VICTORIA UNIVERSITY OF
WELLINGTON
TE HERENGA WAKA

Analysis with Root Locus

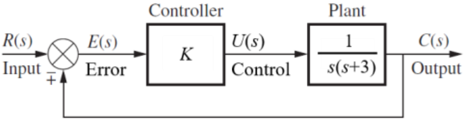
XMUT315 Control System Engineering

Topics

- Sketching root locus.
- Rules in root locus analysis.
- Examples of root locus analysis.
- Refining root locus analysis.
- Break away and break in.
- Imaginary axes crossing.
- Angle of departure and angle of arrival.

Sketching the Root Locus

- Direct solution (e.g. analytical method) for the closed-loop pole locations as a function of K becomes burdensome as the system order increases.

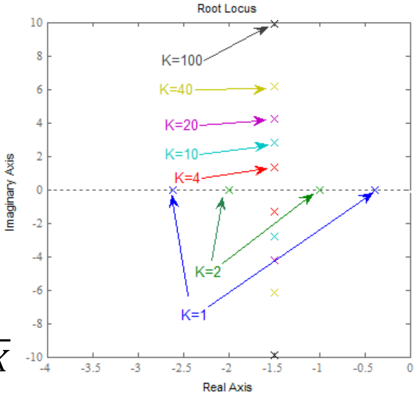


- Open-loop system:

$$G(s) = G_P = \frac{1}{s(s+3)}$$

- Closed-loop system:

$$T(s) = \frac{G_C G_P}{1 + G_C G_P} = \frac{K}{s^2 + 3s + K}$$



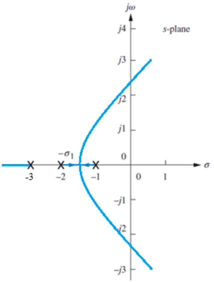
Sketching the Root Locus

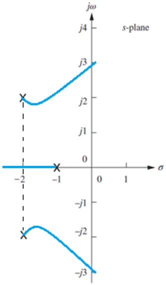
- Walter Evans devised a set of rules that allows sketching of the root locus without brute force calculation.
- The basic root locus diagram is based on five rules -> e.g. Evans rule for sketching root locus.
- These rules are generally sufficient to give a good sense of the shape of the root locus, and consequently its “story” as the gain changes.
- There are additional rules that can be used to refine the shape of the locus when necessary.

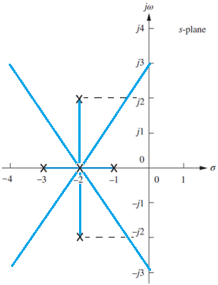
Rule #1 for Sketching a Root Locus

The number of branches of the root locus is equal to the number of closed-loop poles.

- A “branch” is a segment of the root locus traversed (or moved) by a single pole as the gain is varied.
- This rule implies that there is one and only one branch originating at each pole.



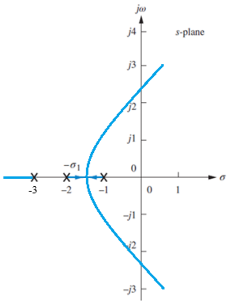


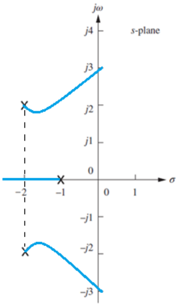


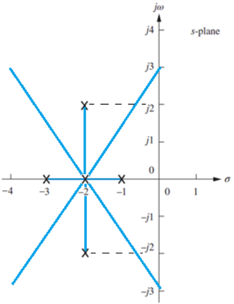
Rule #2 for Sketching a Root Locus

The root locus is symmetric about the real axis.

- Contours of the plots are symmetric about the real axis.
- Any complex roots (poles or zeros) must occur in complex conjugate pairs.
- Therefore, the root locus must be symmetric.







Rule #3 for Sketching a Root Locus

A branch of the root locus will only be on the real axis to the left of an odd number of finite open-loop roots.

- Recall that roots mean both poles and zeros!
- As discussed earlier, the solution of the characteristic equation implies that $\angle G_C G_P = -180°$ at all points on the root locus.
- Now, all roots on the further left than a test point will contribute no phase shift.
- However, both zeros and poles to the right of a test point contribute 180°.
- We therefore only satisfy the characteristic equation if there is an odd number of roots on the real axis that are further to the right than our test point.

