XMUT315 Control Systems Engineering

Tutorial 3: Stability Analysis (Solution)

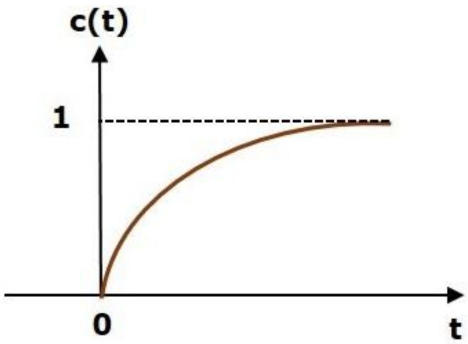
A. Stability Analysis

1. In control system engineering, stability is determined to be one of the criteria of performance of a given control system.

   a. What is stability? [2 marks]

   b. Describe and compare bounded signal and unbounded signal. [4 marks]

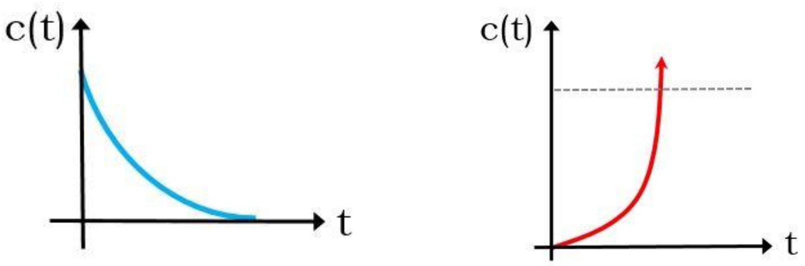
   c. Describe how a system that has a unit-step response as shown in the diagram below is stable. [2 marks]



Solution

a. A system is said to be stable if its output is under control. Otherwise, it is said to be unstable. A stable system produces a bounded output for a given bounded input (BIBO).

b. Bounded value of a signal represents a finite value. More specifically, we can say, the bounded signal holds a finite value of maxima and minima. Thus, if maxima and minima of any signal are finite then this means all the other values between maxima and minima will also be finite.



(a) (b)

The signals whose graph shows continuous rise thereby showing infinite value such as ramp signal are known as unbounded signals. The figure shown below represents the unbounded signal:

Sometimes we come across asymptotically stable systems which are defined as the systems whose output progresses 0, when the input is not present, even when the parameters of the system show variation. It is to be noted here that poles of the transfer function, is a factor defining the stability of the control system.

c. This is the response of first order control system for unit step input. This response has the values between 0 and 1. So, it is bounded output. We know that the unit step signal has the value of one for all positive values of t including zero. So, it is bounded input. Therefore, the first order control system is stable since both the input and the output are bounded.

   As we can see that here, the maxima and minima of the signal represented above is having finite values. Thus, such a signal is said to be bounded and if such an output is provided by a system, then it is said to be a stable system.

   Therefore, conversely, we can say that an unstable system provides an unbounded output when the applied input is bounded in nature.

2. Before performing more specific analysis and design for a given control system, stability analysis is typically performed first.

   a. Explain how you determine stability of a system. [2 marks]
   b. Describe types of systems based on their stabilities. [6 marks]

Solution

a. The stability of a control system is defined as the ability of any system to provide a bounded output when a bounded input is applied to it.

   More specifically, we can say, that stability allows the system to reach the steady-state and remain in that state for that particular input even after variation in the parameters of the system.

b. We can classify the systems based on their stabilities as follows.

- Absolutely Stable System

  If the system is stable for all the range of system component values, then it is known as the absolutely stable system. The open-loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of ‘s’ plane. Similarly, the closed-loop control system is absolutely stable if all the poles of the closed-loop transfer function present in the left half of the ‘s’ plane.

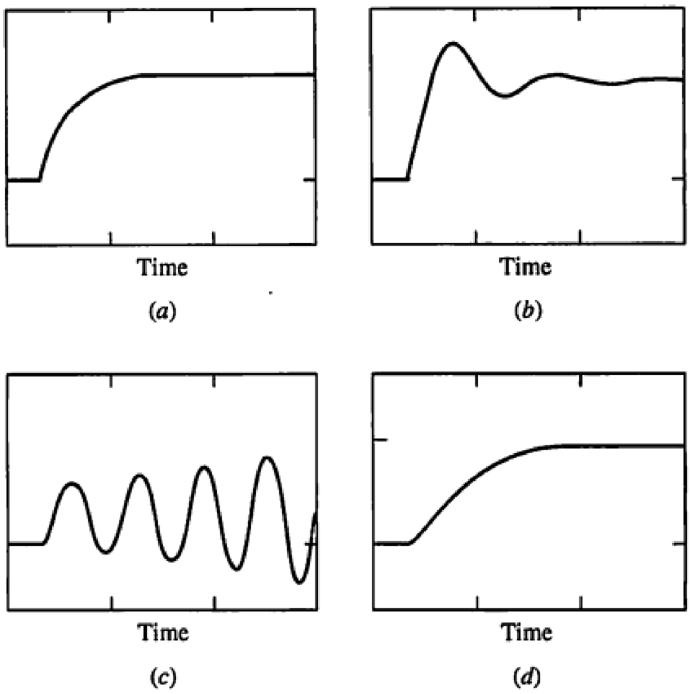
- Conditionally Stable System

  If the system is stable for a certain range of system component values, then it is known as conditionally stable system.

- Marginally Stable System

  If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as marginally stable system. The open-loop control system is marginally stable if any two poles of the open-loop transfer function is present on the imaginary axis. Similarly, the closed-loop control system is marginally stable if any two poles of the closed-loop transfer function is present on the imaginary axis.

3. Describe whether the following systems are stable or not based on their transient responses given in the following figures. [8 marks]



Note:
System (a) has an damped exponential response, system (b) has a damped oscillatory response , system (c) has a growing oscillatory response, and system (d) has a damped exponential response with its time constant is longer than system (a).

