



VICTORIA UNIVERSITY OF
WELLINGTON
TE HERENGA WAKA

Mid-Term Test Review

XMUT315 Control Systems Engineering

Mid-Term Test

- 6 questions, 10-25 marks each.
- 2 hours.
- Bring non-programmable calculator, dictionary and ruler.
- Selected formulas in control system (see revision midterm questions).
- Time and place: T.B.A.

Question Types

Expect 3 types of question in the midterm test:

- Explain...
 - a. Describe the significant of coefficients a and β in feedback system. [5 marks]
- Calculate ...
 - b. Calculate time constant (τ), rise time (T_r), time to peak (t_p), settling time (t_s) and percentage overshoot (%OS) of a second order system. [15 marks]
- Analyse ...
 - c. Determine and sketch an over damped second order system with damping ratio of 1.5 and natural frequency of 2 rad/sec. [10 marks]

Mid-Term Test Topics

1. Physical system modelling
 - Laplace transform.
 - Modelling from physical system.
2. Feedback control system
 - Feedback systems.
 - Block manipulations.
3. Stability of system
 - Concept and criteria of stability.
 - Routh-Hurwitz criterion.
 - Other approaches (e.g. Nyquist, Nichols, Bode and Root Locus).
4. Time-domain analysis
 - First-order systems.
 - Second-order systems.
 - Steady-state analysis.

I. System Modelling

- What is a system?
 - Components acting together to carry out an activity, perform a duty, or solve a problem
- A system is built of:
 - Sub-systems
 - Processes
 - Input and output
- What is our condition of stability?
 - Natural response must eventually approach zero (or oscillate)
- Advantages and disadvantages of open loop systems
 - Does not monitor output so cannot correct for disturbances
 - Simpler and less expensive
- Advantages and disadvantages of closed loop systems:
 - Monitors the output and compares it to the input, correcting for errors
 - More complex and expensive than open loop, can create instability.

I. System Modelling

- When building up a model of a physical system:
 - Components should be easily identifiable.
 - Components should have a simple and clearly defined interaction with other components.
 - Components numbers should be minimised.
- In order to analyse a system:
 1. We identify an input signal [a variable]
 2. Using block diagram components, we combine internal signals [Basic block, summing junction, take-off point]
to produce the output signal [modified variables]
[another variable]

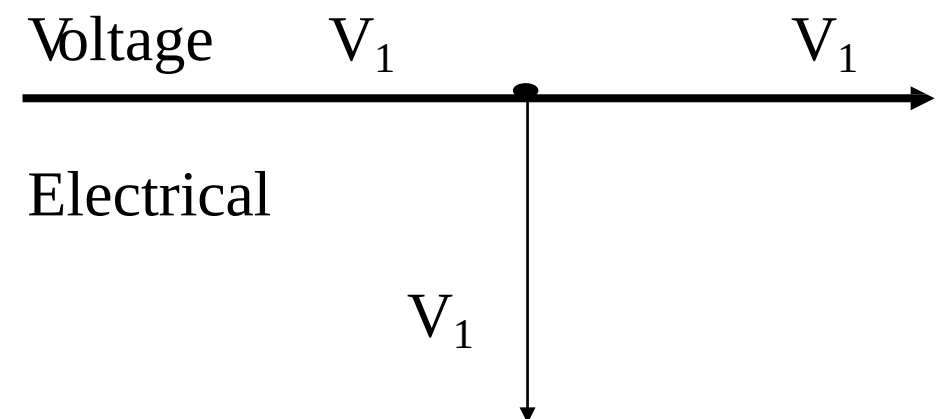
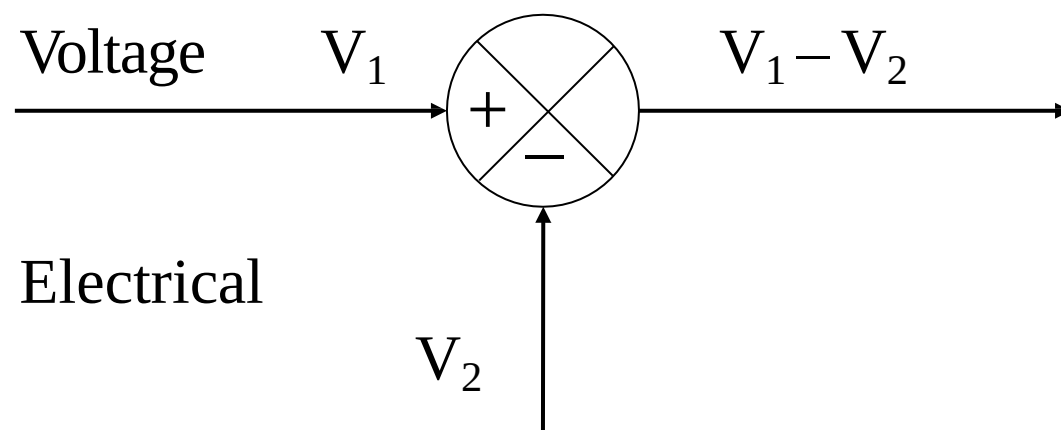
The Input-Output relationship may then be determined

I. System Modelling

1. Signals

- Components are connected together by signal
- Signals have many different forms.
- Must also have direction & name.
- Signals continue until interrupted.
- Signals and components are considered ideal.
- We add other signals and components to alter the properties.

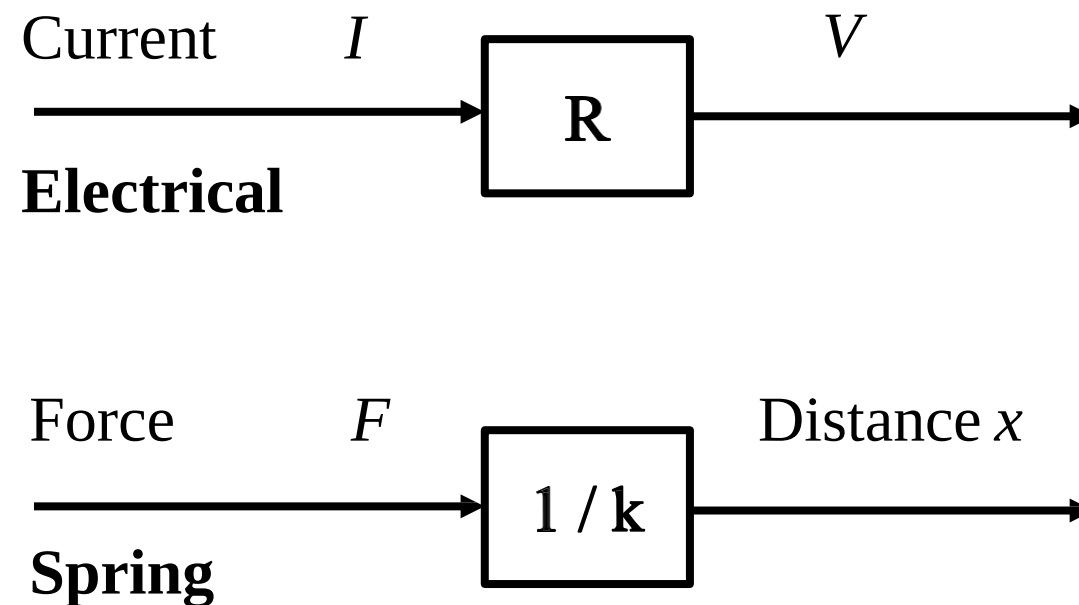
2. Components



I. System Modelling

2. Components (cont..)

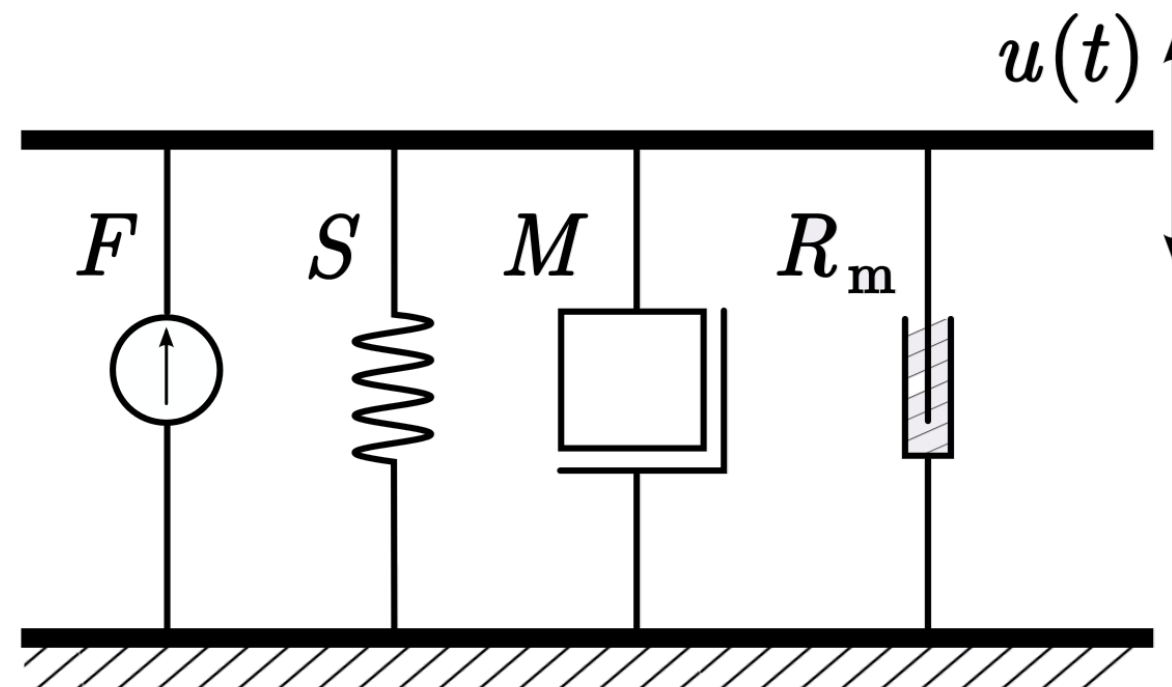
Block is/ or function of the system signal.



Only one input and only one output.

I. System Modelling

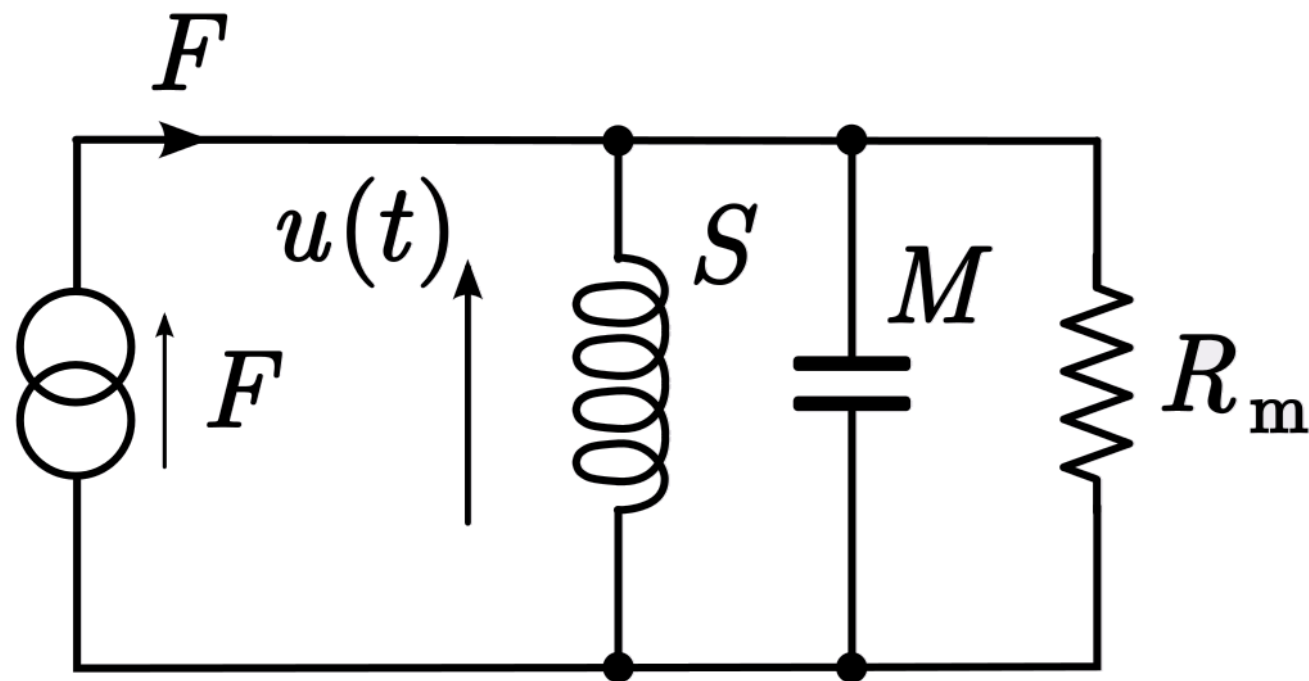
Modelling mechanical systems:



- Newton Law: $\sum F = Ma$
- Mass: Md^2x/dt^2 .
- Spring: kx (k = spring constant).
- Damper: Ddx/dt (D = damper constant).

