XMUT315 Control Systems Engineering

Tutorial 4: Time Domain Analysis (Solution)

A. Time Response Analysis

1. The transient response of a second-order system can be determined from its transfer function equation. Depending on the type of roots in the equation, the response can be categorised as underdamped, critically damped, and overdamped.

   a. Prove the roots of the equation for second order system are: [10 marks]

   $$s_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \quad \text{or} \quad s_2 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$

   b. Prove the time domain equation of the underdamped response of a second-order system when it is given a step input is as shown below. [20 marks]

   $$c(t) = 1 - \frac{e^{-\zeta\omega_t}}{\sqrt{1-\zeta^2}}\sin[\omega_d t + \phi]$$

   Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\sqrt{1-\zeta^2}/\zeta\right)$

Solution

a. Given the general transfer function equation of second order system is given as:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

If the denominator (e.g. characteristic equation of the second order system) is equated to zero, then the equation above becomes.

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Rearrange the equation, it becomes:

$$s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 - \zeta^2\omega_n^2 + \omega_n^2 = 0$$

Then

$$(s + \zeta\omega_n)^2 - \zeta^2\omega_n^2 + \omega_n^2$$

So

$$(s+\zeta\omega_n)^2+\left(\omega_n\sqrt{1-\zeta^2}\right)^2=0$$

The roots of the equation for second order system as shown above are:

$$s_1=-\zeta\omega_n+j\omega_n\sqrt{1-\zeta^2} \quad \text{or} \quad s_2=-\zeta\omega_n-j\omega_n\sqrt{1-\zeta^2}$$

b. For underdamped case ($0<\zeta<1$), the transfer function of the system $C(s)/R(s)$ can be written as follows:

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)}$$

Where: $\omega_d=\omega_n\sqrt{1-\zeta^2}$ (i.e. the frequency ω_d is called the damped natural frequency).

For a step input, $R(s)=1/s$ and $C(s)$ can be written:

$$C(s)=\frac{\omega_n^2}{s(s+\zeta\omega_n+j\omega_d)(s+\zeta\omega_n-j\omega_d)}$$

Applying partial fraction to the equation given above:

$$C(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

Hence the inverse transform of the equation above is:

$$c(t)=1-e^{-\zeta\omega_n t}\left(\cos\omega_d t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)$$

This equation can be simplified into:

$$c(t)=1-\left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\left(\sqrt{1-\zeta^2}\cos\omega_d t+\zeta\sin\omega_d t\right)$$

Say, $\zeta=\cos\phi$, hence $\sqrt{1-\zeta^2}=\sin\phi$ (e.g. for underdamped response $0<\zeta<1$):

$$c(t)=1-\left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)(\sin\phi\cos\omega_d t+\cos\phi\sin\omega_d t)$$

The expression that outlines the underdamped response of the second-order system when it is given a step input function is:

$$c(t)=1-\left(\frac{e^{-\zeta\omega_t}}{\sqrt{1-\zeta^2}}\right)\sin[\omega_d t+\phi]$$

Where: $\omega_d=\omega_n\sqrt{1-\zeta^2}$ and $\phi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$

2. Prove that the time domain equations of the transient responses of the given second order system when it is given a step input are as shown below.

a. For the critically damped response second order system, the time domain equation is:

[20 marks]

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \quad \text{for} \quad t \geq 0$$

b. For the overdamped response second order system, the time domain equation is:

[20 marks]

$$c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right) \quad \text{for} \quad t \geq 0$$

Where: $s_1 = \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$ and $s_2 = \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$

Solution

a. We know that a second order system can be represented with the following standardised transfer function equation.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

Where: $\omega_d = \omega_n\sqrt{1 - \zeta^2}$

Thus, in a critically damped second order system ($\zeta = 1$), the two roots of the $C(s)/R(s)$ are nearly equal.

$$\omega_d = \omega_n\sqrt{1 - (1)^2} = 0$$

As a result, the standardised equation above becomes:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \omega_n)(s + \omega_n)}$$

Furthermore, the system may be approximated by a critical damped one:

$$C(s) = \left[\frac{\omega_n^2}{(s + \omega_n)^2}\right]R(s)$$

For a unit-step input ($R(s) = 1/s$) and $C(s)$ can be written as:

$$C(s) = \frac{\omega_n^2}{(s + \omega_n)^2}\left(\frac{1}{s}\right)$$

Taking the fractional function of the equation given above

$$C(s) = \left(\frac{1}{s}\right) - \frac{\omega_n^2}{(s + \omega_n)^2}$$

The inverse transform of the equation given above is:

$$c(t) = 1 - e^{\omega_n t}(1 + \omega_n t)$$

Thus, the transient response ($c(t)$) of the critically damped second order system is an increase exponential function which is modulated by a time-related function.

b. Knowing that a second order system can be represented with the following standardised transfer function equation.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Thus, in overdamped case ($\zeta > 1$), the two roots of $C(s)/R(s)$ are negative real and unequal. So, the equation above becomes:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{\left(s + \zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)\left(s + \zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}\right)}$$

We know that $j = \sqrt{-1}$, as a result the equation above becomes:

$$C(s) = \left[\frac{\omega_n^2}{\left(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)\left(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)}\right] R(s)$$

For a unit-step input $R(s) = 1/s$, and $C(s)$ can be written as:

$$C(s) = \left[\frac{\omega_n^2}{\left(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)\left(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)}\right]\left(\frac{1}{s}\right)$$

Applying partial fraction to the equation given above:

$$C(s) = \left(\frac{1}{s}\right) + \frac{\omega_n^2}{2\left[s + \omega_n\left(\zeta + \sqrt{\zeta^2-1}\right)\right]} - \frac{\omega_n^2}{2\left[s + \omega_n\left(\zeta - \sqrt{\zeta^2-1}\right)\right]}$$

The inverse Fourier transform of the equation above is:

$$c(t) = 1 + \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta + \sqrt{\zeta^2-1}\right)} e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} - \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta + \sqrt{\zeta^2-1}\right)} e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

Alternatively, the equation given above can be represented as:

$$c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2}\right) \qquad \text{for} \quad t \geq 0$$

Where: $s_1 = \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n$ and $s_2 = \left(\zeta - \sqrt{\zeta^2-1}\right)\omega_n$

Thus, the transient response of the second order system ($c(t)$) for an overdamped case ($\zeta > 1$) has two decaying exponential terms.

3. You are given the following first order systems:

i. System 1:

$$G(s) = \frac{5}{s+5}$$

ii. System 2:

$$G(s) = \frac{20}{s+20}$$

a. For both systems, calculate the time constant, rise time, and settling time for each system.

[12 marks]

b. Based on the results in part (a), comment on the differences in transient and steady-state responses of the first system with the second system. [4 marks]

c. Simulate and describe the transient responses of the systems when each of the systems is subjected to a step input. [10 marks]

d. Simulate and describe the transient responses of the systems when each of the systems is subjected to an impulse input. [10 marks]

Solution

a. The time constant, rise time, and settling time for each system are calculated as follows.

For System (i):

$$G(j\omega) = \frac{5}{(s+5)} = \frac{1}{s/5+1}$$

Time constant:

$$\tau = \frac{1}{5} = 0.2 \text{ second}$$

Rise time:

$$T_r = 2.2\tau = \frac{2.2}{a} = \frac{2.2}{5} = 0.44 \text{ second}$$

Settling time:

$$T_s = 4\tau = \frac{4}{a} = \frac{4}{5} = 0.8 \text{ second}$$

For System (ii):

$$G(s) = \frac{20}{(s+20)} = \frac{1}{s/20+1}$$

Time constant:

$$\tau = \frac{1}{20} = 0.05 \text{ second}$$

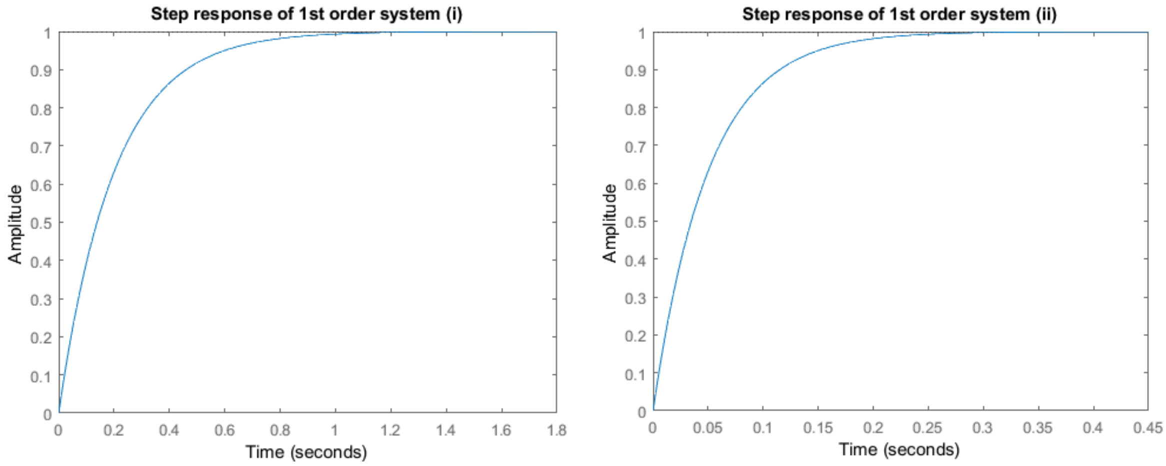
Rise time:

$$T_r = 2.2\tau = \frac{2.2}{a} = \frac{2.2}{20} = 0.11 \text{ second}$$

Settling time:

$$T_s = 4\tau = \frac{4}{a} = \frac{4}{20} = 0.2 \text{ second}$$

b. For transient response of the system, it seems System (i) has considerable longer time constant, rise time and settling time compared with System (ii) e.g. τ = 0.2 second, T_r = 0.44 second and T_s = 0.8 second compared with τ = 0.05 second, T_r = 0.11 second and T_s = 0.2 second respectively.



For steady-state response, the overall response of System (ii) is more responsive e.g. take less time to settle down compared with System (i).

c. Simulate and describe the transient responses of the systems when each of these systems subjected to a step input. For System (i) and a step input of $1/s$:

$$G(s) = \frac{5}{s(s+5)}$$

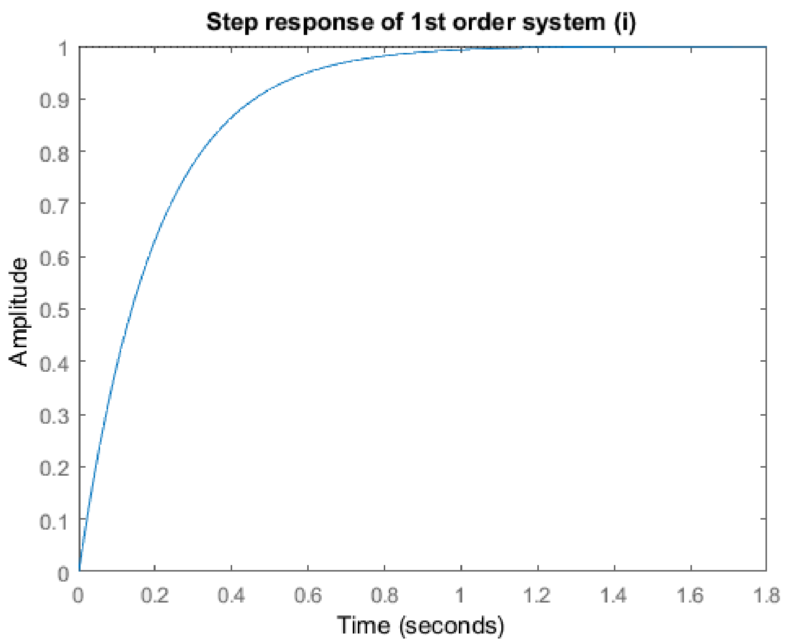
Take partial fraction of the equation given above:

$$G(s) = \frac{1}{s} - \frac{1}{s+5}$$

The inverse transform of the equation given above is:

$$g(t) = 1 - e^{-5t}$$

The transient response of the System (i) is an increasing exponential response with a time constant of 5 seconds.



For System (ii) and a step input of $1/s$:

$$G(s) = \frac{20}{s(s+20)}$$

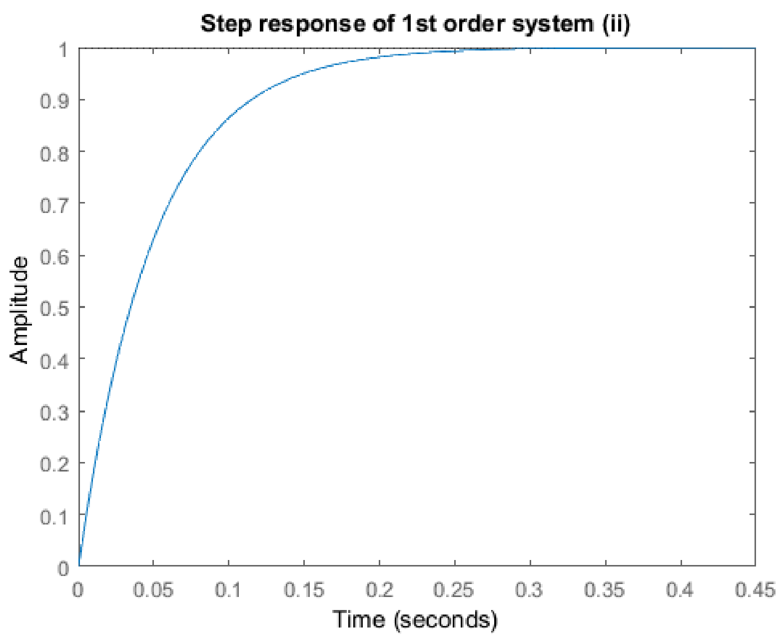
Take partial fraction of the equation above:

$$G(j\omega) = \frac{1}{s} - \frac{1}{s+20}$$

Therefore, the inverse transform of the equation given above is:

$$g(t) = 1 - e^{-20t}$$

The transient response of the System (ii) is an increasing exponential response with a time constant of 20 seconds.



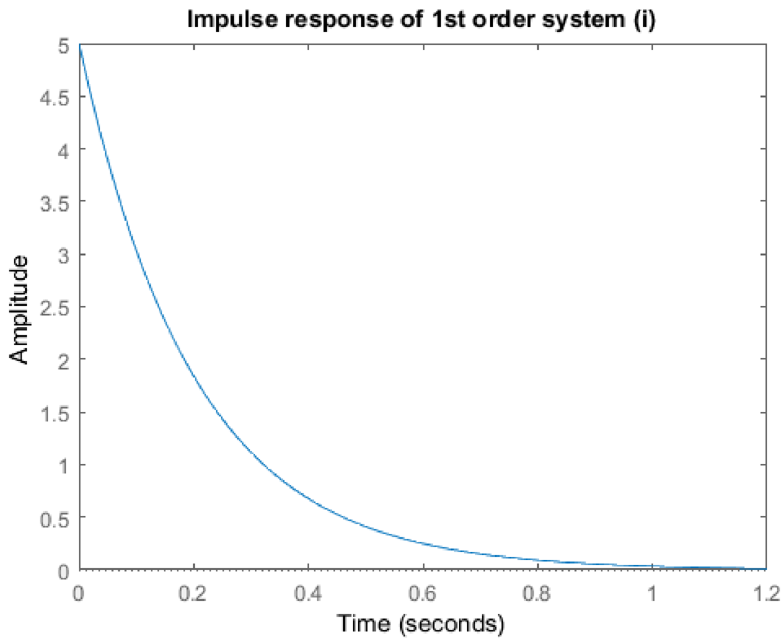
d. Simulate and describe the transient responses of the systems when each of these systems subjected to an impulse input. For System (i) and impulse input:

$$G(s) = \frac{5}{(s+5)}$$

The inverse transform of the equation given above is:

$$g(t) = 5e^{-5t}$$

The transient response of the System (i) is a decreasing exponential response with a time constant of 5 seconds.



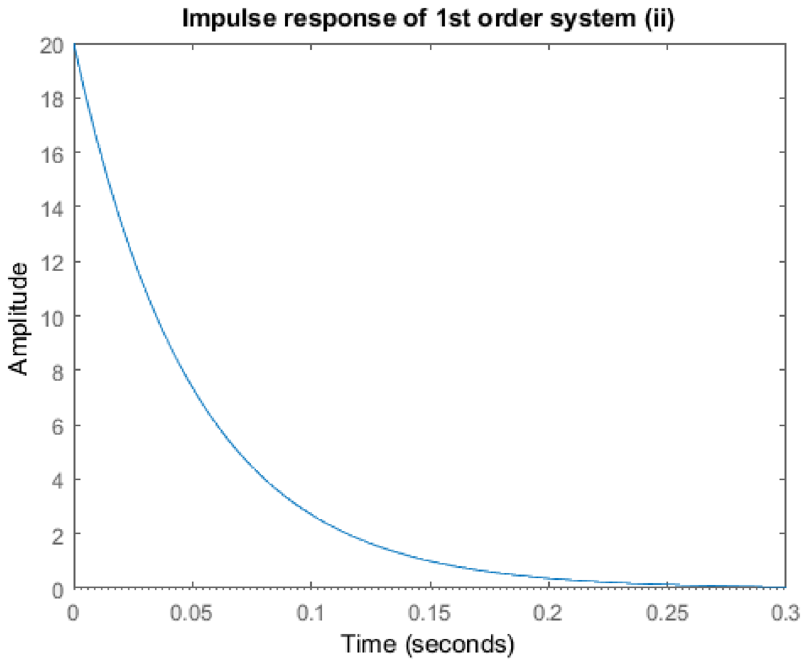
For System (ii) and impulse input:

$$G(s) = \frac{20}{(s+20)}$$

Therefore

$$g(t) = 20e^{-20t}$$

The transient response of the System (ii) is a decreasing exponential response with a time constant of 20 seconds.



