XMUT315 Control Systems Engineering

Tutorial 5: Controllers and Compensators (Solution)

A. Controllers and Compensators (Introduction)

1. Controllers and compensators are typically used for managing and controlling systems in control system.
   a. Describe where controllers and compensators are used in control systems. [4 marks]
   b. List three types of these controllers and compensators. [6 marks]
   c. How can controllers and compensators change the characteristics and behaviours of the system? Give at least three examples. [3 marks]

Solution

a. Applications of controllers and compensators in control systems:
   - Controller: It is used in the control systems as an element whose role is to maintain a physical quantity in a desired level.
   - Compensator: It is used as an element for modification of system dynamics and to improve characteristics of the open-loop plant that can be used with feedback control.

b. Types of compensator and controller:
   - Three main types of controller: Gain/Proportional, Integral/Derivative, and PID (Proportional, Integral, and Derivative).
   - Three main types of compensator: Lag, Lead and Lead-lag.

c. Functions of controllers and compensators:
   - They change the natural response of the system.
   - They adjust the poles of the system.

- They help achieve the desired output from a given input.

2. List and describe various types of controller or compensator in control systems in terms of their transfer functions, functionalities, and characteristics.

[21 marks]

Solution

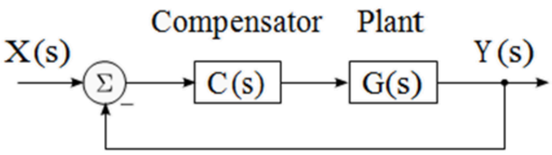
The following table outlines description of various types of standard controllers/compensators in control systems in terms of their transfer functions, functionalities, and characteristics.

Controller/ Compensator	**Function**	**Transfer Function**	**Characteristics**
P	Improve transient response (up to a point)	K	a. Increases gain of the system. b. Often result in non-zero steady-state error. c. Relatively easy to implement.
PI	Improve steady-state error	$K\left(\frac{s+z_c}{s}\right)$	a. Increases system type. b. Error becomes zero. c. Zero at $z_c$is small and negative. d. Active circuits are required to implement.
Lag	Improve steady-state error	$K\left(\frac{s+z_c}{s+p_c}\right)$	a. Error is improved, but not driven to zero. b. Pole at $-p_c$ is small and negative. c. Zero at $-z_c$ is close to, and to the left of, the pole at $-p_c$. d. Active circuits are not required t implement.
PD	Improve transient response	$K(s+z_c)$	a. Zero at $-z_c$ is selected to put design point on root locus. b. Active circuits are required to implement.

			c. It can cause noise and saturation; implement with rate feedback or with a pole (lead).
Lead	Improve transient response	$K\left(\frac{s+z_c}{s+p_c}\right)$	a. Zero at $-z_c$ and pole at $-p_c$ at are selected to put design point on root locus. b. Pole at $-p_c$ is more negative than zero at $-z_c$. c. Active circuits are not required to implement.
PID	Improve steady-state error and transient response	$K\left[\frac{(s+z_{lag})(s+z_{lead})}{s}\right]$	a. Lag zero at $-z_{lag}$ and pole at the origin improve steady-state error. b. Lead zero at $-z_{lead}$ improves transient response. c. Lag zero at $-z_{lag}$ is close to, and to the left of, the origin. d. Lead zero at $-z_{lead}$ is selected to put design point on root locus. e. Active circuits are required to implement. f. It can cause noise and saturation; implement with rate feedback or with an additional pole.
Lag-lead	Improve steady-state error and transient response	$K\left[\frac{(s+z_{lag})(s+z_{lead})}{(s+p_{lag})(s+p_{lead})}\right]$	a. Lag pole at $-p_{lag}$ and lag zero at $-z_{lag}$ are used to improve steady-state error. b. Lead pole at $-p_{lead}$ and lead zero at $-z_{lead}$ are used to improve transient response. c. Lag pole at $-p_{lag}$ is small and negative. d. Lag zero at $-z_{lag}$ is close to, and to the left of, lag pole at $-p_{lag}$

			e. Lead zero at $-z_{lead}$ and lead pole at $-p_{lead}$ are selected to put design point on root locus. f. Lead pole at $-p_{lead}$ is more negative than lead zero at $-z_{lead}$. g. Active circuits are not required to implement.

3. Given the following unity-gain feedback-control system as shown in the diagram below.

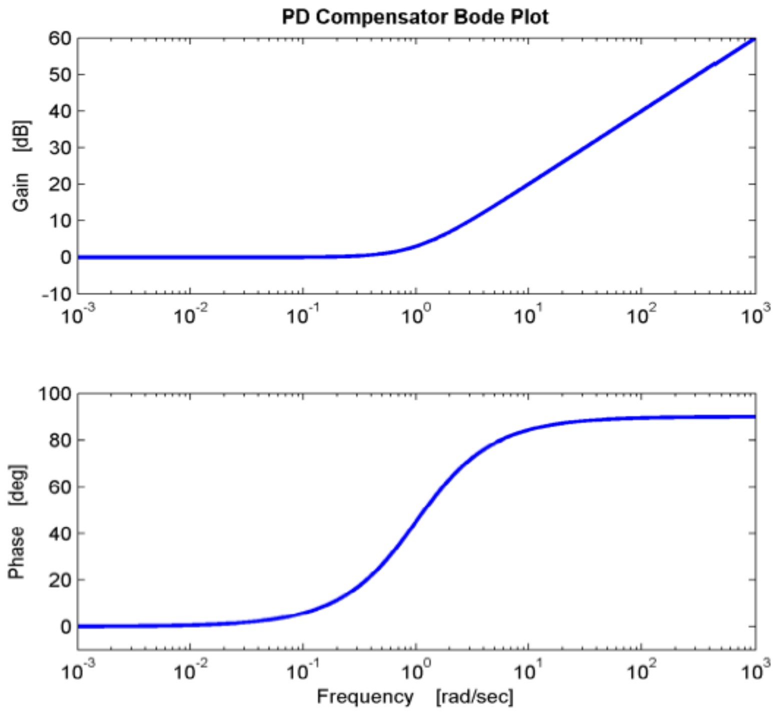


By using Bode plots, outline and describe briefly the frequency responses of the following controllers or compensators: [24 marks]

No	Controller/Compensator (C(s))	Transfer function
a	PD controller	$C(s) = T_D(s+1)$
b	PI controller	$C(s) = \frac{1}{T_D}\left(\frac{T_D s + 1}{s}\right)$
c	Lead compensator	$C(s) = \left(\frac{Ts+1}{\beta Ts+1}\right) \qquad \beta < 1$
d	Lag compensator	$C(s) = \alpha\left(\frac{Ts+1}{\alpha Ts+1}\right) \qquad \alpha > 1$

Solution

a. The frequency responses (gain and phase) of PD controller are shown as below.