Solution

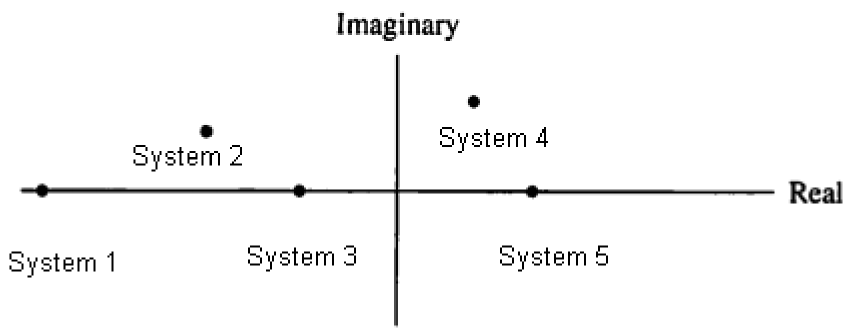
System (a) is stable as the output of the system settles down to a level after period-of-time.

System (b) is stable as the output of the system settles down to a level after period-of-time, although initially it experiences a damped oscillation at the beginning.

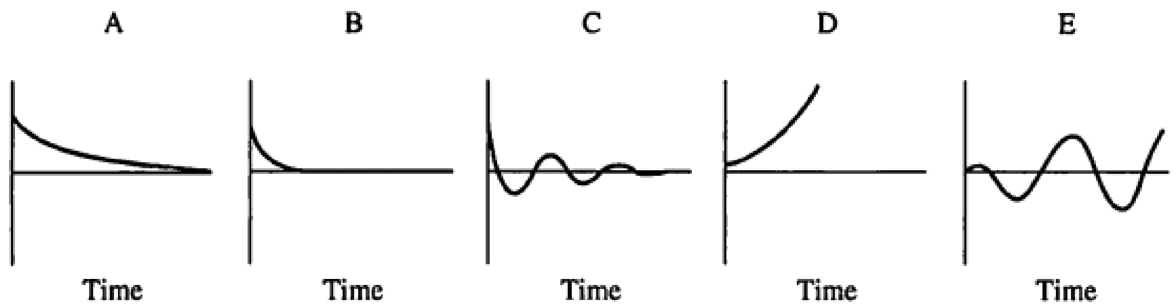
System (c) is not stable as it has a growing oscillation and it does not settle down.

System (d) is stable as the output of the system settles down to a level after period-of-time.

4. Referring to a (pole-zero) s-plane diagram of a number of systems as shown below, answer the following questions.



Which of the unit-step responses given below correspond to which systems as above. [5 marks]



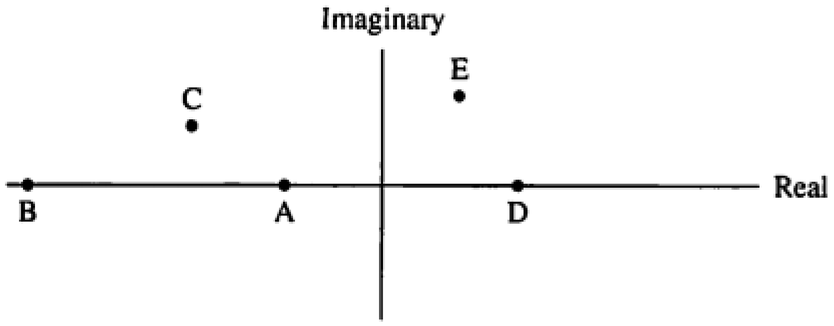
Solution

Response of the systems and their pole locations in the s-domain:

- System 1 - a damped exponential response due to its simple pole (located on the x-axis) with its time response is shorter than the time response of System 3 i.e. response B.
- System 2 - a damped oscillatory response due to its stable complex pole i.e. response C.

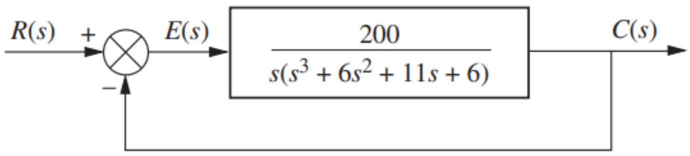
- System 3 - it is also a damped exponential response like the response of System 1, due to its simple pole, although its time response is longer than the time response of System 1 (e.g. closer to the y-axis) i.e. response A.
- System 4 – a growing oscillatory response due to its unstable complex pole i.e. response E.
- System 5 – a growing exponential response due to its unstable simple pole i.e. response D.

The correct matching of the systems with their corresponding unit step responses are illustrated as shown in the figure below.



B. Routh-Hurwitz Stability Criterion

5. Find the number of poles in the left half-plane, the right half-plane, and on the jω-axis for the system of the following figure. [12 marks]



Solution

First, find the closed-loop transfer function as

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

$$= \frac{\left(\frac{200}{s(s^3 + 6s^2 + 11s + 6)}\right)}{\left[1 + \left(\frac{200}{s(s^3 + 6s^2 + 11s + 6)}\right)(1)\right]}$$

$$= \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200}$$

Given the transfer function equation of the closed-loop system.

$$T(s) = \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200}$$

As a result, the characteristic equation of the closed-loop system is:

$$s^4 + 6s^3 + 11s^2 + 6s + 200$$

Thus, we construct the Routh table for the system.