Rule #4 for Sketching a Root Locus

The root locus begins at the finite and infinite poles of $G_C(s)G_P(s)$ *and ends at the finite and infinite zeros of* $G_C(s)G_P(s)$.

- Let's calculate what happens to $T = KG_CG_P/(1 + KG_CG_P)$ for very small and very large values of K.
- Notice the characteristic equations of the closed-loop system is:

$$1 + KG_C(s)G_P(s) = 0$$

- Factor $G_P(s)$ and rewrite the polynomial equation as:

$$1 + KG_C(s)G_P(s) = 1 + K\frac{N_C(s)N_G(s)}{D_C(s)D_G(s)} = 1 + K\frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{m}(s - p_i)}$$

- Therefore, the characteristic equation becomes:

$$\prod_{i=1}^{m}(s - p_i) + K\prod_{i=1}^{m}(s - z_i) = 0$$

Rule #4 for Sketching a Root Locus

- Note that we have allowed for more interesting compensators here, by expanding the previous G_C into KG_C, where G_C now contains any compensator roots.

$$\lim_{k\to 0}\left[\prod_{i=1}^{m}(s-p_i)+K\prod_{i=1}^{m}(s-z_i)\right]$$

- Therefore, the poles of T for small K are the poles of $G_C G_P$. That is, the poles of T start at the open-loop poles of the plant and the open-loop poles of the compensator.

$$\lim_{k\to \infty}\left[\prod_{i=1}^{m}(s-p_i)+K\prod_{i=1}^{m}(s-z_i)]\right]$$

- Therefore, the poles of T for large K are the roots of $G_C G_P$. These are the zeros of $G_C G_P$, which is just the combination of the zeros of G_C and the zeros of G_P.

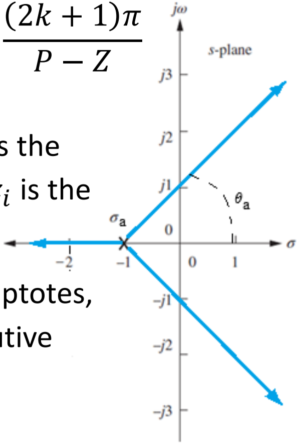
Rule #5 for Sketching a Root Locus

The root locus approaches straight-line asymptotes as the gain approaches infinity. The real-axis intercept, σ_a, and angles, θ_a, are given by:

$$\sigma_a = \frac{\sum_i p_i - \sum_i z_i}{P - Z} \qquad \theta_a = \frac{(2k+1)\pi}{P - Z}$$

Where: P is the number of finite poles, Z is the number of finite zeros, p_i is the i-th pole, z_i is the i-th zero and $k \in Z$.

- In general, a root locus will have *P* - *Z* asymptotes, so you will need to substitute *P* - *Z* consecutive values for k into the θ_a equation.



Example for Root Locus Construction

Sketch the root locus of the transfer-function equation:

$$T(s) = \frac{s+3}{s(s+1)(s+2)(s+4)}$$

a. Work out and determine the locations of the poles and zeros on the system in the s-plane diagram. [2 marks]

b. Calculate the departure point and indicate this on the s-plane diagram. [3 marks]

c. Calculate the angle of asymptotes and sketch these angles on the s-plane diagram. [4 marks]

d. Determine the branches of the root locus and sketch them on the s-plane diagram. [4 marks]

Example for Root Locus Construction

e. Evaluate the construction of the root locus diagram of the system using the rules for sketching a root locus. [10 marks]

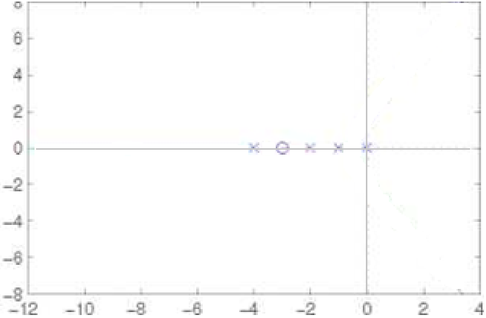
f. Simulate the root locus diagram in MATLAB. [5 marks]

Example for Root Locus Construction

- Knowing the transfer function of the system:

$$T(s) = \frac{s+3}{s(s+1)(s+2)(s+4)}$$

- For the given open-loop system, we have poles at s = 0,-1,-2 and -4 and a zero at s = -3.