4. Given the transfer function as shown below, find damping factor (ζ) and natural frequency (ω_n) of the system. [4 marks]

$$G(s) = \frac{36}{s^2 + 4.2s + 36}$$

Solution

Comparing the above equation to the standard equation of the second order transfer function:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

As a result, $\omega_n^2 = 36$ from which $\omega_n = 6$.

Also, $2\zeta\omega_n = 4.2$. Substituting the value of ω_n, as a result, $\zeta = 0.35$.

5. Given the transfer function as shown below:

$$G(s) = \frac{100}{s^2 + 15s + 100}$$

Determine the parameters of the time response of the system as follow:

a. Natural frequency (ω_n) and damping ratio (ζ). [4 marks]

b. Time-to-peak (T_p), percentage overshoot ($\%OS$) and settling time (T_s). [6 marks]

c. Rise time (T_r) using the following methods: derived equation, alternative equation, and graph of normalised damping ratio. Simulate the system in MATLAB for rise time and determine which method gives the most accurate result. [8 marks]

Solution

a. Comparing the transfer equation of the system with the standard second order equation as given below, ω_n and ζ are calculated as follow.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Thus

$$\omega_n = \sqrt{100} = 10$$

And

$$2\zeta\omega_n = 15$$

With ω_n = 10, the damping ratio is:

$$\zeta = \frac{15}{2(10)} = 0.75$$

As a result, natural frequency ω_n and damping ratio ζ are 10 and 0.75 respectively.

b. Now substitute ζ and ω_n found in part (a) into the following equations. First, measuring the time-to-peak.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

The following equation is used for calculating the percentage overshoot.

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100$$

The equation given below is for calculating the settling time.

$$T_s = \frac{4}{\zeta\omega_n}$$

We found respectively that T_p = 0.475 second, $\%OS$ = 2.838, and T_s = 0.533 second.

c. The rise time (T_r) are determined and calculated using the three methods as follows.

(i) Derived Equation:

Using the derived equation as shown below, the rise time (T_r) is:

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi - 0.722}{10\sqrt{1-(0.75)^2}} = \frac{2.419}{6.614} = 0.366$$

Where:

$$\phi = \arctan\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \arctan\left(\frac{\sqrt{1-(0.75)^2}}{0.75}\right) = 41.4° = 0.722 \text{ rad}$$

(ii) Alternative Equation:

Using the alternative equation as shown below, the rise time (T_r) is:

$$T_r = \frac{(1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)}{\omega_n}$$

$$= \frac{(1.76(0.75)^3 - 0.417(0.75)^2 + 1.039(0.75) + 1)}{10}$$

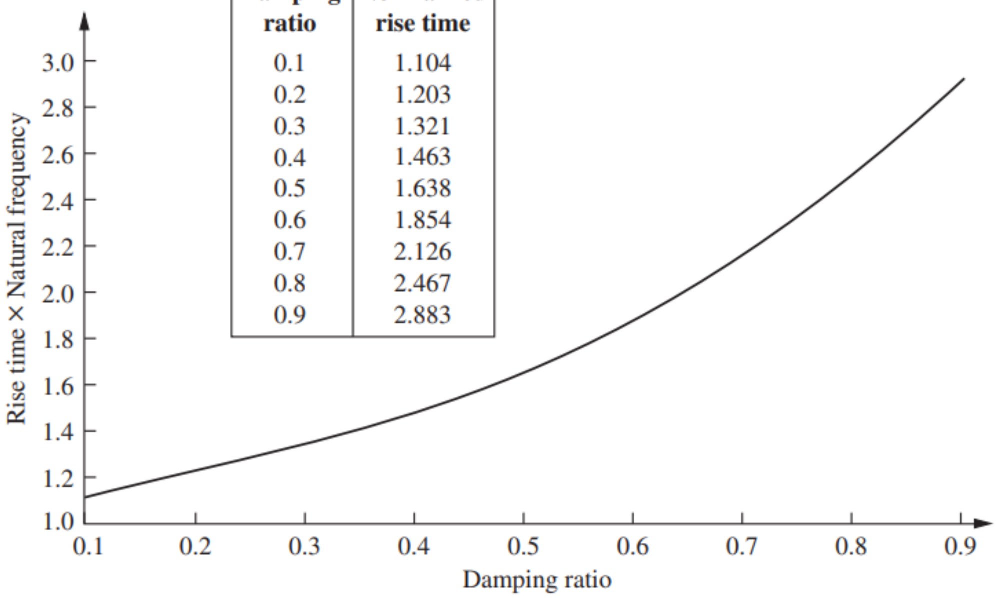
$$= \frac{0.7425 - 0.2345 + 1.77925}{10} = 0.229$$

(iii) Graph of (Normalised) Damping Ratio vs. Rise Time:

The normalised damping ratio vs. rise time graph shown below is constructed from the following equation:

$$t_r\omega_0 = 2.230\zeta^2 - 0.078\zeta + 1.12$$

Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883



Using the table in the figure above, when the damping ratio (ζ) is 0.75, the normalized rise time is approximately 2.3 seconds. Divided this value by ω_n (i.e. 10), this yields:

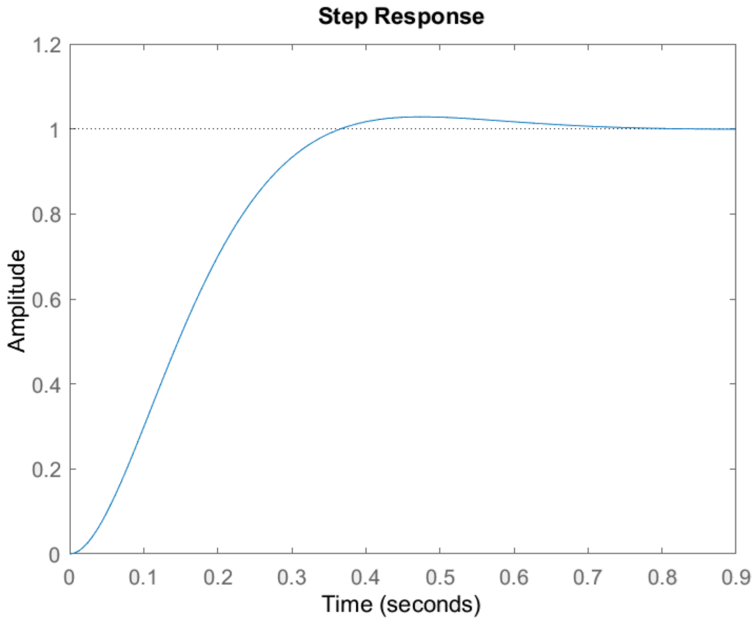
$$T_r = \frac{\zeta}{\omega_n} = \frac{2.3}{10} = 0.23 \text{ s}$$

MATLAB code:

```
sys = tf([100],[1 15 100]);
step(sys)
```

```
S = stepinfo(sys)
```

The results of MATLAB simulation are as shown in the figures below.




```
S =

  struct with fields:

        RiseTime: 0.2288
   TransientTime: 0.5743
    SettlingTime: 0.5743
     SettlingMin: 0.9049
     SettlingMax: 1.0284
       Overshoot: 2.8369
      Undershoot: 0
            Peak: 1.0284
        PeakTime: 0.4728
```

As shown above, the result of simulation in MATLAB gives a rise time (T_r) of 0.2288 s.

The most accurate method for determining rise time (T_r) is using the alternative method as its result is close to the simulation result.

This question demonstrates that we can find T_p, $\%OS$, T_s, and T_r without the tedious task of taking an inverse Laplace transform, plotting the output response, and taking measurements from the plot.