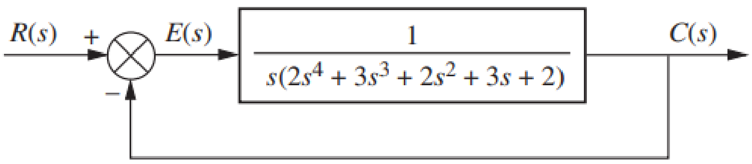
s^4	1	11	200
s^3	$\cancel{6}\quad 1$	$\cancel{6}\quad 1$	0
s^2	$\frac{-\begin{vmatrix}1 & 11\\ 1 & 1\end{vmatrix}}{1} = \cancel{10}\quad 1$	$\frac{-\begin{vmatrix}1 & 200\\ 1 & 0\end{vmatrix}}{1} = \cancel{200}\quad 20$	$\frac{-\begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}}{1} = 0$
s^1	$\frac{-\begin{vmatrix}1 & 1\\ 1 & 20\end{vmatrix}}{1} = -19$	$\frac{-\begin{vmatrix}1 & 0\\ 1 & 0\end{vmatrix}}{1} = 0$	0
s^0	$\frac{-\begin{vmatrix}1 & 20\\ -19 & 0\end{vmatrix}}{-19} = 20$	$\frac{-\begin{vmatrix}1 & 0\\ -19 & 0\end{vmatrix}}{-19} = 0$	0

Then, the Routh table for the denominator of transfer function equation is shown as the following table. For clarity, we leave most zero cells blank.

s^4	1	11	200	+
s^3	$\cancel{6}\quad 1$	$\cancel{6}\quad 1$		+
s^2	$\cancel{10}\quad 1$	$\cancel{200}\quad 20$		+
s^1	-19			-
s^0	20			+

At the s^1 row there is a negative coefficient; thus, there are two sign changes. The system is unstable, since it has two right-half plane poles and two left-half-plane poles. The system cannot have $j\omega$ poles since a row of zeros did not appear in the Routh table.

6. Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the system shown on the following figure. [12 marks]



Solution

The closed-loop transfer function is:

$$T(s) = \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$$

Using the denominator of transfer function equation of the system, the characteristic equation of the closed-loop system is:

$$2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1$$

Form the Routh table shown as standard table using the characteristic equation. The Routh table is as shown below.

s^5	2	2	2
s^4	3	3	1
s^3	$\frac{-\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}}{3} = 0$	$\frac{-\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}}{3} = 4/3$	
s^2	$\frac{-\begin{vmatrix} 3 & 3 \\ 0 & 4/3 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}}{0} = \infty$	
s^1	$\frac{-\begin{vmatrix} 0 & 4/3 \\ \infty & \infty \end{vmatrix}}{\infty} = \mathrm{N/A}$	$\frac{-\begin{vmatrix} 0 & 0 \\ \infty & 0 \end{vmatrix}}{\infty} = 0$	
s^0	$\frac{-\begin{vmatrix} \infty & \infty \\ \mathrm{N/A} & 0 \end{vmatrix}}{\mathrm{N/A}} = \mathrm{N/A}$		

As shown in the table above, a zero appears in the first column of the s^3 row.

Since the entire row is not zero, simply replace the zero with a small quantity, ϵ, and continue the table. Permitting e to be a small, positive quantity, we find that the first term of the s^2row is negative.

Thus, there are two sign changes, and the system is unstable, with two poles in the right half-plane. The remaining three poles are in the left half-plane.

s^5	2	2	2	+
s^4	3	3	1	+
s^3	$\cancel{0}\ \epsilon$	4/3		+
s^2	$\frac{3\epsilon - 4}{\epsilon}$	1		-
s^1	$\frac{12\epsilon - 16 - 3\epsilon^2}{9\epsilon - 12}$			+
s^0	1			+

We also can use the alternative approach, where we produce a polynomial, whose roots are the reciprocal of the original.

Using the denominator of the closed loop transfer function equation, we form a polynomial by writing the coefficients in reverse order:

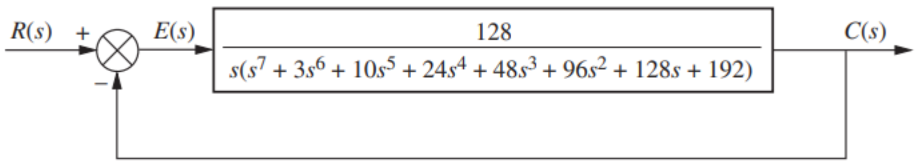
$$s^5 + 2s^4 + 3s^3 + 2s^2 + 3s + 2$$

The Routh table for this polynomial is shown as in the following table. Unfortunately, in this case we also produce a zero only in the first column at the s^2 row. However, the table is easier to work with than the table given above.

The following table yields the same results as the table given above: three poles in the left half-plane and two poles in the right half-plane. The system is unstable.

s^5	1	3	3	+
s^4	2	2	2	+
s^3	2	2		+
s^2	$\cancel{0}$ ϵ	2		+
s^1	$\frac{2\epsilon - 4}{\epsilon}$			-
s^0	2			+

7. For a control system given in the following block diagram, attempt the following tasks.



a. Find the number of poles in the left half-plane, the right half-plane, and on the $j\omega$-axis for the system. [20 marks]

b. Draw conclusions about the stability of the closed-loop system. [6 marks]

Solution

a. The closed-loop transfer function for the system is:

$$T(s) = \frac{128}{s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128}$$

Using the denominator, form the Routh table as shown in the following table. A row of zeros appears in the s^5 row.

Thus, the closed-loop transfer function denominator must have an even polynomial as a factor. Return to the s^6 row and form the even polynomial:

$$P(s) = s^6 + 8s^4 + 32s^2 + 64$$

The following table outlines the Routh table of the system.

