Final Exam Review

XMUT315 Control Systems Engineering

Final Exam Format

- Four questions, 25 marks each.
- Two hours (theoretical, analytical, and design cases).
- Bring to the exam:
  - Calculator (non-programmable).
  - Ruler.
  - Dictionary.
- A list of selected formulas in the control systems engineering is provided.

Final Exam Format

Descriptive, drawing and result of calculation answers:

Q1: Modelling, feedback control system, block diagram manipulation, time domain analysis, and stability.

Q2: Compensator or controller characteristics and analysis.

Q3: Bode plots analysis and Nyquist diagram analysis.

Q4: Root locus analysis.

Question Types

Expect three types of question in the exam:

- Explain…

  (d) Describe the significant of coefficients α and β in feedback system. [5 marks]

- Calculate…

  (a) Calculate time constant (τ), rise time (T_r), time to peak (t_p), settling time (t_s) and percentage overshoot (%OS) of the following second order system. [15 marks]

- Design…

  (c) Suggest a controller/compensator that could eliminate the steady-state error of the given system. [10 marks]

Topics

1. System modelling:
   - Laplace transform.
   - Modelling from physical system.
2. Feedback control system:
   - Feedback system.
   - Block model manipulation.
3. Stability:
   - System stability.
   - Routh-Hurwitz criterion stability analysis.
4. Time responses:
   - First-order transient.
   - Second-order transient.
   - Steady-state.

Topics

5. Controllers or compensators:
   - P, PI, PD, and PID controllers.
   - Lag, Lead, and Lag-Lead compensators.
   - Design of control system with controllers or compensators.
6. Bode plots:
   - Construction of Bode plots.
   - Analysis of control system with Bode plots.
7. Root locus diagram:
   - Construction of Root locus diagram.
   - Analysis of control system with root locus diagram.
8. Nyquist diagram:
   - Construction of Nyquist diagram.
   - Analysis of control system with Nyquist diagram.

1. System Modelling

Laplace transforms:

FUNCTION	LAPLACE TRANSFORMS
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
$f'(t)$	$sL\left[f(t)\right] - f(0)$
$f''(t)$	$s^2L\left[f(t)\right] - sf(0) - f'(0)$

1. System Modelling

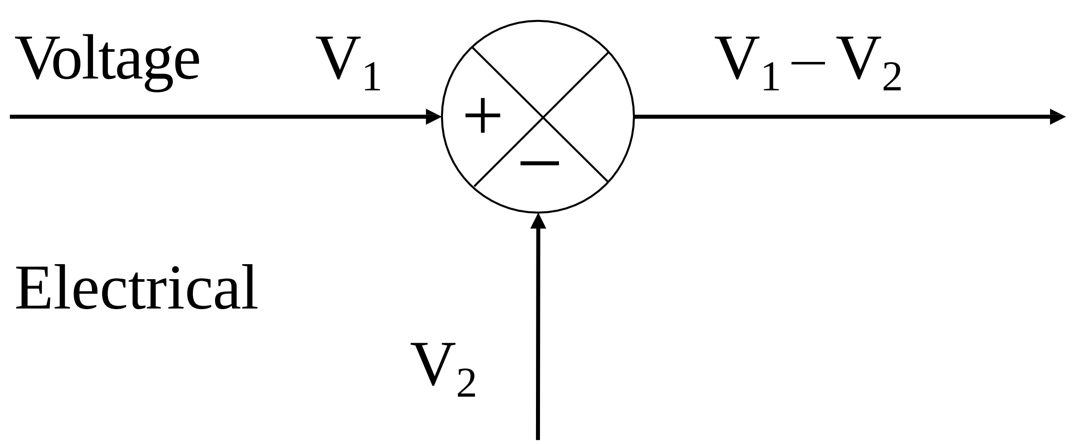
- Modelling system from physical entities:
  - Scaled physical model, mathematical model, and numerical model.
- When building up a model system:
  - Components should be easily identifiable, components should have a simple and clearly defined interaction with other components, and components numbers should be minimised.
- In order to analyse a system:
  1. We identify an input signal. [a variable]
  2. Using block diagram components. [basic block, summing junction, take-off point] [modified variables]
  3. We combine internal signals to produce the output signal. [another variable]

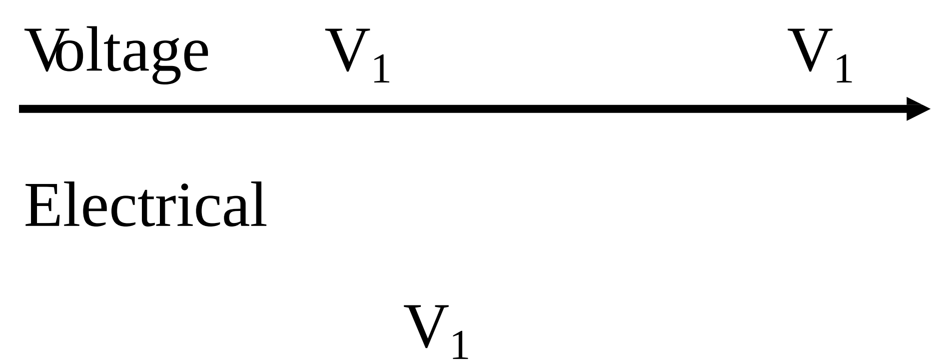
The Input-Output relationship may then be determined.

1. System Modelling

1. Signals:
   - Components are connected together by signals.
   - Signals have many different forms.
   - Must also have direction and name
   - Signals continue until interrupted!
   - Signals and components are considered ideal.
   - We add other signals and components to alter the properties

2. Components (summing junction and take off point)

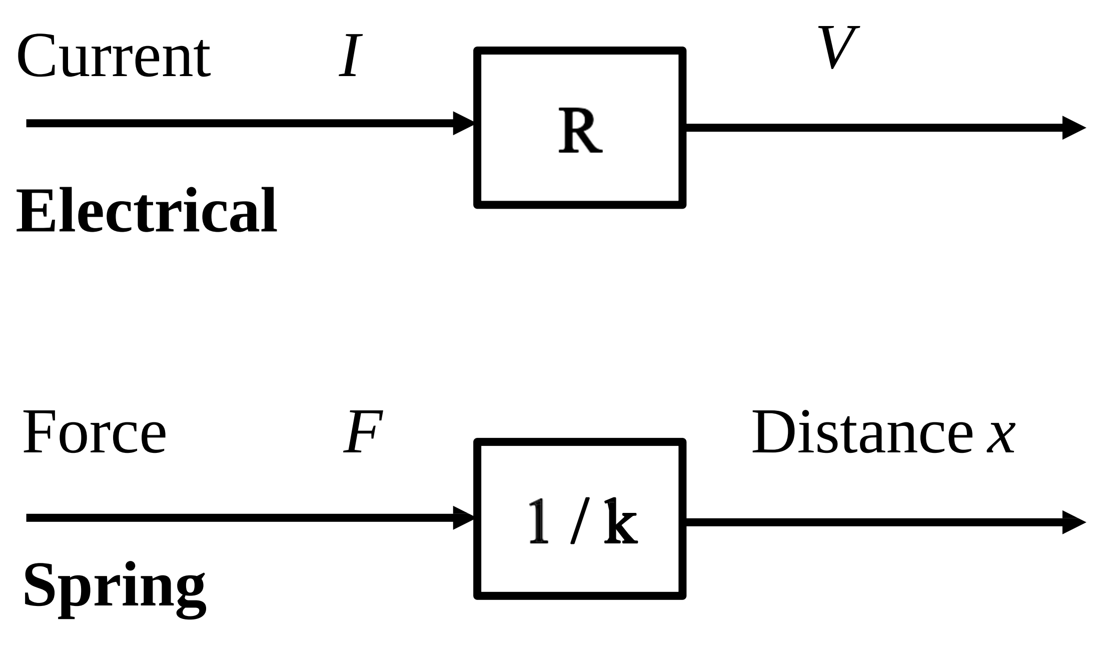




1. System Modelling

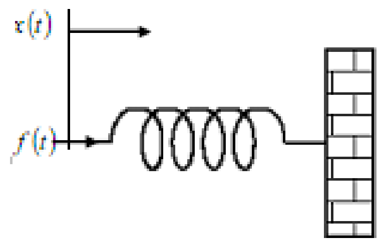
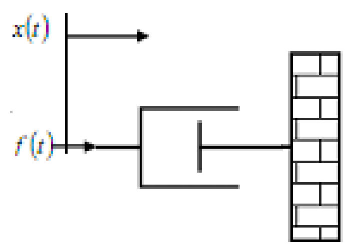
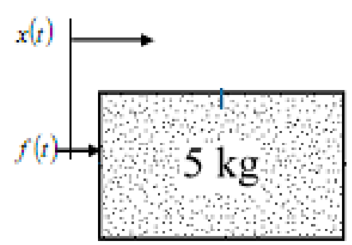
2. Components (block):

- Block is function of the system signal.
- Only one input and only one output (SISO).
- Three types of models: electrical system, mechanical system and electro-mechanical system.



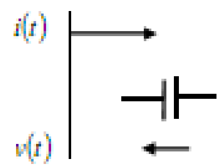
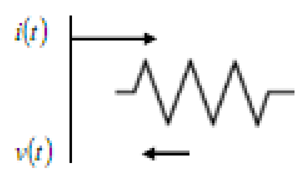
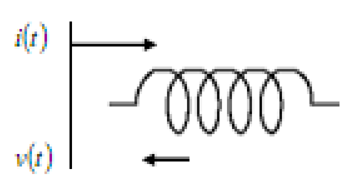
1. System Modelling

Modelling mechanical systems:

	Force - Distance	Impedance $=F(s)/X(s)$
Spring 	$f(t)=Kx(t)$	K
Damper 	$f(t)=D\dfrac{dx(t)}{dt}$	Ds
Mass 	$f(t)=M\dfrac{d^2x(t)}{dt^2}$	Ms^2

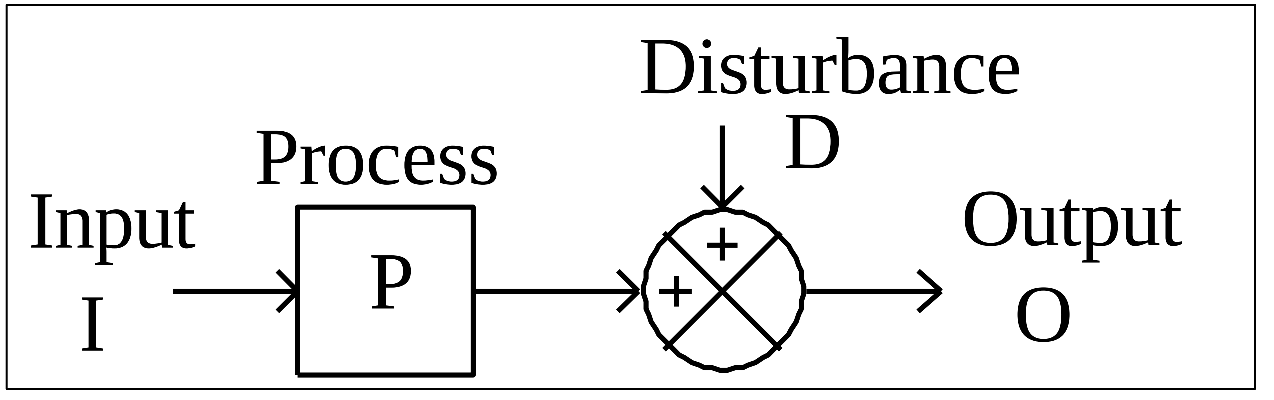
1. System Modelling

Modelling electrical systems:

	Voltage - Current	Impedance $=V(s)/I(s)$
Capacitor 	$v(t)=\frac{1}{C}\int i(t)$	$\frac{1}{C}\times\frac{1}{s}$
Resistor 	$v(t)=Ri(t)$	R
Inductor 	$v(t)=L\frac{di(t)}{dt}$	Ls

2. Feedback Control Systems

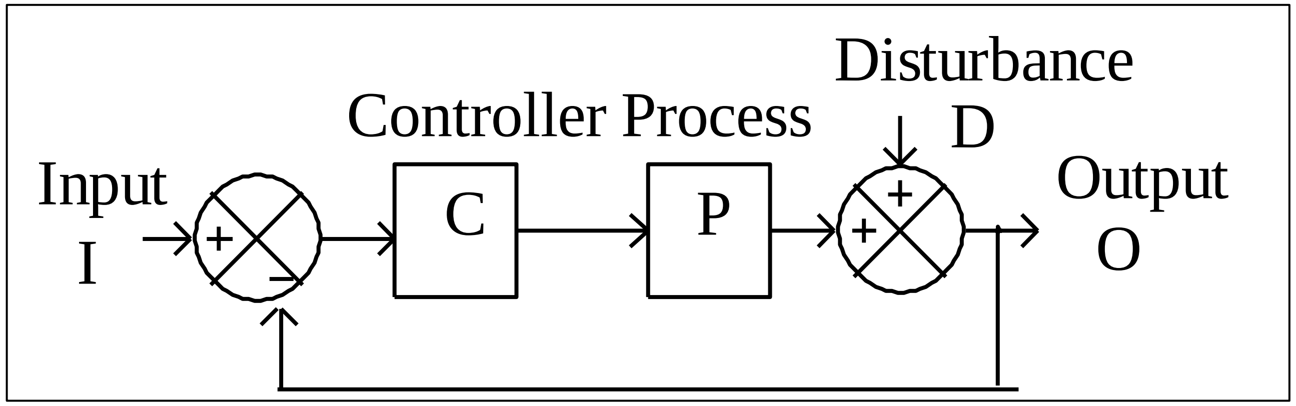
We would like to control the output of a system i.e. have the output resemble the input, despite disturbances.