6. Find the step response of each of the transfer functions shown in the equations given below and compare them. [12 marks]

- System 1:

$$T_1(s) = \frac{24.542}{s^2 + 4s + 24.542}$$

- System 2:

$$T_2(s) = \frac{245.42}{(s + 10)(s^2 + 4s + 24.542)}$$

- System 3:

$$T_3(s) = \frac{73.626}{(s + 3)(s^2 + 4s + 24.542)}$$

Solution

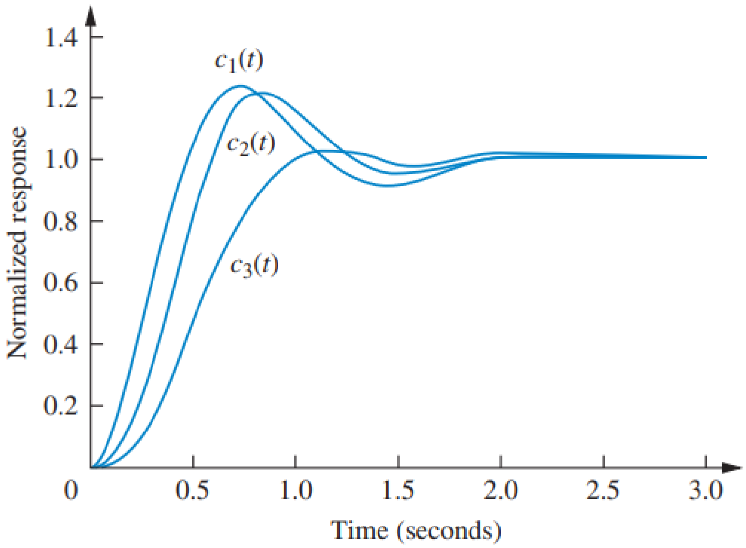
The step response, $C_i(s)$, for the transfer function, $T_i(s)$, can be found by multiplying the transfer function by $1/s$, a step input, and using partial-fraction expansion followed by the inverse Laplace transform to find the response, $c_i(t)$. The results are:

$$c_1(t) = 1 - 1.09e^{-2t}\cos(4.532t + 23.8°)$$

$$c_2(t) = 1 - 0.29e^{-10t} - 1.189e^{-2t}\cos(4.532t - 53.34°)$$

$$c_3(t) = 1 - 1.14e^{-3t} + 0.707e^{-2t}\cos(4.532t + 78.63°)$$

The three responses are plotted in the figure given below (e.g. step responses of system $T_1(s)$, system $T_2(s)$ and System $T_3(s)$).



Notice the first effect of distance of the poles to the imaginary or y-axis: the farther from the y-axis, the quicker to settle down and the closer to the y-axis the slower to settle down.

Observe also the second effect of distance of the poles to the imaginary or y-axis: the farther from the y-axis, the smaller is damping ratio (underdamped) and the closer to the y-axis the bigger is the damping ratio (overdamped).

Here, that $c_2(t)$, with its third pole at -10 and farthest from the dominant poles, is the better approximation of $c_1(t)$, the pure second-order system response; $c_3(t)$, with a third pole close to the dominant poles, yields the most error.

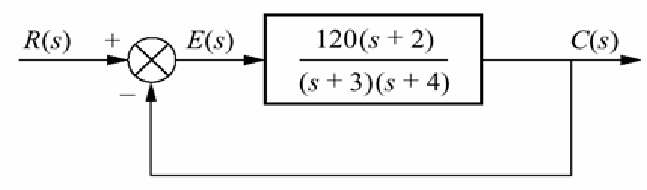
B. Steady-State Analysis

7. For the system shown below, find the steady-state errors for the inputs:

   a. Step input, $5u(t)$. [2 marks]

   b. Ramp input, $5tu(t)$. [2 marks]

   c. Parabolic input, $5t^2u(t)$. [2 marks]



Solution

First, check that closed-loop system is stable. We need the poles are all in the left-hand side of the s-plane for the system is stable.

Apply the equation for the feedback system, the transfer function equation for the closed-loop system is:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

For a unity feedback system, the equation above becomes:

$$T(s) = \frac{G(s)}{1 + G(s)}$$

Entering the $G(s)$ of the system into the equation above, it is now:

$$T(s) = \frac{\frac{120(s+2)}{(s+3)(s+4)}}{1 + \frac{120(s+2)}{(s+3)(s+4)}} = \frac{120(s+2)}{(s+3)(s+4) + 120(s+2)}$$

$$= \frac{120(s+2)}{s^2 + 127s + 252}$$

Applying Routh-Hurwitz method, the closed-loop transfer function of the system is stable.

a. Consider step input $r(t) = 5u(t)$, the Laplace transform of the step input is:

$$R(s) = 5/s$$

The steady-state error of the system is then:

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)} = \frac{s\left(\frac{5}{s}\right)}{1 + \lim_{s\to 0} \frac{120(s+2)}{(s+3)(s+4)}} = \frac{5}{1+20} = \frac{5}{21}$$

b. For the ramp input $r(t) = 5tu(t)$, the Laplace transform is given by:

$$R(s) = 5/s^2$$

The steady-state error of the system is:

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)} = \frac{s\left(\frac{5}{s^2}\right)}{1 + \lim_{s\to 0} G(s} = \frac{5}{\lim_{s\to 0} sG(s)} = \frac{5}{0} = \infty$$

c. For the parabolic input $r(t) = 5t^2u(t)$, the Laplace transform is given by:

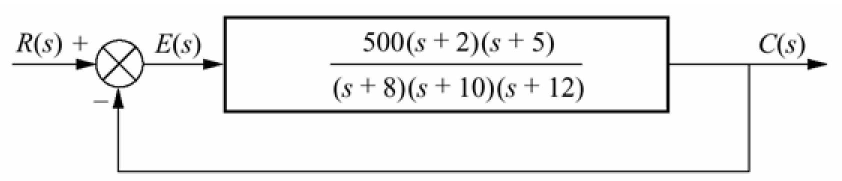
$$R(s) = 10/s^3$$

The steady-state error of the system is:

$$e(\infty) = \lim_{s\to0} sE(s) = \lim_{s\to0} \frac{sR(s)}{1+G(s)} = \frac{s\left(\frac{10}{s^3}\right)}{1+\lim_{s\to0} G(s} = \frac{10}{\lim_{s\to0} s^2G(s)} = \frac{10}{0} = \infty$$

8. For the system below, evaluate the static-error constants and find the steady-state errors for:

- Step input, $5u(t)$. [4 marks]
- Ramp input, $5tu(t)$. [4 marks]
- Parabolic input, $5t^2u(t)$. [4 marks]



Solution

Check first the stability of the closed-loop system. The system is stable if there is no pole in the right-hand side of the s-plane.

For a unity feedback system, the closed-loop transfer function equation is:

$$T(s) = \frac{G(s)}{1+G(s)}$$

Entering the $G(s)$ of the system into the equation above, it is now:

$$T(s) = \frac{\frac{500(s+2)(s+5)}{(s+8)(s+10)(s+12)}}{1+\frac{500(s+2)(s+5)}{(s+8)(s+10)(s+12)}}$$

$$= \frac{500(s+2)(s+5)}{(s+8)(s+10)(s+12)+500(s+2)(s+5)}$$

$$= \frac{500(s+2)(s+5)}{s^3+530s^2+3796s+5960}$$

Applying Routh-Hurwitz method, the closed-loop transfer function of the system is stable.

The static-error constants are:

$$K_p = \lim_{s\to0} G(s) = \frac{(500)(2)(5)}{(8)(10)(12)} = 5.208$$

$$K_v = \lim_{s\to0} sG(s) = 0$$

$$K_a = \lim_{s\to 0} s^2 G(s) = 0$$

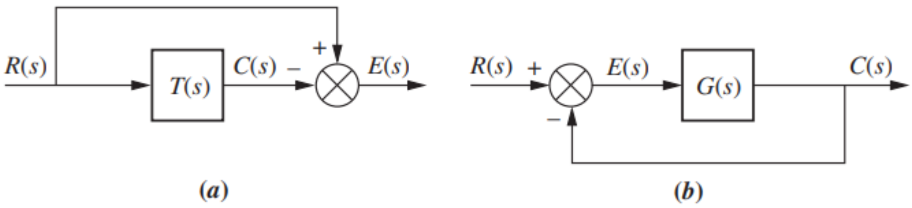
The steady-state errors of the system are:

$$e_{step}(\infty) = \frac{1}{1+K_p} = \frac{1}{1+5.208} = 0.161$$

$$e_{ramp}(\infty) = \frac{1}{K_v} = \frac{1}{0} = \infty$$

$$e_{parabola}(\infty) = \frac{1}{K_a} = \frac{1}{0} = \infty$$

9. Find the steady-state error for the system in the figure given below if $T(s) = 5/(s^2 + 7s + 10)$ and the input is a unit step. [6 marks]



Shown in the diagram above is closed loop control system error, as a general representation of the system given in (a) and a representation for unity feedback system in (b).