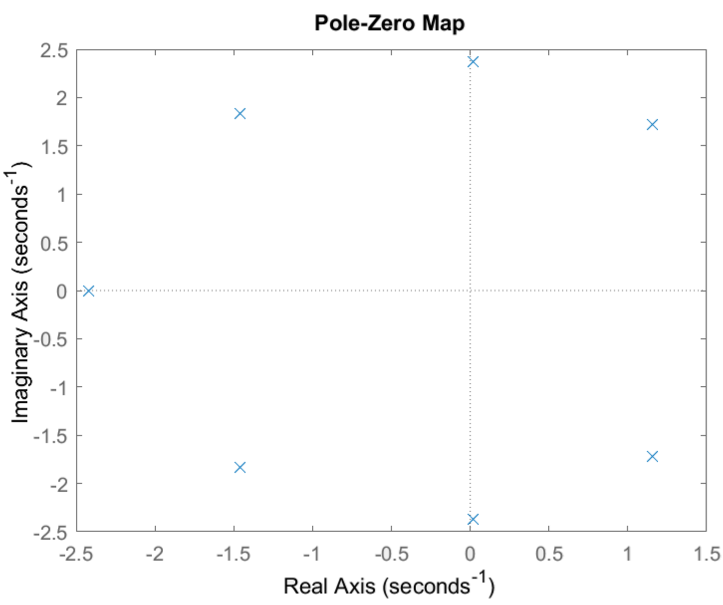
s^8	1	10	48	128	128	+
s^7	~~3~~ 1	~~24~~ 8	~~96~~ 32	~~192~~ 64		+
s^6	~~2~~ 1	~~16~~ 8	~~64~~ 32	~~128~~ 64		+
s^5	~~0~~ ~~6~~ 3	~~0~~ ~~32~~ 16	~~0~~ ~~64~~ 32	~~0~~ ~~0~~ 0		+
s^4	~~8/3~~ 1	~~64/3~~ 8	~~64~~ 24			+
s^3	~~8~~ -1	~~40~~ -5				-
s^2	~~3~~ 1	~~24~~ 8				+
s^1	3					+
s^0	8					+

Differentiate this polynomial with respect to s to form the coefficients that will replace the row of zeros:

$$\frac{dP(s)}{ds} = 6s^5 + 32s^3 + 64s + 0$$

Replace the row of zeros at the s^5 row by the coefficients of equation given above and multiply through by 1/2 for convenience. Then, complete the table.

We note that there are two sign changes from the even polynomial $P(s) = s^6 + 8s^4 + 32s^2 + 64$ at row s^3 evaluating the first column from the s^6 row down to the end of the table. Hence, the even polynomial has two right half-plane poles.



MATLAB Code:

```
H = tf(128,[1 3 10 24 48 96 128 192]);
T=feedback(H,1);
pzmap(T)
```

Because of the symmetry about the origin, the even polynomial must have an equal number of left-half-plane poles. Therefore, the even polynomial has two left-half-plane poles.

Since the even polynomial is of sixth order, the two remaining poles must be on the $j\omega$-axis.

There are no sign changes from the beginning of the table down to the even polynomial at the s^6 row. Therefore, the rest of the polynomial has no right half-plane poles.

b. The results of stability analysis of the given control system are summarised in the table given below. The system has two poles in the right half-plane, four poles in the left half-plane, and two poles on the $j\omega$-axis, which are of unit multiplicity. The closed-loop system is unstable because of the right-half-plane poles.

	Polynomial		
Location	**Even (6th order)**	**Other (2th order)**	**Total (8th order)**
Right half-plane	2	0	2
Left half-plane	2	2	4
$j\omega$-axis	2	0	2

8. For the transfer function tell how many poles are in the right half-plane, in the left half-plane, and on the $j\omega$-axis. [24 marks]

$$T(s) = \frac{20}{s^8 + s^7 + 12s^6 + 22s^5 + 39s^4 + 59s^3 + 48s^2 + 38s + 20}$$

Solution

Use the denominator of the transfer function equation and form the Routh table in the following table. For convenience the s^6 row is multiplied by 1/10, and the s^5 row is multiplied by 1/20. At the s^3 row we obtain a row of zeros. Moving back one row to s^4, we extract the even polynomial, $P(s)$, as:

$$P(s) = s^4 + 3s^2 + 2$$

This polynomial will divide evenly into the denominator of equation of the transfer function of the system and thus is a factor.

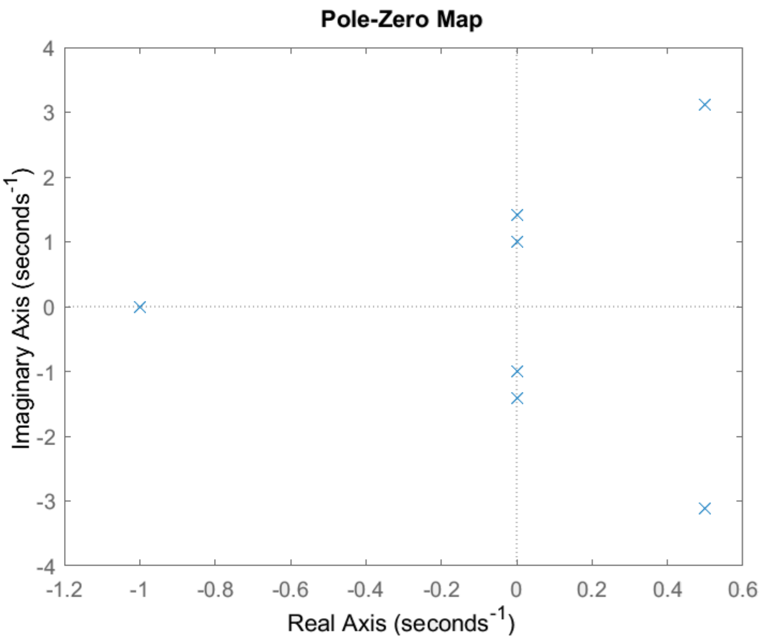
Taking the derivative with respect to s to obtain the coefficients that replace the row of zeros in the s^3 row, we find:

$$\frac{dP(s)}{ds} = 4s^3 + 6s + 0$$

Replace the row of zeros with 4, 6, and 0 and multiply the row by 1/2 for convenience. Finally, continue the table to the s^0 row, using the standard procedure.

s^8	1	12	39	48	20	+
s^7	1	22	59	38	0	+
s^6	~~-10~~ -1	~~-20~~ -2	~~10~~ 1	~~20~~ 2	0	-
s^5	~~20~~ 1	~~60~~ 3	~~40~~ 2	0	0	+
s^4	1	3	2	0	0	+
s^3	~~0~~ ~~4~~ 2	~~0~~ ~~6~~ 3	~~0~~ ~~0~~ 0	0	0	+
s^2	~~3/2~~ 3	~~2~~ 4	0	0	0	+
s^1	1/3	0	0	0	0	+
s^0	4	0	0	0	0	+

How do we now interpret this Routh table? Since all entries from the even polynomial $P(s) = s^4 + 3s^2 + 2$ at the s^4 row down to the s^0 row are a test of the even polynomial, we begin to draw some conclusions about the roots of the even polynomial.