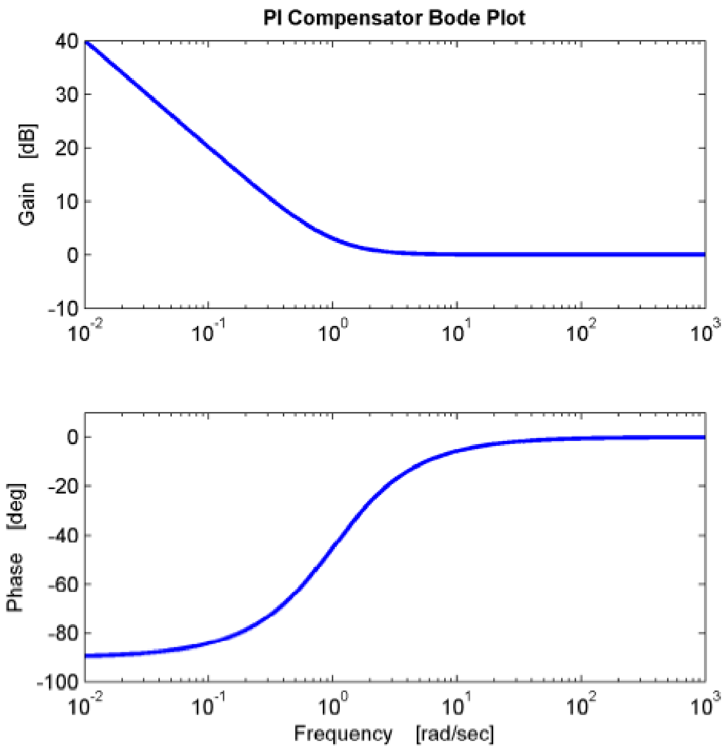


In the PD controller, phase added near (and above) the crossover frequency e.g. an increase of the phase margin and giving a stabilizing effect.

$$C(s) = T_D(s+4)$$

Then, the gain continues to rise at high frequencies, but this causes the sensor noise to be amplified and as a result a lead compensation is usually preferable.

b. The frequency responses (gain and phase) of PI controller are as shown in the diagram below.

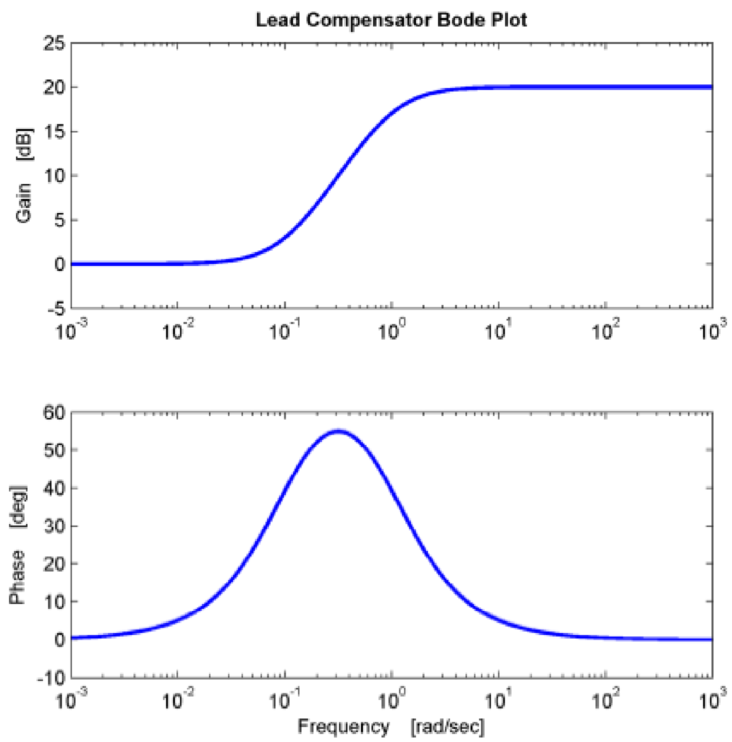


At low frequency, the gain of proportional-integral compensator is infinite at DC (0 rad/s) and this compensator can increase system type of the system.

$$C(s) = \frac{1}{T_D}\left(\frac{T_D s + 1}{s}\right)$$

For frequency above the cut-off frequency of the compensator ($\omega \gg 1/T_D$), the gain of the system is unaffected, there is a slight change in the phase, but phase margin of the system is unaffected. In the end, the given proportional-integral compensator has a tendency to increase low frequency gain of the system.

c. The frequency responses (gain and phase) of lead compensator are shown in the figure below.

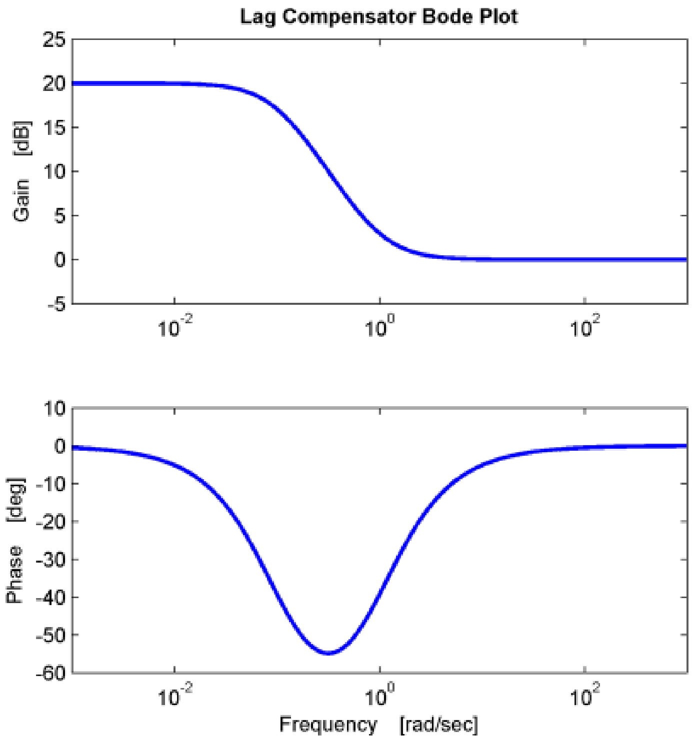


For frequency below the cut-off frequency of the compensator ($\omega \ll 1/T$) the gain is ~ 0 dB and Phase is ~ 0°.

$$C(s) = \left(\frac{Ts + 1}{\beta Ts + 1}\right) \qquad \beta < 1$$

For frequency above the cut-off frequency of the compensator ($\omega \gg 1/\beta T$) the gain is and phase is ~0°. Thus, lead compensator adds phase lead near the crossover frequency and/or alter the crossover frequency.

d. The frequency responses (gain and phase) of lag compensator are as shown in the diagram given below.



For frequency less than cut-off frequency of the compensator ($\omega \ll 1/\alpha T$), the gain of the system is 20 log (α) dB and phase is 0°.

$$C(s) = \alpha\left(\frac{Ts + 10}{\alpha Ts + 1}\right) \qquad \alpha > 1$$

For frequency more than the cut-off frequency of the compensator ($\omega \gg 1/\alpha T$), both the gain and phase are 0 dB and 0° respectively. In short, lag compensator adds a gain of α at low frequencies without affecting phase margin.

4. Describe similarities and differences between the following controllers or compensators.

   a. Lead compensator and PD controller. [4 marks]

   b. Lag compensator and PI controller. [4 marks]

   c. Lead-lag compensator and PID controller. [4 marks]

Solution

a. Lead compensator and PD controller.

   Similarity:

   - Both are applied in the control system to improve the transient performance.
   - Both use some sorts of derivation function in their functions.

   Difference:

- As gain of PD controller keeps increasing at high frequency, it suffers from noise problem as sensor noise is typically amplified, especially the noise due to its high frequency components. On the other hand, lead compensator is not prone for this type of problem.
- Lead compensator only target the dynamic response improvement of the system near the intended design crossover frequency.

b. Lag compensator and PI controller.

Similarity:

- Both are used for improving the steady-state condition of the control system.
- Both use some forms of integration function in their transfer functions.

Difference:

- Proportional-integral controller is susceptible to integrator overflow and on the other hand, lag compensator does not suffer from this type of problem.
- Lag compensator is often preferable than PI controller for solving steady-state problems.

c. Lead-lag compensator and PID controller.

Similarity:

- Both are used for improving both transient response and steady-state condition of the control system.
- Both have integration and differentiation functions in their transfer functions.