Solution

From the problem statement, $R(s) = 1/s$ and $T(s) = 5/(s^2 + 7s + 10)$. Substituting into equation below.

$$E(s) = R(s)[1 - T(s)]$$

This yields:

$$E(s) = \frac{1}{s}\left(1 - \frac{5}{s^2 + 7s + 10}\right) = \frac{s^2 + 7s + 5}{s(s^2 + 7s + 10)}$$

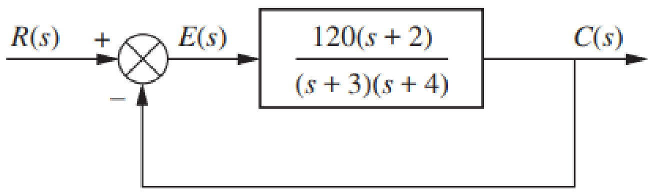
Since $T(s)$ is stable and, subsequently, $E(s)$ does not have right-half-plane poles or jω poles other than at the origin, we can apply the final value theorem. Substituting the equation given above into the following equation.

$$e(\infty) = \lim_{e\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[\frac{s^2 + 7s + 5}{s(s^2 + 7s + 10)}\right] = \frac{5}{10}$$

This gives $e(\infty)$ = 1/2.

10. Match up the steady-state conditions for inputs of $5u(t)$, $5tu(t)$, and $5t^2u(t)$ to the system shown in the following figure with the simulation results. The function $u(t)$ is the unit step.

[12 marks]



Solution

First, we verify that the closed-loop system is indeed stable. For this example, we leave out the details.

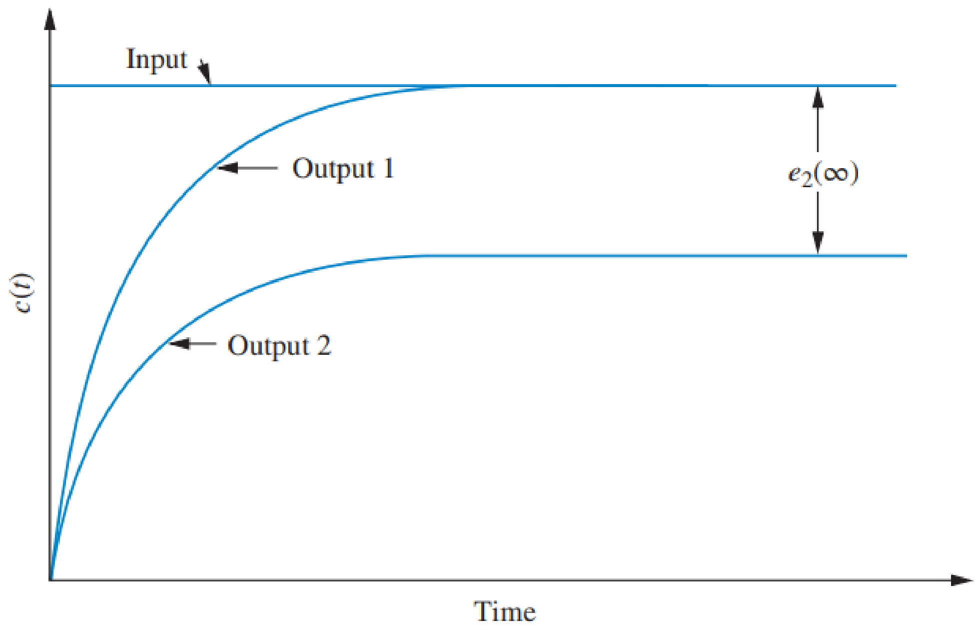
Next, for the step input $5u(t)$, whose Laplace transform is $5/s$, the steady-state error will be five times as large as that given by the equation below.

$$e(\infty) = e_{step}(\infty) = \lim_{s\to 0} \frac{s(5/s)}{1+G(s)} = \frac{5}{1+\lim_{s\to 0} G(s)}$$

Or

$$e(\infty) = e_{step}(\infty) = \frac{5}{1+\lim_{s\to 0} G(s)} = \frac{5}{1+20} = \frac{5}{21}$$

This implies a response similar to output 2 of the following figure.



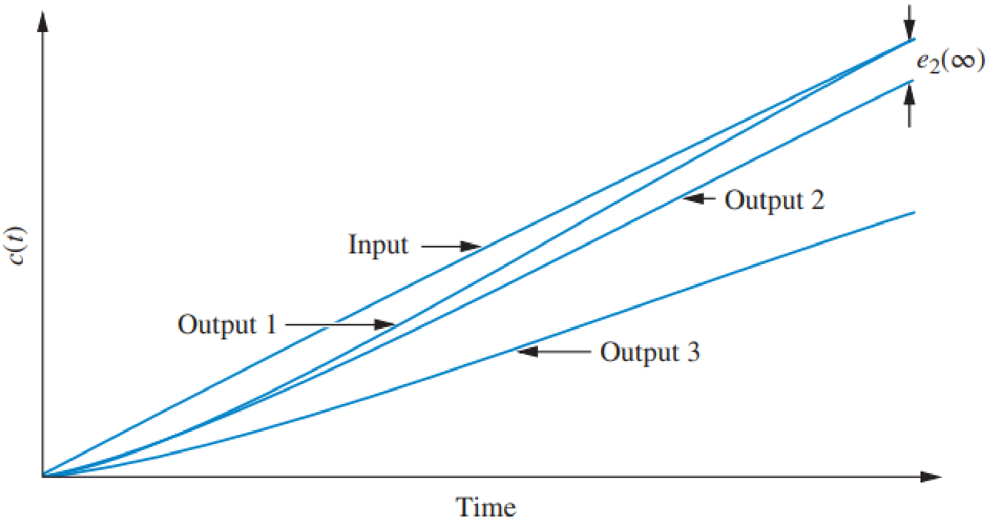
For the ramp input $5tu(t)$, whose Laplace transform is $5/s^2$, the steady-state error will be five times as large as that given by the equation given below.

$$e(\infty) = e_{ramp}(\infty) = \lim_{s\to 0} \frac{s(5/s^2)}{1+G(s)} = \lim_{s\to 0} \frac{5}{s+sG(s)} = \frac{5}{\lim_{s\to 0} sG(s)}$$

Or

$$e(\infty) = e_{ramp}(\infty) = \frac{5}{\lim_{s \to 0} sG(s)} = \frac{5}{0} = \infty$$

This implies a response similar to output 3 of the figure given below.



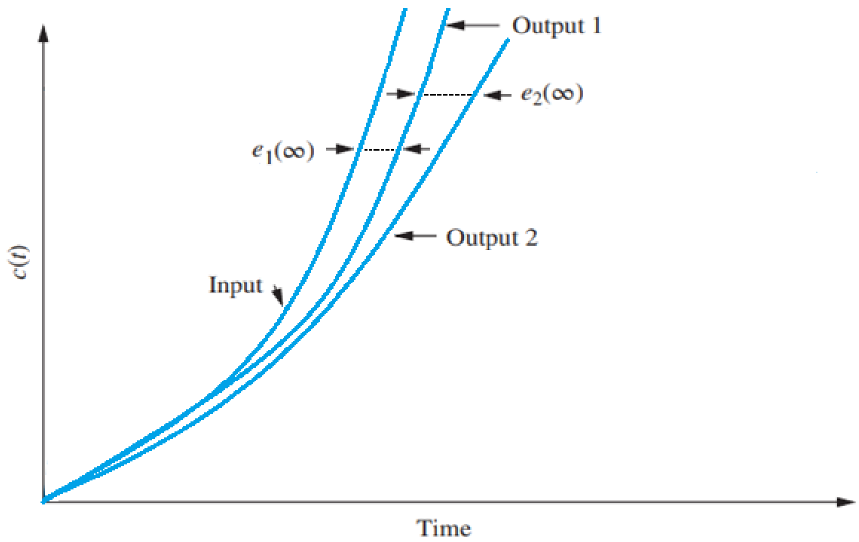
For the parabola input $5t^2u(t)$, whose Laplace transform is $10/s^3$, the steady-state error will be 10 times as large as that given by the equation given below.

$$e(\infty) = e_{parabola}(\infty) = \lim_{s \to 0} \frac{s(10/s^3)}{1+G(s)} = \lim_{s \to 0} \frac{10}{s^2 + s^2G(s)} = \frac{10}{\lim_{s \to 0} s^2G(s)}$$

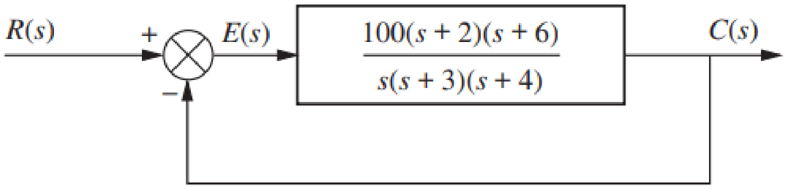
Or

$$e(\infty) = e_{parabola}(\infty) = \frac{10}{\lim_{s \to 0} s^2G(s)} = \frac{10}{0} = \infty$$

This implies a response similar to output 2 of the figure given below.