MATLAB Code:

```
T = tf(20,[1 1 12 22 39 59 48 38 20]);
pzmap(T)
```

No sign changes exist from the s^4 row down to the s^0 row. Thus, the even polynomial does not have right-half-plane poles.

Since there are no right half-plane poles, no left half-plane poles are present because of the requirement for symmetry. Hence, the even polynomial equation $P(s) = s^4 + 3s^2 + 2$ must have all four of its poles on the $j\omega$-axis (note: a necessary condition for stability is that the $j\omega$ roots have unit multiplicity).

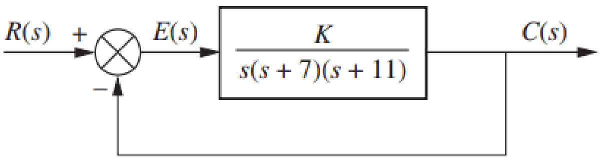
The even polynomial must be checked for multiple $j\omega$ roots. For this case, the existence of multiple $j\omega$ roots would lead to a perfect, fourth-order square polynomial.

Since even polynomial equation is not a perfect square, the four $j\omega$ roots are distinct. These results are summarized in the first column of the table below.

From row with s^8 to s^5, there are two sign changes in the first column. So, there are unstable two roots on the right half-plane. Finally, the remaining two stable poles are located on the left half-plane.

	Polynomial		
Location	**Even (4th order)**	**Other (4th order)**	**Total (8th order)**
Right half-plane	0	2	2
Left half-plane	0	2	2
$j\omega$	4	0	4

9. Find the range of gain, K, for the system given in the following figure that will cause the system to be stable, unstable, and marginally stable. Assume $K > 0$. [20 marks]



Solution

First, find the closed-loop transfer function as:

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

Next, form the Routh table shown as shown in the following table.

$$\begin{array}{c|cc} s^3 & 1 & 77 \\ s^2 & 18 & K \\ s^1 & \dfrac{1386 - K}{18} & \\ s^0 & K & \end{array}$$

Since K is assumed positive, we see that all elements in the first column are always positive except the s^1 row. This entry can be positive, zero, or negative, depending upon the value of K. If $K < 1386$, all terms in the first column will be positive, and since there are no sign changes, the system will have three poles in the left half-plane and be stable.

If $K > 1386$, the s^1 term in the first column is negative. There are two sign changes, indicating that the system has two right-half-plane poles and one left half-plane pole, which makes the system unstable.

If $K = 1386$, we have an entire row of zeros, which could signify $j\omega$ poles. Returning to the s^2 row and replacing K with 1386, we form the even polynomial:

$$P(s) = 18s^2 + 1386$$

Differentiating with respect to s, we have:

$$\frac{dP(s)}{ds} = 36s + 0$$

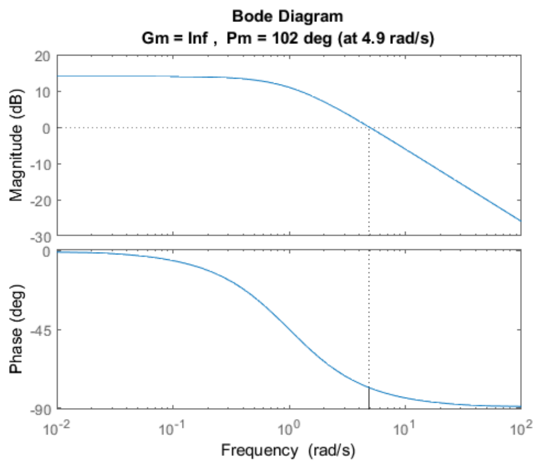
Replacing the row of zeros with the coefficients of equation given above, we obtain the Routh-Hurwitz table shown as in the following table for the case of K = 1,386.

s^3	1	77	+
s^2	18	1386	+
s^1	$\cancel{0}$ 36	$\cancel{0}$ 0	+
s^0	1386		+

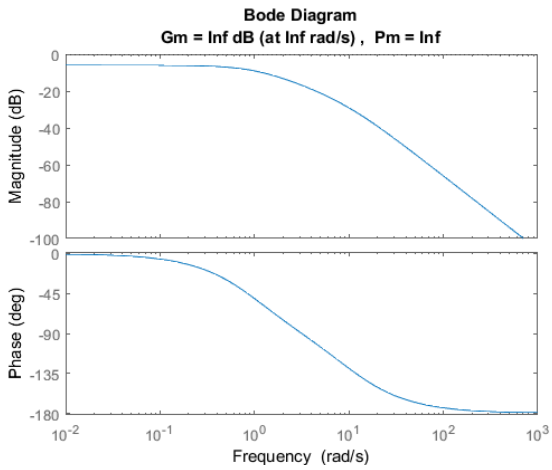
Since there are no sign changes from the even polynomial (s^2 row) down to the bottom of the table, the even polynomial has its two roots on the jω-axis of unit multiplicity. Since there are no sign changes above the even polynomial, the remaining root is in the left half-plane. Therefore, the system is marginally stable.