Difference:

- Compensator is preferred rather than controller for solving the control system problems due to its simplicity in its implementation. It can be created with passive components, not active components like the controller.
- Controller is preferable if you require some sorts of control flexibility in its application compared with the compensator.

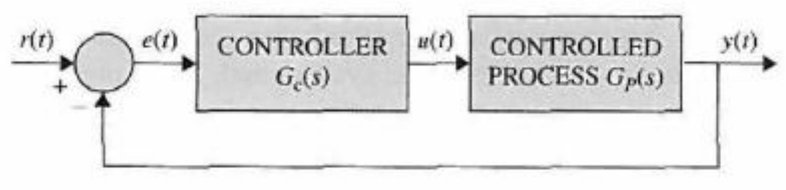
5. The arrangement of compensator or controller is very important to be considered, so the modification and improvement that we wish to apply to the control system could be effectively and efficiently achieved.

   a. Outline and describe a variety of typically arrangements of the controller or compensator in the control systems. [12 marks]

   b. Describe the differences between the arrangement of the controllers or compensators in the control system based on their flexibility in modifying the control system. [4 marks]

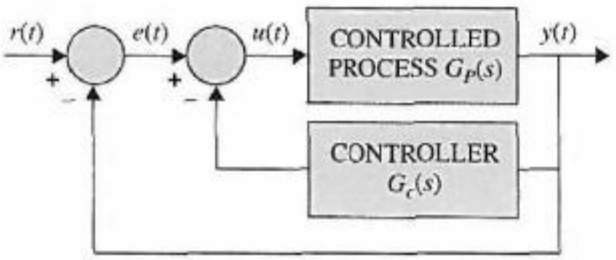
Solution

a. The arrangement of the controller or compensator in the control systems are as follows. [Various controller configurations in control-system compensation, (a) Series or cascade compensation, (b) Feedback compensation, (c) State-feedback control, (d) Series-feedback compensation (two degrees of freedom), (e) Forward compensation with series compensation (two degrees of freedom), and (f) Feedforward compensation (two degrees of freedom)].



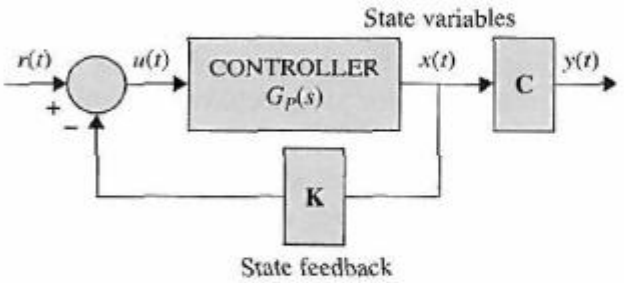
(a)

Series (cascade) compensation: (a) shows the most commonly used system configuration with the controller placed in series with the controlled process, and the configuration is referred to as series or cascade compensation.



(b)

Feedback compensation: In (b), the controller is placed in the minor feedback path, and the scheme is called feedback compensation.

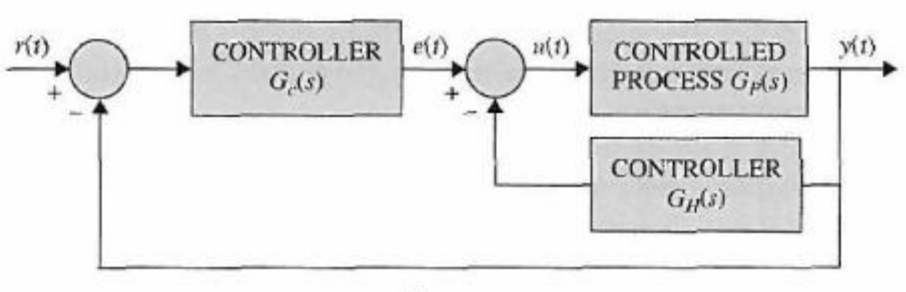


(c)

State-feedback compensation: (c) shows a system that generates the control signal by feeding back the state variables through constant real gains, and the scheme is known as state feedback.

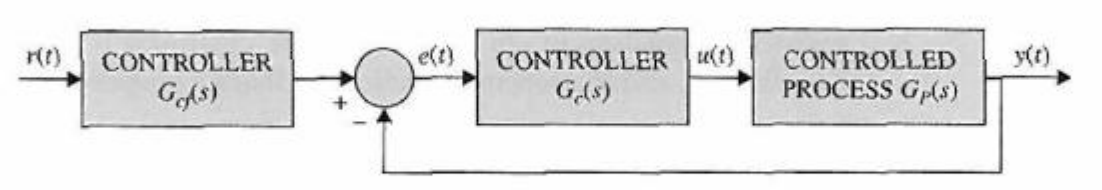
The problem with state-feedback control is that, for high-order systems, the large number of state variables involved would require a large number of transducers to sense the state variables for feedback.

Thus, the actual implementation of the state-feedback control scheme may be costly or impractical. Even for low-order systems, often not all the state variables are directly accessible, and an observer or estimator may be necessary to create the estimated state variables from measurements of the output variables.



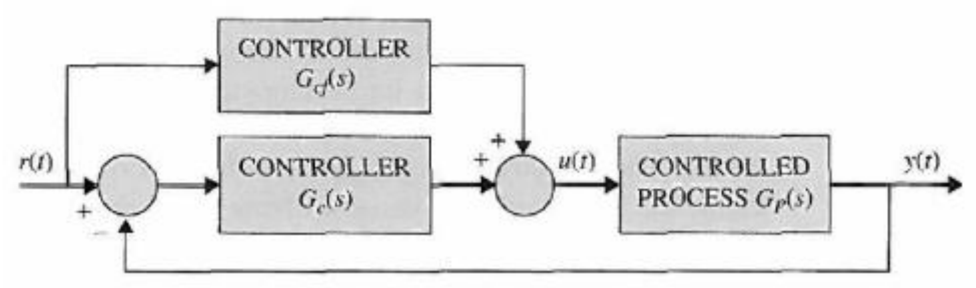
(d)

Series-feedback compensation: (d) shows the series-feedback compensation for which a series controller and a feedback controller are used.



(e)

Feedforward compensation: (e) and (f) show the so-called feedforward compensation. In (e), the feedforward controller $G_{cf}(s)$ is placed in series with the closed-loop system, which has a controller $G_c(s)$ in the forward path. In (f), the feedforward controller $G_{cf}(s)$ is placed in parallel with the forward path. The key to the feedforward compensation is that the controller $G_{cf}(s)$ is not in the loop of the system, so it does not affect the roots of the characteristic equation of the original system. The poles and zeros of $G_{cf}(s)$ may be selected to add or cancel the poles and zeros of the closed-loop system transfer function.



(f)

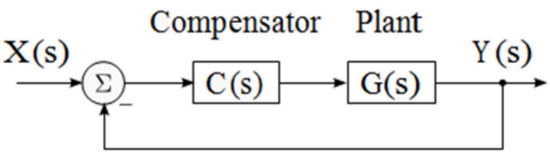
b. The compensation schemes shown in (a), (b), and (c) all have one degree of freedom in that there is only one controller in each system, even though the controller may have more than one parameter that can be varied. The compensation schemes shown in (d), (e), and (f) all have two degrees of freedom.