- Process with transfer function **P** perturbed by a disturbance **D**.
- Suppose **P** is **10** and disturbance **D** is **0**. If the output **O** is to be **1**, we make input **I** = **0.1**.
- But if **P** changes by **10%** to **11** then O changes by 10% to 1.1.
- If disturbance D is 0.1, then O will also change by 0.1.

2. Feedback Control Systems

Principle of superposition: what is superposition?



$$O = \frac{C * P}{1 + C * P} I + \frac{1}{1 + C * P} D$$

$$\frac{O(s)}{I(s)} = \frac{CP}{1 + CP} \quad \text{with } D = 0$$

- If CP large: O ~ I + 0 = I
- So, feedback:
  - Makes output almost same as input,
  - Minimises effects of disturbances, and
  - Reduces effect of change in device.
- This is true because the ‘loop gain’, C * P, is high.

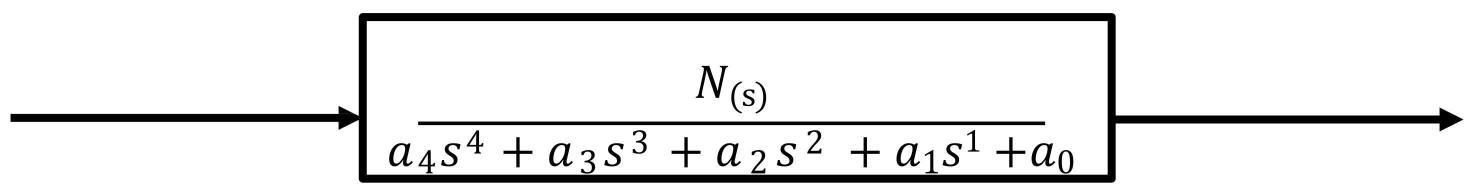
3. Stability

Routh-Hurwitz Criterion:

- Using this method, we can tell how many closed-loop system poles are in the left half-plane, in the right half-plane, and whether the poles are on the jω-axis.
- Cannot tell where, but only how many are in each plane determining the system's stability.
- The method requires two steps:
  - Generate a data table called a Routh table.
  - Interpret the Routh table to tell how many closed-loop system poles are in the left half-plane, the right half-plane, and on the jω -axis.

3. Stability

Generating a Routh table:



Equivalent closed loop transfer function

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

3. Stability

Interpreting a Routh table:

Simply stated, *the Routh-Hurwitz criterion declares that the number of roots of the polynomial that are in the right hand-plane is equal to the number of sign changes in the first column.*

Routh-Hurwitz special cases:

1. Zero in the first row or column.

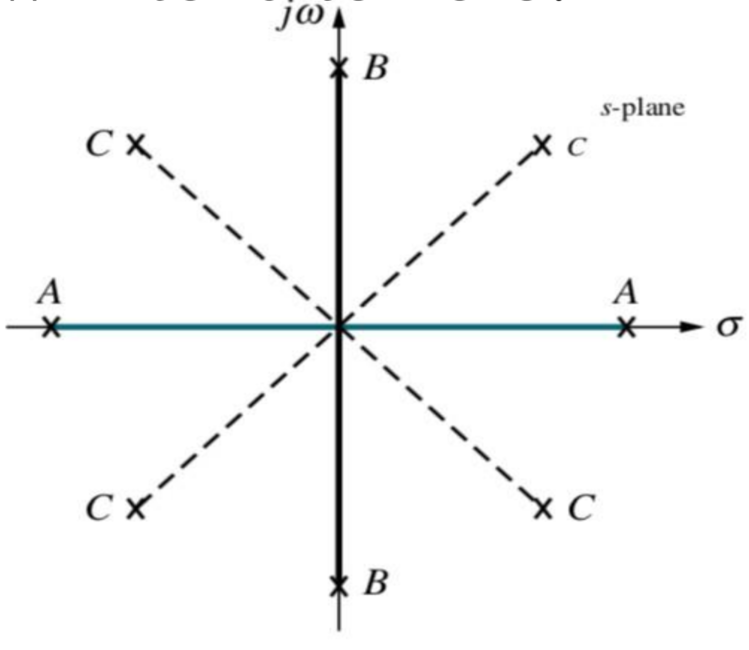
   In this case, zero is replaced with epsilon (ϵ) and will tend to zero.

2. Entire row of zeros:

   A: Real and symmetrical about the origin.

   B: Imaginary and symmetrical about the origin.

   C: Quadrantal and symmetrical about the origin.



4. Time Responses

Estimating system response:

- The systems examined so far can be modelled by transfer functions:

$$\frac{K}{1+sT} \quad \text{or} \quad \frac{K}{As^2+Bs+c}$$

- Given a particular input, what is the system output?
- Can use differential equation techniques.
- Easier to define the approximate response, just from the transfer function.
- We will do this, assuming that the input is a step.

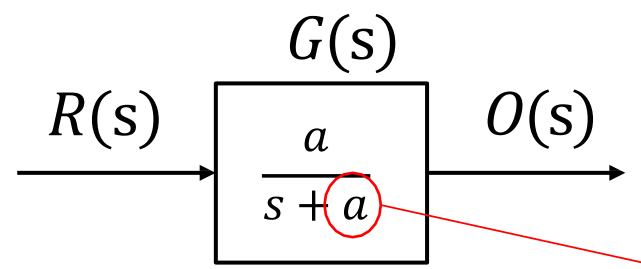
4. Time Responses

- Step response of first order system:

$$O(s) = G(s)R(s)$$

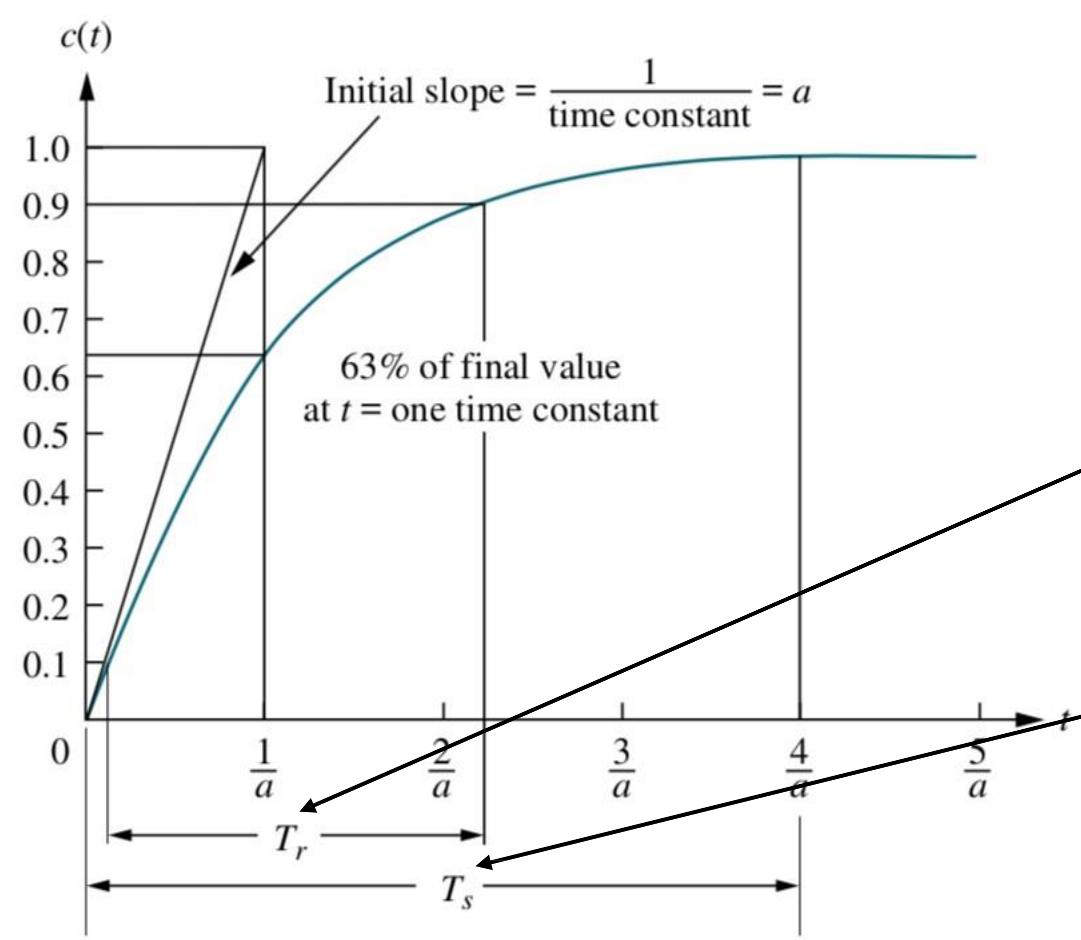
- For R = step input

$$O(s) = G(s)$$



Exponential Frequency

$e^{j\omega}$



Tr = Rise Time (2.2/a)

Ts = Settling time (4/a)

4. Time Responses

Steady-state response of first-order system:

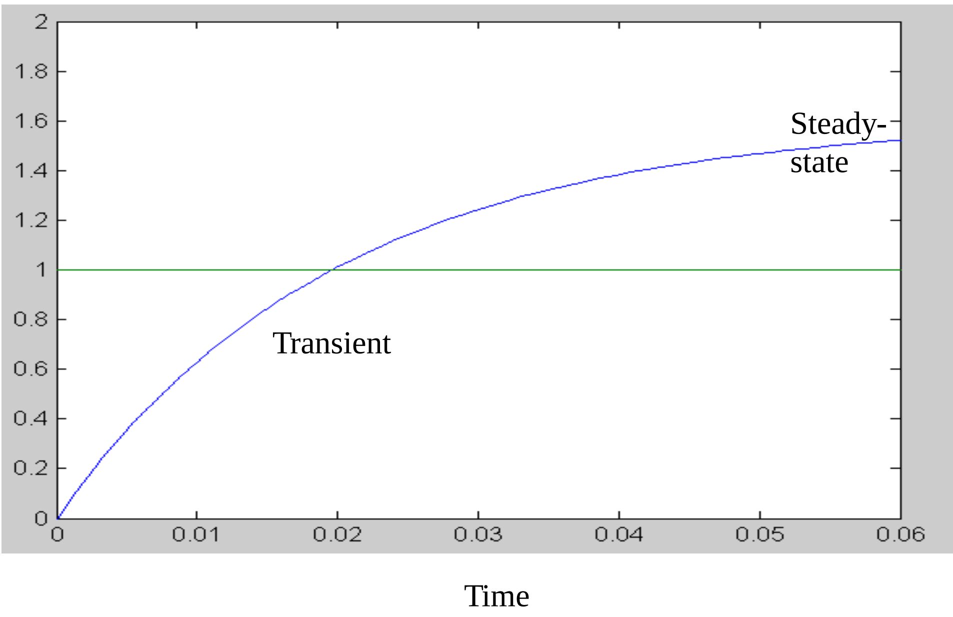
$$o(t) = K\left(1 - e^{-t/T_1}\right) \quad \text{where } t \text{ is large } e^{-\frac{t}{T}-1} \rightarrow 0$$

In s-domain:

$$s \rightarrow 0 \qquad \frac{O(s)}{I(s)} = \frac{K}{1 + T_1 s}$$

Step input:

$$O_{ss}(s) = \frac{K}{1 + T_1(0)} = \frac{K}{1} = K$$



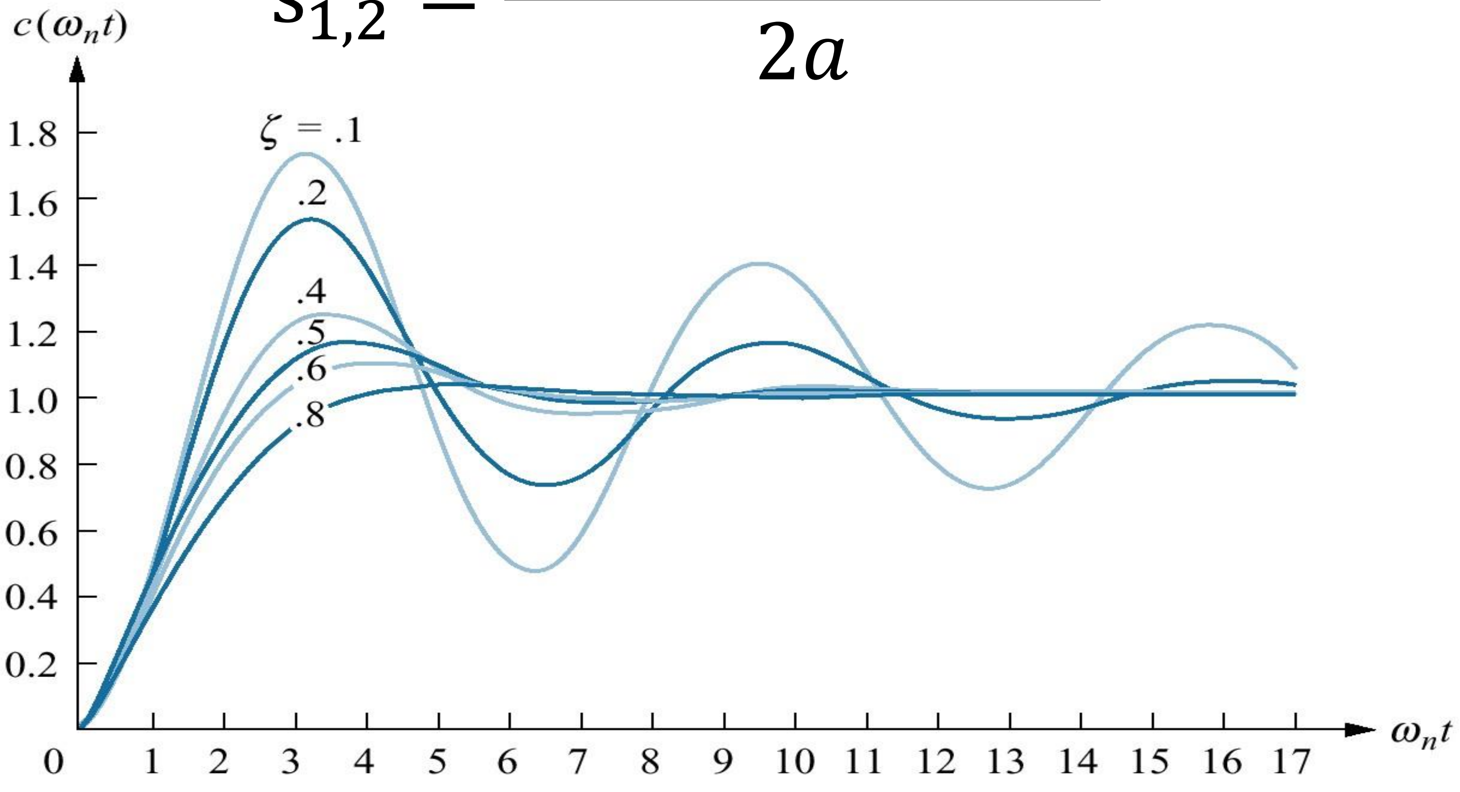
4. Time Responses

- General second-order response:

$$G(s) = \frac{\omega_N^2}{s^2 + 2\omega_N \zeta s + \omega_N^2}$$

- Standard second-order root equation:

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



4. Time Responses

General second-order response:

- Natural frequency (ω_n): Frequency of oscillation of the system without damping.
- Damping ratio (ζ): Quantitatively describe this damped oscillation regardless of the time scale.

$$\zeta = \frac{\text{exponential decay frequency}}{\text{Natural frequency (rad/s)}} = \frac{|\sigma|}{\omega_n} = \frac{\left(\frac{a}{2}\right)}{\omega_n}$$

- For a second order:

$$G(s) = \frac{b}{s^2 + as + b} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

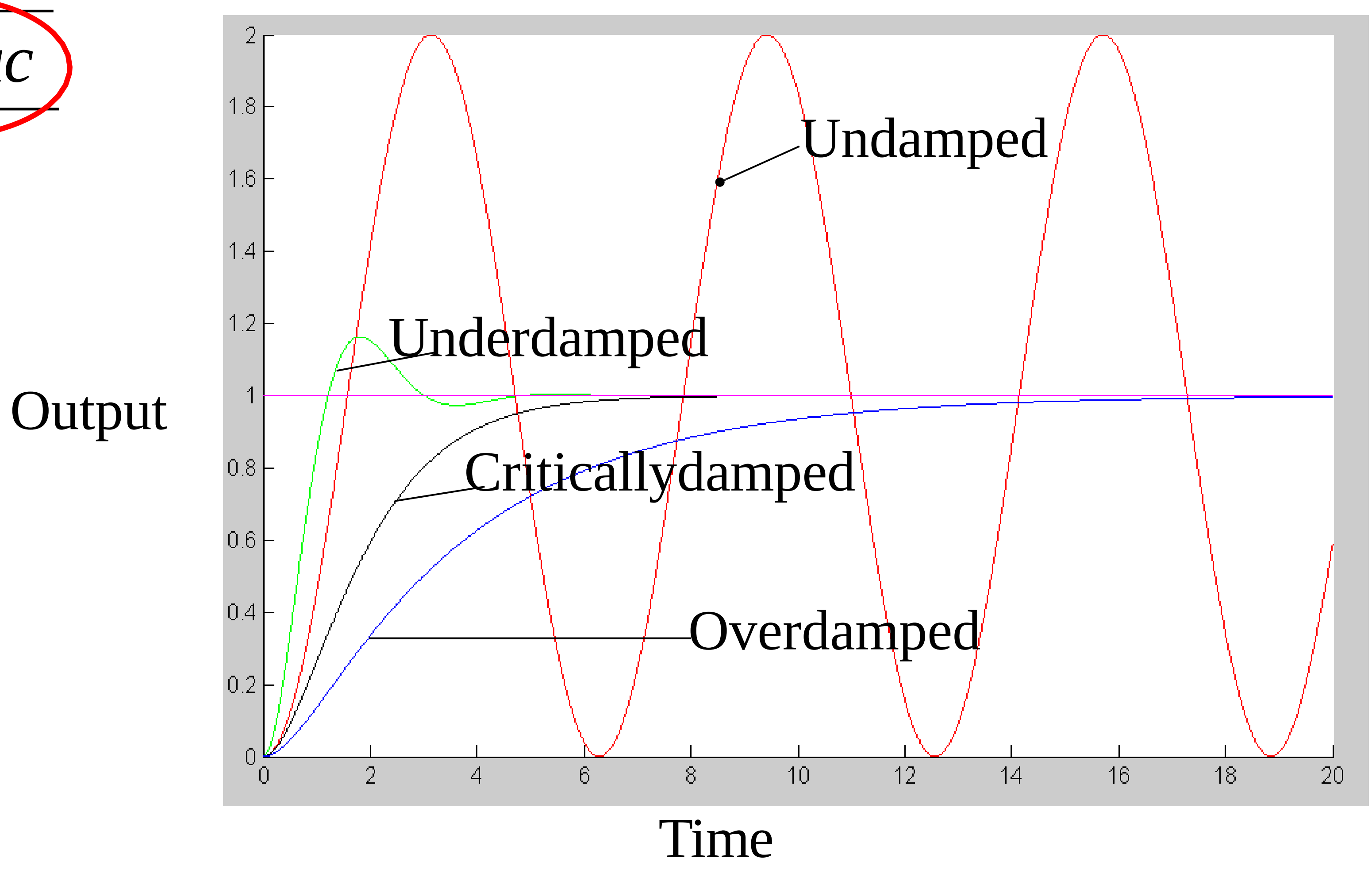
Where:

$$\omega_n = \sqrt{b} \qquad \text{and} \qquad a = 2\zeta\omega_n$$

4. Time Responses

Damping and second order response:

$$R_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



4. Time Responses

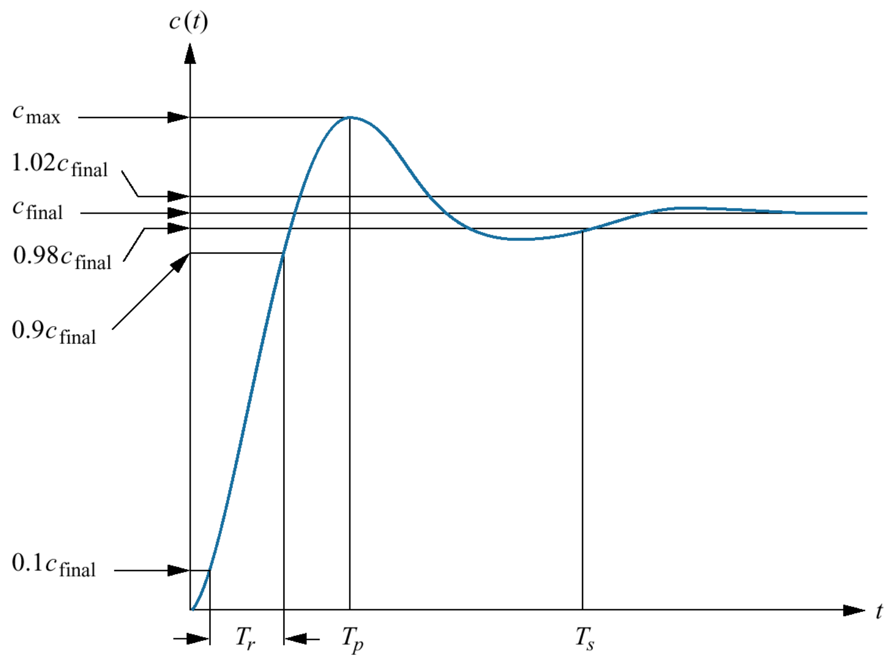
Determine damping of system:

Roots	**Description**	**Response**
$b^2 - 4ac > 0$	Real, different	Overdamped
$b^2 - 4ac = 0$	Real, same	Critically damped
$b^2 - 4ac < 0$	Complex, different	Underdamped
$-4ac > 0$	Complex, same	Undamped

4. Time Responses

Second order time response:

- Rise time, peak time, and settling time yield information about the speed of the transient response



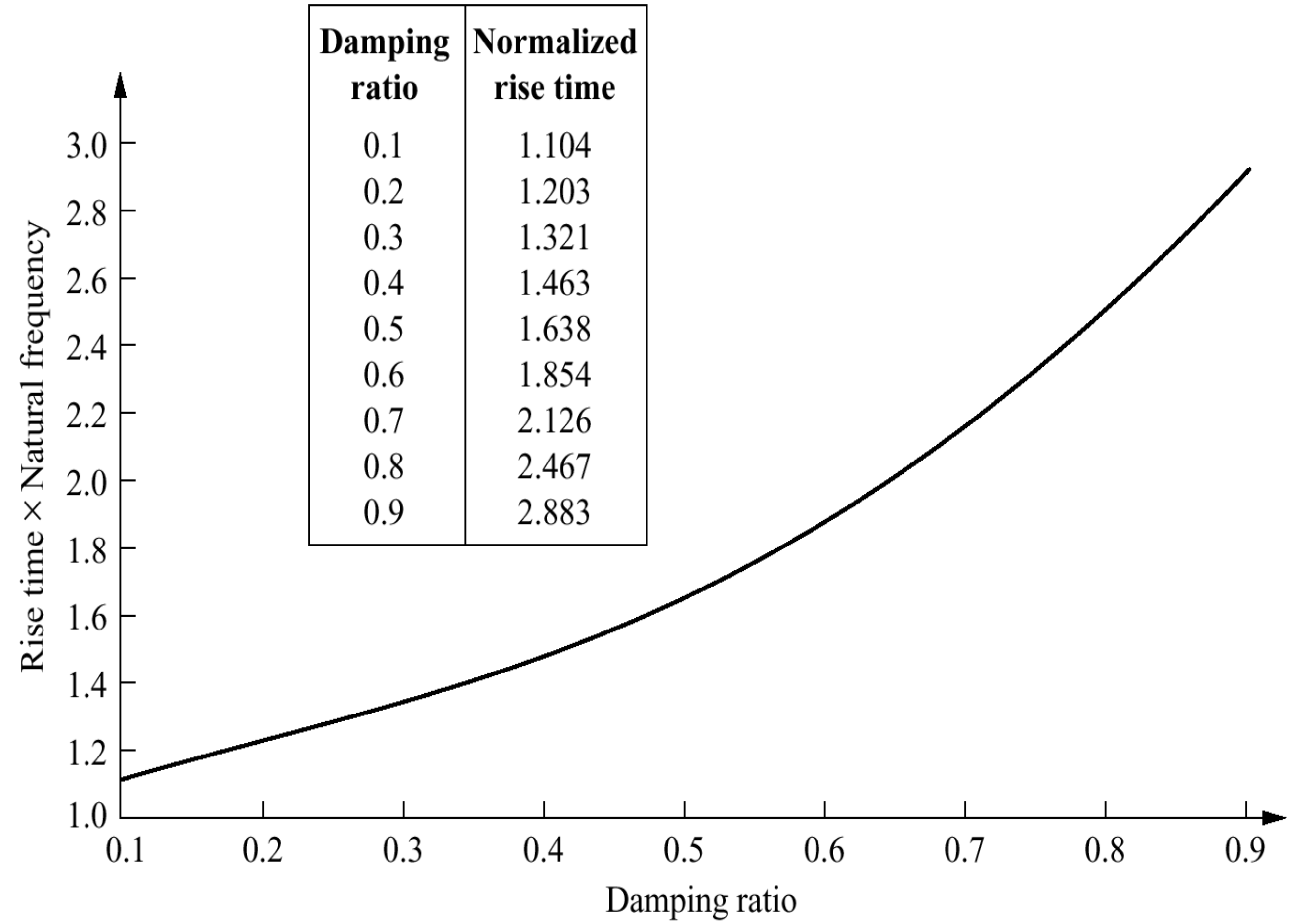
Second order (underdamped)

- Rise time (T_r) - The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.
- Settling time (T_s) - The time required for the transient's damped oscillations to reach and stay within +/- 2% of the steady-state value.
- Time to peak (T_p) - The time required to reach the first, or maximum, peak.

4. Time Responses

Rise time (T_r):

- Normalized rise time vs. damping ratio for a second-order underdamped response.



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

$$T_r = \frac{(1.76\xi^3 - 0.417\xi^2 + 1.039\xi + 1)}{\omega_n}$$

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\xi^2}}$$

Where: $\phi = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$

- For the given graph: $t_r\omega_0 = 2.230\xi^2 - 0.078\xi + 1.12$

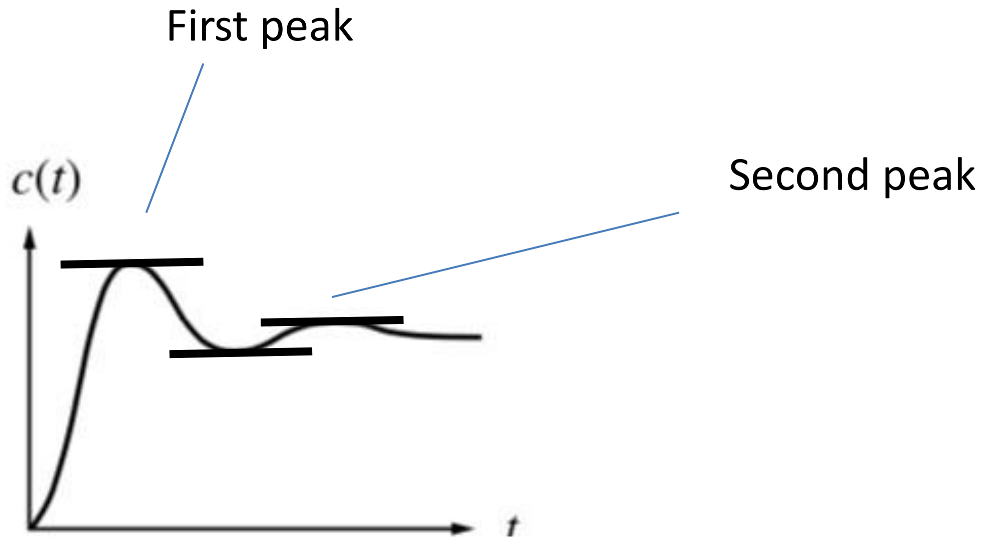
4. Time Responses

Time-to-Peak (T_p):

- Time to reach the first peak of the transient oscillation.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

- We wish to know the time to all *n* peaks, but in most cases only need the first peak i.e. *n* = 1.



4. Time Responses

Percentage overshoot ($\%OS$):

- Ratio of the maximum overshoot and steady-state value.

$$\%OS = \left[\frac{c(\max) + c(\infty)}{c(\infty)}\right] \times 100\%$$

- $C(\max)$ is $C(t)$ evaluated at the peak time $C(T_p)$. Note: $\%OS$ is a function of damping ratio only.

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100\%$$

- Can be rearranged to find damping ratio:

$$\zeta = -\frac{\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

4. Time Responses

Settling time (T_s):

- Time to reach and stay within +/-2% of the steady-state value

$$e^{-\zeta\omega_n t}\frac{1}{\sqrt{1-\zeta^2}} = 0.02$$

$$T_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n}$$

$$T_s = \frac{4}{\zeta\omega_n}$$

4. (Time Responses) Steady-State

- Steady-state error (e(∞):

  Steady-state error is the difference between the input and output for a prescribed test input as $t \to \infty$.

- Steady-state response:

  In the s domain: $s \to 0$.

$$e_{ss}(s) = \frac{k}{a_0 s^2 + b_0 s + c} = \frac{K}{c}$$

4. (Time Responses) Steady-State

- Steady-state error:

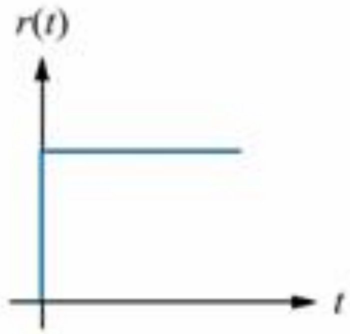
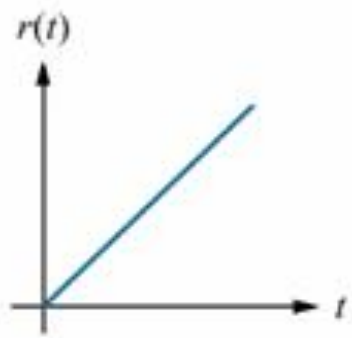
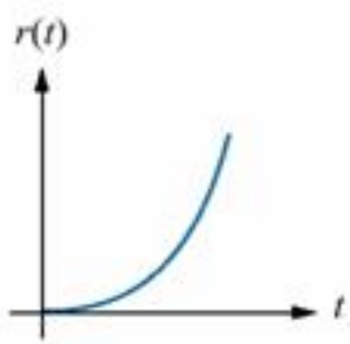
  The system error $e(t)$ for a feedback control system is given by the difference between the demanded output $r(t)$ and the actual output $c(t)$:

$$e(t) = r(t) - c(t)$$

- The steady-state error is then defined as the difference between demanded and actual output when $t \rightarrow \infty$.
- The steady-state error is now defined for specific test inputs:
  - Step.
  - Ramp.
  - Parabola.

4. (Time Responses) Steady-State

Specific test inputs in control systems engineering:

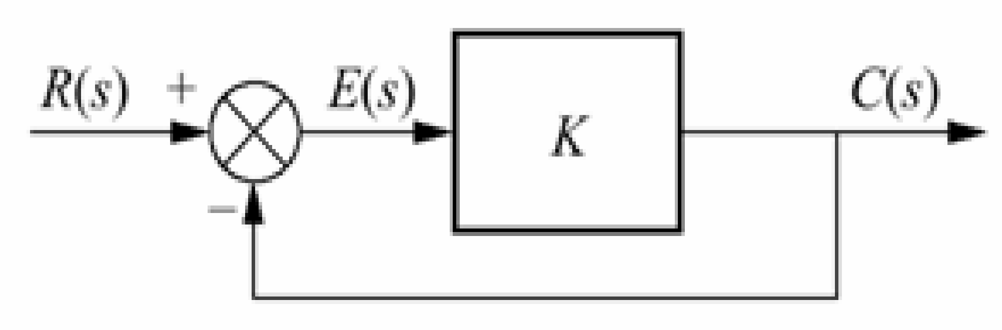
Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

4. (Time Responses) Steady-State

Sources of steady-state error:

- Consider steady-state errors due to system configuration.
- System with **pure gain** element.
- System output:

  $C(s) = K \,.\, E(s)$



- The steady-state error can then never be = 0 or the output of the system will be zero, there will thus always be a steady state error present.
- If c_{ss} is the steady-state value of the output and e_{ss} is the steady-state value of the error, then:

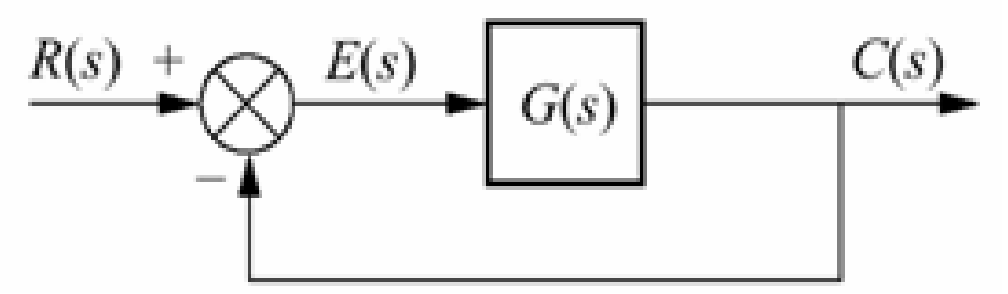
$$c_{ss}(t) = K \,.\, e_{ss}(t)$$ Error

will diminish as K increases.

4. (Time Responses) Steady-State

Steady-state error in terms of G(s):

For the system: $E(s) = R(s) – C(s)$



Thus $E(s) = R(s) – E(s)G(s)$

So that:

$$E(s) = \frac{R(s)}{1+G(s)}$$

From the final value theorem:

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Above equation will thus allow us to calculate the steady-state error given a particular input $R(s)$.

4. (Time Responses) Steady-State

Static error constant and system type:

- The term in the denominator of the definition of the steady state error for each input type is taken to limit the steady state error.
- These are then called the static error constants and are defined as follows:

Position constant, $K_p = \lim_{s \to 0} G(s)$

Velocity constant, $K_v = \lim_{s \to 0} sG(s)$

Acceleration constant, $K_a = \lim_{s \to 0} s^2 G(s)$

- These constants depend on the form of $G(s)$ and determine the steady-state error (i.e. error decreases as static error constant increases).

4. (Time Responses) Steady-State

System type:

- **The system type** is taken to be the number of integration in the feed-forward path. (the value of n in s^n of denominator).
- This value of n (the system type) then determines the steady state error of a unit feedback system for a particular type of input.

		Type 0		Type 1		Type 2	
Input	Steady-state error formula	Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

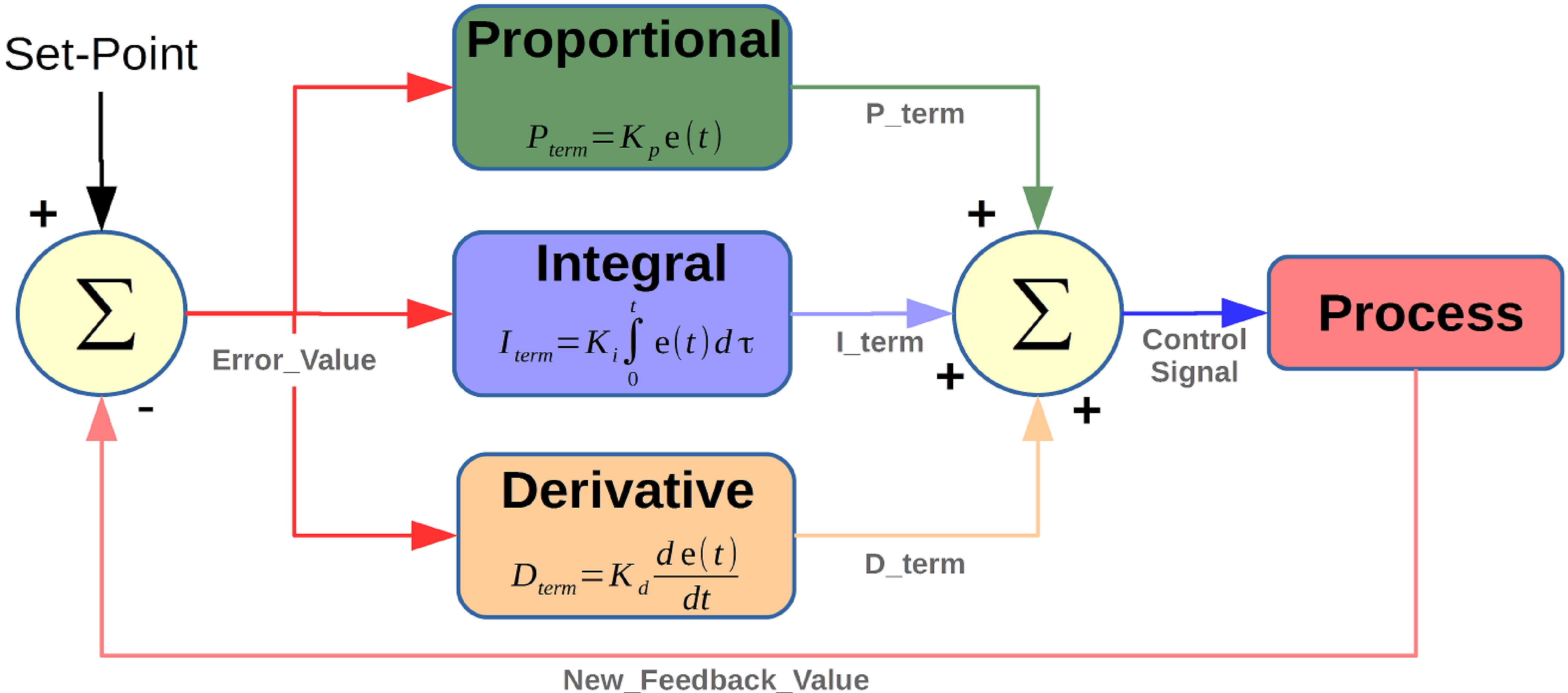
5. Controllers/Compensators

Compensators:

- Lag Compensators, Lead Compensators, and Lag-Lead Compensators.

Controllers:

- Proportional Controllers, Integral Controllers, Derivative Controllers, and PI, PD, PID controllers.



5. Controllers/Compensators

Controller/ Compensator	**Function**	**Transfer Function**	**Characteristics**
P	Improve transient response (up to a point)	K	a. Increases gain of the system. b. Often result in non-zero steady-state error. c. Relatively easy to implement.
PI	Improve steady-state error	$K\left(\frac{s+z_c}{s}\right)$	a. Increases system type. b. Error becomes zero. c. Zero at $z_c$is small and negative. d. Active circuits are required to implement.
Lag	Improve steady-state error	$K\left(\frac{s+z_c}{s+p_c}\right)$	a. Error is improved, but not driven to zero. b. Pole at $-p_c$ is small and negative. c. Zero at $-z_c$ is close to, and to the left of, the pole at $-p_c$. d. Active circuits are not required t implement.

5. Controllers/Compensators

PD	Improve transient response	$K(s+z_c)$	a. Zero at $-z_c$ is selected to put design point on root locus. b. Active circuits are required to implement. c. It can cause noise and saturation; implement with rate feedback or with a pole (lead).
Lead	Improve transient response	$K\left(\frac{s+z_c}{s+p_c}\right)$	a. Zero at $-z_c$ and pole at $-p_c$ at are selected to put design point on root locus. b. Pole at $-p_c$ is more negative than zero at $-z_c$. c. Active circuits are not required to implement.
PID	Improve steady-state error and transient response	$K\left[\frac{(s+z_{lag})(s+z_{lead})}{s}\right]$	a. Lag zero at $-z_{lag}$ and pole at the origin improve steady-state error. b. Lead zero at $-z_{lead}$ improves transient response. c. Lag zero at $-z_{lag}$ is close to, and to the left of, the origin. d. Lead zero at $-z_{lead}$ is selected to put design point on root locus. e. Active circuits are required to implement. f. It can cause noise and saturation; implement with rate feedback or with an additional pole.

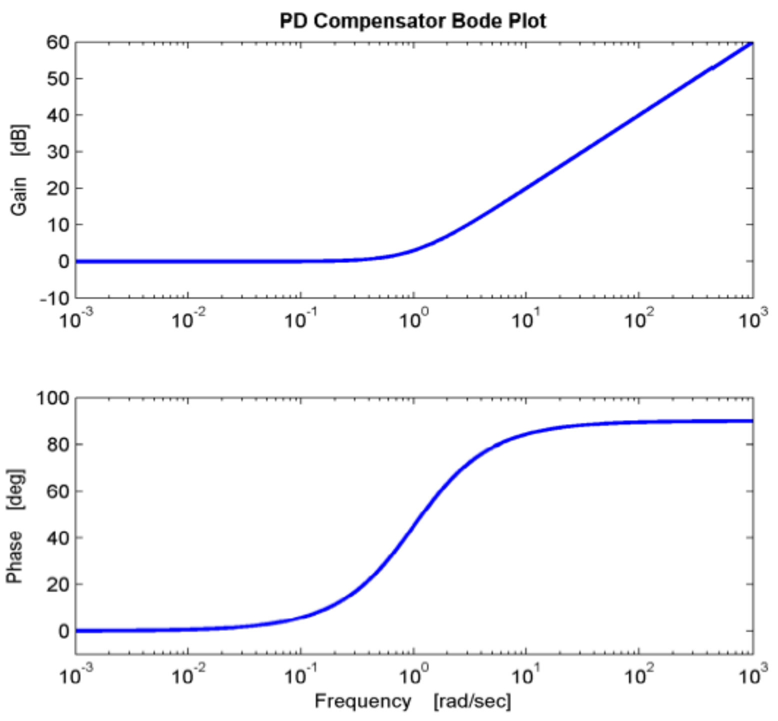
5. Controllers/Compensators

Lag-lead	Improve steady-state error and transient response	$K\left[\frac{(s+z_{lag})(s+z_{lead})}{(s+p_{lag})(s+p_{lead})}\right]$	a. Lag pole at $-p_{lag}$ and lag zero at $-z_{lag}$ are used to improve steady-state error. b. Lead pole at $-p_{lead}$ and lead zero at $-z_{lead}$ are used to improve transient response. c. Lag pole at $-p_{lag}$ is small and negative. d. Lag zero at $-z_{lag}$ is close to, and to the left of, lag pole at $-p_{lag}$ e. Lead zero at $-z_{lead}$ and lead pole at $-p_{lead}$ are selected to put design point on root locus. f. Lead pole at $-p_{lead}$ is more negative than lead zero at $-z_{lead}$. g. Active circuits are not required to implement.

5. Controllers/Compensators

PD Controller:

$$C(s) = T_D(s+4)$$

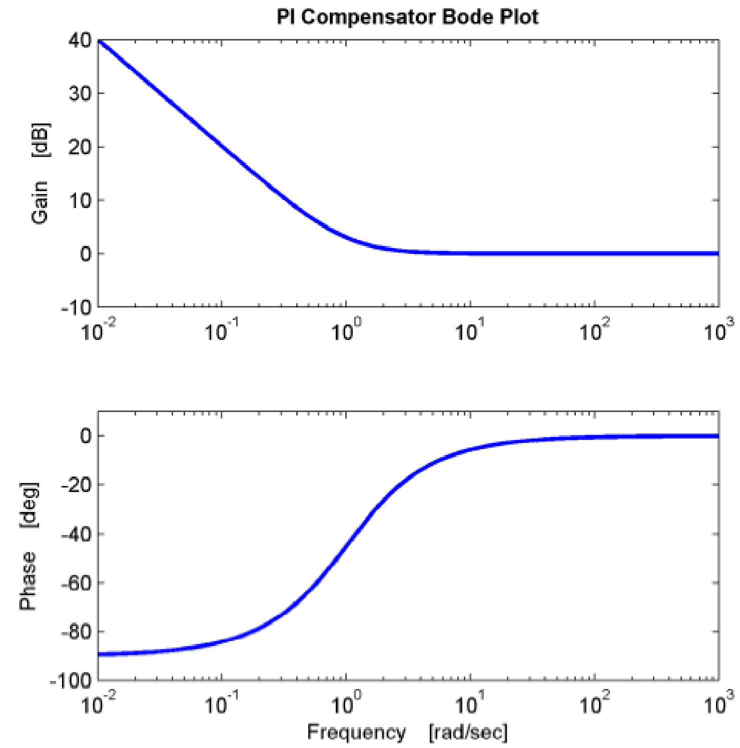


- In the PD controller, phase added near (and above) the crossover frequency e.g. an increase of the phase margin and giving a stabilizing effect.
- Then, the gain continues to rise at high frequencies, but this causes the sensor noise to be amplified and as a result a lead compensation is usually preferable.

5. Controllers/Compensators

PI Controller:

$$C(s) = \frac{1}{T_D}\left(\frac{T_D s + 1}{s}\right)$$



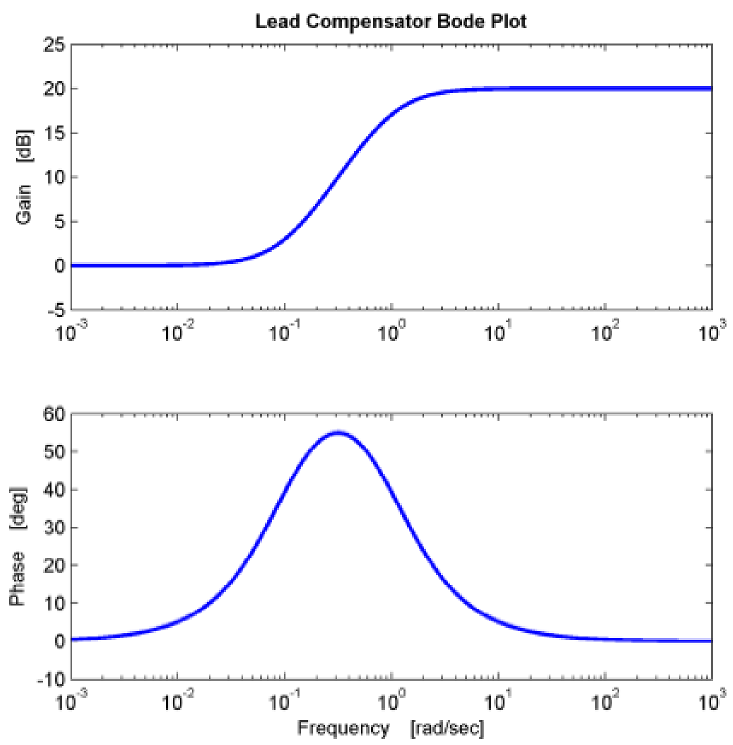
- At low frequency, the gain of proportional-integral compensator is infinite at DC (0 rad/s) and this compensator can increase system type of the system.
- For frequency above the cut-off frequency of the compensator ($\omega \gg 1/T_D$), the gain of the system is unaffected, there is a slight change in the phase, but phase margin of the system is unaffected.
- In the end, the proportional-integral compensator has a tendency to increase low frequency gain of the system.

5. Controllers/Compensators

Lag Compensator:

$$C(s) = \left(\frac{Ts+1}{\beta Ts+1}\right)$$

Where: $\beta < 1$



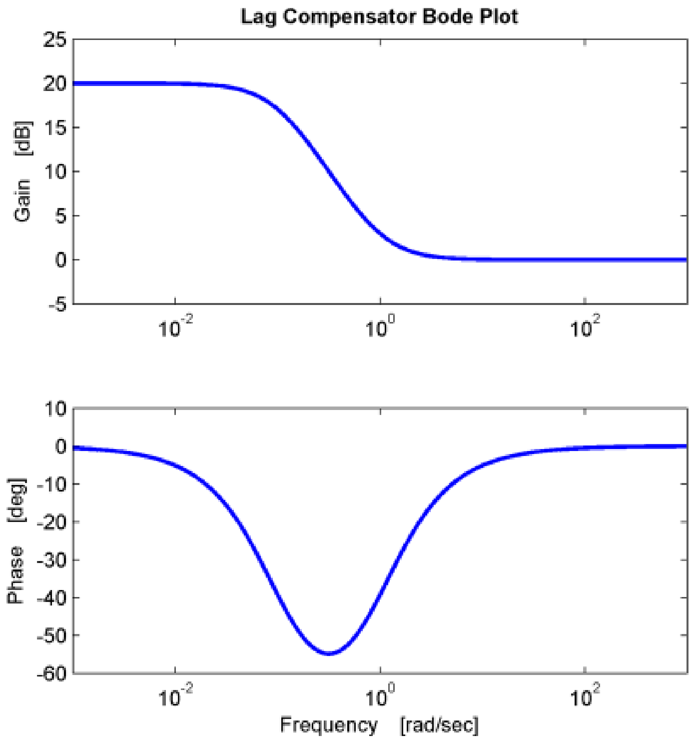
- For frequency below the cut-off frequency of the compensator ($\omega \ll 1/T$) the gain is ~0 dB and Phase is ~0°.
- For frequency above the cut-off frequency of the compensator ($\omega \gg 1/\beta T$) the gain is +20 dB and phase is ~0°.
- Thus, lead compensator adds phase lead near the crossover frequency and/or alter the crossover frequency.

5. Controllers/Compensators

Lag Compensator:

$$C(s) = \alpha\left(\frac{Ts + 10}{\alpha Ts + 1}\right)$$

Where: $\alpha > 1$

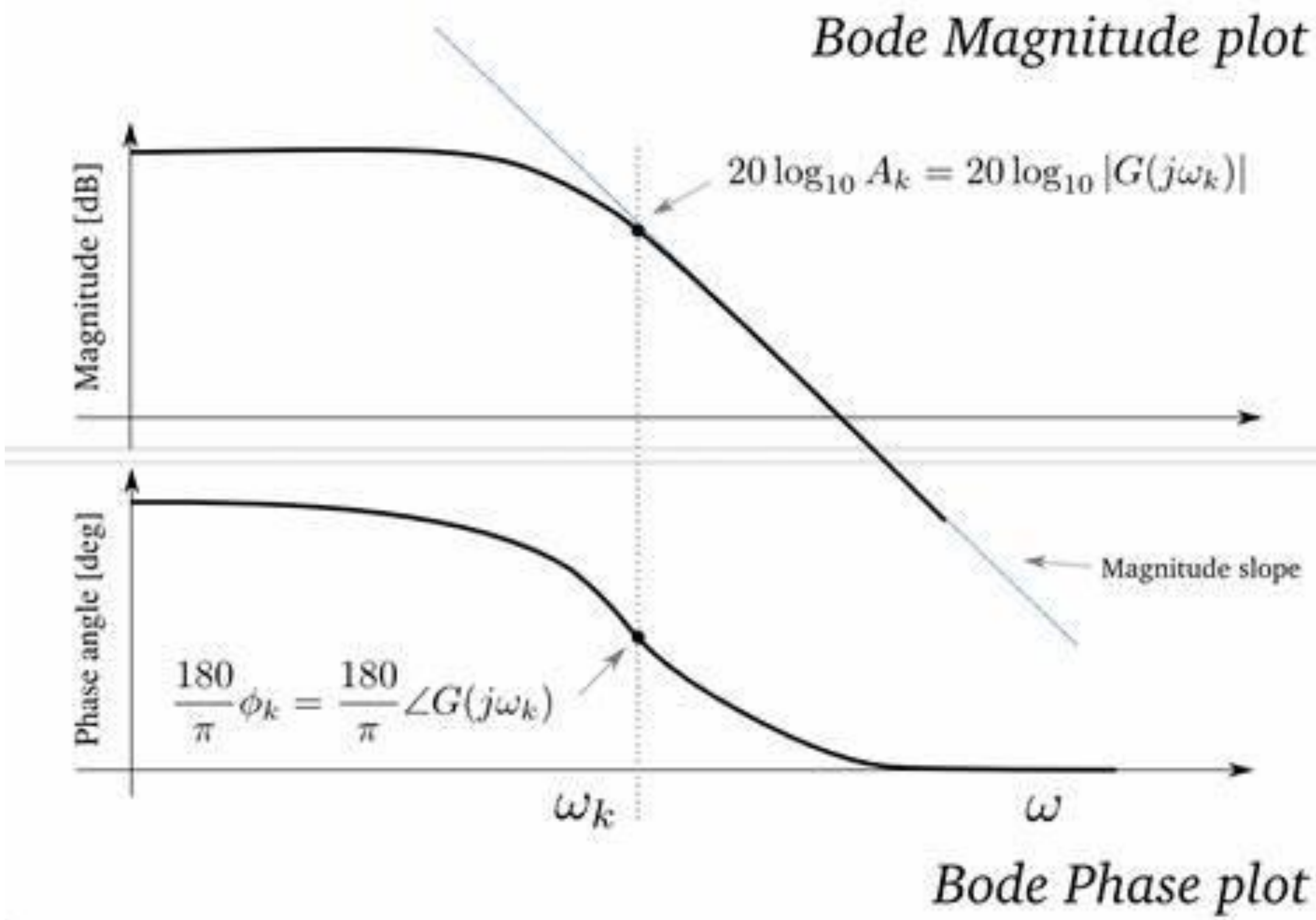


- For frequency less than cut-off frequency of the compensator ($\omega \ll 1/\alpha T$), the gain of the system is 20 log (α) dB and phase is 0°.
- For frequency more than the cut-off frequency of the compensator ($\omega \gg 1/\alpha T$), both the gain and phase are 0 dB and 0° respectively.
- In short, lag compensator adds a gain of α at low frequencies without affecting phase margin.

6. Bode Plots

Bode plots:

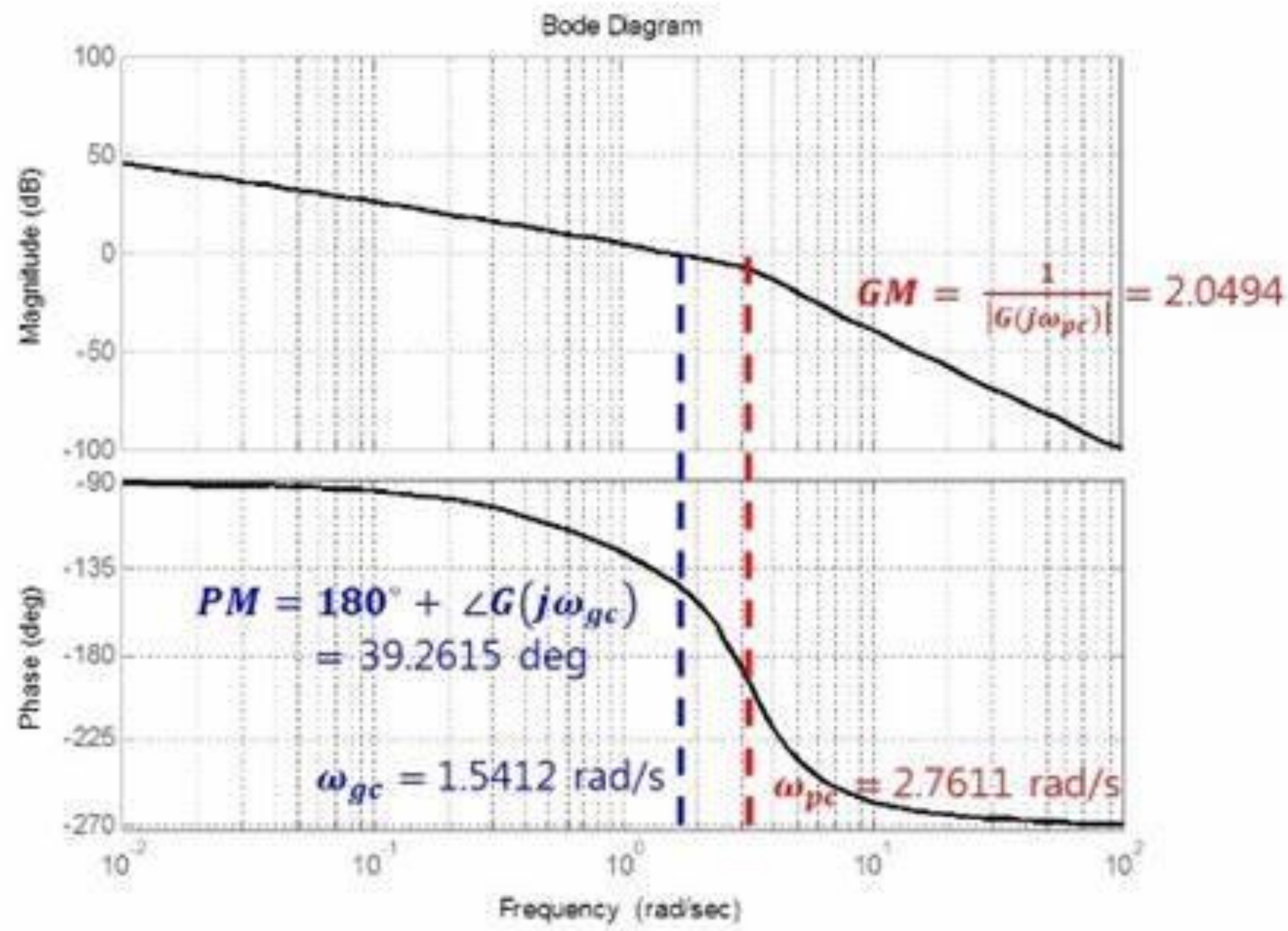
- Frequency response analysis method in control system for measuring: stability, time domain performance, and frequency domain performance.



6. Bode Plots

Components of analysis:

- Gain or magnitude plot.
- Phase shift plot.
- Gain and phase margins.



Analysis of Bode plots:

- Stability: Gain, Phase Shift Angle, Phase Margin, and Gain Margin.
- Transient Performance (e.g. Damping Factor, Rise Time, Settling Time, Time-to-Peak, % Oscillation, Steady State Errors, etc.): break points, slopes, peaking.

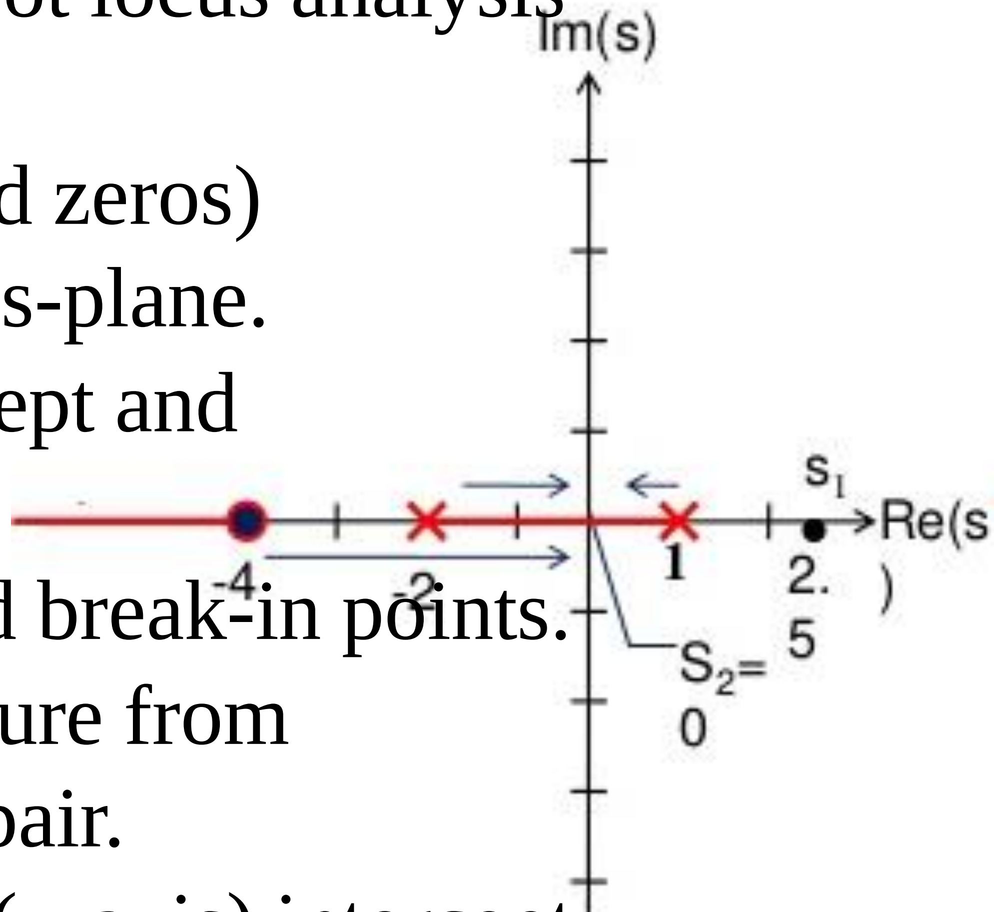
7. Root Locus Diagram

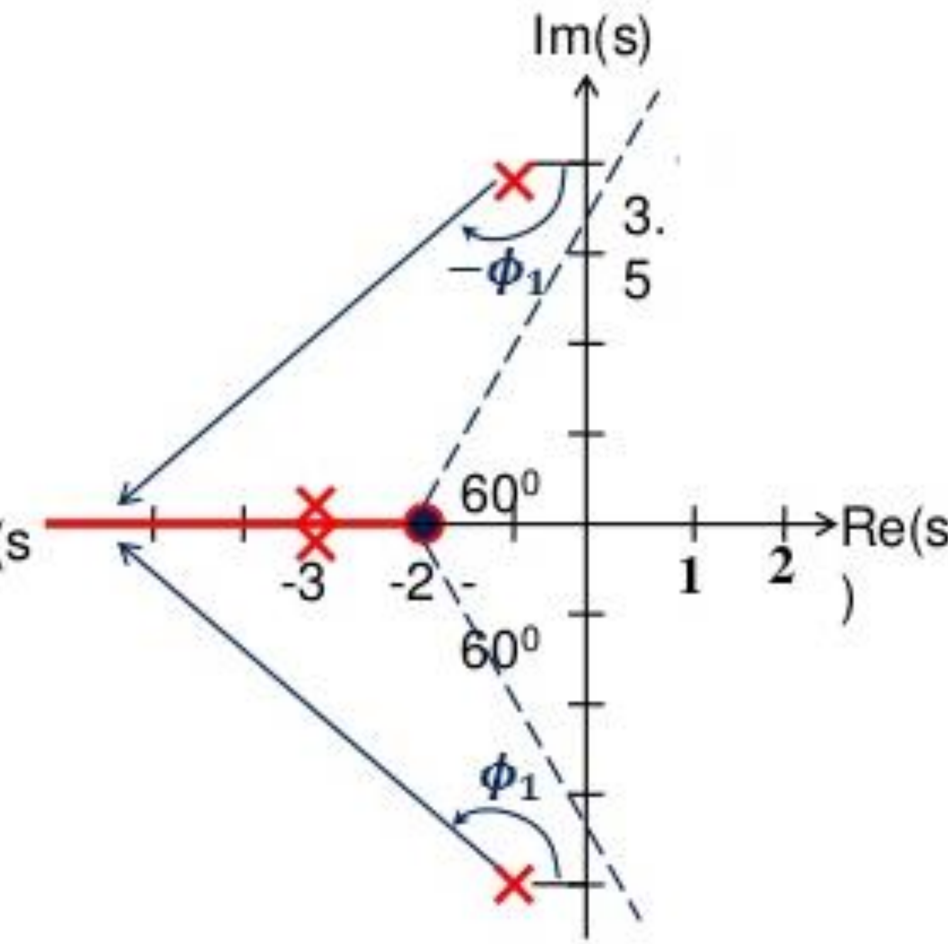
Root locus diagram:

S-plane (roots of transfer function of the system) based analysis method for measuring: stability, time domain performance, and frequency domain performance.

Components of root locus analysis (Evan’s rule):

- Roots (poles and zeros) locations in the s-plane.
- Real axis intercept and asymptotes.
- Break-away and break-in points.
- Angle of departure from complex poles pair.
- Imaginary axis (y-axis) intercept point with the root locus.

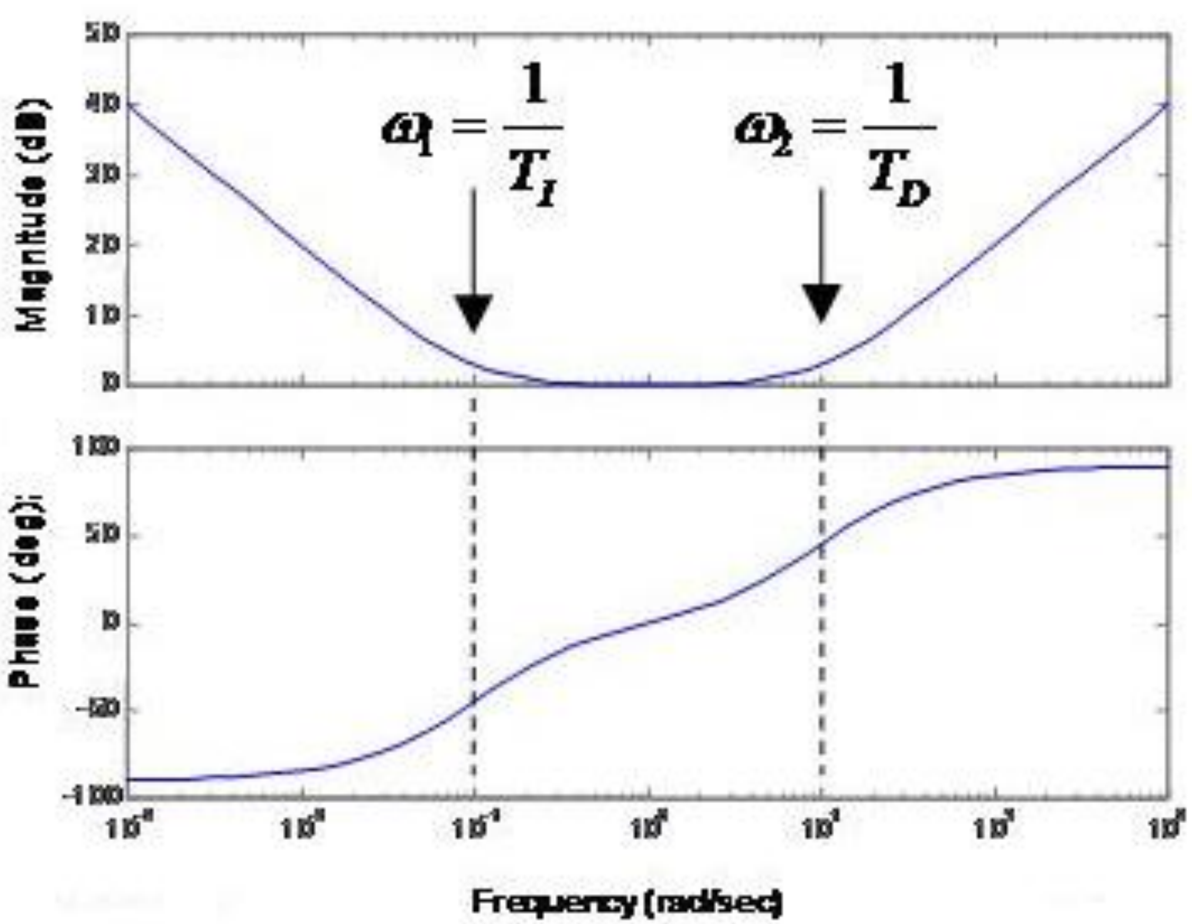


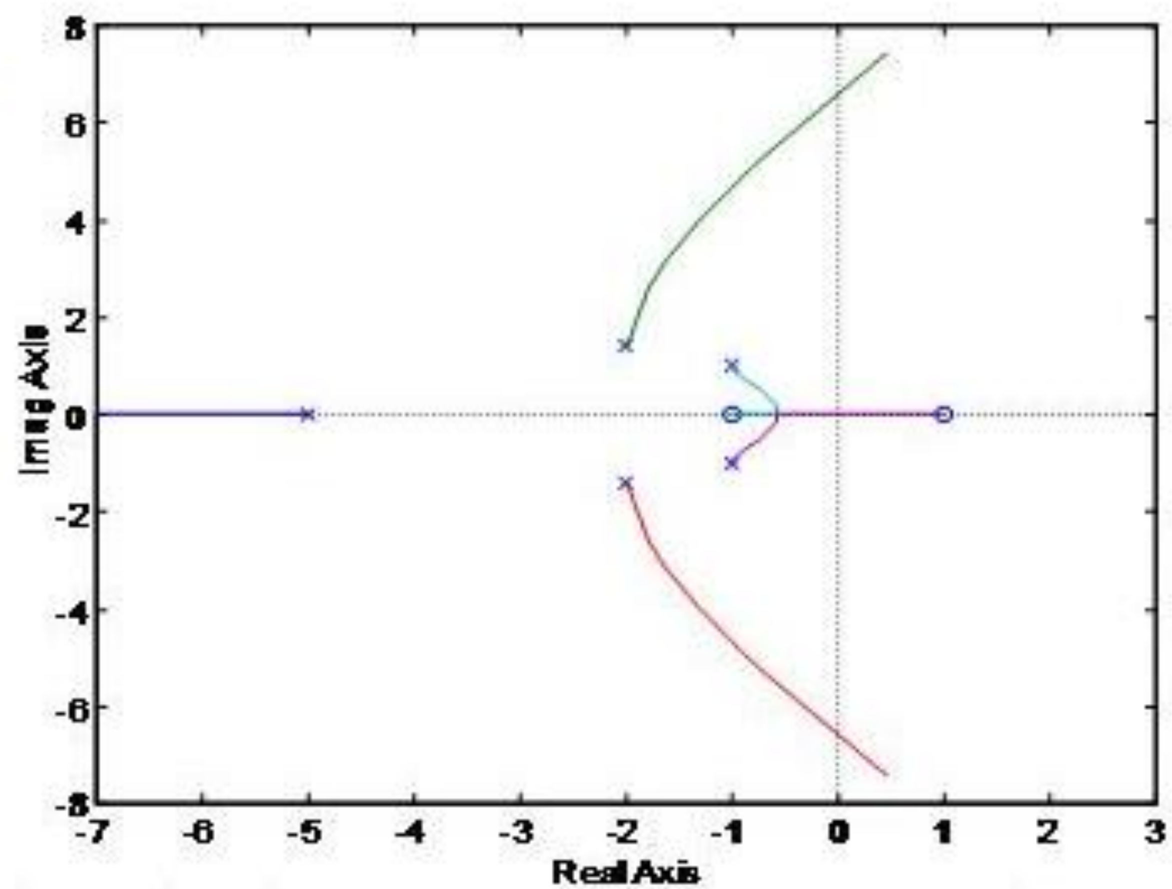


7. Root Locus Diagram

Analysis of root locus:

- Stability: Gain, Phase Shift Angle and Interception with y-axis.
- Transient Performance (Damping Factor, Rise Time, Settling Time, Time-to-Peak, %OS, Steady State Errors, etc.): location of poles and zeros in the diagram.