C. Other Stability Analysis

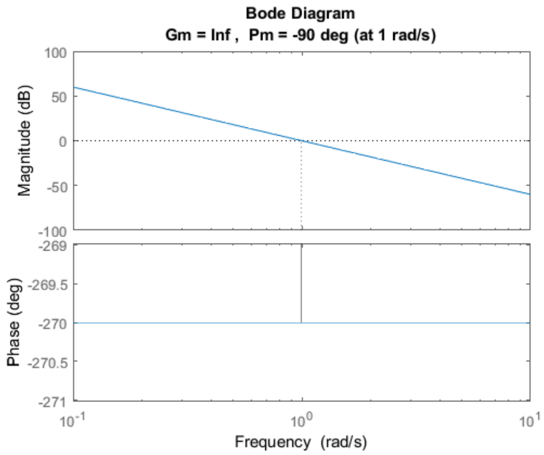
10. Determine the stability of the system using Bode plot if the responses of the system are given in the figures below. [8 marks]



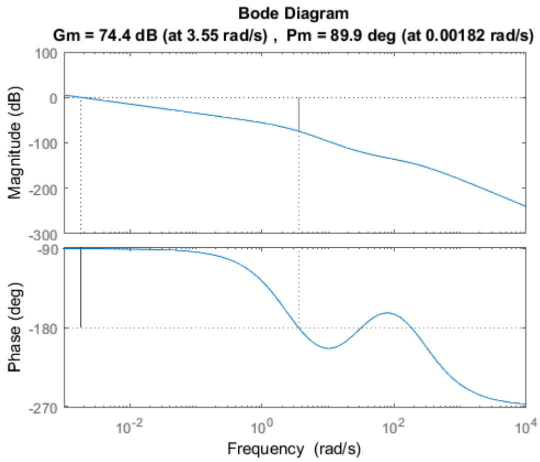
(a) System 1 $G(s) = \frac{5}{s+1}$



(b) System 2 $G(s) = \frac{5}{(s+1)(s+10)}$



(c) System 3 $G(s) = \frac{1}{s^3}$



(d) System 4 $G(s) = \frac{(s+25)(s+35)}{s(s+2)(s+4)(s+200)(s+300)}$

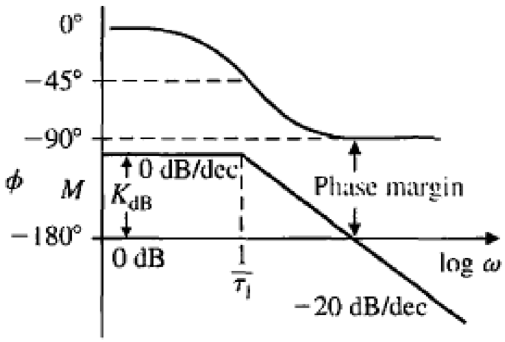
Solution

System 1:

Stable, gain margin = ∞. For the system with transfer function as shown below:

$$G(s) = \frac{K}{s\tau_1 + 1}$$

The Bode plot of the system with the transfer function as given above is:

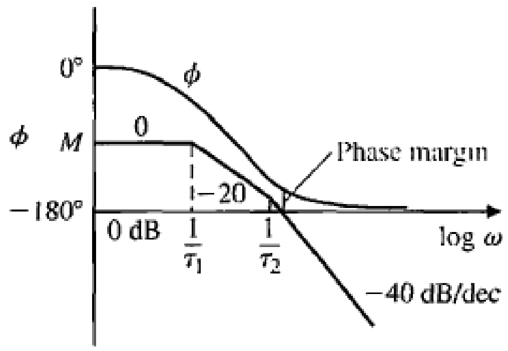


System 2:

Elementary regulator; stable; gain margin = ∞. For the system with transfer function as shown below:

$$(s) = \frac{K}{(s\tau_1 + 1)(s\tau_2 + 1)}$$

The Bode plot of the system with the transfer function as given above is:

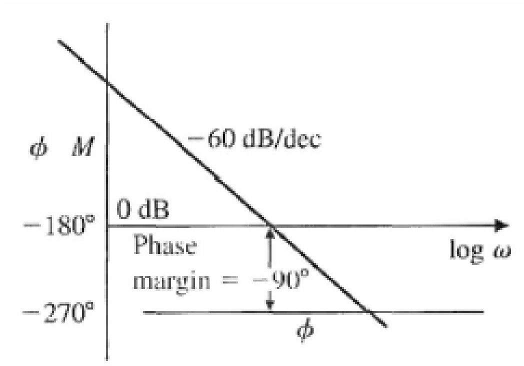


System 3:

Inherently unstable. For the system with transfer function as shown below:

$$G(s) = \frac{K}{s^3}$$

The Bode plot of the system with the transfer function as given above is:

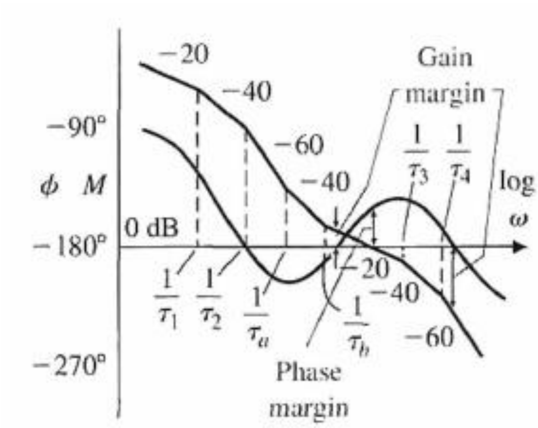


System 4:

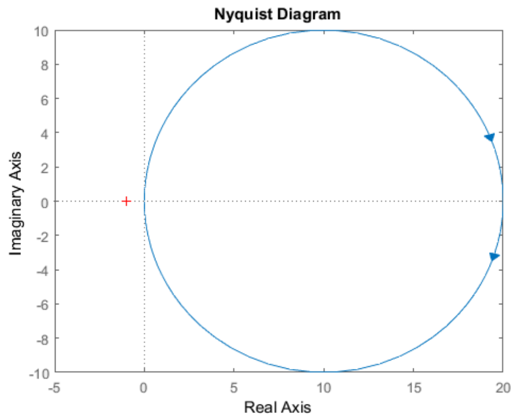
Conditionally stable; stable at low gain, becomes unstable as gain is raised, again becomes stable as gain is further increased, and becomes unstable for very high gains. For the system with transfer function as shown below:

$$G(s) = \frac{K(s\tau_a + 1)(s\tau_b + 1)}{s(s\tau_1 + 1)(s\tau_2 + 1)(s\tau_3 + 1)(s\tau_4 + 1)}$$