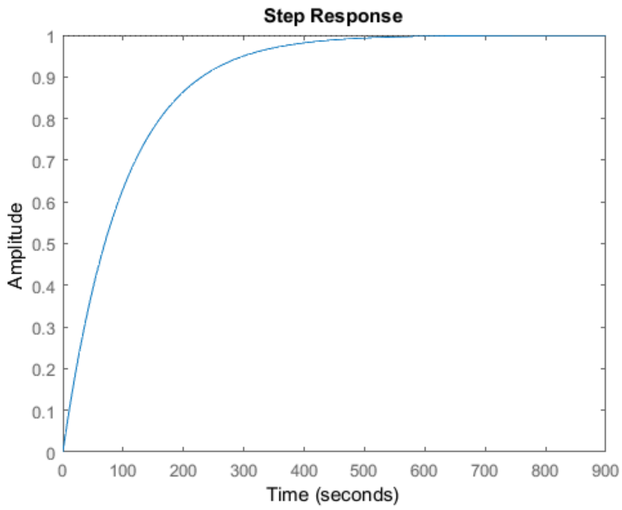
The disadvantage with a one-degree-of freedom controller is that the performance criteria that can be realized are limited. For example, if a system is to be designed to achieve a certain amount of relative stability, it may have poor sensitivity to parameter variations.

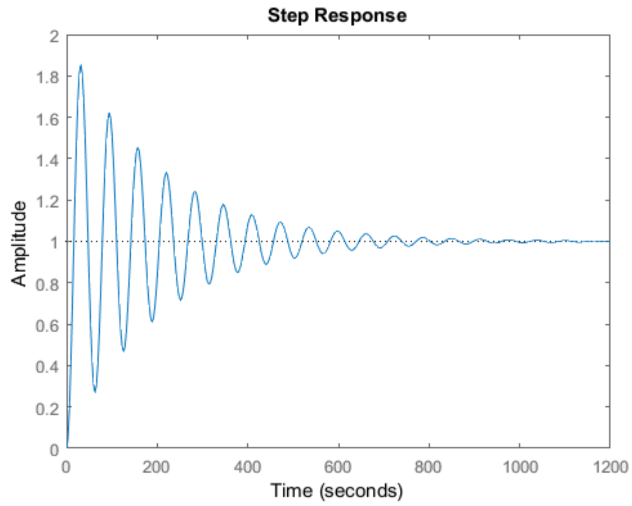
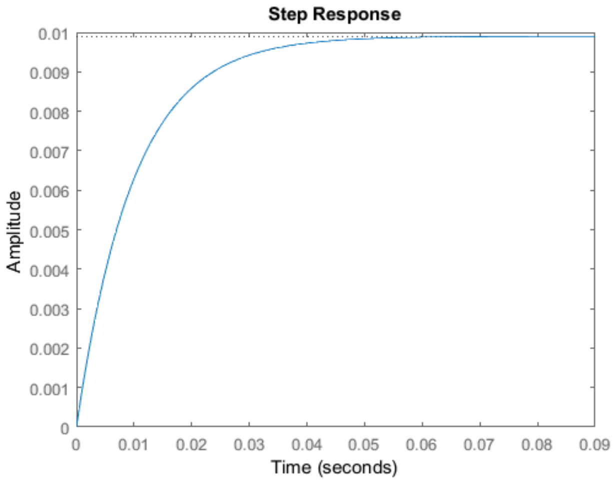
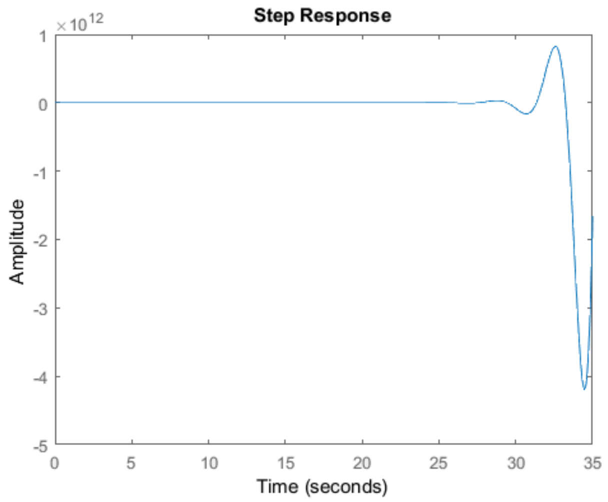
Or if the roots of the characteristic equation are selected to provide a certain amount of relative damping, the maximum overshoot of the step response may still be excessive because of the zeros of the closed-loop transfer function.

B. Controllers and Compensators (Application)

6. Given several control systems with the following transfer functions and the transient responses of the systems after given a step input:



System	Transfer function of Plant ($G(s)$)	Transient response of the closed loop system after given a step input
(i)	$G(s) = \dfrac{10}{s(s+1000)}$	

(ii)	$G(s) = \dfrac{1}{s(s+1)}$	
(iii)	$G(s) = \dfrac{1}{s+100}$	
(iv)	$G(s) = \dfrac{7}{s(s+0.1)(s+0.2)}$	

By referring to the table as shown above, we can conclude that the systems listed above are experiencing some problems.

a. Describe the problem that each of the systems is experiencing. [8 marks]

b. Suggest a controller or compensator that would fix each of the problems. [8 marks]

Solution

a. The problems with all closed loop systems given above are:

System (i) – Sluggish system. The time constant of the system is in the order of 100’s seconds that is considered to be relatively very slow. Moreover, the system takes almost 600 seconds to settle down.

System (ii) – Reactive or damped oscillation. The transient response of the closed loop system indicates a reactive system that has excessive damped oscillation. This damped oscillation makes the system to be difficult for its stabilisation and prevents it from settling down.

System (iii) – Steady-state error. The system suffers from a steady-state error following its transient response. The system settles down at 0.01 that is only 1% of the intended reference point or final value at steady state after it is applied a step input.

System (iv) – Unstable system. The transient response of the system shows an increase oscillation response which is a typical characteristic and behaviour of unstable system. The output of the system is excessively larger than the input.

b. Suggested controller or compensator for solving those problems:

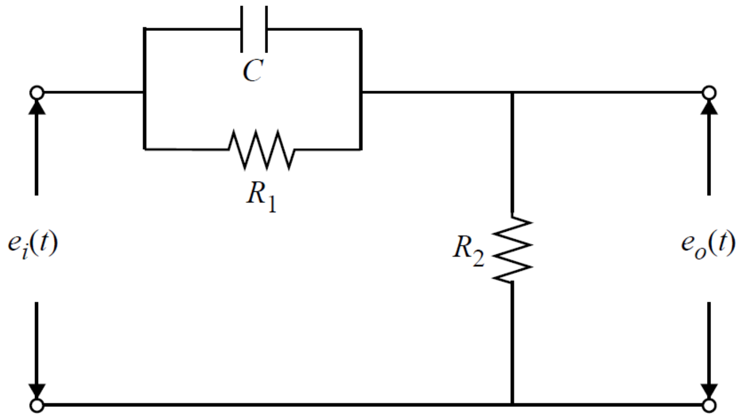
System (i) – The application of proportional controller typically can increase the responsiveness of the system. If this is not working, we need to consider the application of the derivative controller or lead compensator, so the dynamic response of the system can be improved.

System (ii) – In most cases, reducing the gain of the overall system can reduce the reactiveness of the system. With careful design of the gain of the system, the response of the system will be a balance between the response and performance of the system.

System (iii) – Steady state error can be minimalised with application of lag compensator or it can be eliminated with proportional-integral controller.

System (iv) – Unstable system can be made stable using the proportional-integral controller or a compensator that has a integration function in its core. Depending on the test input used and types of the system, the system can be made stable as required.

7. A typical electric network of a lead compensator is as shown in the figure below.



a. Determine the transfer function equation of the lead compensator in terms of the values of the components in the electrical network. [12 marks]

b. When $\alpha = 0.5$ and T = 5, determine the values of the resistors R_1 and R_2 and capacitor C in the electrical network. [4 marks]

Solution

a. The transfer function of the lead compensator is determined as given below.

