Formulas for Control Systems Engineering

A. Common Laplace Transforms

Time Domain	Laplace Domain
$\delta(t)$	1
$\delta^n(t)$	s^n
$u(t)$	$\frac{1}{s}$
t	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\frac{t^n}{n!}e^{at}$	$\frac{1}{(s+a)^{n+1}}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\frac{\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin(\omega_n\sqrt{1-\xi^2}t)$	$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

In all cases above, the symbols have their normal meanings.

B. Properties of the Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt \qquad \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{-st}\,ds$$

Definition:	$f(t) \Leftrightarrow F(s)$
Linearity:	$af(t) + bg(t) \Leftrightarrow aF(s) + bG(s)$
t-scaling	$f(ct) \Leftrightarrow \frac{1}{\lvert c \rvert} F\left(\frac{s}{c}\right)$
t-shifting:	$f(t - t_0)u(t - t_0) \Leftrightarrow e^{-st_0}F(s)$
s-shifting:	$e^{-s_0 t}f(t) \Leftrightarrow F(s - s_0)$
Differentiation in t:	$f'(t) \Leftrightarrow sF(s) - f(0)$ $f''(t) \Leftrightarrow s^2F(s) - sf(0) - f'(0)$
Integration in t:	$f^{(k)} \Leftrightarrow s^kF(s) - s^{k-1}f(0) - s^{k-2}f'(0)\ldots - f^{(k-1)}(0)$ $\int_0^t f(\tau)\,d\tau \Leftrightarrow \frac{1}{s}F(s)$
Differentiation in s:	$tf(t) \Leftrightarrow -F'(s)$
Integration in s:	$\frac{f(t)}{t} \Leftrightarrow \int_s^\infty F(\tilde{s})d\tilde{s}$
Convolution:	$f(t) * g(t) \Leftrightarrow F(s)G(s)$ $f(t)g(t) \Leftrightarrow \frac{1}{2\pi j}F(s) * G(s)$
Periodicity	$F(t) \Leftrightarrow F_1(s)\frac{1}{1 - e^{-sp}}$ For $f_1(t)$ one cycle of $f(t)$ with period p.
Initial value theorem:	$f(0+) = \lim_{t\to\infty} sF(s)$
Final value theorem:	$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} sF(s)$

(for $a, b, t_0, s_0 \in R, c \in R_{++}$).

C. Partial Fractions Expansion

If a partial fraction expansion of $Y(s)$ includes terms,

$$\frac{A_m}{(s-a)^m}\frac{A_{m-1}}{(s-a)^{m-1}} + \cdots + \frac{A_1}{s-a}$$

then the coefficients of factors having multiplicity $m > 1$ are given by the following expressions, where $k \neq m$.

$$A_m = \lim_{s\to a}(s-a)^m Y(s)$$

$$A_k = \frac{1}{(m-k)!}\lim_{s\to a}\frac{d^{m-k}}{ds^{m-k}}(s-a)^m Y(s)$$

D. Trigonometric Identities

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \Longrightarrow \begin{cases} \sin(\theta + \pi/2) = \cos(\theta) \\ \sin(\theta - \pi/2) = -\cos(\theta) \end{cases}$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi \Longrightarrow \begin{cases} \cos(\theta + \pi/2) = -\sin\theta \\ \cos(\theta - \pi/2) = \sin\theta \end{cases}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

E. First Order Systems

For a first order system with transfer function:

$$G(s) = \frac{1}{(s\ + a)}$$

Time constant is:

$$\tau = 1/a$$

Rise time (10-90%) is:

$$t_r = 2.2\tau$$

Settling time (to 2% of final value standard) is:

$$t_s = 4\tau$$

F. Second Order Systems

For an underdamped second order system, the following relationships hold.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The rise time

$$T_r = \frac{(1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)}{\omega_n}$$

Or

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} \qquad \text{where: } \phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

The settling time (i.e. 2% of final value standard):

$$T_s = \frac{4}{\zeta\omega_n}$$

The time taken to reach the peak value (n = #peak) is:

$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

The percentage overshoot is related to damping ratio by:

$$\%OS = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \times 100$$

Damping ratio.

$$\zeta = -\frac{\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}}$$

G. Steady State

Steady-state errors.