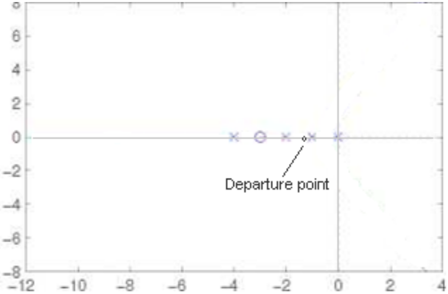
Example for Root Locus Construction

- Let’s first determine the parameters of the asymptotes. First, the departure point:

$$\sigma_a = \frac{\sum_i p_i - \sum_i z_i}{P - Z}$$

$$= \frac{(0 - 1 - 2 - 4) - (-3)}{4 - 1}$$

$$= \frac{-7 + 3}{3} = -\frac{4}{3}$$



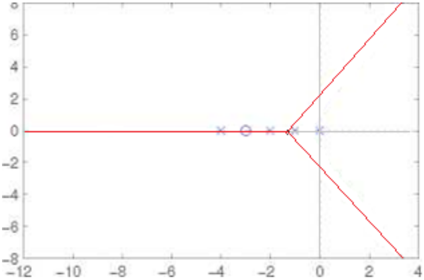
- We can therefore indicate this point in the root locus.

Example for Root Locus Construction

- Work on the angle of the asymptotes:

$$\theta_a = \frac{(2k+1)\pi}{P-Z} = \frac{(2k+1)\pi}{4-1}$$

$$\theta_a = \begin{cases} \dfrac{\pi}{3} & \text{for } k = 0 \\ \pi & \text{for } k = 1 \\ -\dfrac{\pi}{3} & \text{for } k = -1 \end{cases}$$



- From the departure point there are three asymptotes: at

  π/3 (60°), -π/3 (-60°), and π (-180°).

- We can therefore sketch in the asymptotes.

Example for Root Locus Construction

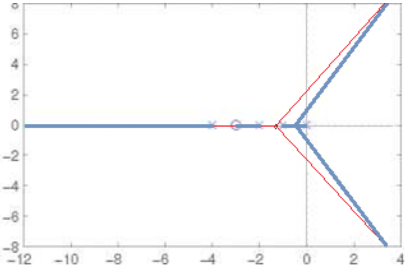
- With the asymptotes in place, we can sketch in the various branches of the root locus.
- Asymptotes are:

  Asymptote 1: Pole (0,0) -> +∞

  Asymptote 2: Pole (-1,0) -> -∞

  Asymptote 3: Pole (-2,0) -> Zero (-3,0)

  Asymptote 4: Pole (-4,0) -> -∞



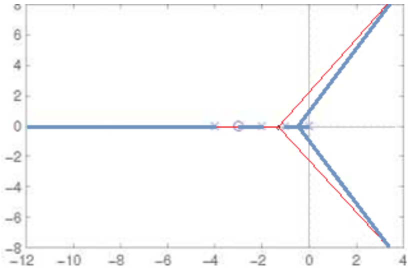
Example for Root Locus Construction

- R1 (*The number of branches of the root locus is equal to the number of closed-loop poles)* :

  We expect four branches to the locus.

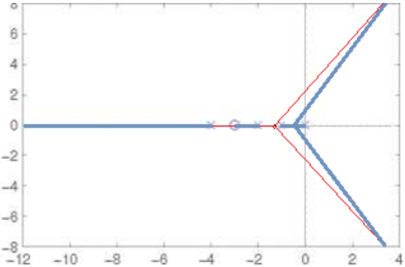
- R2 (*The root locus is symmetric about the real axis*) :

  The sketch is symmetric about the real axis.



Example for Root Locus Construction

- R3 (*A branch of the root locus will only be on the real axis to the left of an odd number of finite open loop roots)* :
  - The locus will pass along the real axis between the slowest two poles and between the third slowest pole and the zero.
  - There is a branch on the real axis beyond the fastest pole.