Let Z_1 be the equivalent impedance of the parallel combination of R_1 and C.

$$Z_1 = \frac{R_1}{(1 + R_1 Cs)}$$

And

$$Z_2 = R_2$$

The transfer function of the lead compensator is:

$$G_c(s) = \frac{E_o(s)}{E_i(s)} = \frac{Z_2}{Z_2 + Z_1}$$

$$= \frac{R_2}{R_2 + \dfrac{R_1}{1 + \left(\dfrac{R_1}{1 + R_1 Cs}\right)}}$$

$$= \frac{R_2(1 + R_1 Cs)}{(R_1 + R_2) + R_1 R_2 Cs}$$

$$= \left(\frac{R_2}{R_1 + R_2}\right)\left(\frac{1 + R_1 Cs}{1 + \dfrac{R_2 R_1 Cs}{R_1 + R_2}}\right)$$

Considering that the transfer function of the lead compensator is:

$$\frac{E_o(s)}{E_i(s)} = \alpha\left(\frac{1 + Ts}{1 + \alpha Ts}\right)$$

Thus:

$$\alpha = \frac{R_2}{R_1 + R_2} < 1$$

And

$$T = R_1 C$$

b. When α = 0.5 and T = 5, the values of the components in the electrical network are determined as follows.

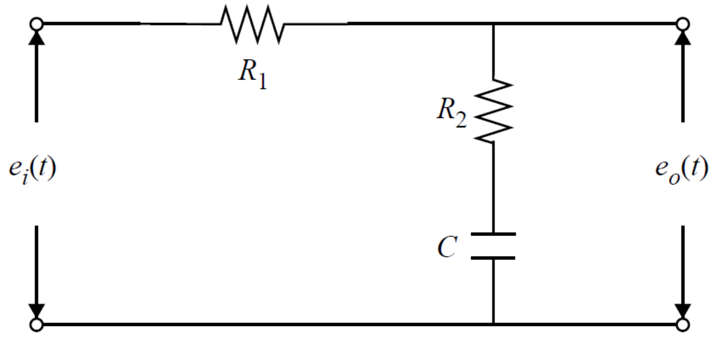
Choosing C = 100 nF, the value of R_1 is:

$$R_1 = \frac{T}{C} = \frac{5}{100 \times 10^{-9}} = 50\ \mathrm{M\Omega}$$

Then, the value of R_2 is:

$$R_2 = \frac{\alpha R_1}{1 - \alpha} = \frac{(0.5)(50 \times 10^6)}{1 - 0.5} = 50\ \mathrm{M\Omega}$$

8. The electric network circuit of a lag compensator is as shown in the figure below.



a. Derive the expression for the lag compensator in terms of the values of the components in the electrical network given above. [12 marks]

b. Like part (a), derive the expression for lag compensator in the pole-zero form. [6 marks]

c. For the following transfer function equation of a lag compensator, determine the values of the components in the electrical network. [6 marks]

$$G_c(s) = \frac{0.1s + 5}{2s + 2}$$

Solution

a. For the given lag compensator circuit, its impedances are:

$$Z_1(s) = R_1$$

And

$$Z_2(s) = R_2 + \frac{1}{Cs} = \frac{1 + R_2 Cs}{Cs}$$

Thus, the transfer function of the electrical network of the lag compensator is:

$$\begin{aligned} G_c(s) &= \frac{E_o(s)}{E_i(s)} \\ &= \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \\ &= \frac{\left(\frac{1 + R_2 Cs}{Cs}\right)}{R_1 + \left(\frac{1 + R_2 Cs}{Cs}\right)} \\ &= \frac{1 + R_2 Cs}{1 + (R_1 + R_2)Cs} \end{aligned}$$

Since the transfer function of the lag compensator is:

$$G_c(s) = \frac{1 + \alpha T s}{1 + Ts}$$

Thus

$$\alpha = \frac{R_2}{R_1 + R_2} < 1$$

And

$$T = (R_1 + R_2)C$$

b. The transfer function equation of the lag compensator can be expressed in the pole-zero form as:

$$G_c(s) = \frac{\alpha(s + z)}{s + p}$$

Where:

$$z = \frac{1}{R_2 C}$$

And

$$p = \alpha z = \frac{1}{(R_1 + R_2)C}$$

c. For the given transfer function equation of the lag compensator

$$G_c(s) = \frac{0.1s + 5}{s + 2}$$

Thus, the values of α and T are p = 2, z = 50 and α = 0.1.

Considering that C = 100 μF, the values of the R_2 is:

$$R_2 = \frac{1}{zC} = \frac{1}{(50)(100 \times 10^{-6})} = 200\ \Omega$$

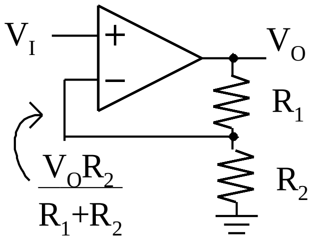
Then, the value of R_1 is:

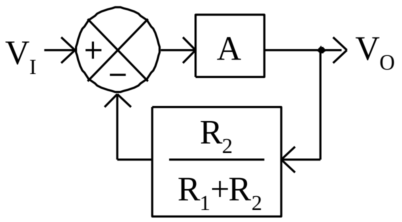
$$R_1 = \frac{1}{pC} - R_2 = \frac{1}{(2)(100 \times 10^{-6})} - 200 = 4.8\ \text{k}\Omega$$

9. Describe how a proportional (P) controller with a proportional gain (K_p) of 2.5 is implemented in practice. Consider E12 standards and the open loop gain of the op amp is 8 x 10^5. [12 marks]

Solution

P controller is implemented in practice as a non-inverting amplifier with open loop gain of the op amp A and the feedback path component of $R_2/(R_1 + R_2)$.





The transfer function of the amplifier circuit which is its voltage gain is:

$$A_v = \frac{V_o}{V_i} = \frac{A}{1 - \left(-\frac{AR_2}{R_1 + R_2}\right)} = \frac{A(R_1 + R_2)}{R_1 + R_2 + AR_2}$$

If the loop gain $AR_2/(R_1 + R_2)$ is large, $AR_2 \gg R_1 + R_2$:

$$A_v = \frac{V_o}{V_i} = \frac{A(R_1 + R_2)}{AR_2} = \frac{R_1 + R_2}{R_2}$$

In the above, $\beta = R_2/(R_1 + R_2)$, so if loop gain (A) is large:

$$A_v = \frac{V_o}{V_i} = \frac{1}{\beta} = \frac{R_1 + R_2}{R_2}$$

For example, given an operational amplifier based non-inverting amplifier with op amp with gain A = 10^5 and feedback resistors: R_1 = 6 kΩ, R_2 = 4 kΩ.

Then, since $AR_2 = 32 \times 10^8 \gg R_1 + R_2 = 10^4$, as a result the gain of the amplifier is:

$$A_v = \frac{V_o}{V_i} = \frac{R_1 + R_2}{R_2} = \frac{6\ \text{k}\Omega + 4\ \text{k}\Omega}{4\ \text{k}\Omega} = 2.5$$

In practice, using E12 component standard, we have for two options. These are option 1: 5.6 kΩ and 3.9 kΩ and option 2: 6.8 kΩ and 4.7 kΩ.