11. For the control system shown in the figure below, attempt the following tasks.



a. Determine the steady-state errors for inputs of $5u(t)$, $5tu(t)$, and $5t^2u(t)$ to the system shown in the figure above and match them up with simulation results. The function $u(t)$ is the unit step. [6 marks]

b. Compare the steady-state conditions of the system for the inputs given. [3 marks]

c. Describe the role of integral component towards steady-state characteristics of the system. Compare the results with the steady-state characteristics of the system from the previous question. [5 marks]

Solution

a. First, verify that the closed-loop system is indeed stable. For this question, we leave out the details.

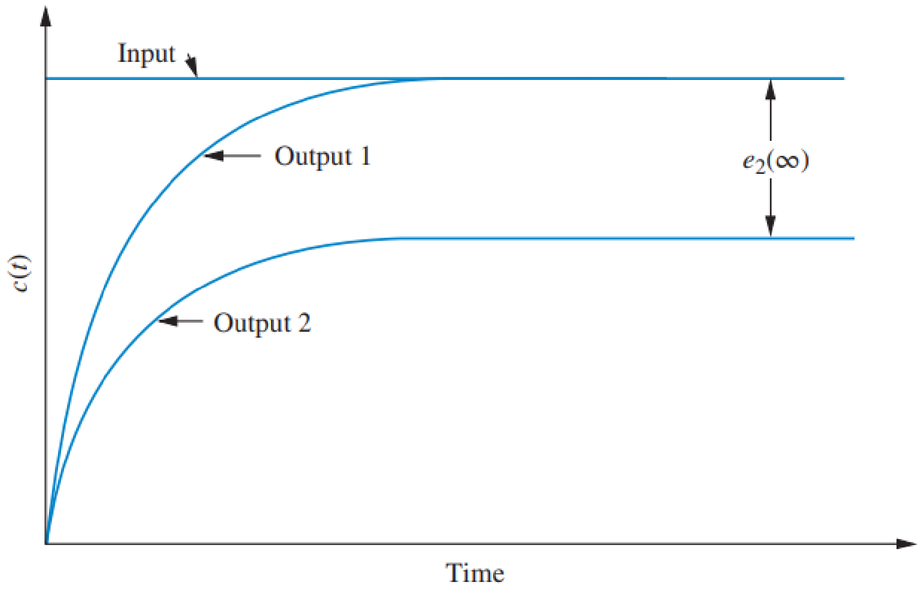
For the step input $5u(t)$, whose Laplace transform is $5/s$, the steady-state error will be five times as large as that given by the following equation.

$$e(\infty) = e_{step}(\infty) = \lim_{s\to 0}\frac{s(5/s)}{1+G(s)} = \frac{5}{1+\lim_{s\to 0} G(s)}$$

Or

$$e(\infty) = e_{step}(\infty) = \frac{5}{1+\lim_{s\to 0} G(s)} = \frac{5}{\infty} = 0$$

This implies a response similar to output 1 of the figure given below.



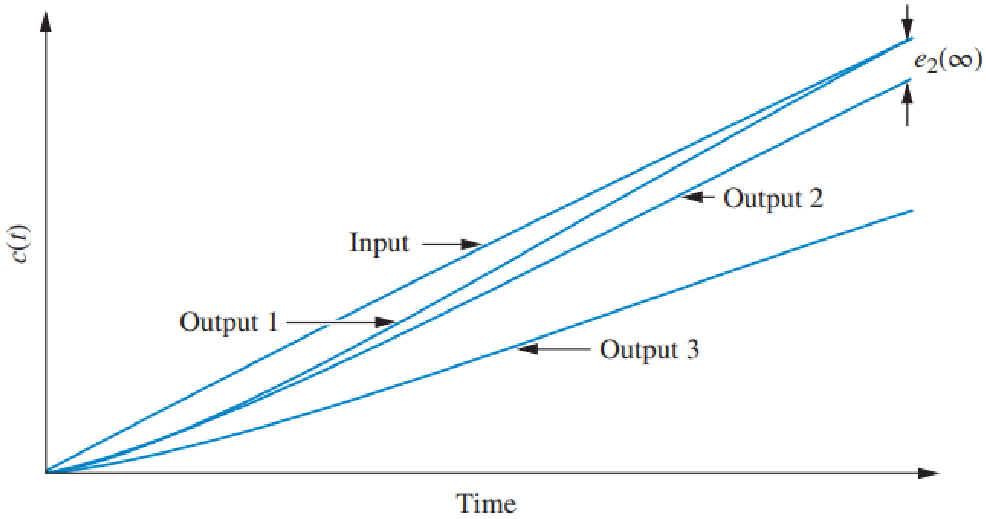
For the ramp input $5tu(t)$, whose Laplace transform is $5/s^2$, the steady-state error will be five times as large as that given by the equation given below.

$$e(\infty) = e_{ramp}(\infty) = \lim_{s\to 0}\frac{s(5/s^2)}{1+G(s)} = \lim_{s\to 0}\frac{5}{s+sG(s)} = \frac{5}{\lim_{s\to 0} sG(s)}$$

Or

$$e(\infty) = e_{ramp}(\infty) = \frac{5}{\lim_{s\to 0} sG(s)} = \frac{5}{100} = \frac{1}{20}$$

This implies a response similar to output 2 of the figure below.



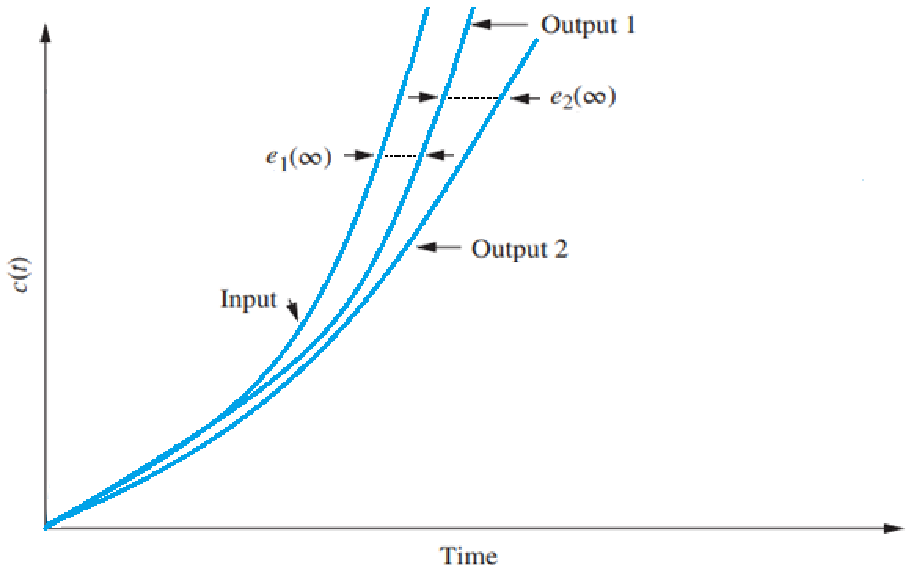
For the parabola input, $5t^2u(t)$, whose Laplace transform is $10/s^3$, the steady-state error will be 10 times as large as that given by the equation below.

$$e(\infty) = e_{parabola}(\infty) = \lim_{s\to 0}\frac{s(10/s^3)}{1+G(s)} = \lim_{s\to 0}\frac{10}{s^2+s^2G(s)} = \frac{10}{\lim_{s\to 0} s^2G(s)}$$

Or

$$e(\infty) = e_{parabola}(\infty) = \frac{10}{\lim_{s \to 0} s^2 G(s)} = \frac{10}{0} = \infty$$

Response of system is similar to output 2.



