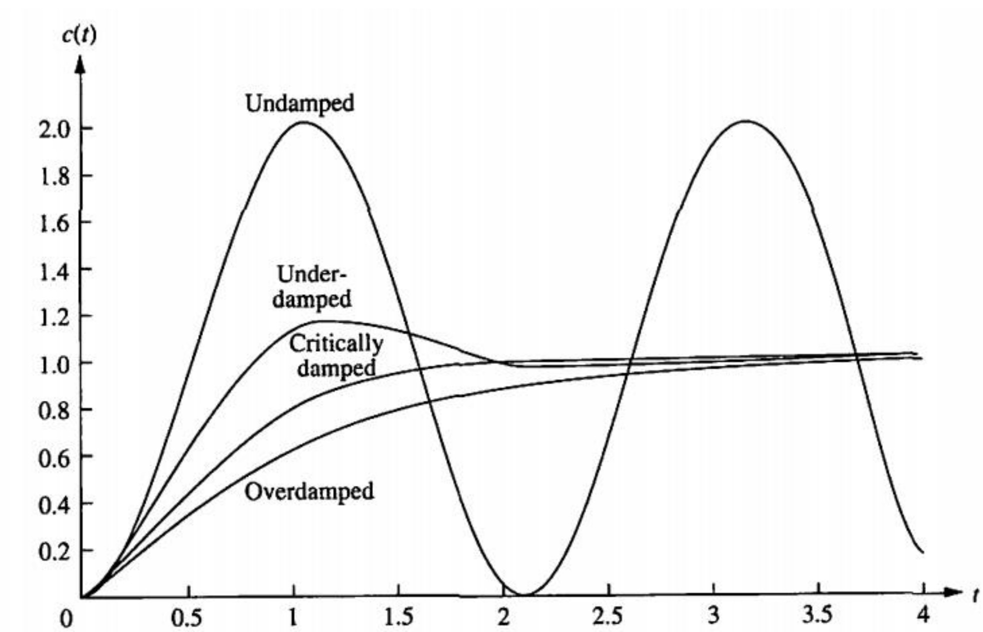


Figure 17: Transient responses of various systems with different response characteristics

6. Trends in Time Responses of Second-Order System

The following diagram shows the locations of the pole in the s-plane and their corresponding transient response. Notice the difference between the response of the poles on the left-hand side (stable response) with the response of the poles on the right-hand side (unstable response).

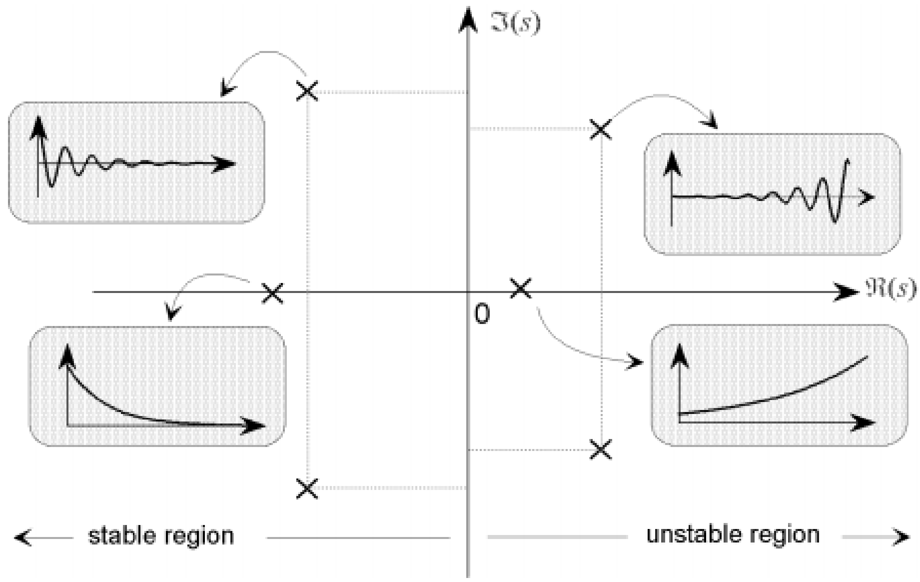


Figure 18: Locations of poles in s-plane and their transient responses

It can be observed in the figure above the difference between the response of a real pole e.g. along the x-line (damped response) with the response of a pair of complex poles (oscillatory response).

Then, in the following sections, we evaluate step responses of second-order systems as poles move with:

- constant real part.
- constant imaginary part.
- constant damping ratio.

6.1. Moving Poles in a Vertical Direction

Frequency increases, but the envelope remains the same since the real part of the pole is not changing. Constant exponential envelope, even though the sinusoidal response is changing frequency. Since all curves fit under the same exponential decay curve, the settling time is virtually the same for all waveforms. Overshoot increases, the rise time decreases.

