



Introduction to Root Locus

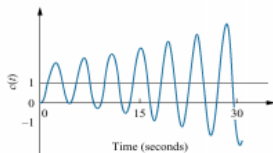
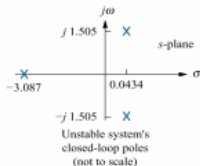
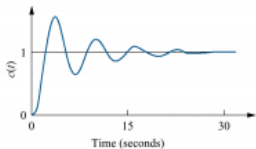
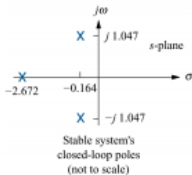
XMUT315 Control System Engineering

Topics

- Introduction to Pole- Zero Diagrams
- Poles and Zeros of Systems
- System Performance
- Closed-Loop Poles and Zeros
- Examples of Root Locus Diagram
- Design using Root Locus Diagram
- Formalisations of Root Locus Diagram
- Examples of Plots in Root Locus.

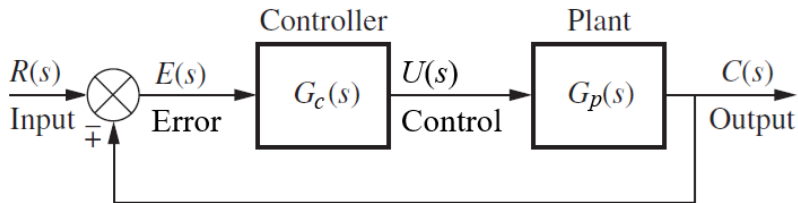
Introduction

- We know that the location of the system poles and zeros determines the response of a LTI system.
- Poles on the y -axis \rightarrow undamped response.
- Complex poles \rightarrow underdamped response.
- Single double poles on the x -axis \rightarrow critical damped response.
- Poles on the x -axis \rightarrow overdamped response.



Introduction

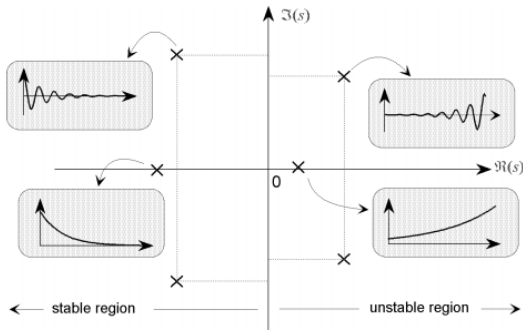
- To change the system response, we therefore need to change these root (pole and zero) location(s).
- We will do this by applying feedback and including a compensator (or controller) into the system.



- Application of feedback enables us to manage the system.
- Compensator (or controller) enables us to determine the specific response of the system.

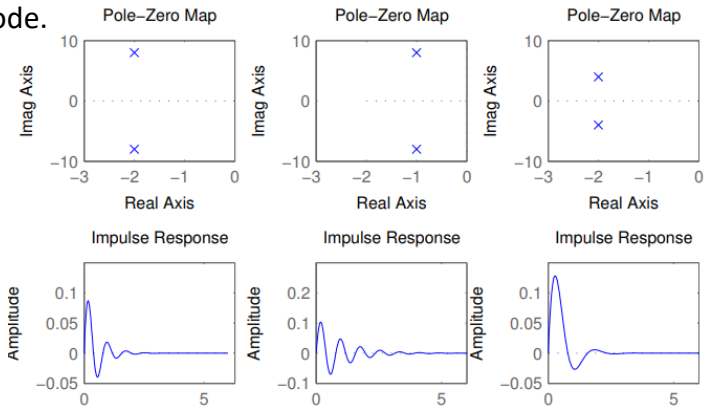
Introduction

- The root locus is a visual presentation of the way the roots of a system change as we vary some parameters of our compensator.
- We will usually be interested in how altering the gain affects the system response, but we can use the root locus for other parameters too.



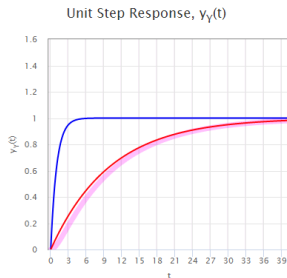
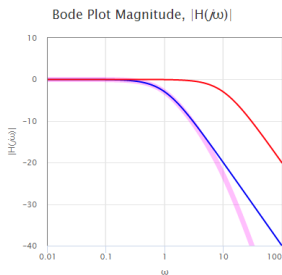
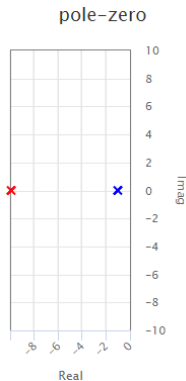
Poles on the Pole-Zero Map

- We know that pole location determines which modes will be present in a time response.
- The real part of a pole location determines the damping of a mode, and the imaginary part determines the natural frequency of the mode.



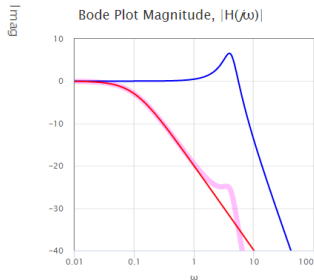
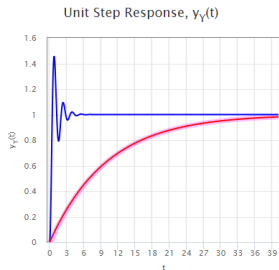
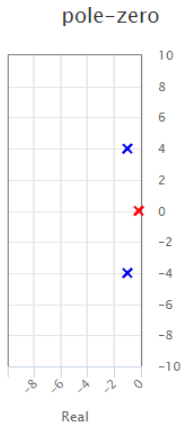
Dominant Poles

- The pole with the slowest response will dominate the response of any system.
- The pole near the origin (blue – with the slowest response) compared with the pole further from the origin (red).



Dominant Poles

- The pole with the slowest response will dominate the response of any system.
- The pole at the origin (red – with the slowest response) compared with complex poles (blue), further from the origin.

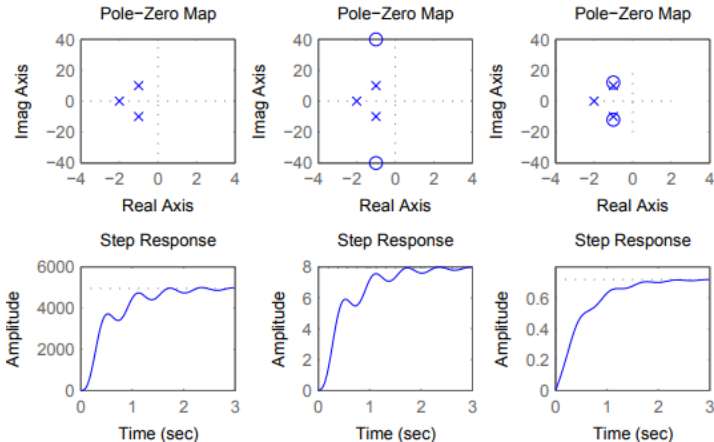


Dominant Poles

- We can therefore often approximate the response of a system as either a first-order or a second-order system.
- We can then use the location of the dominant poles to determine the system's settling time, overshoot, damping, natural frequency, etc.
- So, if the dominant pole (pair) is sufficiently far away from any other poles, then we can ignore the other poles when designing our control system.
- This approximation is normally safe if other poles are at least a factor of three further to the left of the s -plane.

Zeros on the Pole-Zero Map

- Zeros near a pole suppress the mode related to the nearby pole.
- In the extreme case where the zeros and poles are co-located, we get complete cancellation of the corresponding mode.



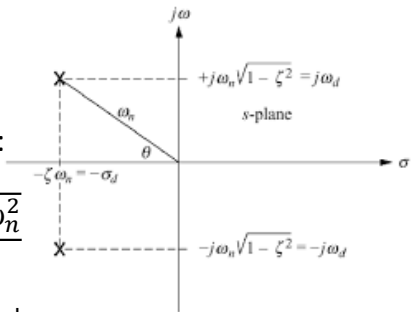
System Performance - Settling Time

- The further a pole is to the left of the s-plane, the faster the corresponding mode decays.
- Therefore, if we are given a specification for system response time, we can convert this to a requirement on pole position.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

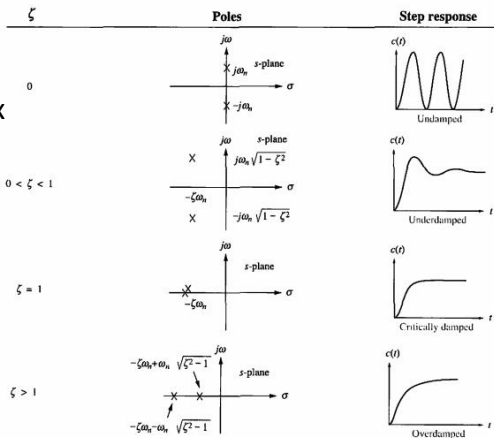
- Roots of the characteristic equation:

$$s = -\frac{2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2}$$
$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\sigma \pm \omega_d$$



System Performance - Settling Time

- For the given roots of the characteristic equation:
- The two roots are imaginaries when $\zeta = 0$.
- The two roots are complex conjugate when $0 < \zeta < 1$.
- The two roots are real and equal when $\zeta = 1$.
- The two roots are real but not equal when $\zeta > 1$.



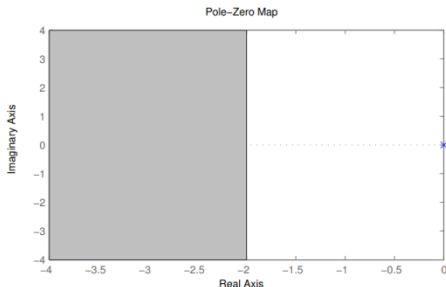
System Performance - Settling Time

- The settling time of a second-order system (T_s) is approximately (e.g. for the 2% steady-state settling time standard):

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma}$$

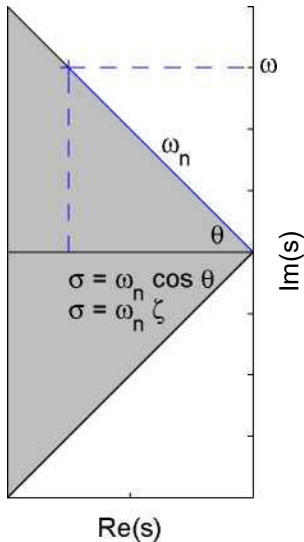
Where: σ is the real part of the pole pair.

- So, for example, if we have a requirement that settling time, $T_s < 2$ seconds, then we must have the dominant pole further left than $s = -2$.



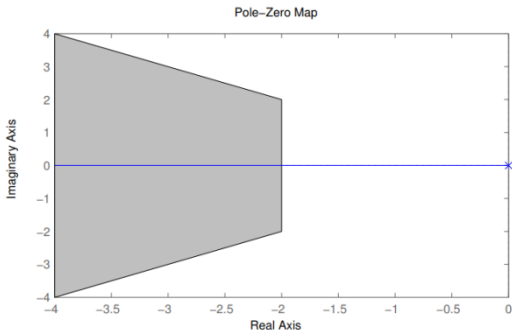
System Performance - Damping

- We are often also given a requirement on the damping ratio.
- Recall that a damping ratio ζ occurs on the pole-zero diagram as a straight line with an angle of $\theta = \cos^{-1} \zeta$ to the negative real axis.
- We can thus similarly construct an allowed region for the poles if we are given a damping specification.
- For example, if we require $\zeta > 0.707$, then the poles must lie less than $\theta = \cos^{-1} 0.707 \approx 45^\circ$ from the negative real axis.



System Performance - Allowed Pole Region

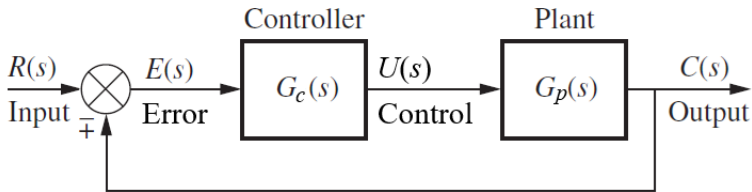
- We are often given specifications on both settling time and damping, so we must combine the constraints imposed by the two.
- We therefore obtain a composite region in which we can place the poles.



- So, if the closed-loop system poles lie within the allowed region then we have satisfied the design specification.

Closed-Loop Poles

- If we have a plant described by an open-loop transfer function $G(s)$, it will have a certain set of open-loop poles and zeros.



- We now enclose the plant within a unity-gain feedback loop including a compensator with a transfer function $G_C(s)$, which results in a closed-loop transfer function:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G_C(s)G_P(s)}{1 + G_C(s)G_P(s)}$$

Closed-Loop Poles

- Notice that the closed-loop transfer function will not have the same pole locations as the open-loop transfer function.
- There is no reason that the denominator of $G(s)$ and the denominator of $T(s)$ would have the same roots.
- Closing the loop has moved the poles, but we do not yet know to where it is heading.

- Open-loop transfer function:

$$G(s) = \frac{1}{(s + 10)}$$

- Root of characteristic equation:

$$s = -10$$

- Closed-loop transfer function:

$$T(s) = \frac{\frac{1}{(s + 10)}}{1 + \frac{1}{(s + 10)}} = \frac{1}{s + 11}$$

- Root of characteristic equation:

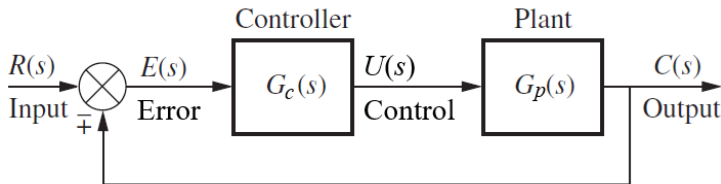
$$s = -11$$

Closed-Loop Poles

- We will find is that the location of the closed-loop poles will depend on the DC gain K or the system. In this case:

$$K = G_C(s)G_P(s) \Big|_{s=0}$$

- As we can change $G_C(s)$, we can use K as a tuning parameter to move the closed-loop poles to a desired location.
- The desired location for the poles will, in turn, be determined by the performance specification for our control system.



Closed-Loop Zeros

- Let's consider the effect that feedback has on zeros.
- Again, consider a plant with a transfer function $G_P(s)$ that we have controlled by adding a series compensator $G_C(s)$.
- Again, the closed-loop transfer function is:

$$T(s) = \frac{G_C(s)G_P(s)}{1 + G_C(s)G_P(s)}$$

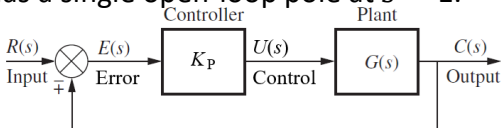
- The only way that $T(s)$ can be zero is if either $G_C(s)$ or $G_P(s)$ is zero (or both).
- Thus, the set of zeros of the closed-loop system is the combination of the zeros of the plant and the compensator.

It is not possible to move the zeros of a plant using feedback.

Example #1

- Consider a system that has a single open-loop pole at $s = -1$.

$$G(s) = \frac{1}{s + 1}$$



- Let us use a proportional compensator $G_C(s) = K_p$ and see what effect this has on the pole.

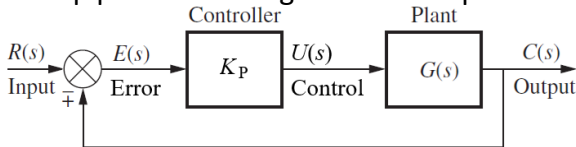
$$T(s) = \frac{G_C(s)G_P(s)}{1 + G_C(s)G_P(s)} = \frac{\left(\frac{K_p}{s + 1}\right)}{1 + \left(\frac{K_p}{s + 1}\right)} = \frac{K_p}{s + (1 + K_p)}$$

- Thus, the closed-loop pole is at $s = -1 - K_p$, compared to the open-loop pole at $s = -1$.
- We can choose the closed-loop pole position by choosing K_p , though only along a constrained path (e.g. a locus).

Example #1

- Depending on the value of proportional compensator, K , the locations of the closed-loop poles are along a constraint path (e.g. a locus).

- Closed-loop system:



$$T(s) = \frac{K_p}{s + (1 + K_p)}$$

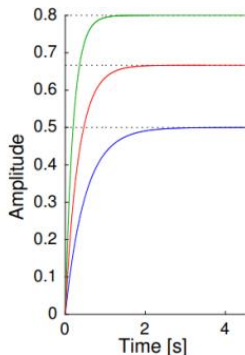
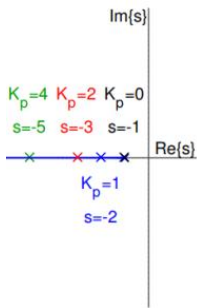
- These loci are:

$$K_p = 0 \rightarrow s = -1$$

$$K_p = 1 \rightarrow s = -2$$

$$K_p = 2 \rightarrow s = -3$$

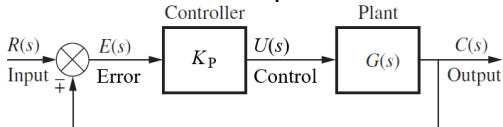
$$K_p = 4 \rightarrow s = -5$$



Example #2

- Consider a DC motor with a transfer function equation:

$$G(s) = \frac{1}{s(s+1)}$$



- Let us add a proportional controller $G_C(s) = K_p$ to this plant.

$$T(s) = \frac{\left[\frac{K_p}{s(s+1)} \right]}{1 + \left[\frac{K_p}{s(s+1)} \right]} = \frac{K_p}{s^2 + s + K_p}$$

- Now, the closed-loop poles are located where the denominator is equal to zero.
- Thus, we find them by solving $s^2 + s + K_p = 0$ (or more generally), by solving the characteristic equation, $1 + G_C G_P = 0$.

Example #2

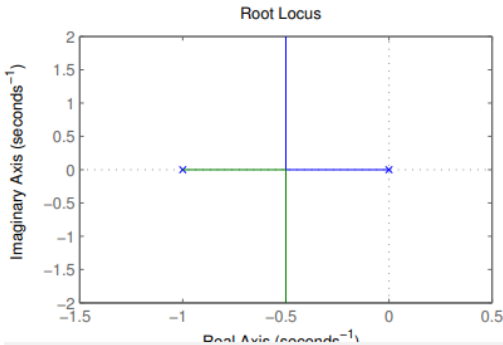
- Let's find roots of $s^2 + s + K_p = 0$ using the quadratic equation.

$$s = \frac{-1 \pm \sqrt{1 - 4K_p}}{2}$$

- For $K_p < 1/4$, we get two real roots at:

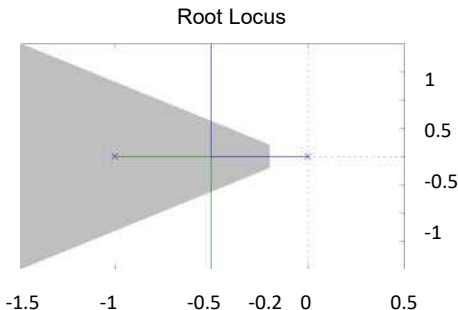
$$s = -\frac{1}{2} \pm \frac{\sqrt{1 - 4K_p}}{2}$$

- For $K_p > 1/4$ we change from a first order to a second order (oscillatory) response as the poles become complex.



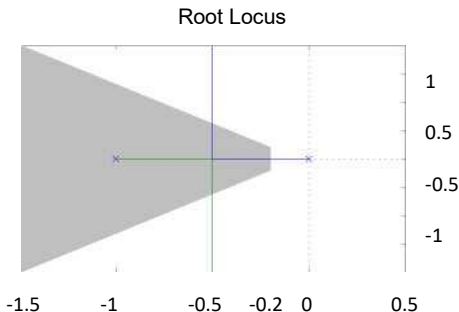
Design using the Root Locus

- Let us imagine that we have been asked to design a controller for the dc motor so that it has a settling time of less than 20 seconds and a damping ratio better than $\zeta = 0.707$.
- As we need $T_s = 4/\sigma < 20$ the dominant pole must be further left than $s = -0.2$ (e.g. $\sigma = 4/20 = 0.2$).
- Damping ratio, $\zeta > 0.707$ requires that the angle from the negative real axis be no greater than $\theta = \cos^{-1} 0.707 = 45^\circ$.



Design using the Root Locus

- In this example, there are a range of possible K values that would satisfy the design specification.
- Let's work out the minimum and maximum gains that would be acceptable.
- The dominant pole crosses into the allowed region at $s = -0.2$.



Design using the Root Locus

- We know that while the roots are real, they are located at:

$$s = -\frac{1}{2} \pm \frac{\sqrt{1 - 4K_p}}{2}$$

- So, in this case:

$$s = -0.5 \pm \frac{\sqrt{1 - 4K_p}}{2} = -0.2$$

- Solving the equation given above:

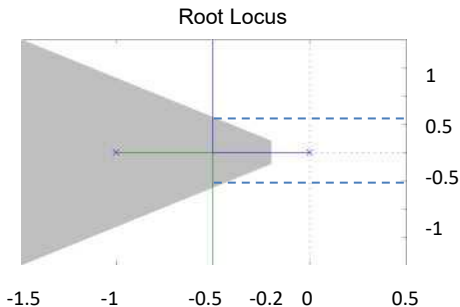
$$\frac{\sqrt{1 - 4K_p}}{2} = 0.3$$

- As a result, the proportional gain of the system is:

$$K_p = 0.16$$

Design using the Root Locus

- By inspection of the diagram, we can see that the poles leave the allowed region at $s = -0.5 \pm j0.5$.
- These points are the maximum values that meet the design specification of the system.



Design using the Root Locus

- We previously calculated that the complex poles are located at:

$$s = -\frac{1}{2} \pm j \frac{\sqrt{4K_p - 1}}{2}$$

- So, $\sigma = -0.5$ and again we can equate to find the value of K_p on the boundary:

$$\frac{\sqrt{4K_p - 1}}{2} = 0.5$$

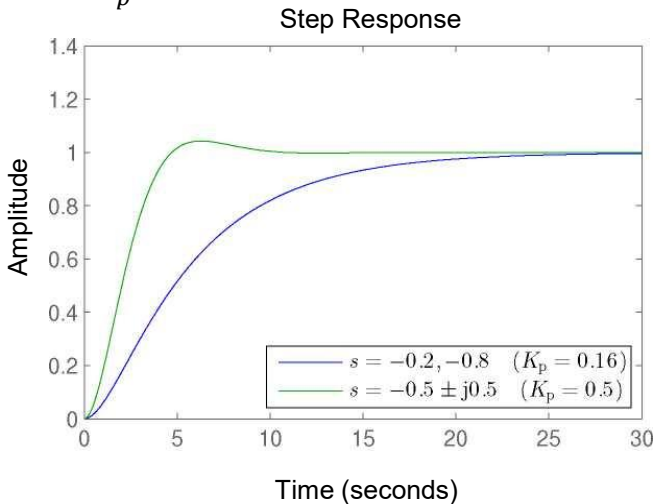
- So, the proportional gain of the system is:

$$K_p = 0.5$$

- So, we have found that the design specification will be met for $0.16 < K_p < 0.5$.

Design using the Root Locus

- The graph shows the step responses of the system when $K_p = 0.16$ and $K_p = 0.5$.



The Evan's Root Locus

- We can always determine the root locus using mathematical analysis, as in the examples above.
- However, this becomes tedious as the number of poles increases.
- The Evan's root locus is a graphical technique that automates the mathematics to provide a method to draw the locus directly.
- Examination of the root locus allows us to:
 - a. Determine the stability of a system as the gain changes.
 - b. Choose an appropriate gain to produce a desired closed-loop response.
 - c. Modify the form of $G_C(s)$, if an adequate closed-loop response cannot be achieved.

Root Locus Formalities

- The root locus is a pole-zero diagram showing “tracks” taken by the system poles as some parameters (i.e. gain in our case) are varied.
- Formally, the root locus shows the locus traced out by the roots of the characteristic equation, $1 + G_C(s)G_P(s) = 0$, as the gain is varied.
- To find a root locus, we are therefore searching for values of s that satisfy the characteristic equation:

$$1 + G_C(s)G_P(s) = 0$$

- Rearrange the equation:

$$G_C(s)G_P(s) = -1$$

- Thus

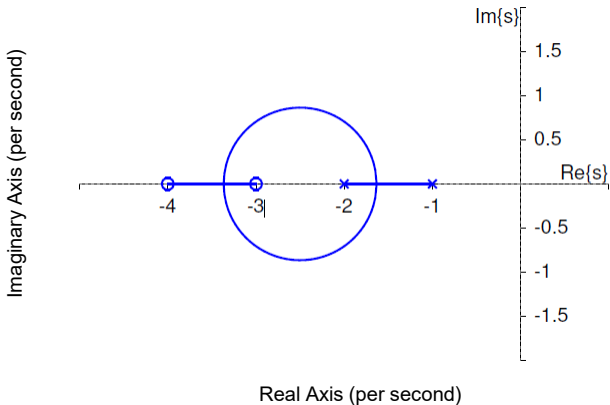
$$G_C(s)G_P(s) = 1 \angle (2k + 1)180^\circ \quad \text{for } k \in \mathbb{Z}$$

Example #3

$$G(s) = \frac{(s + 3)(s + 4)}{(s + 1)(s + 2)}$$

- Zeros: -3, -4.
- Poles: -1, -2.

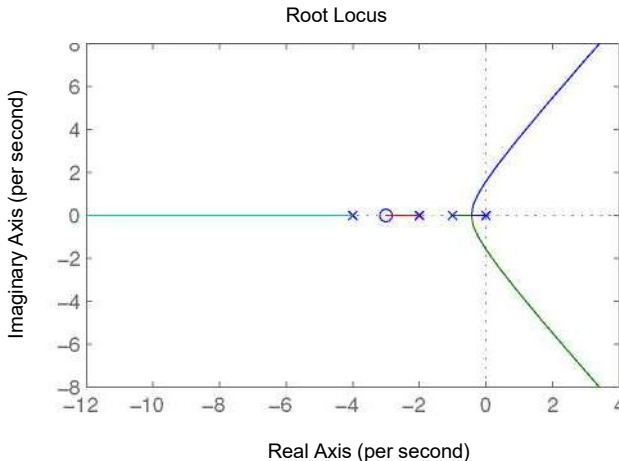
Root Locus



Example #4

$$G(s) = \frac{s + 3}{s(s + 1)(s + 2)(s + 4)}$$

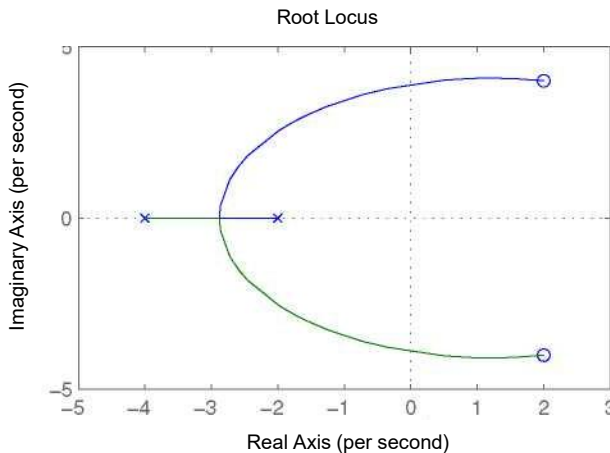
- Zeros: -3.
- Poles: 0, -1, -2, -4.



Example #5

$$G(s) = \frac{(s - 2)^2 + 16}{(s + 2)(s + 4)}$$

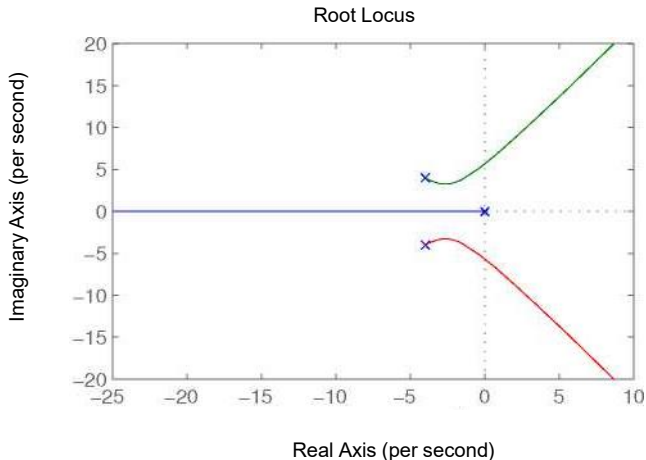
- Zeros: $2 \pm j4$.
- Poles: $-2, -4$.



Example #6

$$G(s) = \frac{1}{s[(s + 4)^2 + 16]}$$

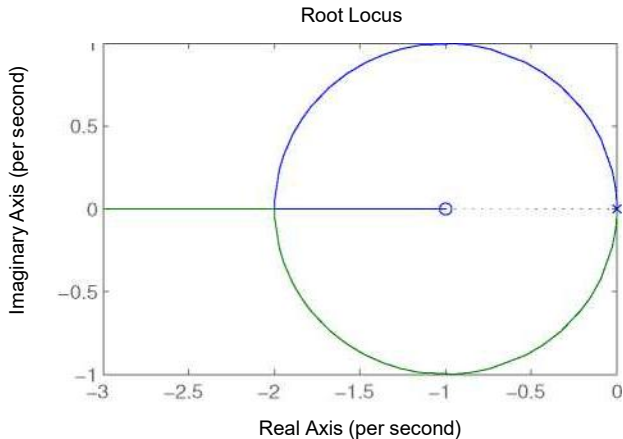
- Zeros:
- Poles: 0, $-4 \pm j4$



Example #7

$$G(s) = \frac{1 + s}{s^2}$$

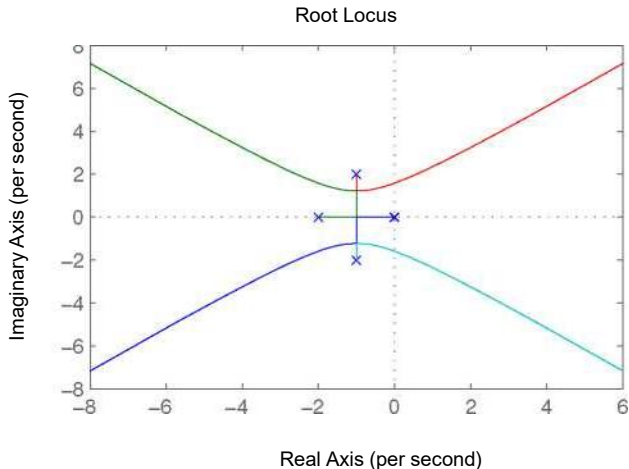
- Zeros: -1.
- Poles: -0 (double poles at origin).



Example #8

$$G(s) = \frac{1}{s(s+2)[(s+1)^2 + 4]}$$

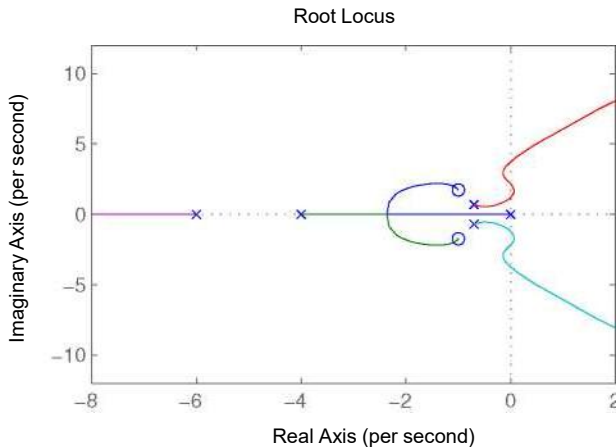
- Zeros:
- Poles: 0, -2, -1 ± j2



Example #9

$$G(s) = \frac{s^2 + 2s + 4}{s(s + 4)(s + 6)(s^2 + 1.4s + 1)}$$

- Zeros: $-1 \pm j\sqrt{3}$.
- Poles: $0, -4, -6, -0.7 \pm j\sqrt{0.51}$



Things to Notice

These are the guidelines for the construction of a root-locus diagram:

1. Each branch of the root locus begins at an open-loop pole.
2. Each branch of the root locus either terminates at a zero or goes to (complex) infinity.
3. One and only one branch leaves each pole.
4. One and only one branch enters each zero.
5. Like any pole-zero diagram, the root locus is always symmetric about the real axis (and complex poles always come in conjugate pairs).