Option 1:

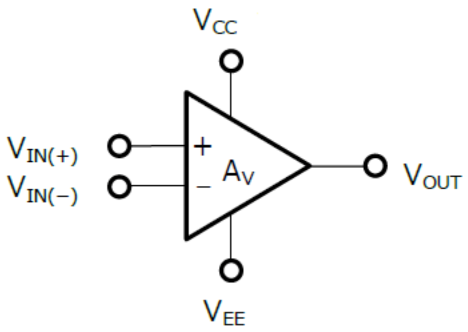
$$A_{v1} = \frac{V_o}{V_i} = \frac{R_1 + R_2}{R_2} = \frac{5.6\ \text{k}\Omega + 3.9\ \text{k}\Omega}{3.9\ \text{k}\Omega} = 2.436$$

Option 2:

$$A_{v2} = \frac{V_o}{V_i} = \frac{R_1 + R_2}{R_2} = \frac{6.8\ \text{k}\Omega + 4.7\ \text{k}\Omega}{4.7\ \text{k}\Omega} = 2.447$$

So, the second option is closer to gain value of 2.5 of the calculated amplifier circuit.

10. For an implementation of a given PID controller using operational amplifier in practice:

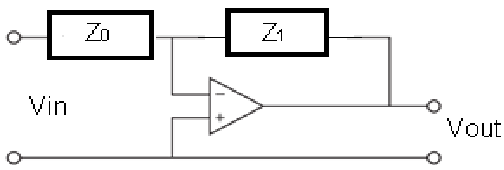


a. Outline the process for designing the PID controller using an op-amp based circuit. [12 marks]

b. How do you modify the op amp-based circuit for PID controller as in part (a) to be a circuit for PI controller? [6 marks]

Solution

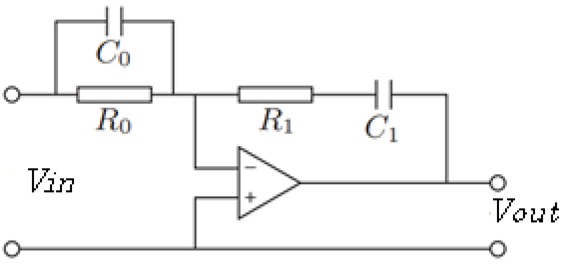
a. For outlining the design process of the PID controller, we will use the approximate relation between the input voltage V_{in} and the output voltage V_{out} of an op-amp based amplifier circuit.



For the above amplifier circuit, the relationship between output voltage with the input voltage is given as follows:

$$V_{out} = -\left(\frac{Z_1(s)}{Z_0(s)}\right)V_{in}$$

Where: Z_0 is the impedance between the negative input of the amplifier and the input voltage V_{in}, and Z_1 is the impedance between the zero input of the amplifier and the output voltage V_{out}.



The impedances of the amplifier circuit are given by:

$$Z_0(s) = \frac{R_0}{1 + R_0 C_0 s}$$

$$Z_1(s) = R_1 + \frac{1}{C_1 s}$$

We find the following relation between the input voltage V_{in} and the output voltage V_{out}:

$$V_{out} = -\left(\frac{Z_1}{Z_0}\right) V_{in} = -\frac{R_1 + \frac{1}{C_1 s}}{\frac{R_0}{1 + R_0 C_0 s}}$$

Knowing that the transfer function of PID controller is:

$$C_{PID}(s) = K\left[\frac{(s + z_{lag})(s + z_{lead})}{s}\right] \quad \text{and} \quad C_{PID}(s) = K_p + \frac{K_i}{s} + K_d s$$

Rearranging and expanding the equation given above into transfer function.

$$\frac{V_{out}}{V_{in}} = -\left(\frac{R_1}{R_0}\right)\left[\frac{(1 + R_0 C_0 s)(1 + R_1 C_1 s)}{R_1 C_1 s}\right]$$

Then, if we normalise the transfer function equation above, the input-output relation for a PID controller on the form as given above with parameters:

$$K_p = \frac{R_1}{R_0}; \qquad T_i = R_1 C_1; \qquad T_d = R_0 C_0$$

b. The corresponding results for a PI controller are obtained by setting $C_0 = 0$ on the circuit as shown in the part (a) as reproduced below.

$$\frac{V_{out}}{V_{in}} = -\left(\frac{R_1}{R_0}\right)\left[\frac{(1 + R_0 C_0 s)(1 + R_1 C_1 s)}{R_1 C_1 s}\right]$$

The equation above becomes as follows:

$$V_{out} = -\left(\frac{Z_1}{Z_0}\right) V_{in} = -\left(\frac{R_1}{R_0}\right)\left[\frac{(1 + R_1 C_1)}{R_1 C_1}\right] V_{in}$$

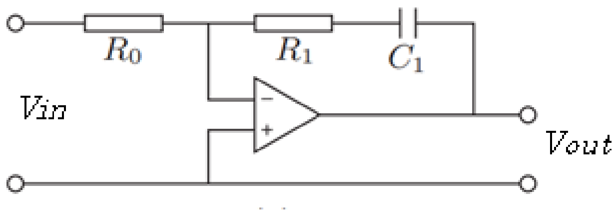
Knowing the PI transfer function equation of:

$$C(s) = K_p + \frac{K_i}{s}$$

Then

$$K_p = \frac{R_1}{R_0}; \qquad T_i = R_1 C_1$$

The circuit for the PI controller is as shown in the figure given below.



11. Tuning a controller or compensator is required for optimal operation of a controller or compensator in control systems.

    a. Describe what is tuning for a controller or compensator. [2 marks]

    b. For a given controller or compensator, list three types of tuning method. [6 marks]

    c. Describe how you tune in a PID controller using Ziegler-Nichols rule. [6 marks]

Solution

a. For tuning of a controller or compensator, the main objective is to adjust the reactions of the controllers or compensator to set point changes and unmeasured disturbances such that variability of control error is minimized. The controllers or compensators are implemented primarily for the purpose of holding measured process value at a set point, or desired value.

b. Three types of tuning methods are:

   Manual tuning of controller - With enough information about the process being controlled, it may be possible to calculate optimal values of gain, reset and rate for the controller. Often the process is too complex, but with some knowledge, particularly about the speed with which it responds to error corrections, it is possible to achieve a rudimentary level of tuning.

   Manual tuning is done by setting the reset time to its maximum value and the rate to zero and increasing the gain until the loop oscillates at a constant amplitude. (When the response to an error correction occurs quickly a larger gain can be used. If response is slow a relatively small gain is desirable). Then set the gain of the controller to half of that value and adjust the reset time so it corrects for any offset within an acceptable period. Finally, increase the rate of the control system loop until overshoot is minimized.

   Tuning Heuristics - Many rules have evolved over the years to address the question of how to tune a control system loop. Probably the first, and certainly the best known are the Zeigler-Nichols (ZN) rules.

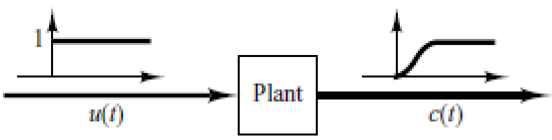
   First published in 1942, Zeigler and Nichols described two methods of tuning a controller. These work by applying a step change to the system and observing the resulting response.

The first method entails measuring the lag or delay in response and then the time is taken to reach the new output value. The second depends on establishing the period of a steady-state oscillation. In both methods, these values are then entered into a table to derive the values for gain, reset time and rate for the control system.
ZN is not without issues.

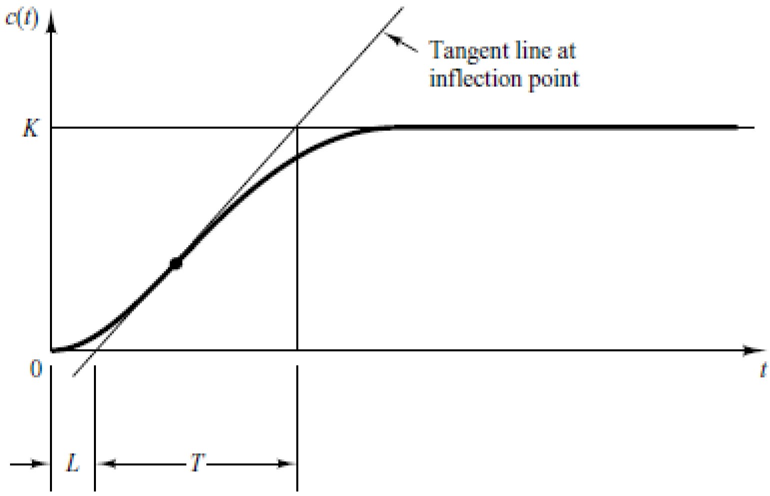
In some applications, it produces a response considered too aggressive in terms of overshoot and oscillation. Another drawback is that it can be time-consuming in processes that react only slowly. For these reasons, some control practitioners prefer other rules such as Tyreus-Luyben or Rivera, Morari and Skogestad.

Auto Tune - Most process controllers sold today incorporate auto-tuning functions. Operating details vary between manufacturers, but all follow rules like those described above. Essentially, the controller "learns" how the process responds to a disturbance or change in set point and calculates appropriate controller settings. By observing both the delay and rate with which the change is made it calculates optimal P, I and D settings, which can then be fine-tuned manually if needed.

c. For tuning a PID controller using Ziegler-Nichols rule, first of all, we obtain experimentally the response of the plant to a unit-step input, as shown in the figure below.



If the plant involves neither integrator (s) nor dominant complex-conjugate poles, then such a unit-step response curve may look S-shaped, as shown in the figure below.



This method applies if the response to a step input exhibits an S-shaped curve. Such step-response curves may be generated experimentally or from a dynamic simulation of the plant.

The S-shaped curve may be characterized by two constants, delay time (L) and time constant (T). The delay time and time constant are determined by drawing a tangent line at the inflection point of the S-shaped curve and determining the intersections of the tangent line with the time axis and line $c(t) = K$, as shown in the figure above.

Type of Controller	K_p	T_i	T_d
P	T/L	∞	0
PI	$0.9T/L$	$L/0.3$	0
PID	$1.2T/L$	$2L$	$0.5L$

The transfer function $C(s)/U(s)$ may then be approximated by a first-order system with a transport lag as follows:

$$\frac{C(s)}{U(s)} = \frac{Ke^{-Ls}}{Ts+1}$$

Ziegler and Nichols suggested to set the values of K_p, T_i, and T_d according to the formula shown in the table. Notice that the PID controller tuned by the Ziegler–Nichols rules give:

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

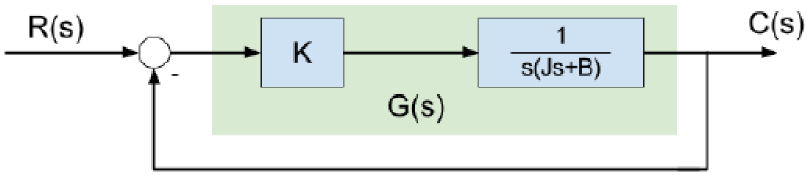
Entering the suggested values into the equation:

$$G_c(s) = 1.2\frac{T}{L}\left(1 + \frac{1}{2Ls} + 0.5Ls\right) = 0.6T\frac{\left(s + \frac{1}{L}\right)^2}{s}$$

Thus, the PID controller has a pole at the origin and double zeros at $s = -1/L$.

C. Controllers and Compensator (Analysis)

12. This case illustrates the application of gain controller in a given control system. Consider a servo control system as given in the block diagram below. Perform the following tasks:



a. For the steady-state error analysis, determine the steady-state error of the system to unit ramp input. [8 marks]

b. For the transient response analysis, derive equation for damping ratio of the system. [6 marks]

c. Evaluate whether proportional controller would meet the desirable steady-state and transient response behaviours. [4 marks]

Solution

a. For the steady-state error analysis, evaluate the loop transfer function of the system:

$$G(s) = \frac{K}{s(Js+B)}$$

So, the system above is Type 1, as a result, its steady-state error $e(\infty)$ is 0 for a step input $r(t) = u(t)$.

But, for a unit ramp input, the steady-state error of the system is:

$$e(\infty) = \frac{1}{K_v}$$

The value of the velocity error constant is:

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\left[\frac{K}{s(Js+B)}\right] = \frac{K}{B}$$

Thus, the steady-state error ($e(\infty)$) when system is given a unit ramp is:

$$e_{ramp}(\infty) = \frac{1}{K_v} = \frac{B}{K}$$

This implies small steady-state error requires large gain K.

b. For transient response analysis, the closed-loop transfer function of the system:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s(sJ+B)}}{1+\dfrac{K}{s(Js+B)}} = \frac{K/J}{s^2 + B/Js + K/J}$$

Comparing with standard second-order system's case:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

So

$$\omega_n = \sqrt{K/J} \qquad \text{and} \qquad 2\omega_n\zeta = B/J$$

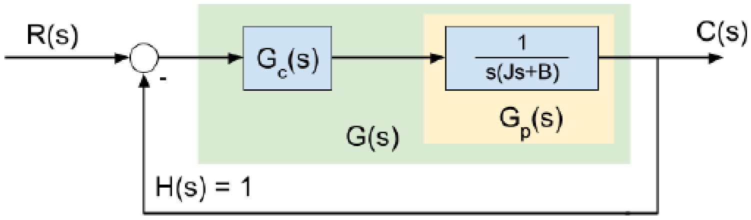
Thus, the damping ratio is:

$$\zeta = \frac{B}{2\sqrt{KJ}}$$

c. Since B and J cannot be tweaked (motor parameters), large K leads to reduced damping ratio (ζ). So, this results in large overshoot. As a result, the system is less stable.

Thus, a simple proportional (gain) controller (K) would not produce desirable steady-state and transient response behaviour as a compromise between small steady-state error and good relative stability and fast response cannot be achieved.

13. For a proportional-derivative (PD) controller implemented in a given control system shown below.



Controller transfer function $G(s) = K_P + K_D s$, where K_P is the proportional constant, and K_D is the derivative constant. It is expected that the inserted PD controller could improve the steady-state error to unit ramp and the transient response will have damping ratio 0.5 < ζ < 0.8. Perform the following tasks:

a. For steady-state error analysis, would it be possible to mitigate steady-state error to a unit ramp using the proportional controller of the given controller? [6 marks]

b. For transient response analysis, derive the equation for the damping ratio of the system. [6 marks]

c. Suggest the setup of the PD controller for the given system that will meet the stated design specifications. [4 marks]

d. Discuss feasibility of PD controller for the improving the control system. [2 marks]

Solution

a. For the steady-state error analysis, the loop transfer function of the system is:

$$G(s) = G_c(s)G_p(s) = \frac{K_P + K_D s}{s(Js + B)}$$

Thus, the system is Type 1, so its steady-state error is 0 for a step input, $r(t) = u(t)$.