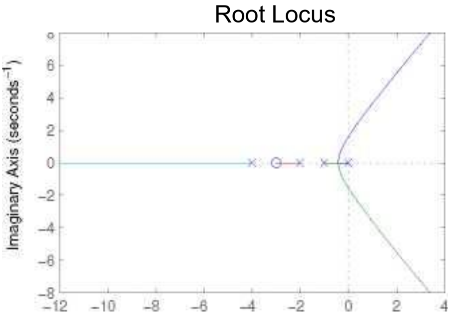
Example for Root Locus Construction

- *R4 (The root locus begins at the finite and infinite poles of $G_C(s)G_P(s)$ and ends at finite and infinite zeros of $G_C(s)G_P(s)$*:

  The locus are from poles to infinities and from a pole at -2,0 to zero at -3,0.

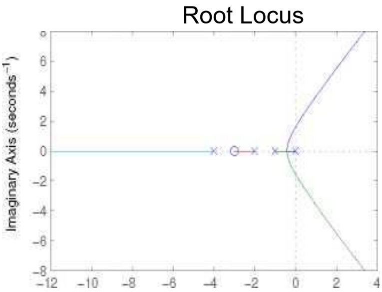
- *R5 (The root locus approaches straight line asymptotes as the gain approaches infinity at the real-axis intercept, σ_a, and angles, θ_a)* :

- There are three asymptotes from (-1.33,0) at $\pi/3$, $-\pi/3$, and π.

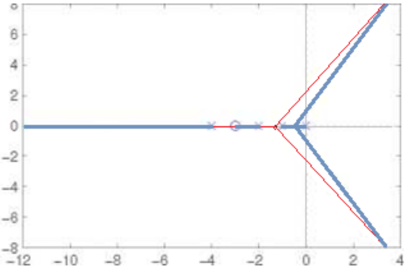


Example for Root Locus Construction

- Root locus diagram of the given system from MATLAB simulation.
- The result shows a similar likeness with the result of the manual sketch.



MATLAB simulation



Sketch

Example Exercises of Root Locus Construction

For each of the systems with the following transfer function equations:

$$G_1(s) = \frac{s^2 - 4s + 24}{(s+2)(s+4)}$$

$$G_2(s) = \frac{1}{s[(s+4)^2 + 16]}$$

$$G_3(s) = \frac{1+s}{s^2}$$

$$G_4(s) = \frac{1}{s(s+2)[(s+1)^2 + 4]}$$

Example Exercises of Root Locus Construction

a. Determine the poles and zeros of the system. [2 marks]

b. Sketch the poles and zeros of the system in the s-plane diagram. [4 marks]

c. Determine the asymptotes of the root locus. [2 marks]

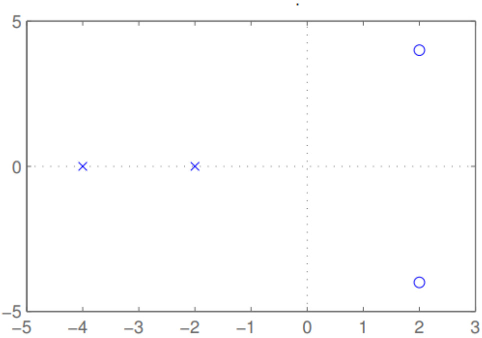
d. Sketch the asymptotes on the s-plane diagram. [4 marks]

e. Simulate the root locus diagram of the system in MATLAB. [5 marks]

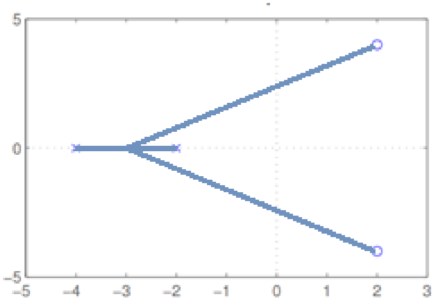
Root Locus Construction Exercise #1

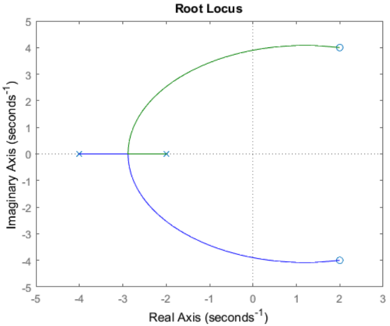
$$G_1(s) = \frac{s^2 - 4s + 24}{(s+2)(s+4)}$$

- Pair of complex zero at $s_{1,2}$ = -2±j$\sqrt{20}$.
- Real pole at s = -2.
- Real pole at s = -4.