Type	Input		
	Step	**Ramp**	**Parabola**
0	$e_{ss} = \frac{1}{1+K_p}$	∞	∞
1	$e_{ss} = 0$	$\frac{1}{K_v}$	∞
2	$e_{ss} = 0$	0	$\frac{1}{K_a}$

Steady-state error constants.

$$K_p = \lim_{s\to 0} G(s) \qquad K_v = \lim_{s\to 0} sG(s) \qquad K_a = \lim_{s\to 0} s^2G(s)$$

H. Compensator Topologies

Proportional Compensator.

$$C(s) = K_p$$

Proportional-Integral (PI) Compensator.

$$C(s) = K_p\left(\frac{s+\omega_b}{s}\right) = K_p\left(\frac{\frac{s}{\omega_b}+1}{\frac{s}{\omega_b}}\right)$$

Proportional-Derivative (PD) Compensator.

$$C(s) = K_p\left(\frac{s}{\omega_b} + 1\right)$$

Lag Compensator (where $\alpha > 1$).

$$C(s) = K_p\left(\frac{s + \omega_b}{s + \dfrac{\omega_b}{\alpha}}\right) = K_p\left(\frac{\dfrac{s}{\omega_b} + 1}{\dfrac{\alpha s}{\omega_b} + 1}\right)$$

Lead Compensator (where $\alpha < 1$).

$$C(s) = \frac{K_p}{\alpha}\left(\frac{s + \omega_b}{s + \dfrac{\omega_b}{\alpha}}\right) = K_p\left(\frac{\dfrac{s}{\omega_b} + 1}{\dfrac{\alpha s}{\omega_b} + 1}\right)$$

The maximum phase lead of $\phi_{max} = \sin^{-1}[(1-\alpha)/(1+\alpha)]$ occurs at a frequency $= \omega_b/\sqrt{\alpha}$. Consequently:

$$\alpha = \frac{1 - \sin(\phi_{max})}{1 + \sin(\phi_{max})}$$

I. Graphical Analysis Techniques

Bode Plot:

Magnitude and phase shift.

$$|G(j\omega)| = K\frac{\prod_{i=1}^{m}|Z_i|}{\prod_{i=1}^{m}|P_i|} \qquad \text{and} \qquad \angle G(j\omega) = \sum_{i=1}^{m}\angle Z_i - \sum_{i=1}^{m}\angle P_i$$

Phase margin – damping ratio relationship (for $\zeta \leq 0.6$).

$$\text{PM} = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}\right) \approx 100\zeta$$

The frequency response has a peak magnitude that occurs at frequency $\omega_P = \omega_n\sqrt{1 - 2\zeta^2}$.

$$M_p = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}$$

Bandwidth of standardized control systems.

$$\omega_{BW} = \omega_n\sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

Nyquist Plot:

Poles and zeros.

$$Z = P + N$$

Note:

Z = unstable closed-loop pole.

P = unstable open-loop poles.

N = # encirclement at (-1+j0).

Phase and gain margins.

$$\mathrm{PM} = 180 + \arg[G(j\omega)(H(j\omega)]$$
$$\mathrm{GM(in\ dB)} = 20\log[1/G(j\omega)H(j\omega)]$$

Root Locus Diagram:

Real-axis intercept of asymptote.

$$\sigma_{asymptote} = \frac{\sum_{n=1}^{k}(s+p_n) - \sum_{n=1}^{k}(s+p_n)}{\#n_p - \#n_z} = \frac{\sum_i p_i - \sum_i z_i}{P - Z}$$

Angle of asymptote.

$$\theta_{asymptote} = \pm\frac{(2k+1)\pi}{\#n_p - \#n_z} = \frac{(2k+1)\pi}{P - Z}$$

Where: $k = 0, \pm 1, \pm 2, \ldots$

Location of pole break-away/break-in.

$$\sum_{i=1}^{Z}\frac{1}{\sigma_b - z_i} = \sum_{j=1}^{P}\frac{1}{\sigma_b - p_j}$$

Where: p_i and z_i are the pole and zero values of CG, where we have Z total zeros and P total poles.

Angle of pole break-away/break-in.

$$\angle\theta = \frac{180°}{n}$$

Where: n is the number of poles breaking away/in.