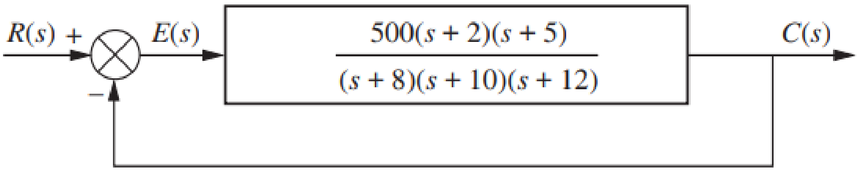
b. Comparing the steady-state responses of the system with various inputs, the steady-state responses of the system are:

- Step input – steady state error is zero.
- Ramp input – steady state error is a constant (1/20).
- Parabolic input - steady-state response is infinity.

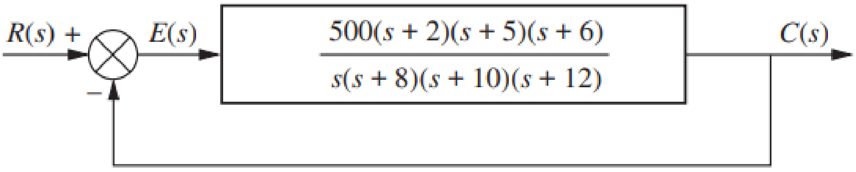
c. Since there is an integration in the forward path, it is expected that the steady-state errors for some of the input waveforms will be less than those found in the previous question. Having integral function in the forward path reduces significantly the stead-state error in the given system.

- For step input, the integration in the forward path yields zero stead-state error, rather than finite steady-state error found in the previous question.
- For ramp input, the integration in the forward path yields a finite steady-state error, rather than the infinite steady-state error found in the previous question.
- For the parabolic input, the integration in the forward path does not yield any improvement in steady-state error over that found in the previous question for a parabolic input.

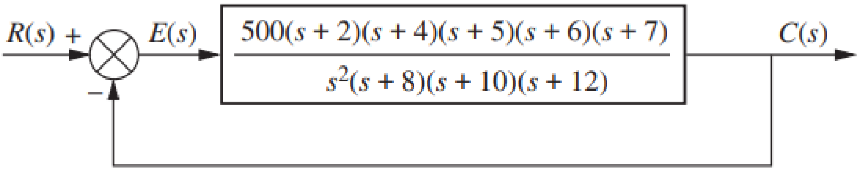
12. For each system of the figure below, evaluate the static error constants and find the expected error for the standard step, ramp, and parabolic inputs. [18 marks]



(a)



(b)



(c)

Solution

First, verify that all closed-loop systems shown are indeed stable. For this example, we leave out the details.

a. Next, for the figure (a), the static-error constants of the system (a) are:

$$K_p = \lim_{s\to 0} G(s) = \frac{500 \times 2 \times 5}{8 \times 10 \times 12} = 5.208$$

$$K_v = \lim_{s\to 0} sG(s) = 0$$

$$K_a = \lim_{s\to 0} s^2 G(s) = 0$$

As a result, the steady-state errors of the system (a) for various inputs:

Step input	**Ramp input**	**Parabolic input**
$e(\infty) = \dfrac{1}{1+K_p} = 0.161$	$e(\infty) = \dfrac{1}{K_v} = \infty$	$e(\infty) = \dfrac{1}{K_a} = \infty$

b. Now, for the figure (b), the static-error constants of the system (b) are:

$$K_p = \lim_{s\to 0} G(s) = \infty$$

$$K_v = \lim_{s\to 0} sG(s) = \frac{500 \times 2 \times 5 \times 6}{8 \times 10 \times 12} = 31.25$$

$$K_a = \lim_{s\to 0} s^2 G(s) = 0$$

As a result, the steady-state errors of the system (b) for various inputs:

Step input	Ramp input	Parabolic input
$e(\infty) = \dfrac{1}{1+K_p} = 0$	$e(\infty) = \dfrac{1}{K_v} = \dfrac{1}{31.25} = 0.032$	$e(\infty) = \dfrac{1}{K_a} = \infty$

c. Finally, for the figure (c), the static-error constants of the system (c) are:

$$K_p = \lim_{s\to 0} G(s) = \infty$$

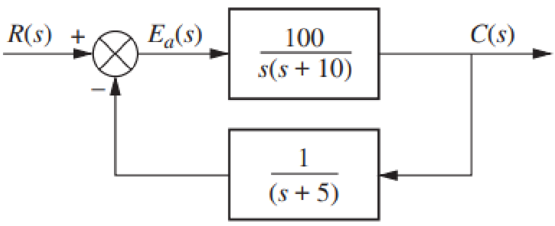
$$K_v = \lim_{s\to 0} sG(s) = \infty$$

$$K_a = \lim_{s\to 0} s^2 G(s) = \frac{500 \times 2 \times 4 \times 5 \times 6 \times 7}{8 \times 10 \times 12} = 875$$

As a result, the steady-state errors of the system (c) for various inputs:

Step input	Ramp input	Parabolic input
$e(\infty) = \dfrac{1}{1+K_p} = 0$	$e(\infty) = \dfrac{1}{K_v} = 0$	$e(\infty) = \dfrac{1}{K_a} = \dfrac{1}{875} = 1.14 \times 10^{-3}$

13. For the system shown in the figure below, find the system type, the appropriate error constant associated with the system type, and the steady-state error for a unit step input. Assume input and output units are the same. [12 marks]

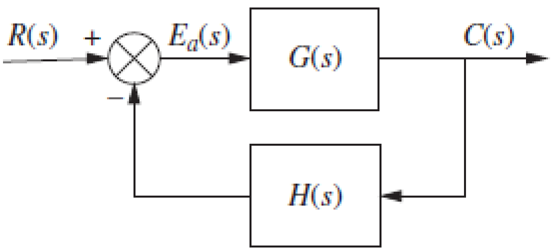


Solution

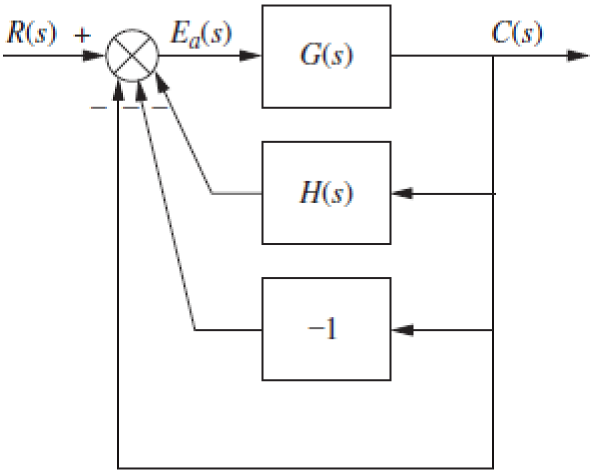
After determining that the system is indeed stable, one may impulsively declare the system to be Type 1. This may not be the case, since there is a non-unity feedback element, and the plant's actuating signal is not the difference between the input and the output.

The first step in solving the problem is to convert the system of the figure above into an equivalent unity feedback system. Using the equivalent forward transfer function of the figure given above along with.

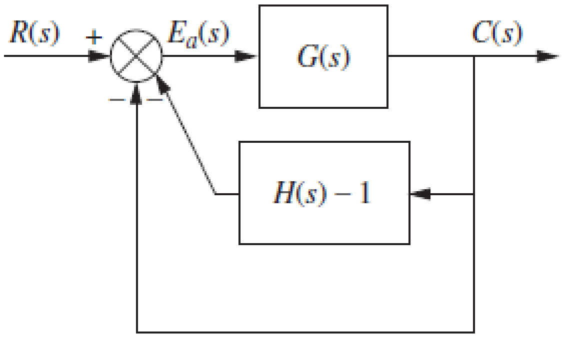
Consider the non-unity feedback control system.



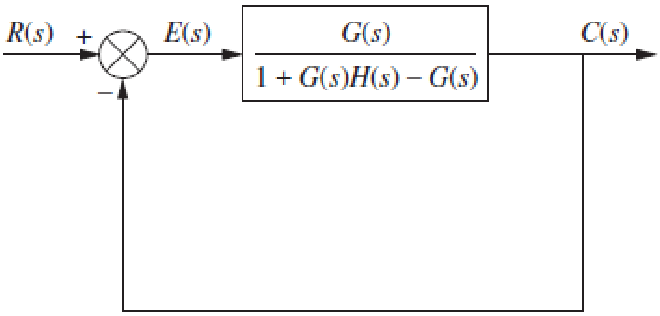
Form a unity feedback system by adding and subtracting unity feedback paths.



Combine $H(s)$ with the negative unity feedback.