But, for a unit ramp input $e(\infty) = 1/K_v$ and velocity error constant is:

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} s\left[\frac{K_P + K_D s}{s(Js + B)}\right] = \frac{K_P}{B}$$

So, the steady-state error when the system is given a unit ramp is:

$$e(\infty) = \frac{B}{K_P}$$

In this case, it is possible to make steady-state error $e(\infty)$ to a unit ramp as small as possible by increasing proportional gain K_P.

b. For the transient-response analysis, the closed-loop transfer function of the system is:

$$\frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)} = \frac{\frac{K_P + K_D s}{s(Js + B)}}{1 + \frac{K_P + K_D s}{s(Js + B)}} = \frac{\frac{(K_P + K_D s)}{J}}{s^2 + \frac{(B + K_D)}{J}s + \frac{K_P}{J}}$$

Again, comparing with standard second-order system's case,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

The natural frequency is:

$$\omega_n = \sqrt{\frac{K_P}{J}}$$

And

$$2\omega_n\zeta = \frac{B + K_D}{J}$$

As a result, the damping ratio is:

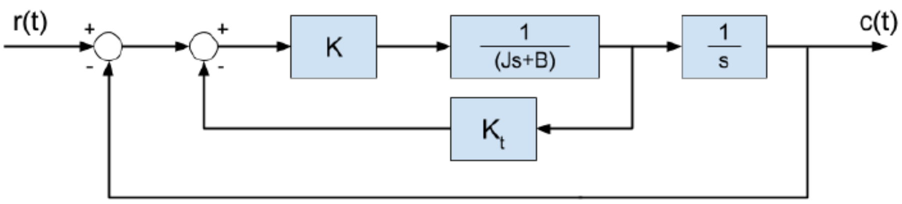
$$\zeta = \frac{B + K_D}{2\sqrt{K_P J}}$$

c. Thus, we can choose to do the following:

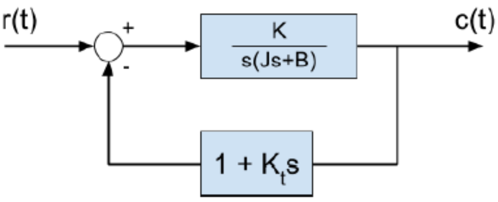
- Large K_P for small steady-state error $e(\infty)$ to unit ramp, and
- Appropriate K_D to have the value of the damping ratio of 0.5 < ζ < 0.8.

d. PD controller adds a zero at $s = -K_P/K_D$ which could have an impact in changing the shape of the response to unit step. Additionally, PD controller is susceptible to noise and difficult to realize.

14. For a given implementation of tachometer as a controller, we design a rate feedback (tachometer) control as shown below.



Alternatively, the above block diagram can be reduced as shown below to the typically used tachometer control system.



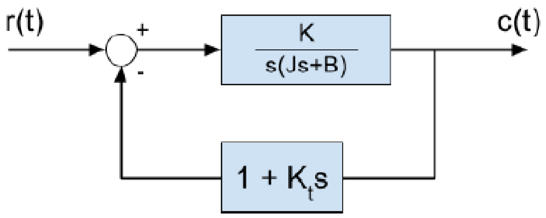
For the given system, perform the following tasks:

a. For the steady-state error analysis, determine whether the given control system can reduce the steady-state error to unit ramp. [6 marks]

b. For the transient response analysis, suggest the setup of the control system, so its steady-state error to unit ramp is improved and damping ratio between 0.5 < ζ < 0.8. [8 marks]

Solution

a. The block diagram given above can be reduced as shown below to the typically used tachometer control system.



It is a non-unity feedback control system, and the equivalent transfer function of its unity feedback control system is:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s) - G(s)}$$

Thus

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(Js+B)}}{1 + \frac{K}{s(Js+B)}(1 + K_t s) - \frac{K}{s(Js+B)}} = \frac{K}{s[(Js+B) + KK_t]}$$

For steady-state error analysis, the unity feedback transfer function of the system:

$$\frac{C(s)}{R(s)} = \frac{K}{s[(Js+B) + KK_t]}$$

The closed-loop system is Type 1, so its steady-state error, $e(\infty)$ is 0 for step input $r(t) = u(t)$.

But, for a unit ramp input $e(\infty) = 1/K_v$ and velocity error constant is:

$$K_v = \lim_{s \to 0} sG(s) = \lim_{s \to 0} s\left[\frac{K}{s[(Js+B) + KK_t]}\right] = \frac{K}{B + KK_t}$$

So, the steady-state error to unit ramp is:

$$e(\infty) = \frac{1}{K_v} = \frac{B + KK_t}{K}$$

In this case, it is possible to make steady-state error $e(\infty)$ to a unit ramp as small as possible by increasing open-loop gain K and decreasing feedback gain K_t.

b. For the transient response analysis, the closed-loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{K/J}{s^2 + \left(\frac{B + KK_t}{j}\right)s + \frac{K}{J}}$$

Comparing with standard second-order system's case,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

Then, the natural frequency is:

$$\omega_n = \sqrt{\frac{K}{J}}$$

And

$$2\omega_n\zeta = \frac{B + KK_t}{J}$$

So, the damping ratio is:

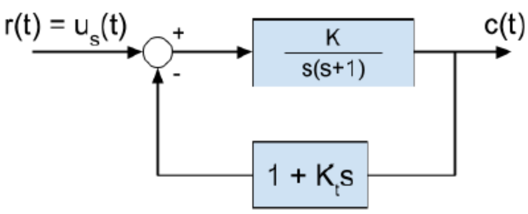
$$\zeta = \frac{B + KK_t}{2\sqrt{KJ}}$$

Thus, we can choose:

- Large gain K for small steady-state error $e(\infty)$ to unit ramp, and
- Appropriate K_t have $0.5 < \zeta < 0.8$.

Note: Tachometer control does not have the same issues of the PD controller. Hence, used widely for servo control.

15. For the tacho control system given below, attempt the following tasks:



a. Find K and K_t such that maximum overshoot, M_p, to unit step is 0.2 and time-to-peak is 1 second. [10 marks]

b. Then, using these values of K and K_t obtained in part (a), determine the values of rise time (T_r) and settling time (T_s). [4 marks]

Solution

a. The transfer equation of the closed-loop system is:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$= \frac{\frac{K}{s(s+1)}}{1 + \frac{K}{s(s+1)}(1 + K_t s)}$$

$$= \frac{K}{s(s+1) + K(1 + K_t s)}$$

$$= \frac{K}{s^2 + (1 + KK_t)s + K}$$

The maximum overshoot of the transient response of the system is calculated from:

$$M_p = e^{-\zeta\pi\sqrt{1-\zeta^2}}$$

Rearrange the equation above, thus, when M_p = 0.2 the damping ratio is:

$$\zeta = \frac{-\ln 0.2}{\sqrt{\pi^2 + [\ln 0.2]^2}} = 0.456$$

The time-to-peak is determined from:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Rearrange the equation given above. As a result, when T_p = 1 second, the natural frequency is:

$$\omega_n = \frac{\pi}{\sqrt{1-(0.456)^2}} = 3.53 \text{ rad/s}$$

Comparing the transfer function of the closed-loop system with the standard equation for second-order system given below:

$$M(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

The gain of the system is calculated from:

$$K = \omega_n^2 \qquad \text{or} \qquad \omega_n = \sqrt{K}$$

So, when ω_n = 3.53 rad/s, the gain is:

$$K = (3.53)^2 = 12.5$$

Also

$$2\omega_n\zeta = 1 + KK_t$$

As a result, when ζ = 0.456, the value of K_t:

$$K_t = \frac{(0.456)2\omega_n - 1}{K} = \frac{(0.456)(2)(3.53) - 1}{12.5} = 0.178$$

b. The rise time (T_r) and settling time (T_s) of the transient response of the system are determined as follows:

$$T_r = \frac{1 + 1.1\zeta + 1.4\zeta^2}{\omega_n} = \frac{1 + 1.1(0.456) + 1.4(0.456)^2}{3.53} = 0.55 \text{ second}$$

And

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{(0.456)(3.53)} = 2.48 \text{ seconds}$$