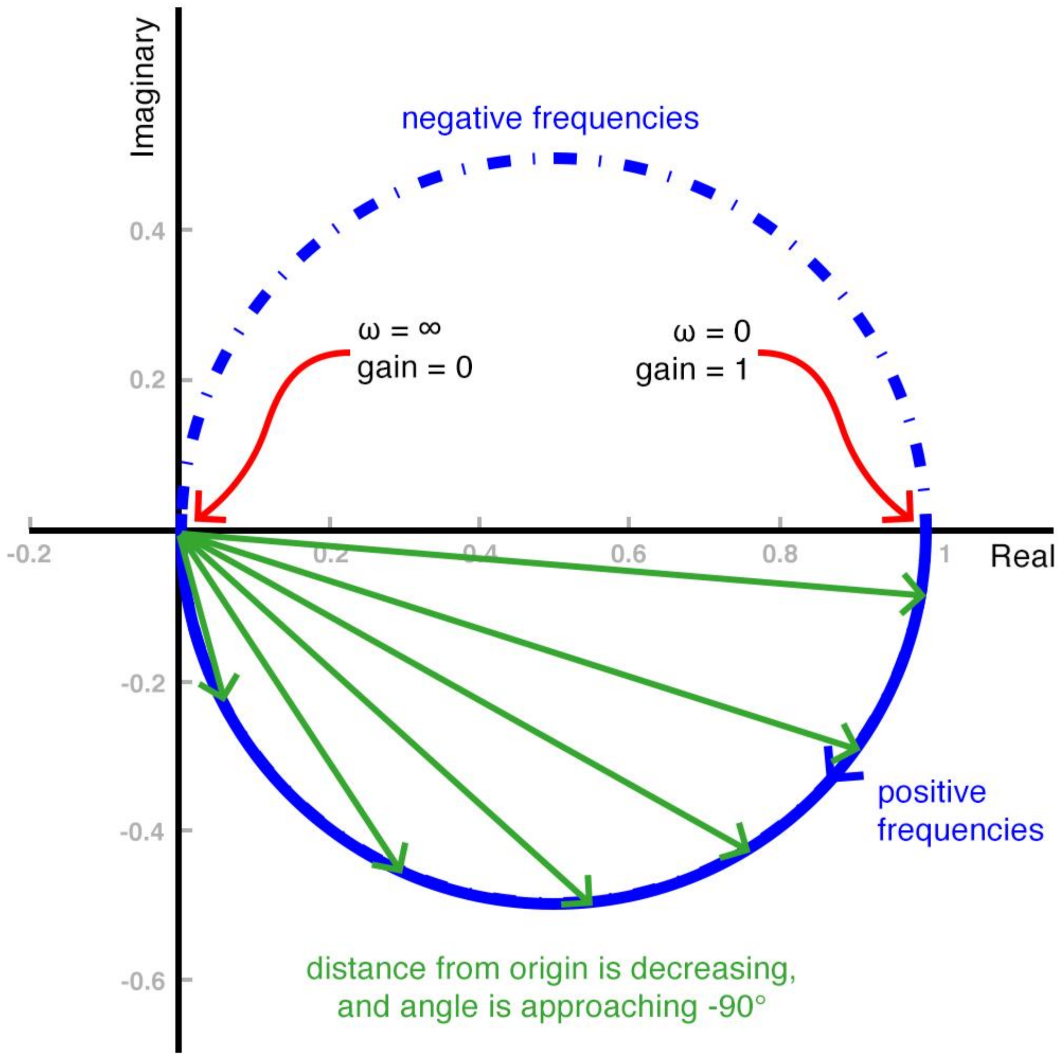


8. Nyquist Diagram

Nyquist Diagram

- Frequency domain analysis method for measuring: stability, time domain performance, and frequency domain performance.

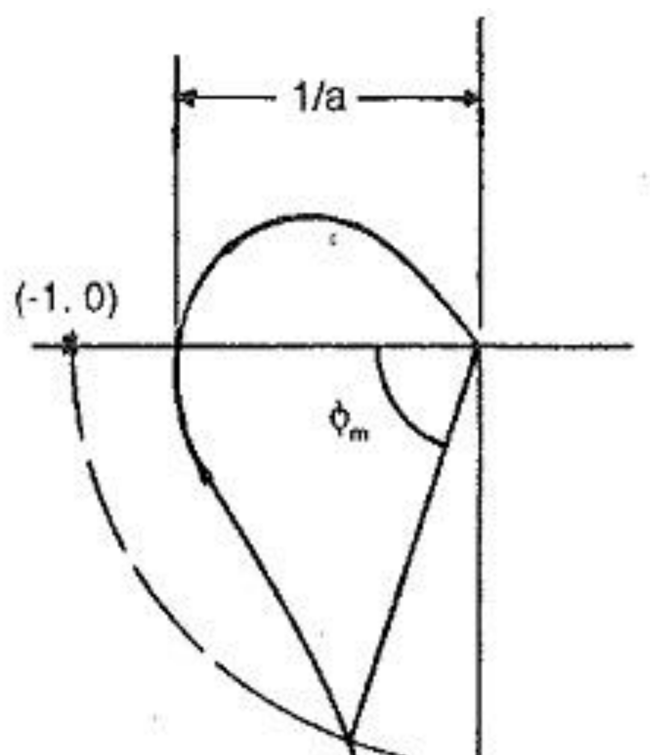
- Component of analysis:
    - Magnitude.
    - Phase shift angle.
    - Encirclement at test point (-1,0).
    - Gain and phase margins.



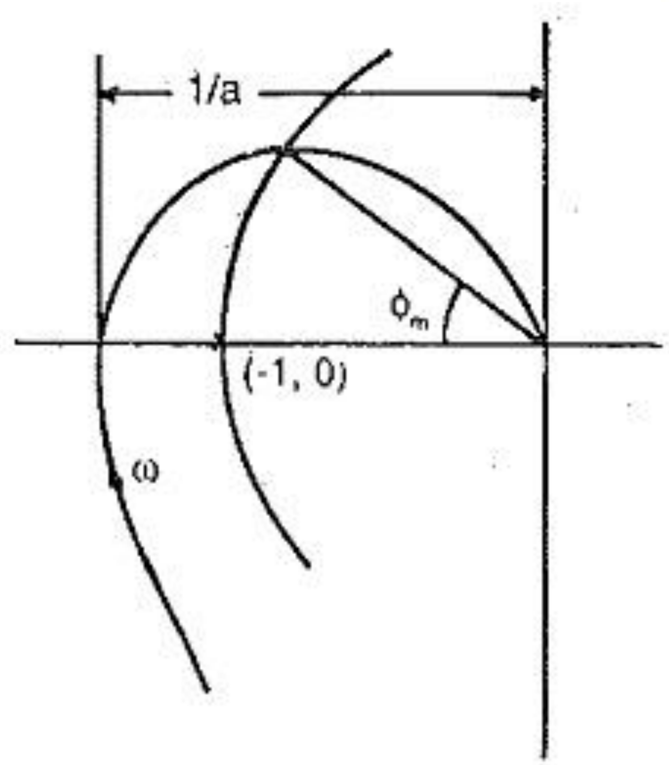
8. Nyquist Diagram

Analysis of Nyquist diagram:

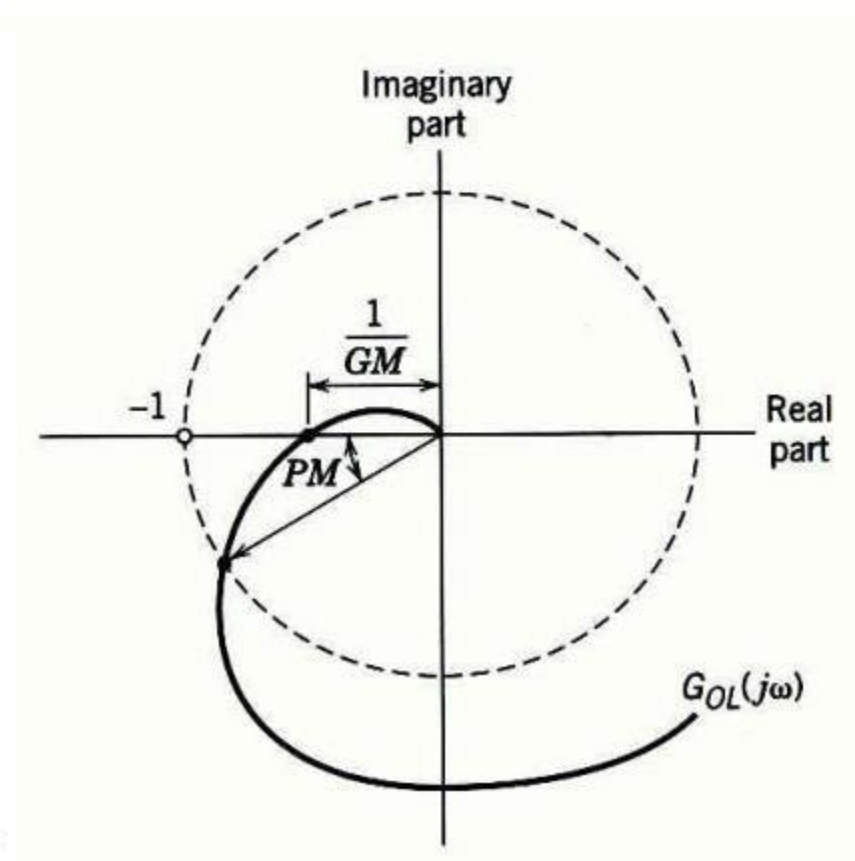
- Stability: Gain, Phase Shift Angle, Phase Margin, Gain Margin and encirclement at test point (-1,0).
- Transient Performance (e.g. Damping Factor, Rise Time, Settling Time, Time-to-Peak, %OS, Steady-State Errors, etc.): encirclement, type of contour, and detour on contour.



Stable system a>1, ϕ_m positive



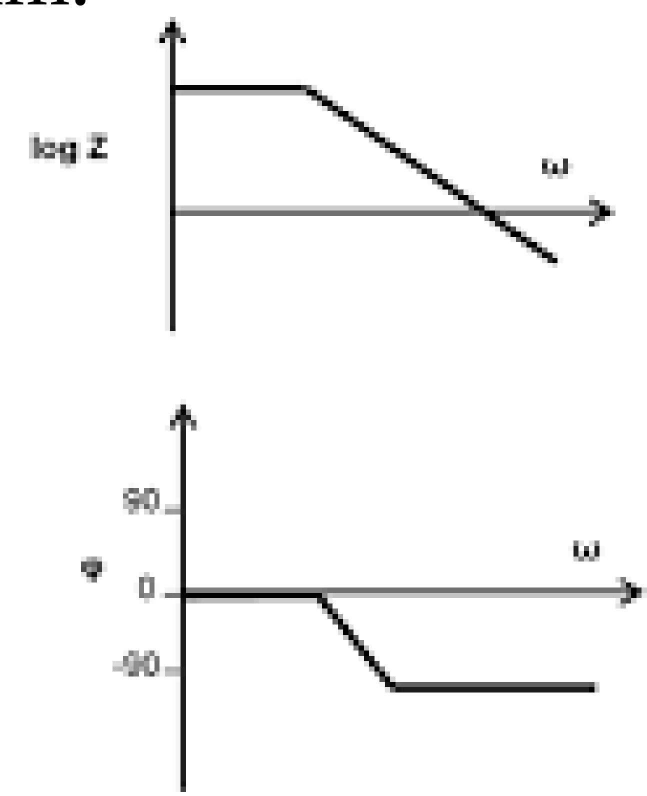
Unstable system a<1, ϕ_m negative

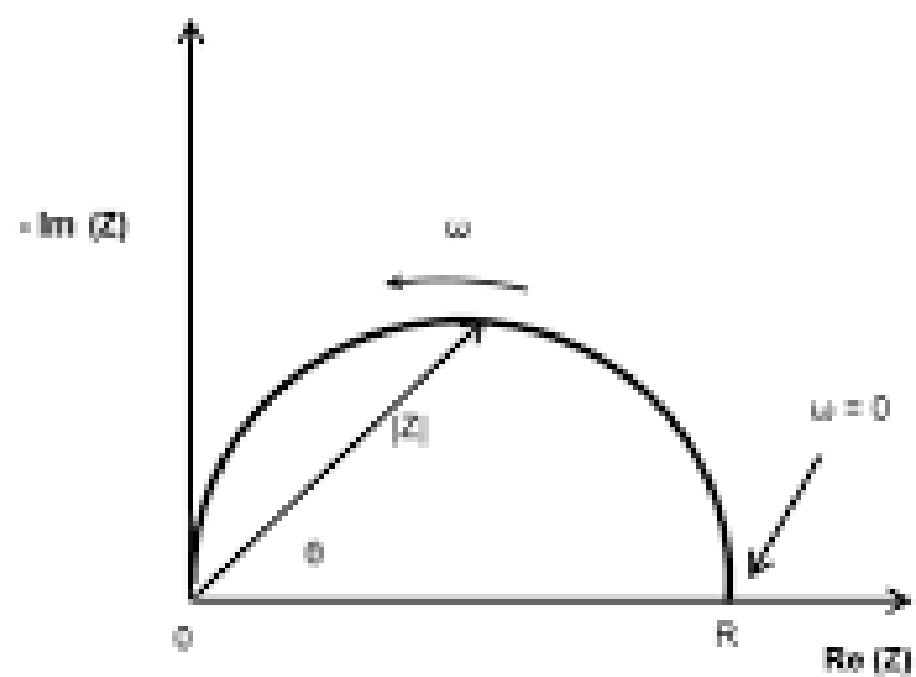


8. Nyquist Diagram

Derivation of Nyquist diagram:

- From Bode plots: straight plotting of the points of interest in the graph from the gain or magnitude and phase shift of the Bode plots to the Nyquist diagram.
- Although the gain or magnitude needs to be converted from logarithmic scale (dB) in Bode plots to linear scale in Nyquist diagram.





8. Nyquist Diagram

Derivation of Nyquist diagram:

- From root locus diagram: Conversion of the sum of the magnitude or gain contribution at unity gain and sum of phase shift angle contribution at -180 of all poles and zeros in the root locus diagram to magnitude or gain and phase shift in the Nyquist diagram.