I. System Modelling

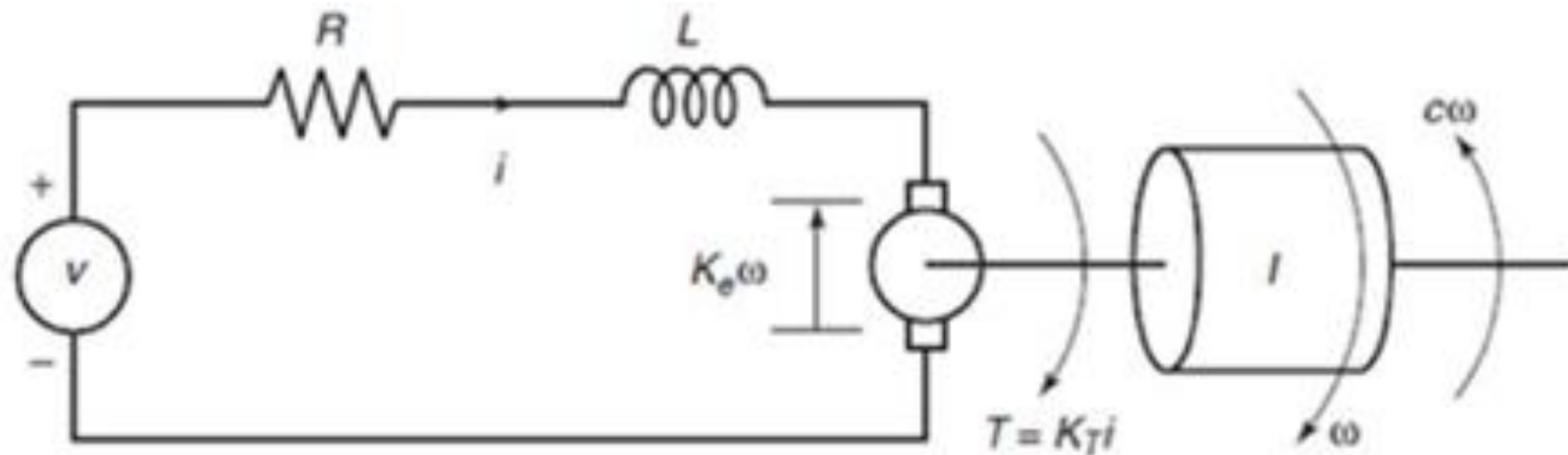
Modelling electrical systems:



- Ohm Law: $I = V/R$
- Kirchhoff laws: $I_{total} = I_a + I_b + \dots$ and $V_{total} = V_a + V_b + \dots$
- Resistor: R
- Capacitor: $1/j\omega C$
- Inductor: $j\omega L$

I. System Modelling

Modelling electromechanical systems:



Electromechanical model of dc motor.

Mechanical Part:

- Newton Law: $\sum F = Ma$
- Mass: Md^2x/dt^2
- Spring: kx (k = spring constant)
- Damper: Ddx/dt (D = damper constant)

Electrical Part:

- Ohm Law: $I = V/R$
- Kirchhoff laws: $I_{total} = I_a + I_b + \dots$ and $V_{total} = V_a + V_b + \dots$
- Resistor: R
- Capacitor: $1/j\omega C$
- Inductor: $j\omega L$

I. System Modelling

Laplace transforms:

- Simple Real Poles

$$Y(s) = \frac{s + 1}{s^3 + s^2 - 6s} = \frac{A}{s} + \frac{B}{s - 2} + \frac{C}{s + 3}$$

- Unique Complex Factors

$$Y(s) = \frac{1}{s(s^2 + s + 1)} = \frac{1}{s} + \frac{A_2s + A_3}{s^2 + s + 1}$$

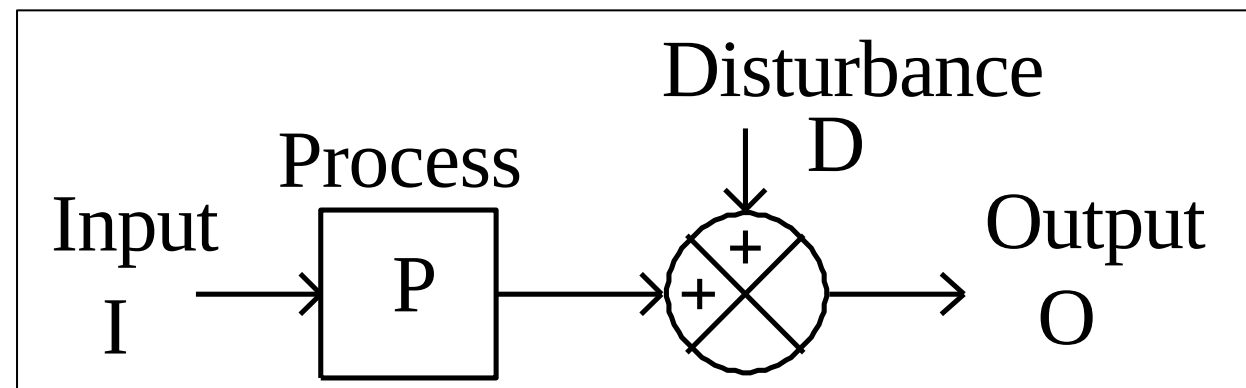
- Repeated Real Factors

$$Y(s) = \frac{3s + 8}{(s + 2)^2} = \frac{A_2}{(s + 2)^2} + \frac{A_1}{s + 2}$$

2. Feedback Control Systems

We would like to control the output of a system

i.e. we have the output resemble the input, despite disturbances



Open loop system (no feedback)

Process with transfer function $\mathbf{P}$ perturbed by a disturbance $\mathbf{D}$.

Suppose $\mathbf{P}$ is $\mathbf{10}$ and disturbance $\mathbf{D}$ is $\mathbf{0}$.

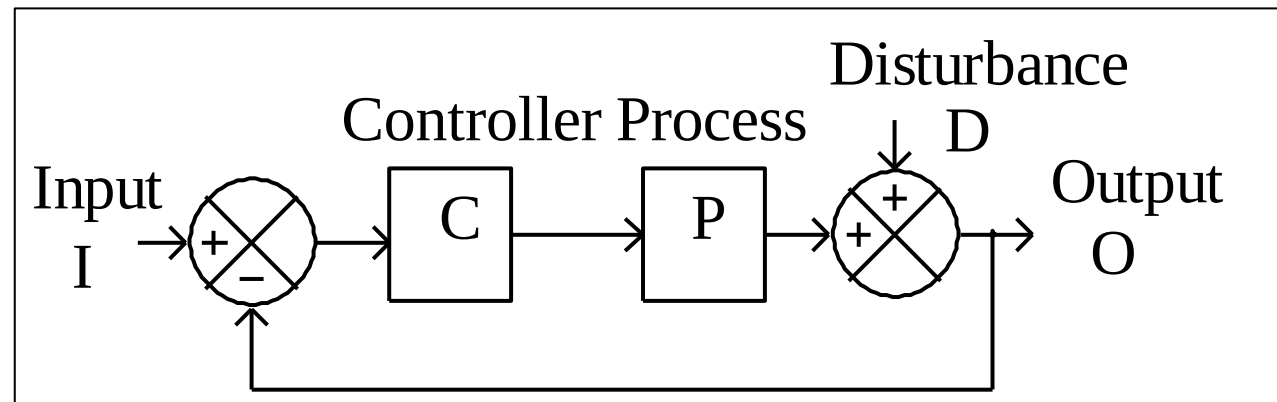
If the output $\mathbf{O}$ is to be $\mathbf{1}$, we make input $\mathbf{I} = \mathbf{0.1}$. But, if $\mathbf{P}$ changes by $\mathbf{10\%}$ to $\mathbf{11}$ then $\mathbf{O}$ changes by $\mathbf{10\%}$ to $\mathbf{1.1}$.

If disturbance $\mathbf{D}$ is $\mathbf{0.1}$, then $\mathbf{O}$ will also change by $\mathbf{0.1}$.

2. Feedback Control Systems

Closed loop system: Feedback added

Now consider the 'closed-loop' system below:



Closed loop system

P represents the device being controlled; **C** is the controller.

We will (initially) ignore disturbances (**D = 0**), by forward over 1 plus loop rule

Let **P = 10**, as before, and **C = 10**;

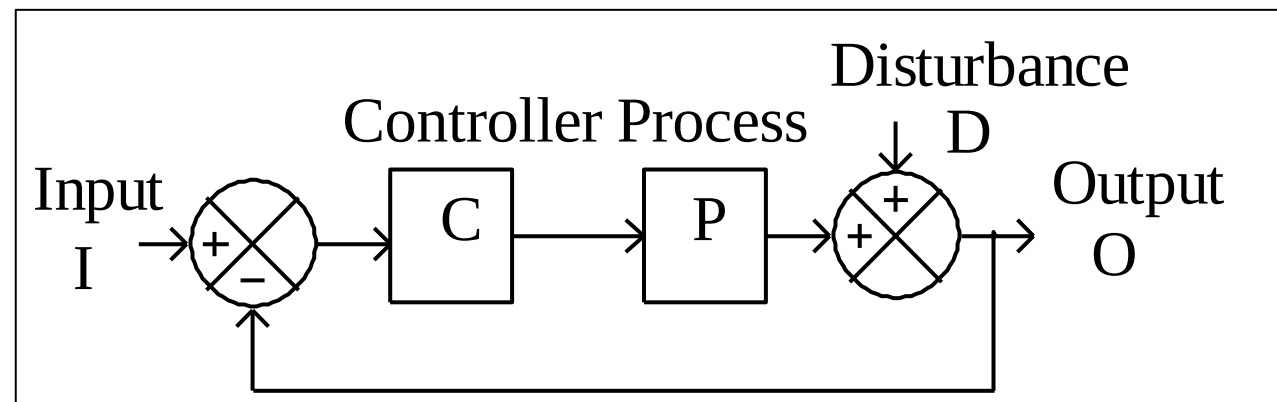
- If **I** is **1**, then **O** is **0.99** (within 1% of being 1)
- If **P** changes to **11**; $O = I * 110/111$;
- if **I** is 1, **O** is still about 0.99.

$$O = \frac{C * P}{1 + C * P} I$$

2. Feedback Control Systems

Closed loop system: Disturbance control

To see the effect of disturbances, assume I is 0.



We will see:

If $C = 10$, $P = 10$ and $D = 0.1$

Negative Feedback

$$\frac{O}{D} = \frac{\text{Forward}}{1 + \text{Loop}} = \frac{1}{1 + C * P}$$

$$O = \frac{1}{1 + 10 * 10} 0.1 = \frac{0.1}{101} = 0.00099$$

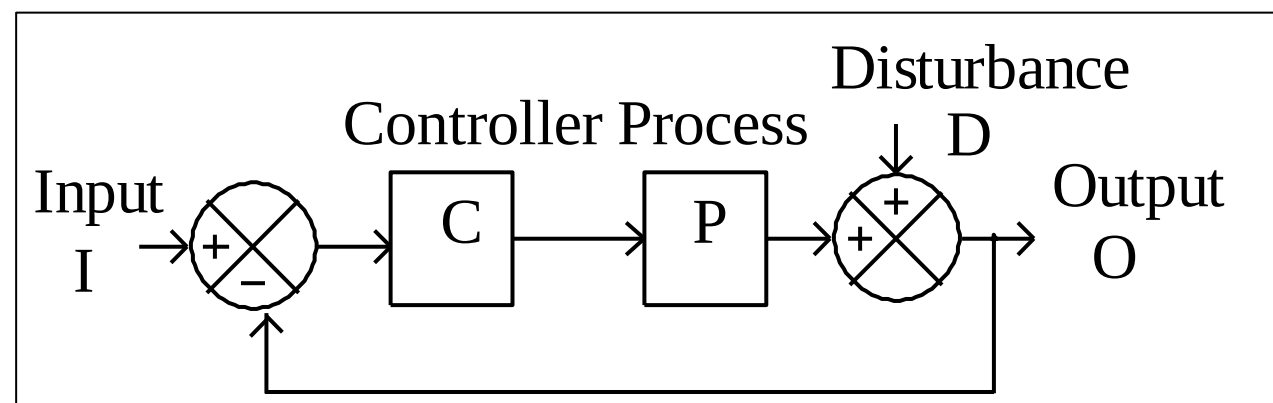
- if
- **Reduces effects of disturbances on output**
 - **Reduces effects of parameter changes on output**

|closed loop gain| < |open loop gain|

2. Feedback Control Systems

Principle of Superposition

What is superposition?



If CP large: $O \sim I + 0 = I$

$$O = \frac{C * P}{1 + C * P} I + \frac{1}{1 + C * P} D$$

So feedback:

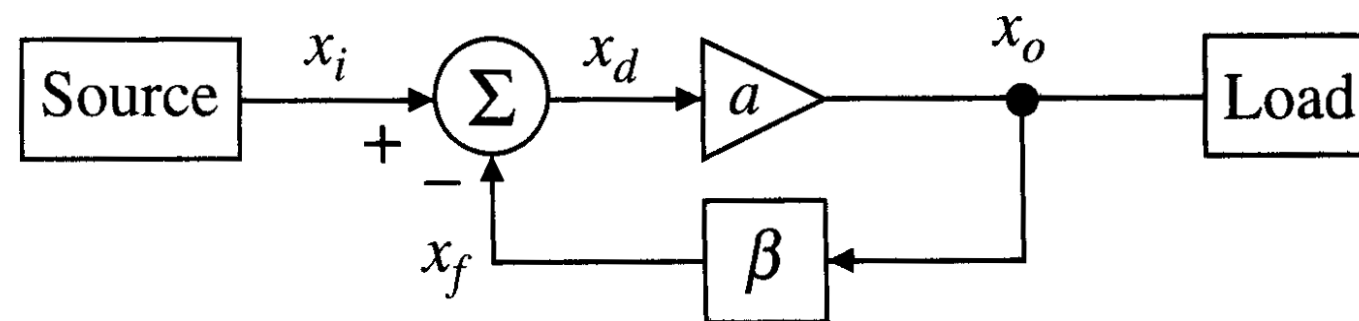
- Makes output almost same as input.
- Minimises effects of disturbances.
- Reduces effect of change in device.

This is true because the 'loop gain', $C * P$ is high.

2. Feedback Control Systems

Feedback Equation

You are given a given feedback system as shown in the following diagram.



The equation that represents the feedback system is:

$$\frac{x_o(s)}{x_i(s)} = \frac{a}{1 + a\beta}$$

2. Feedback Control Systems

Compensator is needed for improving the control systems:

- In order to obtain the desired performance of the system, we use compensators.
- Compensators are applied to the system in the form of feed forward path gain adjustment.
- Compensate an unstable system to make it stable.
- A compensating network is used to minimise overshoot.
- Compensators could increase the steady state accuracy of the system.
- An important point to be noted here is that the increase in the steady state accuracy brings instability to the system.
- Compensator could also introduce poles and zeros in the system thereby causes changes in the transfer function of the system.
- Due to this, performance specifications of the system change.

2. Feedback Control Systems

The simplest compensating networks used for compensators:

- Lead compensators.
- Lag compensators.
- Lead-lag compensators.

Types of controller for managing or controlling specific behavior/ characteristics of the control systems in control systems engineering:

- Proportional controller
- Integral controller
- Derivative controller
- PID controller
- Fuzzy logic controller

3. Stability

Stability:

- A linear, time-invariant system is stable if: the natural response approaches zero as time approaches infinity.
- A linear, time-invariant system is stable if: the natural response grows without bound as time approaches infinity.
- A linear, time-invariant system is marginally stable if: the natural response neither decays nor grows but remains constant, or oscillates as time approaches infinity.

We want to build up a relationship between the total response and stability

- if input is bounded and $C(t)$ does not approach infinity (∞) as t approaches ∞ , natural response is not approaching ∞ .
- if input is unbounded, we can not conclude stability.

A system is stable if every bounded input yields a bounded output, or Bounded-input, bounded-output (BIBO).

3. Stability

Stability (cont..)

1. A system is stable if every bounded input yields a bounded output, or Bounded-input, bounded-output (BIBO)

We want to build up a relationship between the total response and instability:

- If the input is bounded but the output, $C(t)$ is unbounded, the system is unstable.
- If input is unbounded, we can not conclude instability.

2. A system is unstable if any bounded input yields an unbounded output.

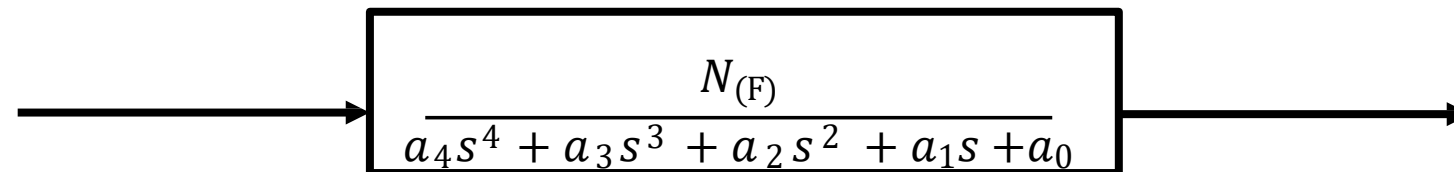
3. Stability

Routh-Hurwitz Criterion

- Using this method, we can tell how many closed-loop system poles are in the left half-plane, in the right half-plane, and on the $j\omega$ -axis.
- We can not tell where, but only how many are in each plane determining the system stability.
- The method requires two steps:
 - Generate a data table called a Routh table and
 - Interpret the Routh table to tell how many closed-loop system poles are in the left half-plane, the right half-plane, and on the $j\omega$ -axis.

3. Stability

Generating a Routh Table



Equivalent closed loop transfer function

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

3. Stability

Interpreting a Routh Table:

Simply stated, *the Routh-Hurwitz criterion declares that the number of roots of the polynomial that are in the right hand-plane is equal to the number of sign changes in the first column.*

3. Stability

Routh-Hurwitz Special Cases

1. Zero in the first row/column

In this case, zero is replaced with epsilon (ϵ) and will tend to zero

$$\frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$$

s^5	2	+		
s^4	3	+		
s^3	ϵ	+		
s^2	$\frac{3\epsilon - 4}{\epsilon}$	-		
s^1	$\frac{12\epsilon - 16 - 3\epsilon}{9\epsilon - 12}$	+		
s^0	1	+		

3 LH poles and 2 RH poles

3. Stability

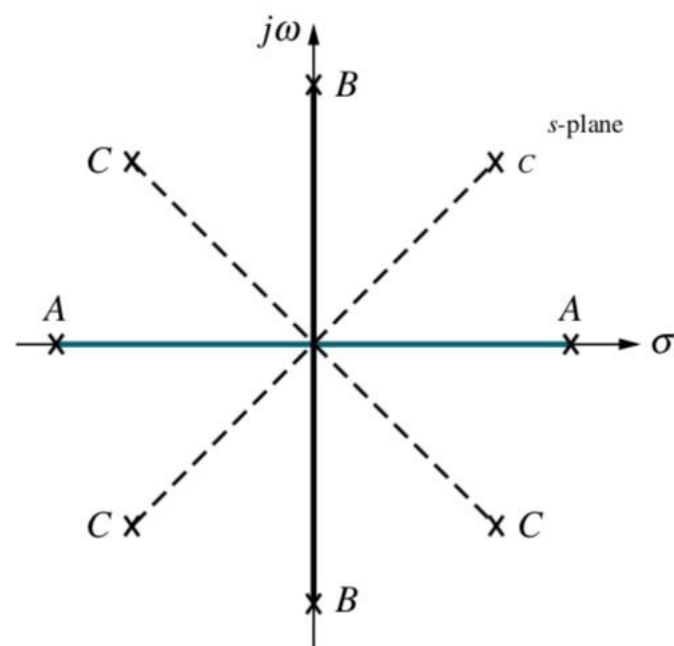
Routh-Hurwitz Special Cases (cont.)

2. Entire row of zeros

A: Real and symmetrical about the origin

B: Imaginary and symmetrical about the origin

C: Quadrantal and symmetrical about the origin



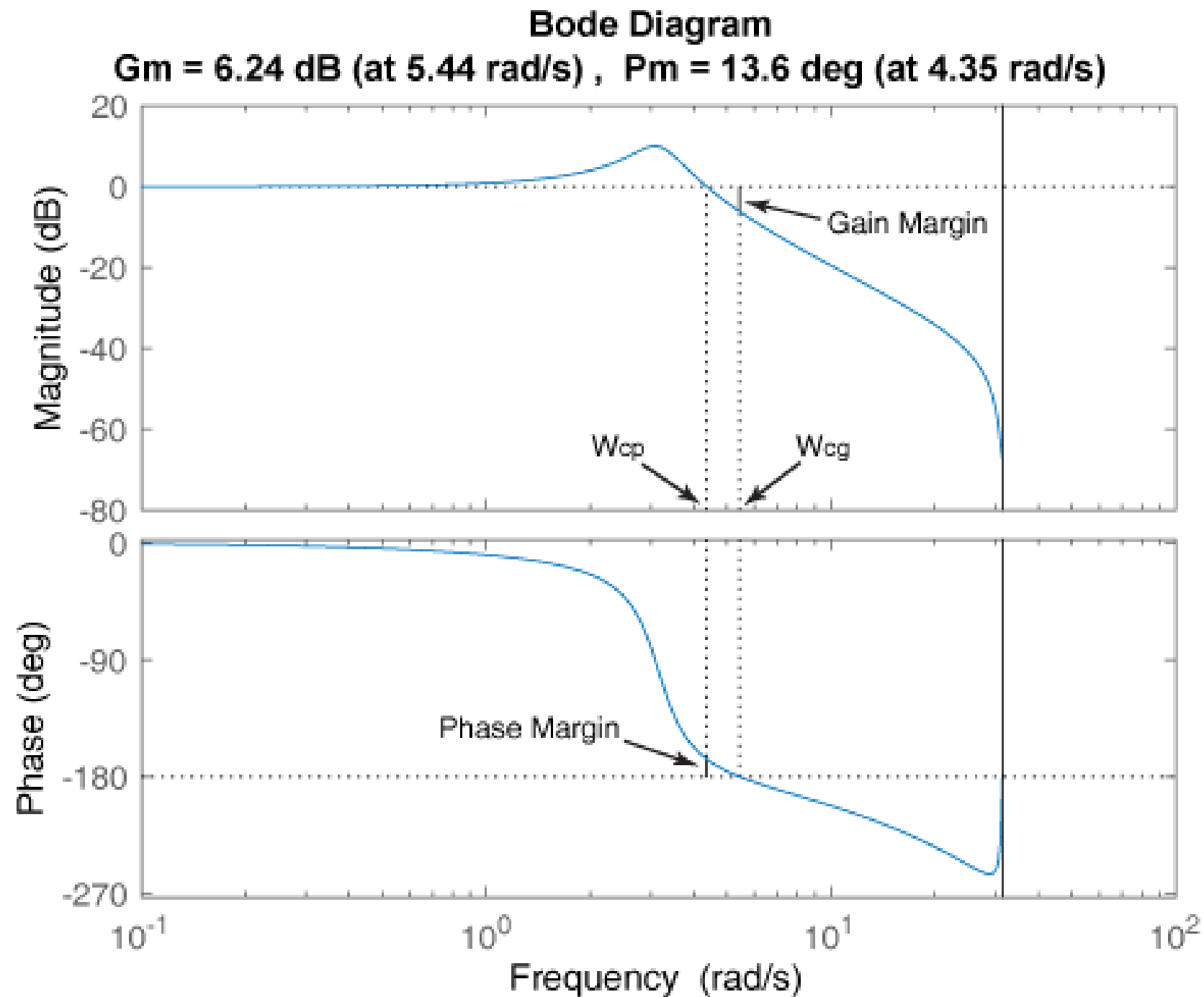
$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

Routh table

s^5	1	6	8	
s^4	7 1	42 6	56 8	
s^3	0	0	0	
s^2	3	8	0	
s^1	$\frac{1}{3}$	0	0	
s^0	8	0	0	

3. Stability

Bode plot:

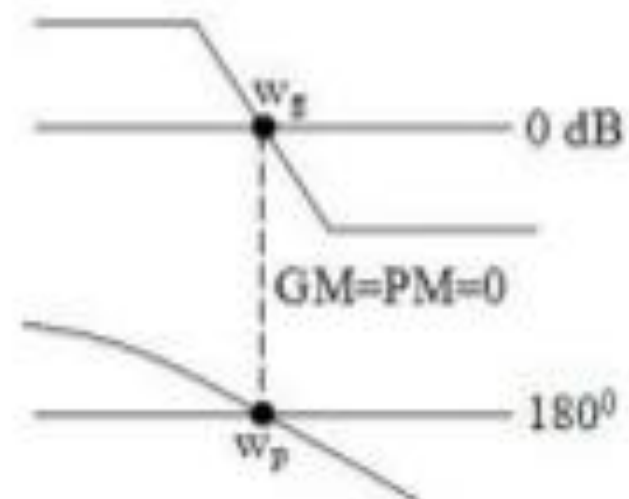
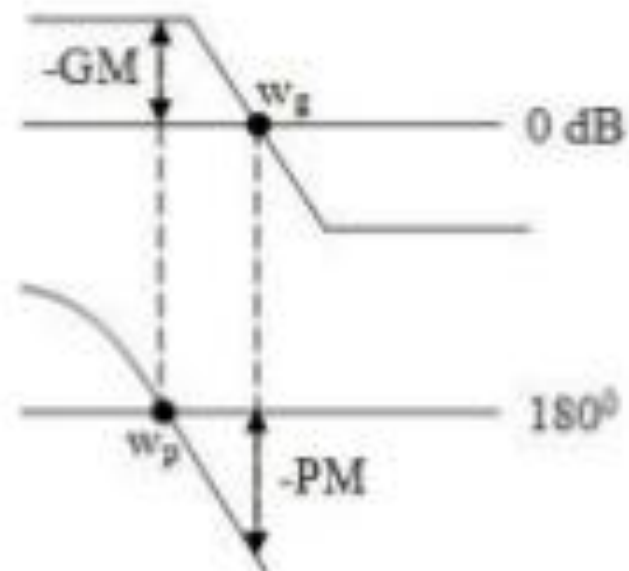
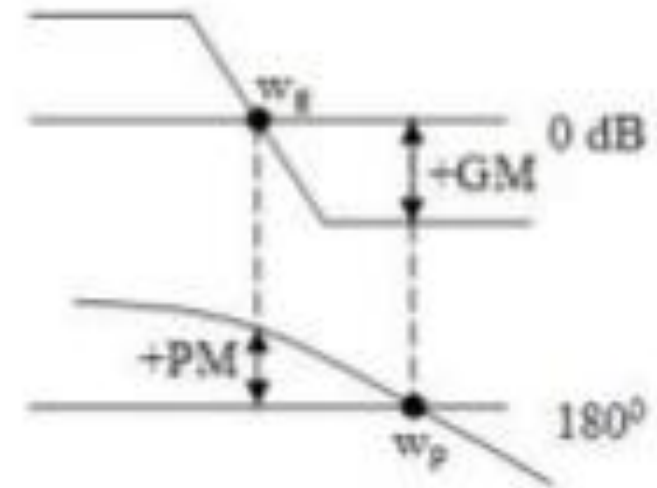


3. Stability

Gain Margin and Phase Margin in the Bode plot

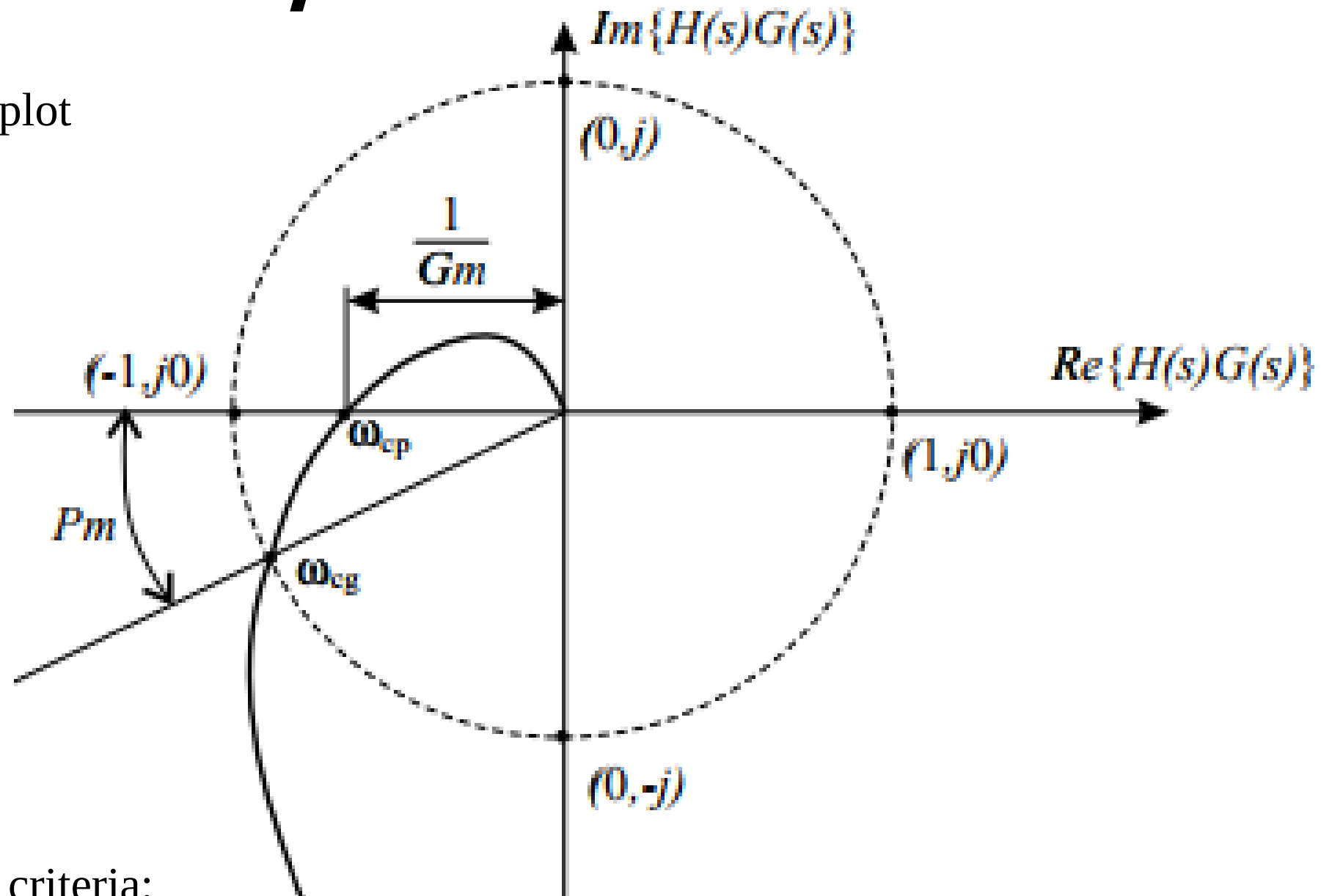
Stability criteria:

- Stable Systems: Gain Margin and Phase Margin are (+)
- Unstable Systems: Gain Margin and Phase Margin are (-)
- Marginally Stable Systems: Gain Margin and Phase Margin are 0



3. Stability

Nyquist plot

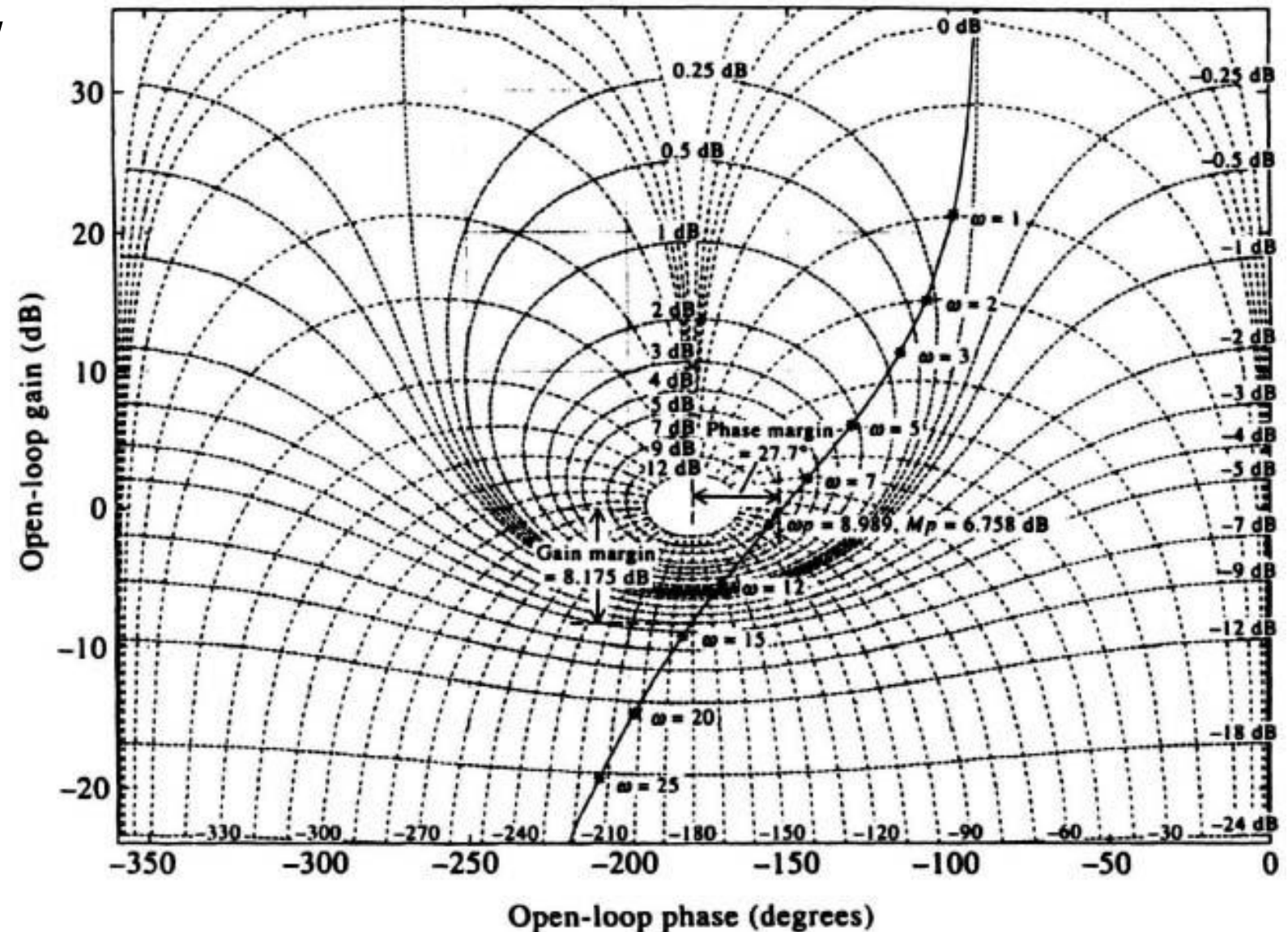


Stability criteria:

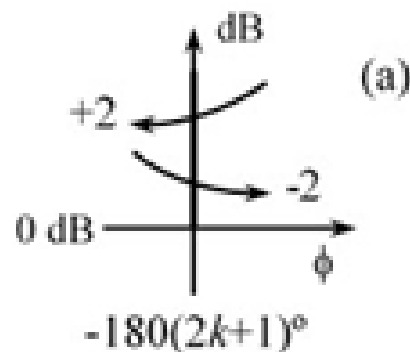
Unity gain point in the plot $(-1, j0)$ is not encircled by the plot

3. Stability

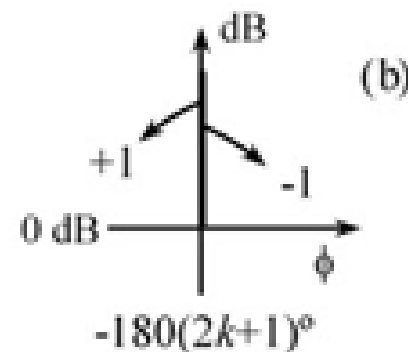
Nichols chart



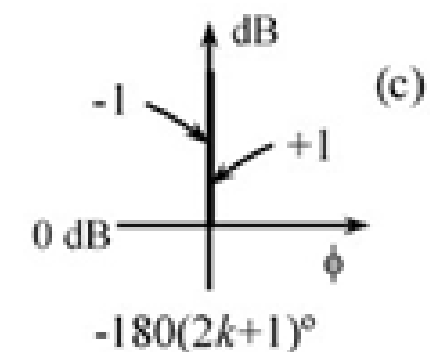
Stability criteria:



Crossing at $\omega = \omega_k$, $0 < \omega_k < +\infty$
 $|L(j\omega_k)| > 0$ dB, finite at crossing
 Add $N_a = \pm 2$ at each $-180(2k+1)^\circ$



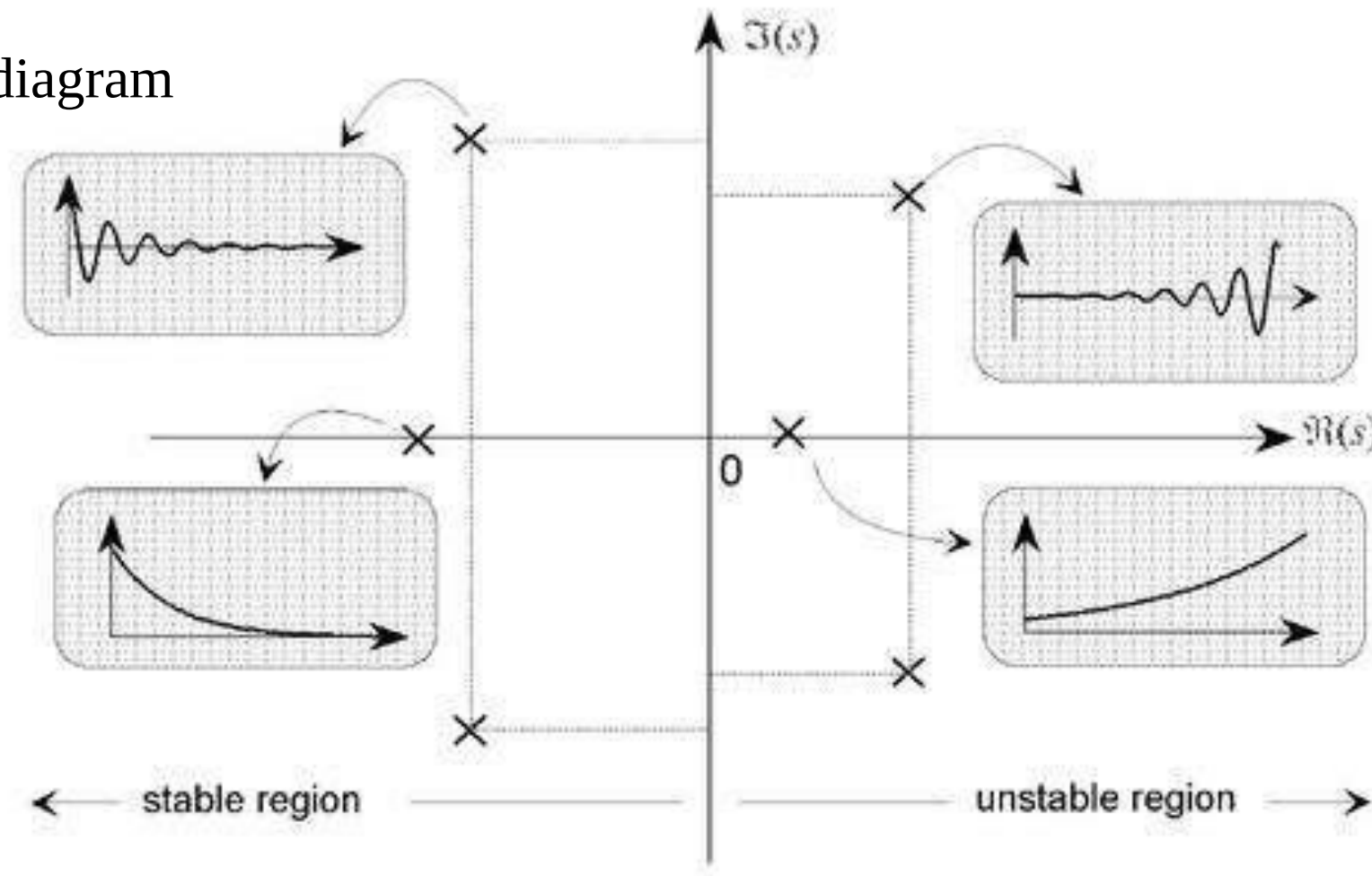
Crossing at $\omega = \omega_0 = 0$
 $|L(j\omega_0)| > 0$ dB, finite at crossing
 Add $N_b = \pm 1$



Crossing at $\omega = \omega_\infty = +\infty$
 $|L(j\omega_\infty)| > 0$ dB, finite at crossing
 Add $N_c = \pm 1$

3. Stability

Pole and zero diagram



Stability Criteria:

- Pole in the Left Hand Side: Stable systems
- Pole in the Imaginary- or Y-axis: Marginal stable systems
- Pole in the Right Hand Side: Unstable system

4. Time Domain Analysis

Estimating System Response

- The systems examined so far can be modelled by transfer functions:

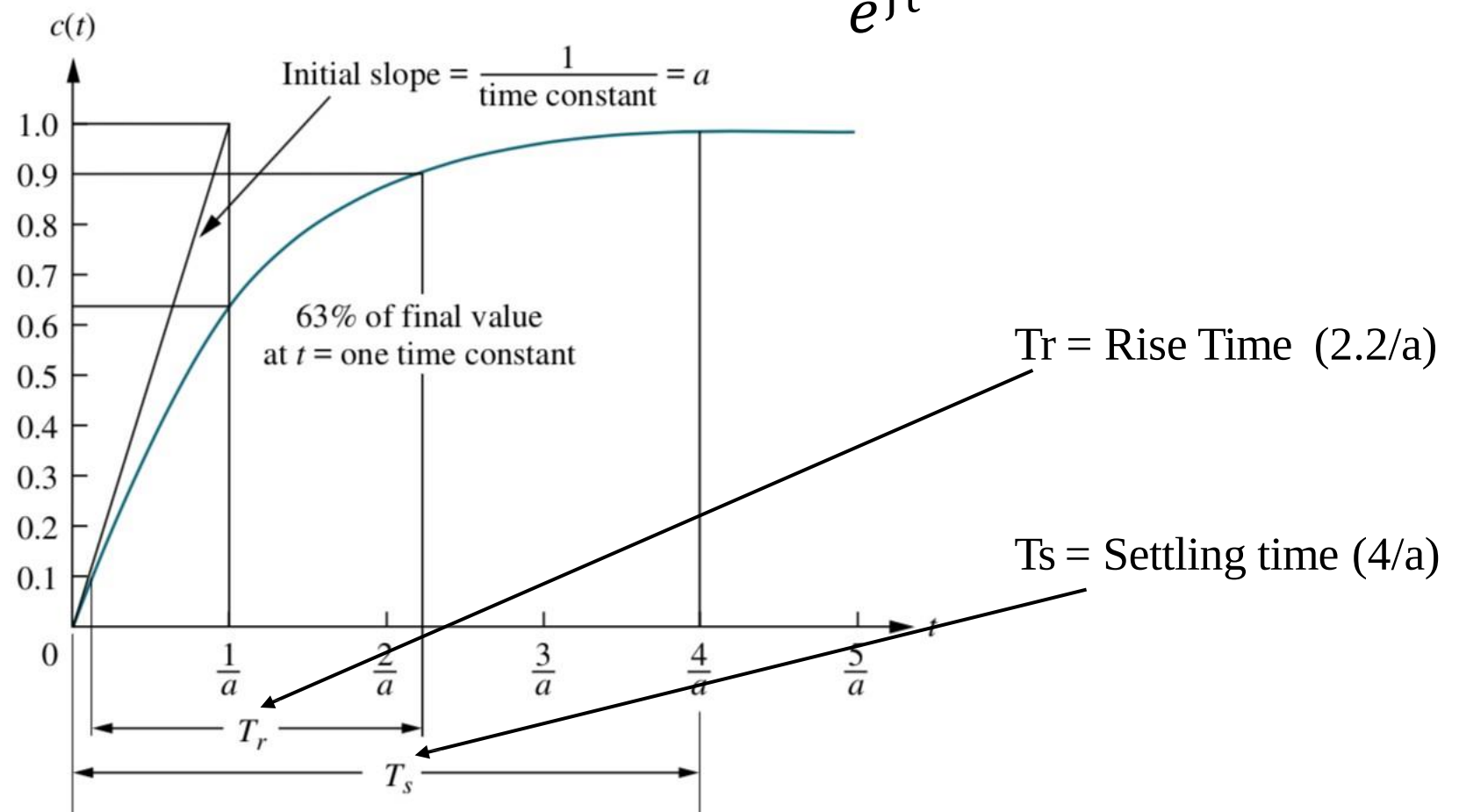
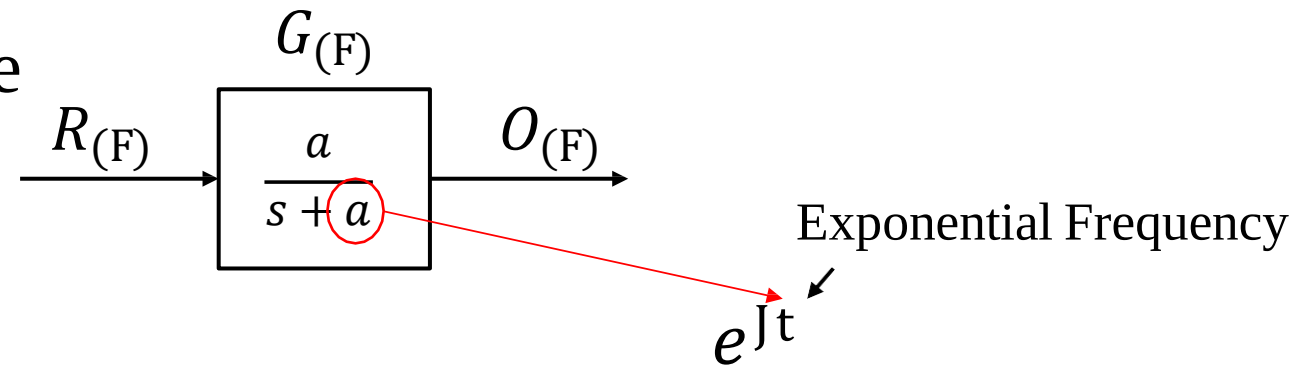
$$\frac{K}{1 + sT} \quad \text{or} \quad \frac{K}{As^2 + Bs + C}$$

- Given a particular input, what is the system output? We can use differential equation techniques.
- Easier to define the approximate response, just from the transfer function.
- We will do this, assuming that the input is a step.

4. Time Domain Analysis

First Order: Step Response

For R = step input:



4. Time Domain Analysis

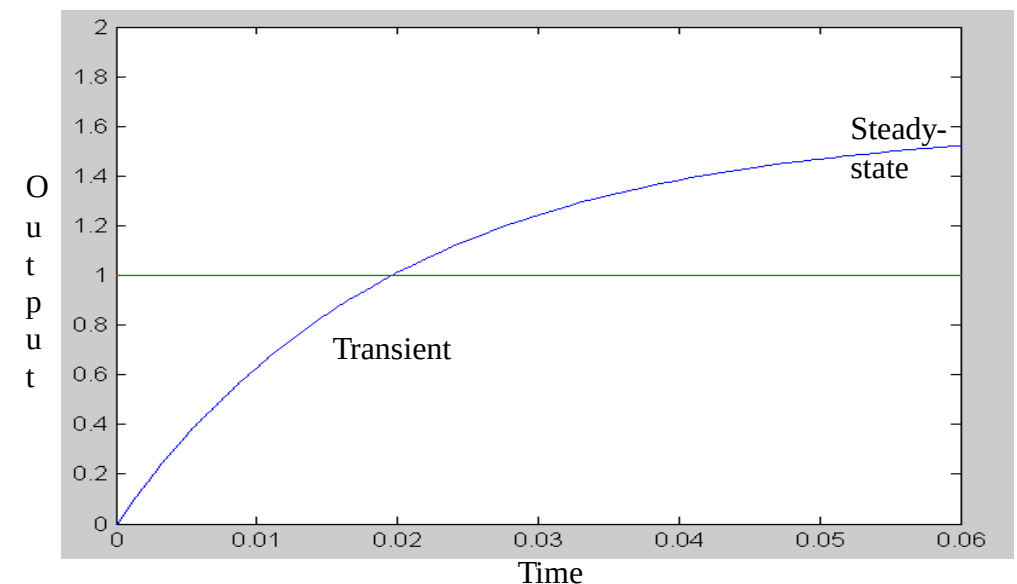
First Order: Steady-State Response

$$O(t) = K \left(1 - e^{-\frac{1}{T_1}t} \right) \quad \text{where } t \text{ is large: } e^{-\frac{1}{T_1}t} \rightarrow 0$$

In s domain, $s \rightarrow 0$

$$\frac{O(s)}{I(s)} = \frac{K}{1 + T_1 s}$$

Step input: $O_{ss}(s) = \frac{K}{1 + T_1 \times 0} = \frac{K}{1} = K$



4. Time Domain Analysis

General second order response

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

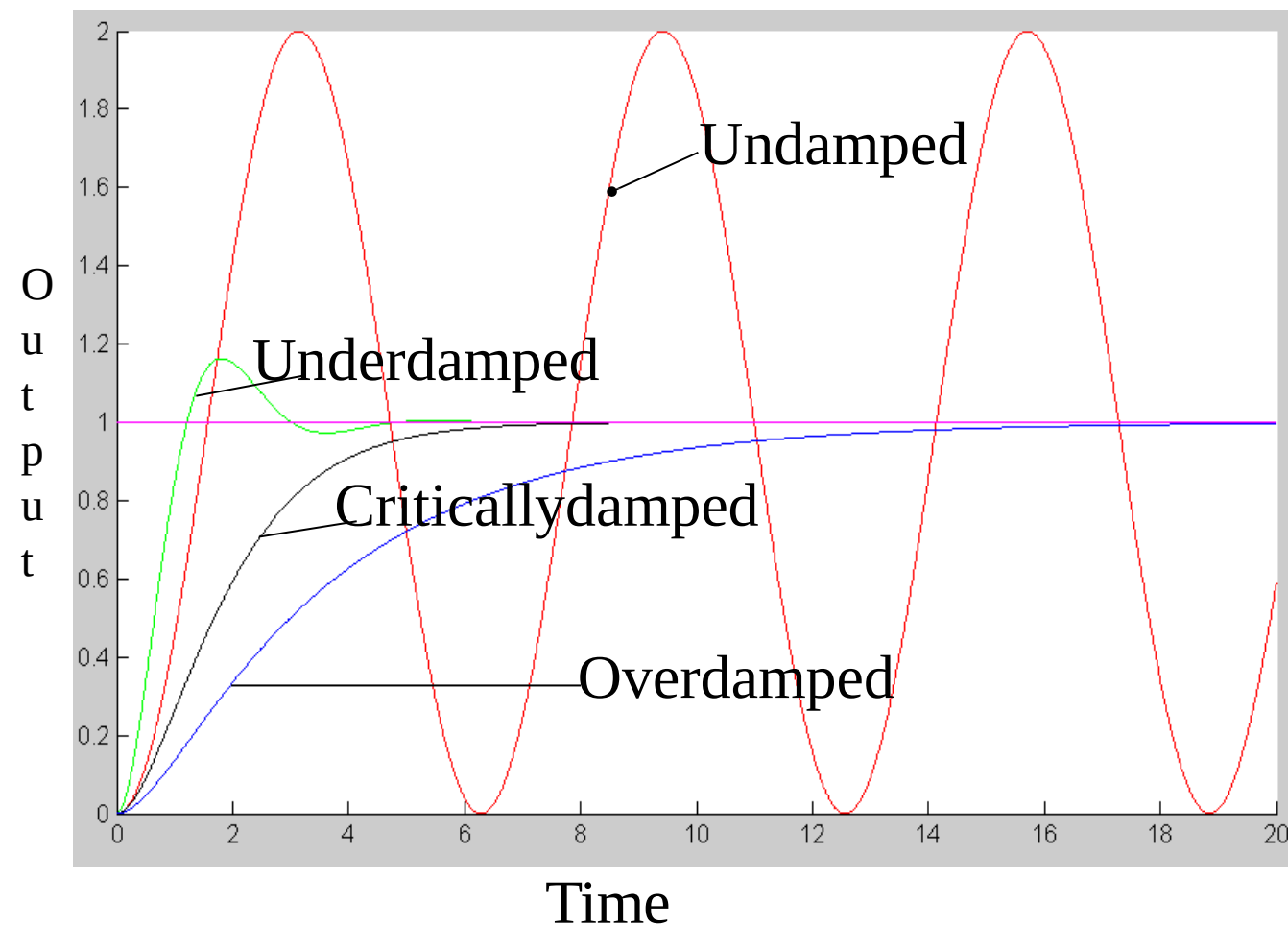
- Natural frequency: ω_n
It is the frequency of oscillation of the system without damping
- Damping ratio: ζ
It is the quantitatively describe this damped oscillation regardless of the time scale
- Steady-state response. In the s domain: $s \rightarrow 0$

4. Time Domain Analysis

Damping of second order system response

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency (rad/second)}}$$

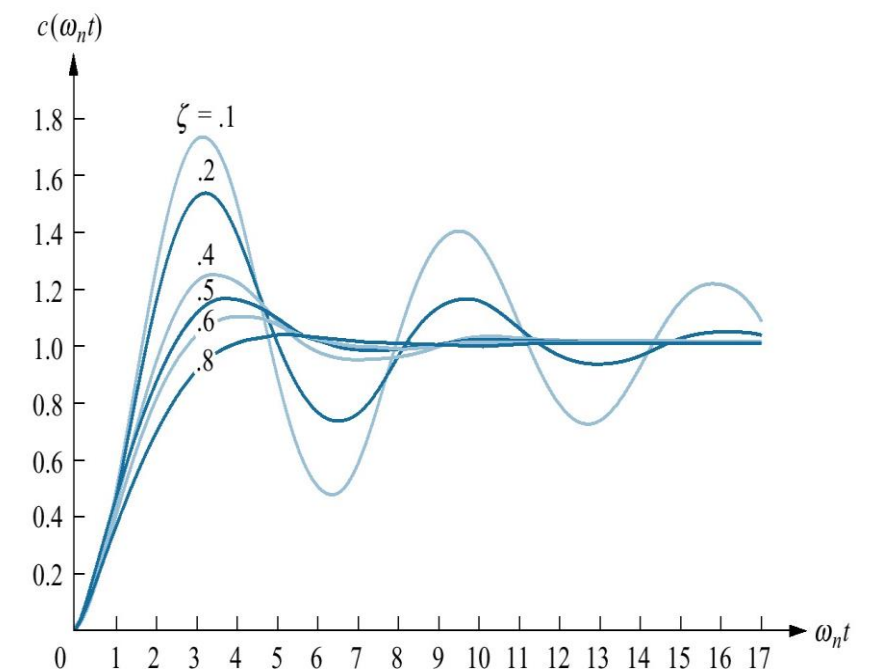
$$R_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$



4. Time Domain Analysis

Determine Damping

Determinant	Roots	Responses
$b^2 - 4ac > 0$	Real, different	Overdamped
$b^2 - 4ac = 0$	Real, same	Critically damped
$b^2 - 4ac < 0$	Complex, different	Underdamped
$-4ac > 0$	Complex, same	Undamped

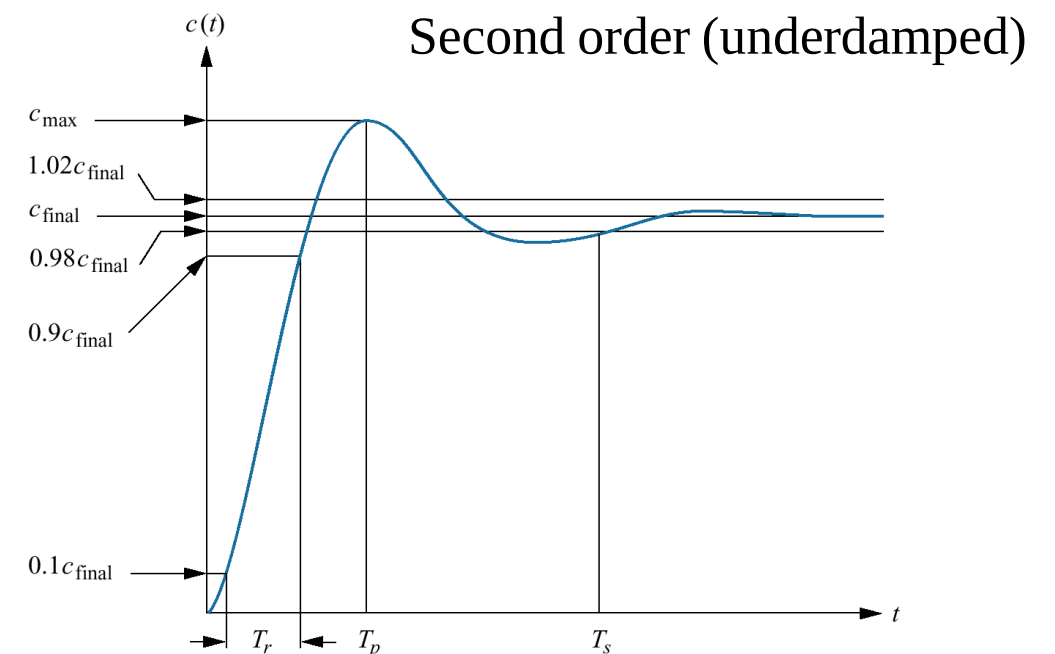


4. Time Domain Analysis

Second Order Time Response

We would like to consider these parameters for time domain analysis of second order system:

- Rise time.
- Settling time.
- Time-to-peak.
- Percentage overshoot (%OS).
- Steady-state error.



4. Time Domain Analysis

Second Order Time Response:

Rise time, peak time, and settling time yield information about the speed of the transient response.

- Rise time (T_r) - The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.
- Settling time (T_s) - The time required for the transient's damped oscillations to reach and stay within $\pm 2\%$ of the steady-state value.
- Time-to-peak (T_p) - The time required to reach the first, or maximum, peak.

4. Time Domain Analysis

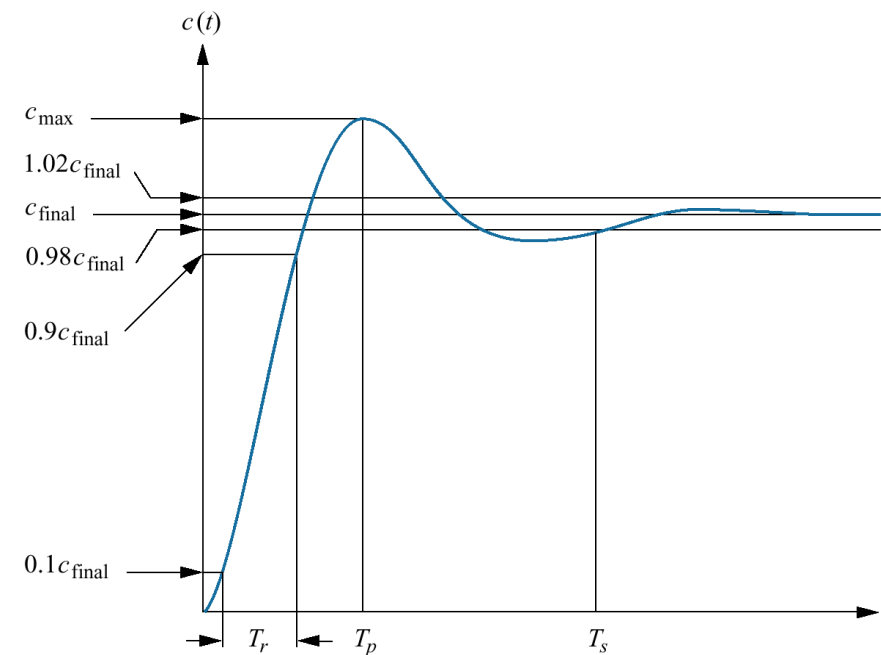
Rise time (T_r):

- Time taken for 0.1 of the final value to 0.9 of the final value.

$$T_r = \frac{(1.76\xi^3 - 0.417\xi^2 + 1.039\xi + 1)}{\omega_n}$$

$$T_r = \frac{\pi - \phi}{\omega_n \sqrt{1 - \xi^2}}$$

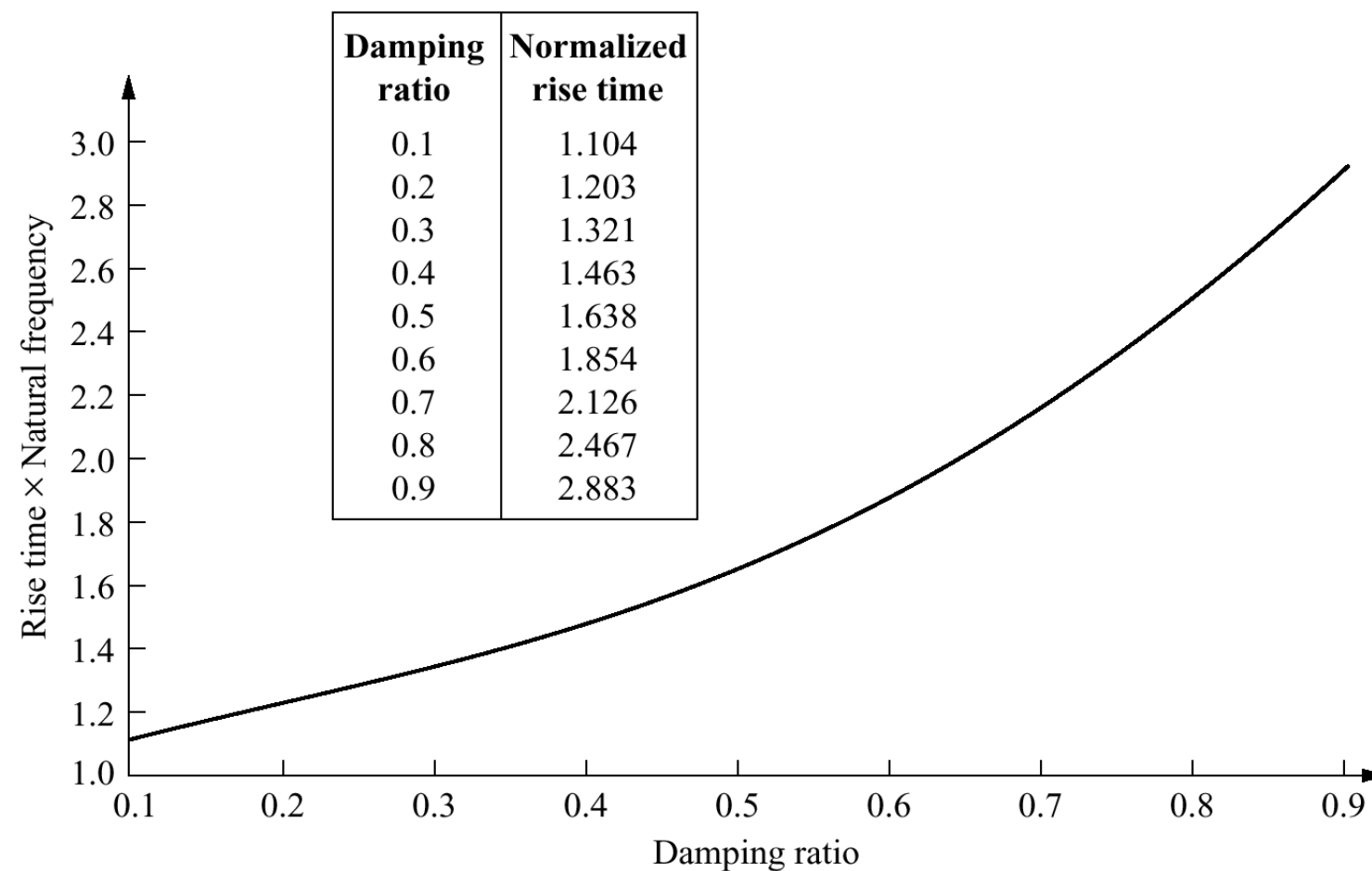
$$\text{Where: } \phi = \tan^{-1} \left(\frac{\sqrt{1 - \xi^2}}{\xi} \right)$$



4. Time Domain Analysis

Rise time (T_r):

- Normalized rise time vs. damping ratio for a second-order underdamped response



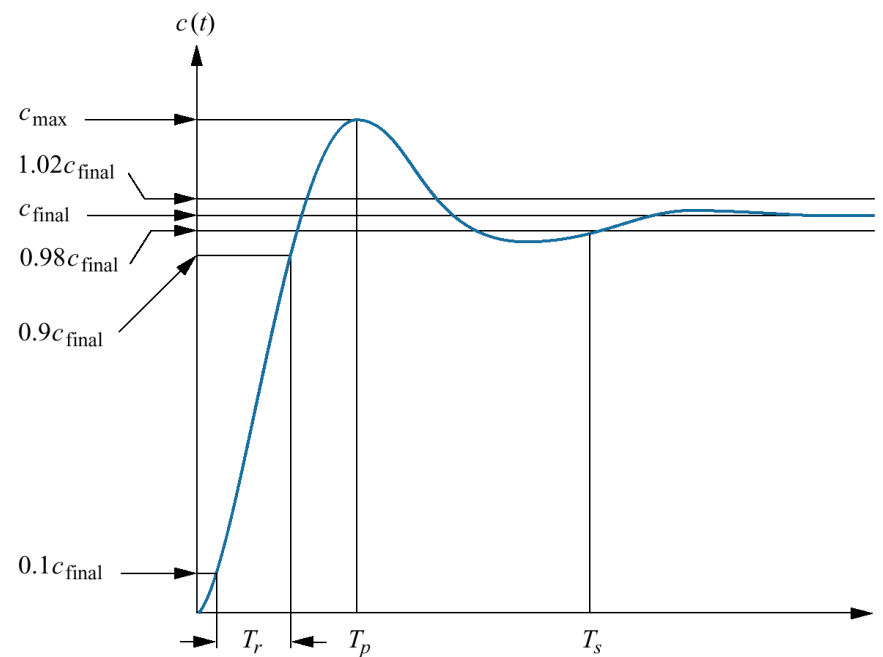
4. Time Domain Analysis

Settling time (T_s):

- Time to reach and stay within +/-2% of the steady-state value

$$T_s = \frac{-\ln(0.02\sqrt{1-\zeta^2})}{\zeta\omega_n}$$

$$T_s = \frac{4}{\zeta\omega_n}$$

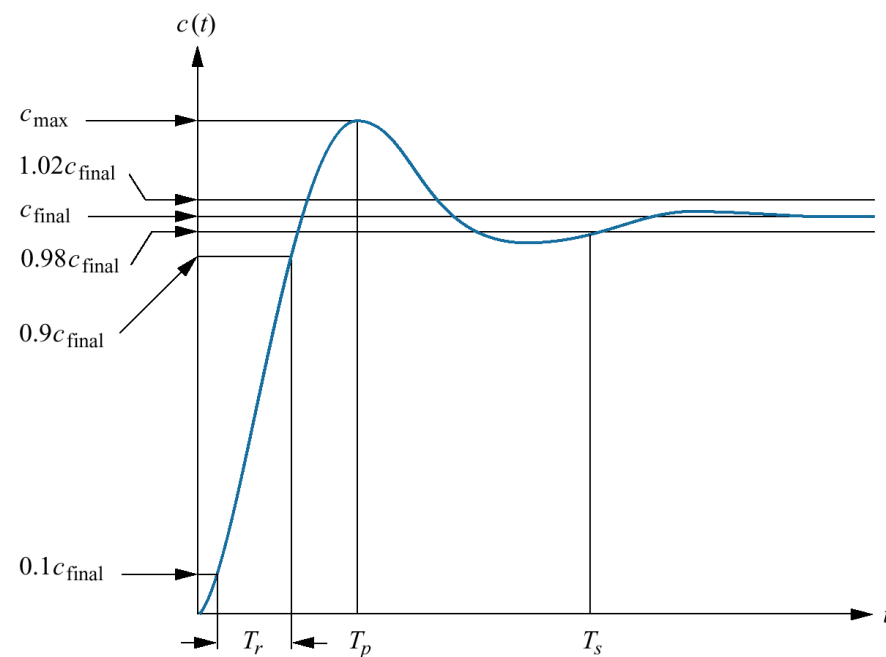
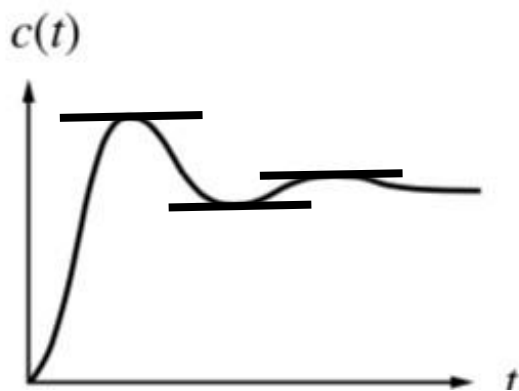


4. Time Domain Analysis

Time-to-Peak (T_p):

- Time to reach the peak of oscillation of underdamped second order system.
- We have the time to all n peaks, but only need $n=1$:

$$T_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$



4. Time Domain Analysis

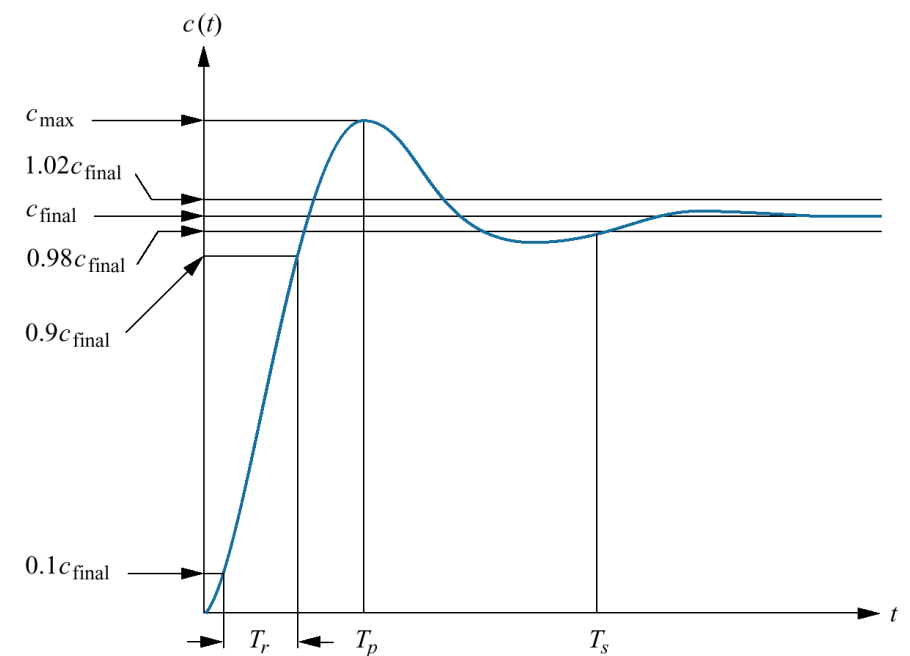
Percentage overshoot (%OS) :

- Ratio of the maximum overshoot and steady-state value
- $C(\max)$ is $C(t)$ evaluated at the peak time $C(T_p)$:

$$\% \text{overshoot} = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \times 100\%$$

- Note %OS is a function of damping ratio only.
- %OS can be rearranged to find damping ratio.

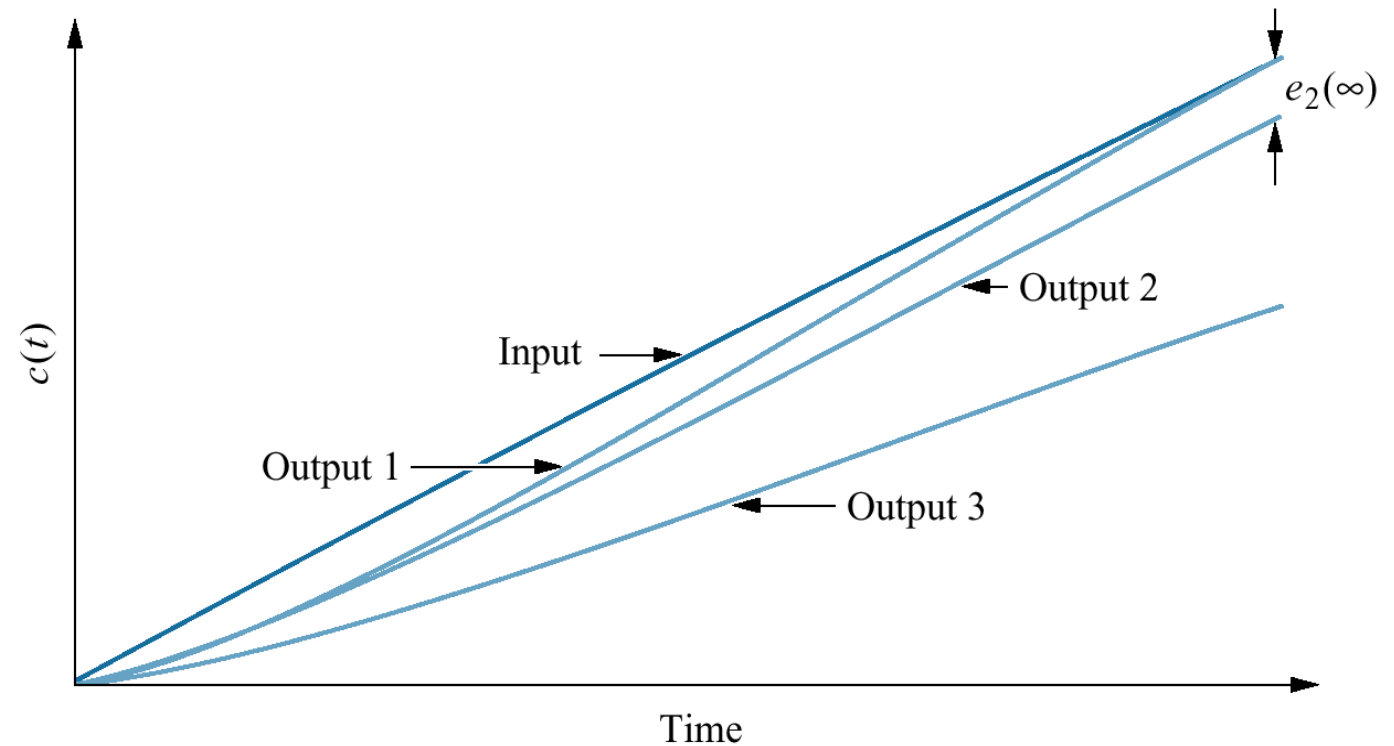
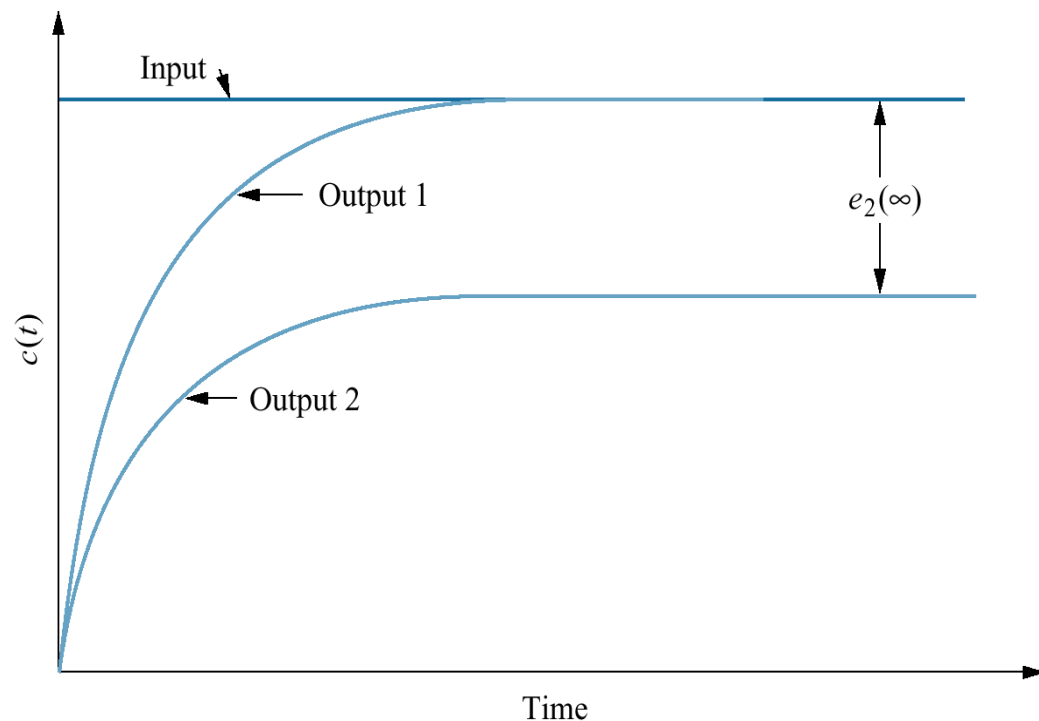
$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$



4. Time Domain Analysis

Steady-state error ($e(\infty)$):

- Steady-state error, is the difference between the input and output for a prescribed test input as $t \rightarrow \infty$
- We will look in a little bit more detail.



4. Time Domain Analysis

Steady state error:

- The system error $e(t)$ for a feedback control system is given by the difference between the demanded output $r(t)$ and the actual output $c(t)$:

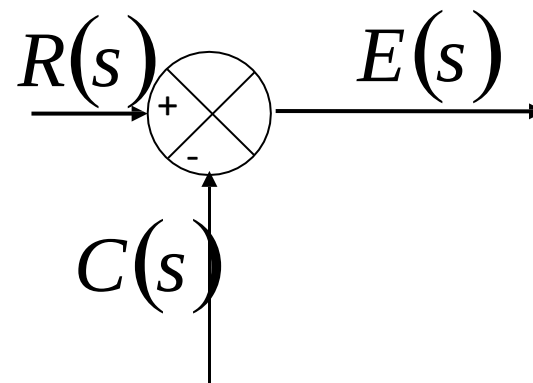
$$e(t) = r(t) - c(t)$$

- The steady state error is then defined as the difference between demanded and actual output when $t \rightarrow \infty$.
- The steady state error is now defined for specific test inputs:

- * Step

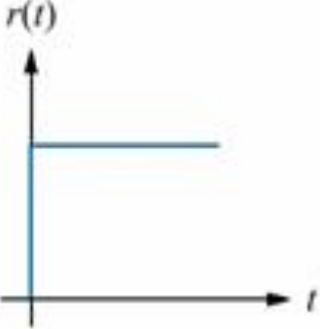
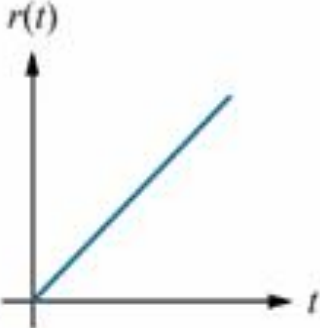
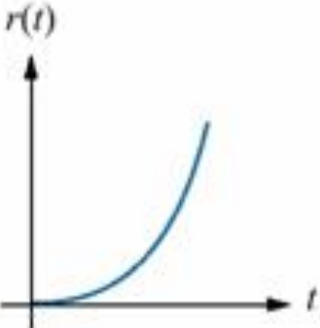
- * Ramp

- * Parabola



4. Time Domain Analysis

Various test inputs used for evaluating and testing control systems are listed below:

Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

4. Time Domain Analysis

Sources of steady state error:

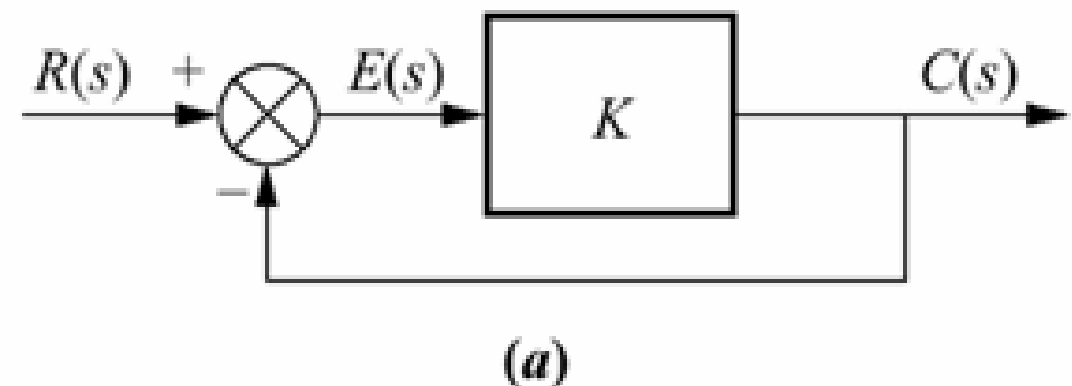
- Consider steady-state errors due to system configuration.

System output, $C(s) = KE(s)$

- The steady-state error can then never be $= 0$ or the output of the system will be zero, there will thus always be a steady state error present.
- If c_{ss} is the steady-state value of the output and e_{ss} is the steady-state value of the error, then:

$$c_{ss}(t) = K \cdot e_{ss}(t)$$

- Error will diminish as K increases.



4. Time Domain Analysis

Steady-state error in terms of $G(s)$:

For the system: $E(s) = R(s) - C(s)$

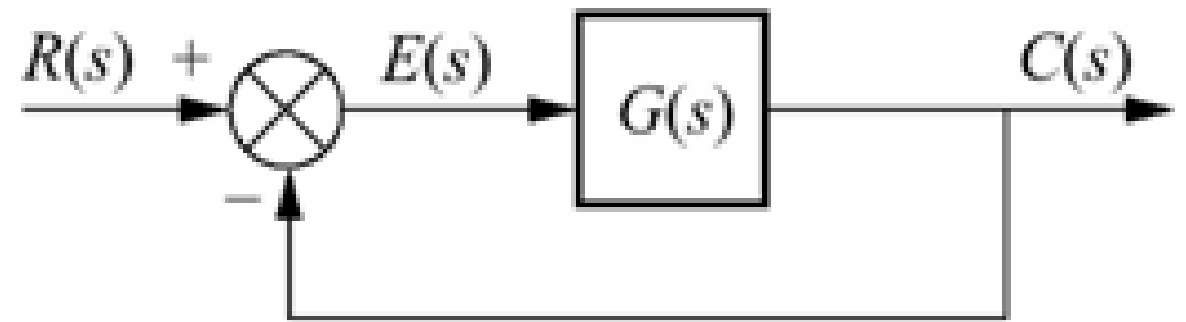
Thus $E(s) = R(s) - E(s)G(s)$

so that: $E(s) = \frac{R(s)}{1 + G(s)}$

From the final value theorem:

$$e(\infty) = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

Above equation will thus allow us to calculate the steady-state error given a particular input $R(s)$.



4. Time Domain Analysis

Static error constant:

- The term in the denominator of the definition of the steady-state error for each input type is taken to limit the steady state error.
- These are then called the static error constants and are defined as follows:

$$\text{Position constant } K_p = \lim_{s \rightarrow 0} G(s)$$

$$\text{Velocity constant } K_v = \lim_{s \rightarrow 0} sG(s)$$

$$\text{Acceleration constant } K_a = \lim_{s \rightarrow 0} s^2 G(s)$$

- These constants depend on the form of $G(s)$ and will determine the value of the steady-state error. (Error decreases as the value of the static error constant increases).

4. Time Domain Analysis

System type:

- The system type is taken to be the number of integration in the feed-forward path (the value of n in s^n of denominator).
- This value of n (the system type) then determines the steady-state error of a unit feedback system for a particular type of input.

Input	Steady-state error formula	Type 0		Type 1		Type 2	
		Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t)$	$\frac{1}{1 + K_p}$	$K_p =$ Constant	$\frac{1}{1 + K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$