Combine the feedback system consisting of $G(s)$ and $[H(s) - 1]$, leaving an equivalent forward path and a unity feedback.



Knowing that:

$$G(s) = \frac{100}{s(s+10)}$$

And

$$H(s) = \frac{1}{(s+5)}$$

We find:

$$G_e(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)} = \frac{100(s+5)}{s^3 + 15s^2 - 50s - 400}$$

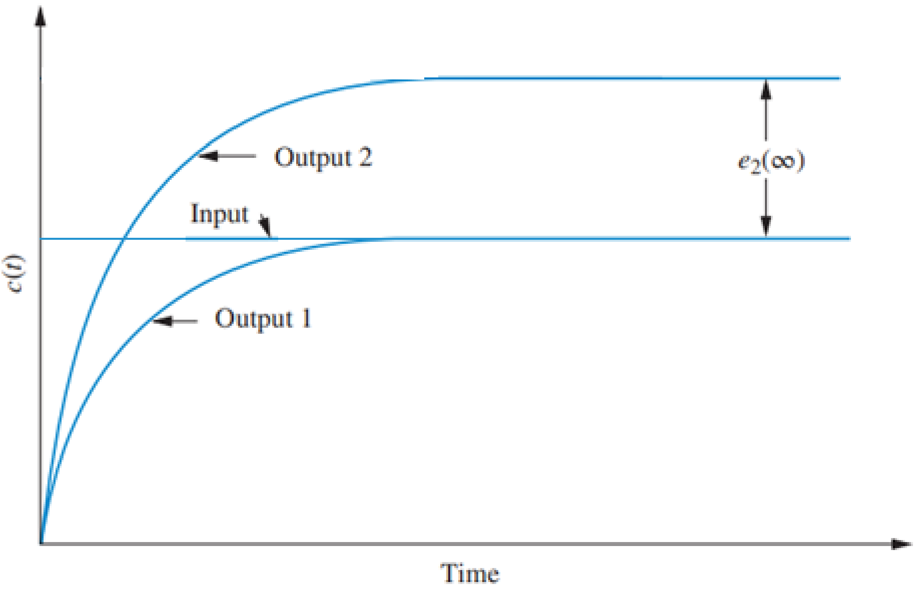
Thus, the system is Type 0 since there are no pure integrations in the equation above. The appropriate static-error constant is then K_p, whose value is:

$$K_p = \lim_{s\to 0} G_e(s) = \frac{100 \times 5}{-400} = -\frac{5}{4}$$

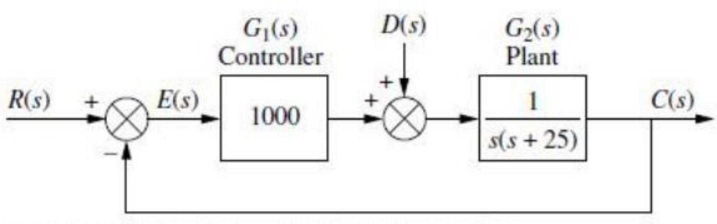
The steady-state error, $e(\infty)$, is:

$$e(\infty) = \frac{1}{1+K_p} = \frac{1}{1-(5/4)} = -4$$

The negative value for steady-state error implies that the output step is larger than the input step.



14. Find the steady-state errors component due to a step disturbance for the system given below. Comment on the result. [4 marks]



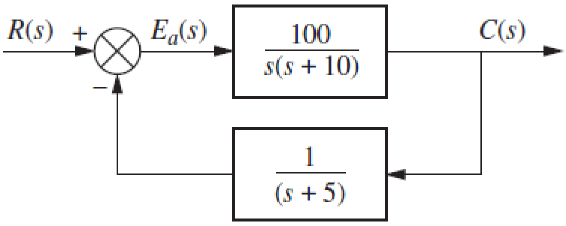
Solution

By inspection, the system is stable. Using the transformation steps for converting system with disturbance, we find:

$$e_D(\infty) = -\frac{1}{\lim\limits_{s\to 0}\frac{1}{G_2(s)} + \lim\limits_{s\to 0} G_1(s)} = -\frac{1}{0+1000} = -\frac{1}{1000}$$

The result shows that the steady-state error produced by the step disturbance is inversely proportional to the DC gain of $G_1(s)$. The DC gain of $G_2(s)$ is infinite in this example.

15. For the system shown below, perform the following tasks.



a. Find the system type, the appropriate error constant associated with the system type, and the steady-state errors for a unit step input. Assume input and output units are the same. [8 marks]

b. Find the steady-state actuating signal for the system ($E_a(s)$) for a unit step input. Repeat for a unit ramp input. [8 marks]

Solution

a. The system is indeed stable. Although the system looks like is a Type 1, since there is a non-unity feedback element, and the plant's actuating signal is not the difference between the input and output.

Convert the system into an equivalent unity feedback system. Using the equivalent forward transfer function:

$$G(s) = \frac{100}{s(s+10)} \quad \text{and} \quad H(s) = \frac{1}{s+5}$$

We find:

$$G_e(s) = \frac{G(s)}{1 + G(s)H(s) - G(s)} = \frac{100(s+5)}{s^3 + 15s^2 - 50s - 400}$$

Thus, the system is actually a Type 0, since there is no integral in the above equation.

The appropriate static error constant is then K_p, whose value is:

$$K_p = \lim_{s \to 0} G_e(s) = \frac{(100)(5)}{(-400)} = -\frac{5}{4}$$

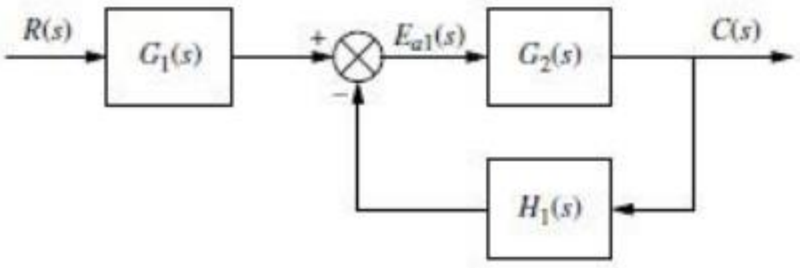
The steady-state error, $e(\infty)$, is:

$$e(\infty) = \frac{1}{1 + K_p} = \frac{1}{1 - (5/4)} = -4$$

The negative value for steady-state error implies that the output step is larger than the input step.

b. With $R(s) = 1/s$, a unit step input, $G_1(s) = 1$; $G_2(s) = 100/[s(s+10)]$, and $H_1(s) = 1/(s+5)$.

The following diagram shows the activating steady-state error in a non-unity feedback control system with disturbance.



Thus, the activating steady-state error of the system given above is:

$$e_{a1}(\infty) = \lim_{s\to 0} \frac{sR(s)G_1(s)}{1 + G_2(s)H_1(s)}$$

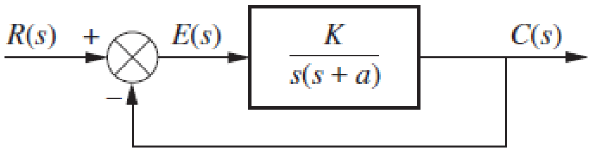
Also, realize that $e_{a1}(\infty) = e_a(\infty)$, since $G_1(s) = 1$. Thus,

$$e_a(\infty) = \lim_{s\to 0} \frac{s\left(\frac{1}{s}\right)}{1 + \left(\frac{100}{s(s+10)}\right)\left(\frac{1}{s+5}\right)} = 0$$

Now, with $R(s) = 1/s^2$, a unit ramp input, and obtain:

$$e_a(\infty) = \lim_{s\to 0} \frac{s\left(\frac{1}{s^2}\right)}{1 + \left(\frac{100}{s(s+10)}\right)\left(\frac{1}{s+5}\right)} = \frac{1}{2}$$

16. Given the system of the figure below, calculate the sensitivity of the closed-loop transfer function to changes in the parameter a. How would you reduce the sensitivity? [6 marks]



Solution

The closed-loop transfer function is:

$$T(s) = \frac{K}{s^2 + as + K}$$

The sensitivity is given by:

$$S_{T:a} = \frac{a}{T}\frac{\delta T}{\delta a} = \frac{a}{\left(\frac{K}{s^2 + as + K}\right)}\left(-\frac{Ks}{(s^2 + as + K)^2}\right)$$

Thus, the sensitivity of the given system is:

$$S_{T:a} = \frac{-as}{s^2 + as + K}$$

This is, in part, a function of the value of s.

For any value of s, however, an increase in K reduces the sensitivity of the closed-loop transfer function to changes in the parameter a.