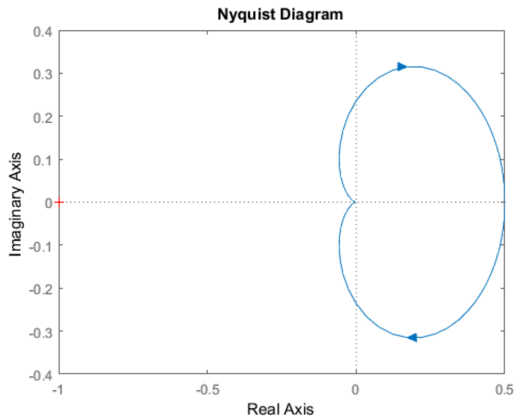
The Bode plot of the system with the transfer function as given above is:



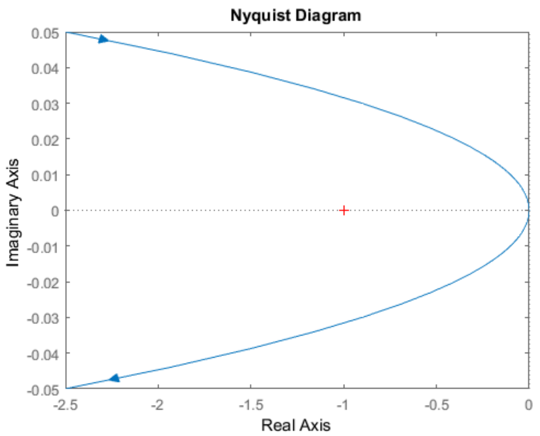
11. Determine the stability of the system using Nyquist plot if the responses of the system are given in the figures below. [8 marks]



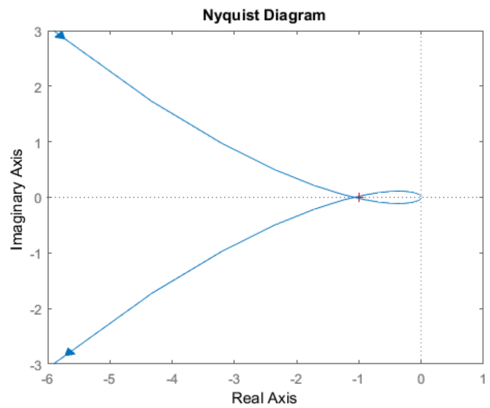
(a) System 1 $G(s) = \frac{100}{s+5}$



(b) System 2 $G(s) = \frac{1}{(s+1)(s+2)}$



(c) System 3 $G(s) = \frac{1}{s^2(s+10)}$



(d) System 4 $G(s) = \frac{(s+1)(s+12)}{s^3}$

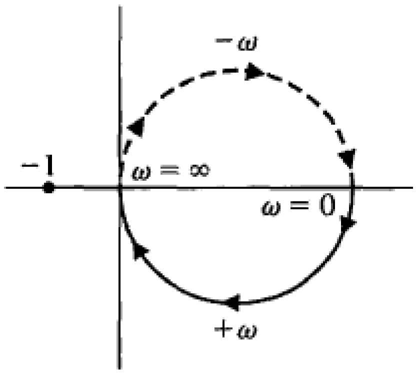
Solution

System 1:

Stable, gain margin = ∞. For the system with transfer function as shown below:

$$G(s) = \frac{K}{s\tau_1 + 1}$$

The Nyquist plot of the system with the transfer function as given above is:

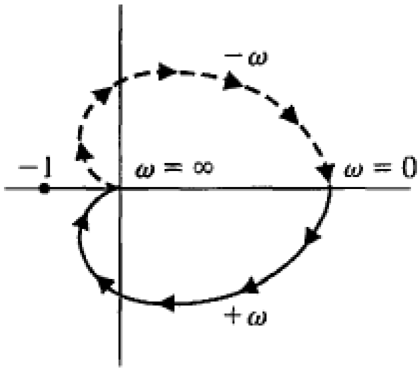


System 2:

Elementary regulator; stable; gain margin = ∞. For the system with transfer function as shown below:

$$G(s) = \frac{K}{(s\tau_1 + 1)(s\tau_2 + 1)}$$

The Nyquist plot of the system with the transfer function as given above is:

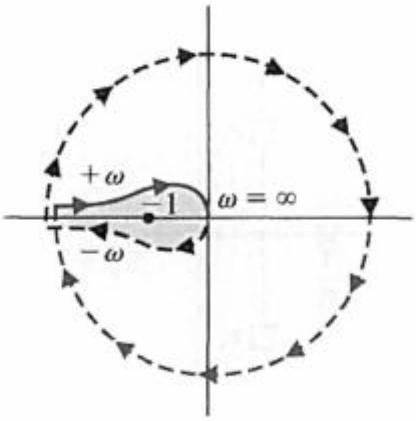


System 3:

Inherently unstable; must be compensated. For the system with transfer function as shown below:

$$G(s) = \frac{K}{s^2(s\tau_1 + 1)}$$

The Nyquist plot of the system with the transfer function as given above is:

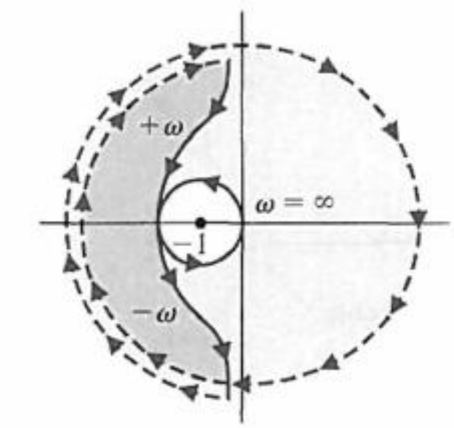


System 4:

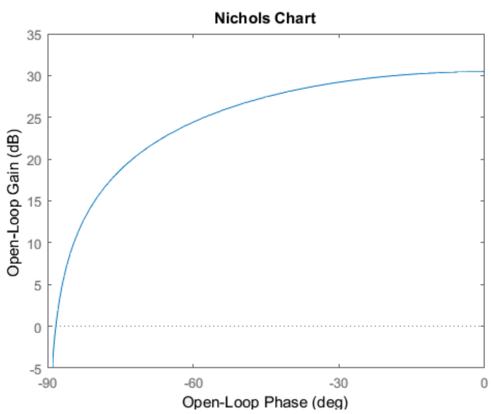
Conditionally stable; becomes unstable if gain is too low. For the system with transfer function as shown below:

$$G(s) = \frac{K(s\tau_a + 1)(s\tau_b + 1)}{s^3}$$

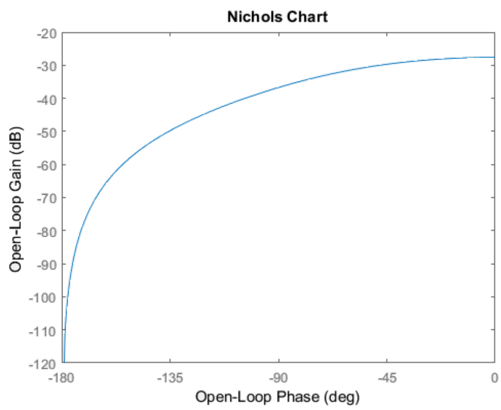
The Nyquist plot of the system with the transfer function as given above is:



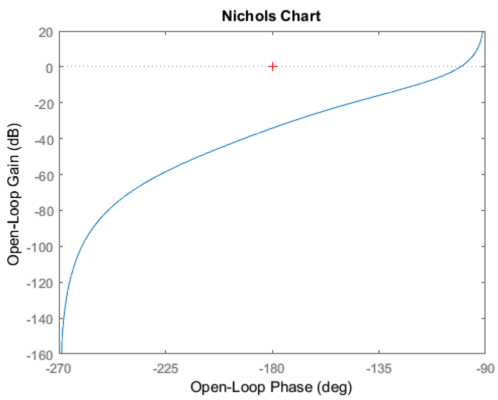
12. Determine the stability of the system using Nichols plot if the responses of the system are given in the figures below. [8 marks]

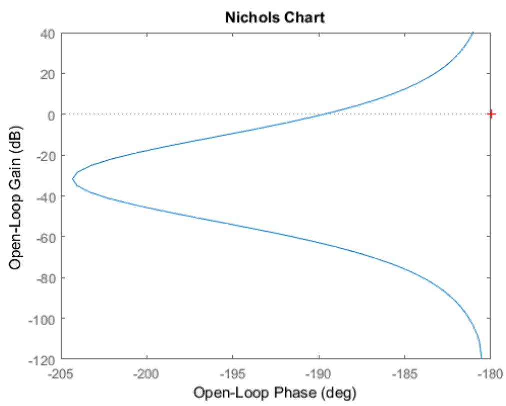


(a) System 1 $G(s) = \frac{100}{s+3}$



G(b) System 2 $(s) = \frac{1}{(s+2)(s+12)}$





(c) System 3 $G(s) = \frac{10}{s(s+2)(s+15)}$

(d) System 4 $G(s) = \frac{(s+12)}{s^2(s+5)}$ $\tau_a > \tau_1$

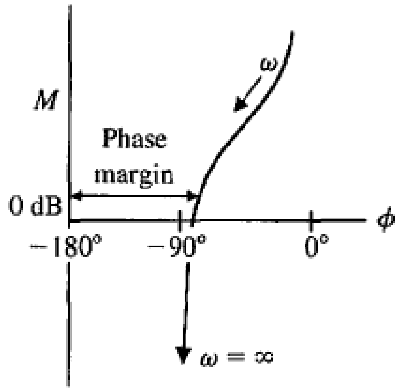
Solution

System 1:

Stable, gain margin = ∞. For the system with transfer function as shown below:

$$G(s) = \frac{K}{s\tau_1 + 1}$$

The Nichols plot of the system with the transfer function as given above is:

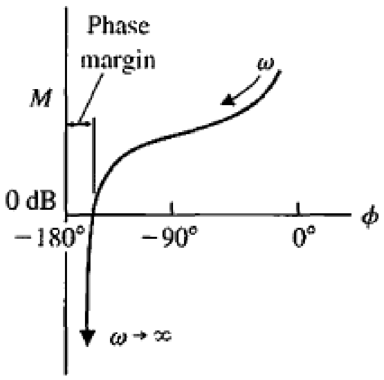


System 2:

Elementary regulator; stable; gain margin = ∞. For the system with transfer function as shown below:

$$G(s) = \frac{K}{(s\tau_1 + 1)(s\tau_2 + 1)}$$

The Nichols plot of the system with the transfer function as given above is:

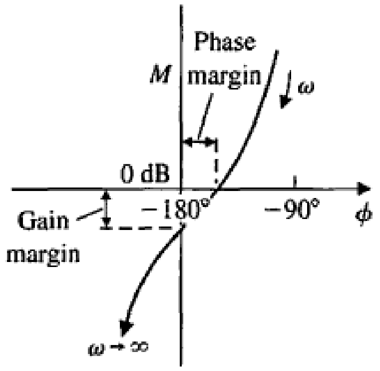


System 3:

Instrument servo with field control motor or power servo with elementary Ward-Leonard drive; stable as shown but may become unstable with increased gain. For the system with transfer function as shown below:

$$G(s) = \frac{K}{s(s\tau_1 + 1)(s\tau_2 + 1)}$$

The Nichols plot of the system with the transfer function as given above is:



System 4:

Stable for all gains. For the system with transfer function as shown below:

$$G(s) = \frac{K(s\tau_a + 1)}{s^2(s\tau_1 + 1)} \qquad \tau_a > \tau_1$$

The Nichols plot of the system with the transfer function as given above is: