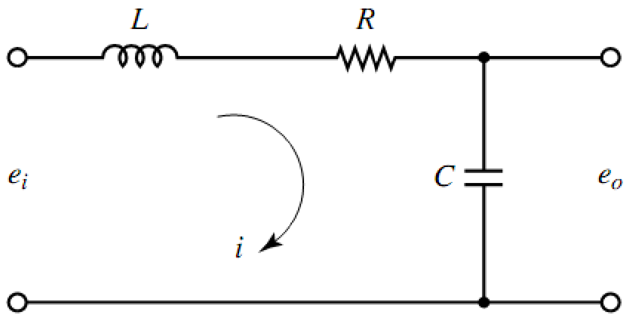
# XMUT315 Control Systems Engineering

# Mid-Term Test Revision Questions

### A. System Modelling

1. Consider the electrical circuit shown in figure below.

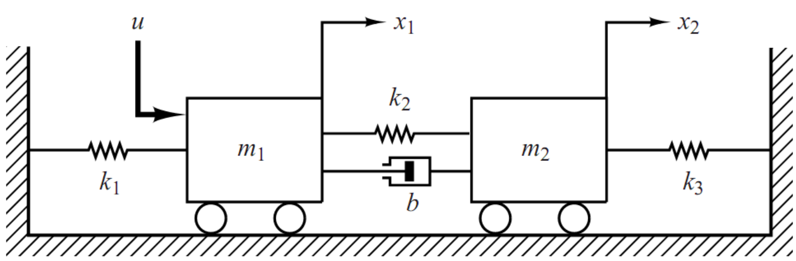



   a. Describe three types of modelling techniques in control systems. [3 marks]

   b. The circuit consists of an inductance $L$ (Henry), a resistance $R$ (Ohm), and a capacitance $C$ (Farad). Determine the system's transfer function. [10 marks]

   c. If $L$ = 1 mH, $R$ = 1 kΩ, and $C$ = 10 μF, calculate the roots of the characteristics equation of the system. Predict the time response of the system. [6 marks]

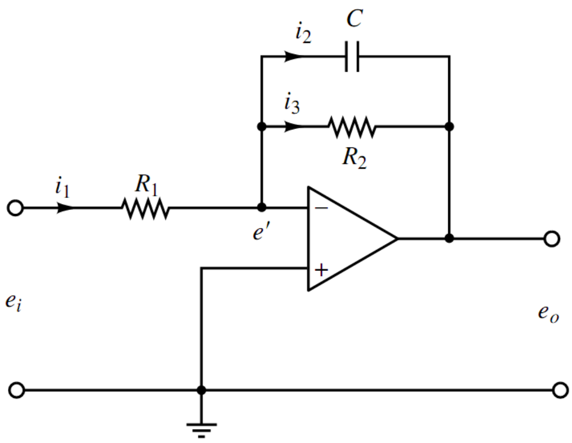
2. You are given a mechanical system that consists of two interconnected moving carts as shown in the figure below. Note that $u$ = force, $m_1$ = $m_2$ = masses, $k_1$ = $k_2$ = $k_3$ = spring constants, $b$ = damper constant, and $x_1$ = $x_2$ = displacements. Assume zero initial conditions of the system.



a. Describe the significant of signal in control systems. [2 marks]

b. Obtain the transfer functions of the system. [10 marks]

3. Figure below shows an electrical circuit involving an operational amplifier.

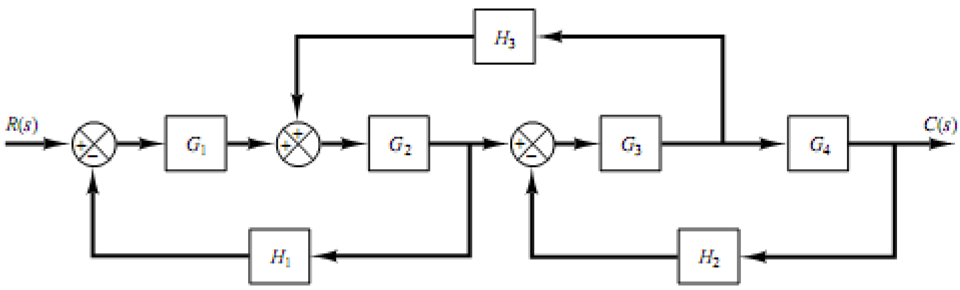



a. What is the two main goals of modelling physical systems? [2 marks]

b. Obtain the transfer function equation of the circuit $e_o(t)/e_i(t)$. [10 marks]

**B. Feedback Control Systems**

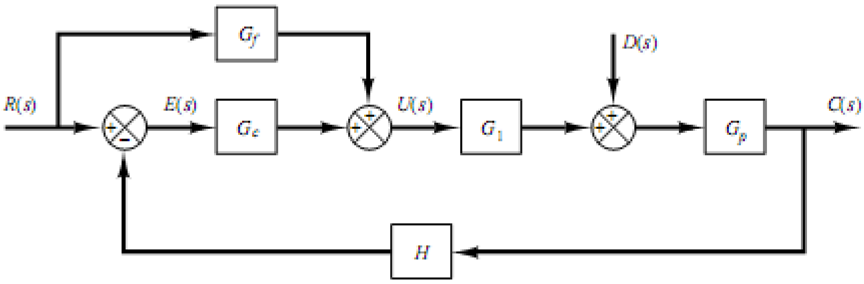
4. Given a physical system represented as a block diagram given below.




a. Simplify the block diagram shown in figure below. Then, obtain the closed-loop transfer function $C(s)/R(s)$. [10 marks]

b. The blocks $H_1$, $H_2$, and $H_3$ are identified as compensators. Why do we need compensators in control systems? [2 marks]

5. Referring to the system shown in the figure below, perform the following tasks.




a. Obtain transfer functions $C(s)/R(s)$ and $C(s)/D(s)$. [10 marks]

b. What are things required for analyzing a system? [2 marks]

**C. Stability Analysis**

6. Apply Routh Hurwitz stability criterion for system given in the following equation:

$$s^3 + 2s^2 + s + 2 = 0$$

a. Determine stability and the roots of the system. [10 marks]

b. Compare the stability and roots of the system in part (a) with the following system: [5 marks]

$$s^3 - 3s + 2 = 0$$

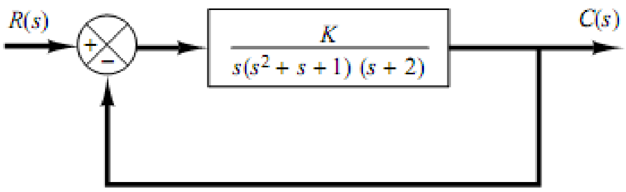
7. Given a control system that is represented by the following equation:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

a. By applying Routh Hurwitz criterion, evaluate the stability of a system. [10 marks]

b. Determine the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis. [5 marks]

8. Consider the system shown below.




The closed-loop transfer function of the system is given as:

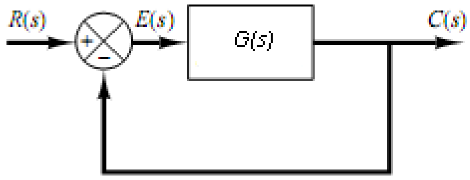
$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2 + s + 1)(s + 2) + K}$$

a. Determine the range of $K$ for stability. [10 marks]

b. Define stability of the system by contrasting stable condition with unstable condition. [2 marks]

**D. Time Responses & Steady State Analysis**

9. Consider the second order system shown in the figure given below.

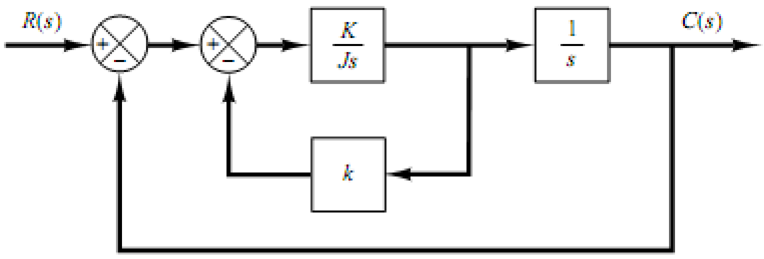



a. If the transfer function of the plant $G(s)$ is as given below, determine the damping ratio ($\zeta$) and natural frequency of the system ($\omega_n$). [5 marks]

$$G(s) = \frac{100}{4s^2 + 24s + 100}$$

b. Obtain the rise time ($T_r$), peak time ($T_r$), maximum overshoot ($M_p$), and settling time ($T_s$) when the system is subjected to a unit-step input. [10 marks]

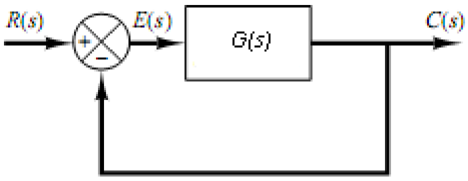
10. Given a system in the figure below.




a. Determine the values of $K$ and $k$ of the closed-loop system shown in the figure below, so that the maximum overshoot in unit-step response is 25% and the peak time is 2 sec. Assume that $J$ = 1 kg-m$^2$ . [10 marks]

b. What is the time response of a system when its poles are moved along a constant radial line? [2 marks]

11. Consider the closed-loop feedback system shown in the figure given below.




a. Derive the equation for steady-state error of the system. [4 marks]

b. Define steady-state error and describe three types of input used for testing steady-state condition of a system. [4 marks]

c. If the transfer function of the plant $G(s)$ is as stated below, calculate the static-error constants of the system ($K_p$, $K_v$, and $K_a$). [6 marks]

$$G(s) = \frac{s+5}{s(s+2)(s+10)}$$

d. Based on the results in part (c), calculate the steady-state error of the system whenever they are subjected to the following test inputs:

   a. Step input. [2 marks]

   b. Ramp input. [2 marks]

   c. Parabolic input. [2 marks]

**Formulas for Control Systems Engineering**

## A. Common Laplace Transforms

| Time Domain | Laplace Domain |
|---|---|
| $\delta(t)$ | $1$ |
| $\delta^n(t)$ | $s^n$ |
| $u(t)$ | $\frac{1}{s}$ |
| $t$ | $\frac{1}{s}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $e^{-at}$ | $\frac{1}{s+a}$ |
| $te^{-at}$ | $\frac{1}{(s+a)^2}$ |
| $\frac{t^n}{n!}e^{at}$ | $\frac{1}{(s+a)^{n+1}}$ |
| $\sin(\omega t)$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos(\omega t)$ | $\frac{s}{s^2+\omega^2}$ |
| $e^{-at}\sin(\omega t)$ | $\frac{\omega}{(s+a)^2+\omega^2}$ |
| $e^{-at}\cos(\omega t)$ | $\frac{s+a}{(s+a)^2+\omega^2}$ |
| $\frac{\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin(\omega_n\sqrt{1-\xi^2}t)$ | $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$ |

In all cases above, the symbols have their normal meanings.

## B. Properties of the Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt \qquad \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{-st}\,ds$$

| | |
|---|---|
| Definition: | $f(t) \Leftrightarrow F(s)$ |
| Linearity: | $af(t)+bg(t) \Leftrightarrow aF(s)+bG(s)$ |

| | |
|---|---|
| t-scaling | $f(ct) \Leftrightarrow \frac{1}{\lvert c\rvert} F\left(\frac{s}{c}\right)$ |
| t-shifting: | $f(t-t_0)u(t-t_0) \Leftrightarrow e^{-st_0}F(s)$ |
| s-shifting: | $e^{-s_0 t}f(t) \Leftrightarrow F(s-s_0)$ |
| Differentiation in t: | $f'(t) \Leftrightarrow sF(s) - f(0)$<br>$f''(t) \Leftrightarrow s^2F(s) - sf(0) - f'(0)$ |
| Integration in t: | $f^{(k)} \Leftrightarrow s^kF(s) - s^{k-1}f(0) - s^{k-2}f'(0)\ldots - f^{(k-1)}(0)$<br>$\int_0^t f(\tau)\, d\tau \Leftrightarrow \frac{1}{s}F(s)$ |
| Differentiation in s: | $tf(t) \Leftrightarrow -F'(s)$ |
| Integration in s: | $\frac{f(t)}{t} \Leftrightarrow \int_s^\infty F(\tilde{s})d\tilde{s}$ |
| Convolution: | $f(t) * g(t) \Leftrightarrow F(s)G(s)$<br>$f(t)g(t) \Leftrightarrow \frac{1}{2\pi j} F(s) * G(s)$ |
| Periodicity | $F(t) \Leftrightarrow F_1(s)\frac{1}{1-e^{-sp}}$<br>For $f_1(t)$ one cycle of $f(t)$ with period $p$. |
| Initial value theorem: | $f(0+) = \lim_{t\to\infty} sF(s)$ |
| Final value theorem: | $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} sF(s)$ |

(for $a, b, t_0, s_0 \in R, c \in R_{++}$ ).

## C. Partial Fractions Expansion

If a partial fraction expansion of $Y(s)$ includes terms,

$$\frac{A_m}{(s-a)^m}\frac{A_{m-1}}{(s-a)^{m-1}} + \cdots + \frac{A_1}{s-a}$$

then the coefficients of factors having multiplicity $m > 1$ are given by the following expressions, where $k \neq m$.

$$A_m = \lim_{s\to a}(s-a)^m Y(s)$$

$$A_k = \frac{1}{(m-k)!}\lim_{s\to a}\frac{d^{m-k}}{ds^{m-k}}(s-a)^m Y(s)$$

## D. Trigonometric Identities

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \Longrightarrow \begin{cases} \sin(\theta + \pi/2) = \cos(\theta) \\ \sin(\theta - \pi/2) = -\cos(\theta) \end{cases}$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi \Longrightarrow \begin{cases} \cos(\theta + \pi/2) = -\sin\theta \\ \cos(\theta - \pi/2) = \sin\theta \end{cases}$$

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

## E. First Order Systems

For a first order system with transfer function:

$$G(s) = \frac{1}{(s\ + a)}$$

Time constant is:

$$\tau = 1/a$$

Rise time (10-90%) is:

$$t_r = 2.2\tau$$

Settling time (to 2% of final value standard) is:

$$t_s = 4\tau$$

## F. Second Order Systems

For an underdamped second order system, the following relationships hold.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The rise time

$$T_r = \frac{(1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)}{\omega_n}$$

Or

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} \qquad \text{where: } \phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

The settling time (i.e. 2% of final value standard):

$$T_s = \frac{4}{\zeta\omega_n}$$

The time taken to reach the peak value (n = #peak) is:

$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

The percentage overshoot is related to damping ratio by:

$$\%OS = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \times 100$$

Damping ratio.

$$\zeta = -\frac{\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}}$$

## G. Steady State

Steady-state errors.

| **Type** | **Input** | | |
|---|---|---|---|
| | **Step** | **Ramp** | **Parabola** |
| 0 | $e_{ss} = \frac{1}{1+K_p}$ | $\infty$ | $\infty$ |
| 1 | $e_{ss} = 0$ | $\frac{1}{K_v}$ | $\infty$ |
| 2 | $e_{ss} = 0$ | 0 | $\frac{1}{K_a}$ |

Steady-state error constants.

$$K_p = \lim_{s\to 0} G(s) \qquad K_v = \lim_{s\to 0} sG(s) \qquad K_a = \lim_{s\to 0} s^2G(s)$$