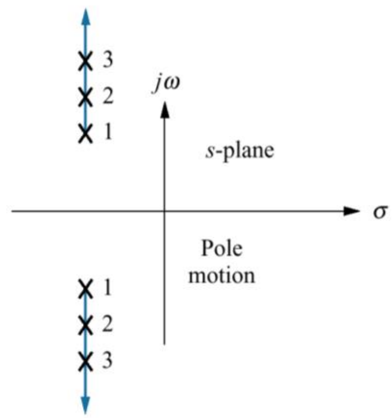


Figure 19: Systems with poles moving in a vertical direction

The following figure shows the transient response of the system based on the movement of the poles in a vertical direction in the s-plane as outlined above.

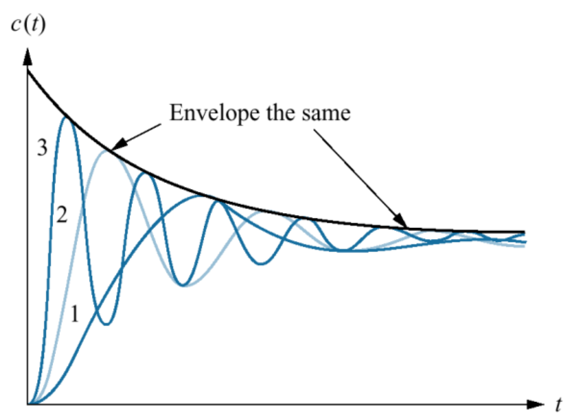


Figure 20: Transient response of systems with poles moving in a vertical direction

6.2. Moving Poles to the Right or Left

Imaginary part is now constant. Frequency is constant over the range of variation of the real part. As the poles move to the left, the response damps out more rapidly, while the frequency remains the same. Peak time is the same for all waveforms because the imaginary part remains the same.

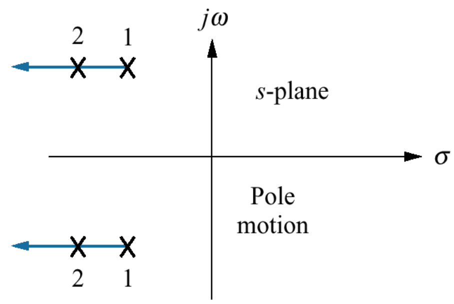


Figure 21: Systems with poles moving from right or left

The following figure shows the transient response of the system based on the movement of the poles from right or light in the s-plane as outlined above.

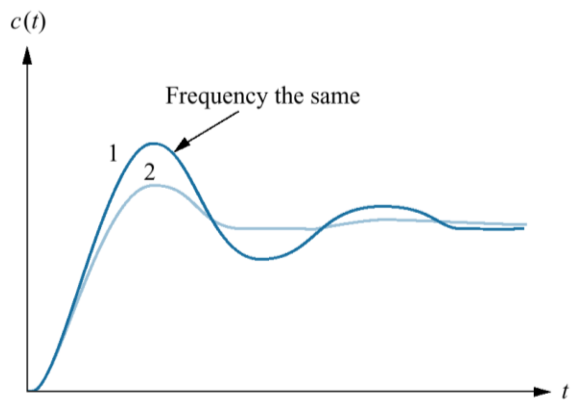


Figure 22: Transient response of systems with poles moving from right or left

6.3. Moving Poles along a Constant Radial Line

Percent overshoot remains the same. The responses look exactly alike, except for their speed. The farther the poles are from the origin, the more rapid the response.

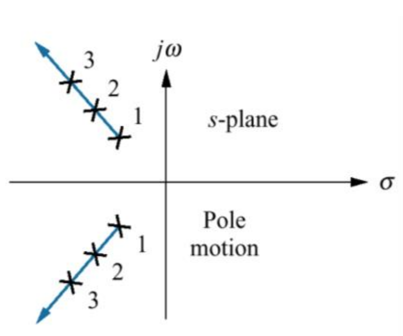


Figure 23: Systems with poles moving along a constant radial line

The following figure shows the transient response of the system based on the movement of the poles along a constant radial line in the s-plane as outlined above.

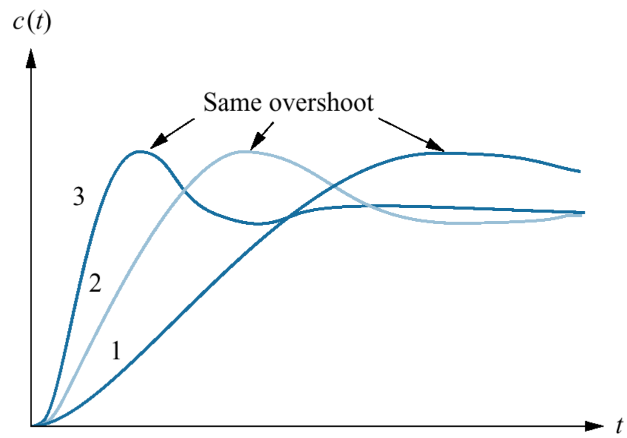


Figure 24: Transient response of systems with poles moving along a constant radial line