Root Locus Construction Exercise #1





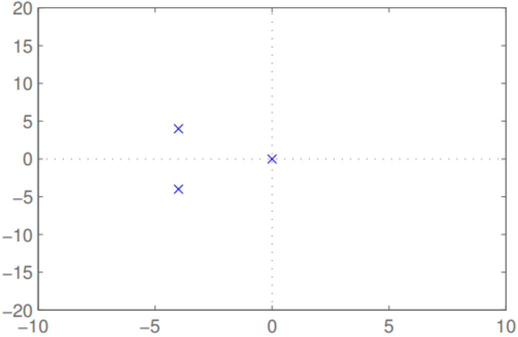
Asymptotes:

- Pole at s=-2 -> zero at 2+j$\sqrt{20}$
- Pole at s=-4 -> zero at 2-j$\sqrt{20}$

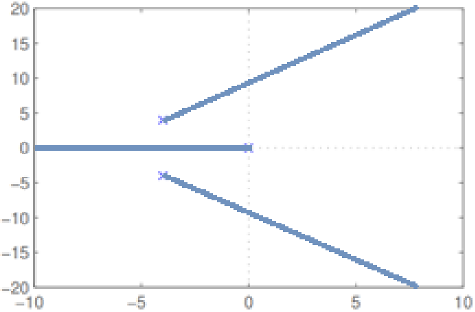
Root Locus Construction Exercise #2

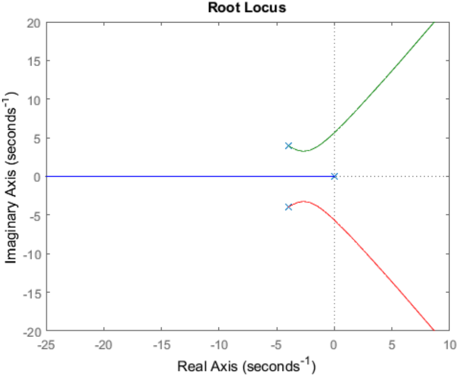
$$G_2(s) = \frac{1}{s[(s+4)^2 + 16]}$$

- No zero.
- A pole at origin.
- Pair of complex poles at $s_{1,2}$ = -4 ± j4.



Root Locus Construction Exercise #2





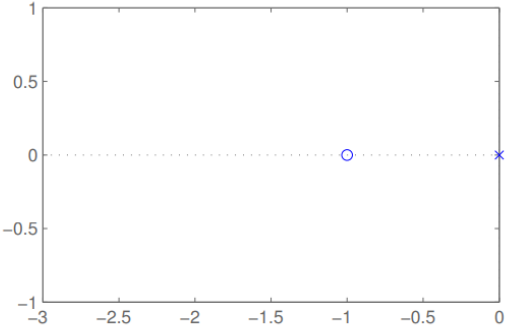
Asymptotes:

- Pole at origin -> -∞
- Pole at *s* = -4 + j4 -> +∞ (+j∞)
- Pole at *s* = −4 − j4 −> -∞ (-j∞)

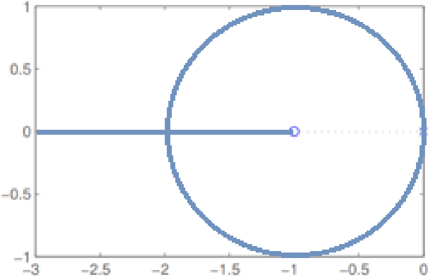
Root Locus Construction Exercise #3

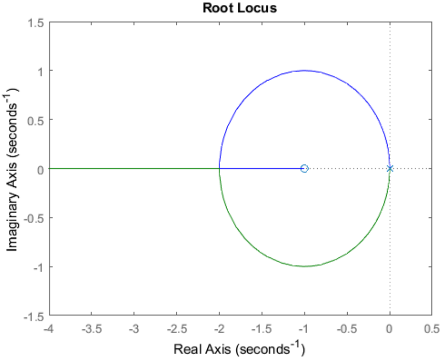
$$G_3(s) = \frac{1+s}{s^2}$$

- A real zero at s = -1.
- Double poles at origin.



Root Locus Construction Exercise #3





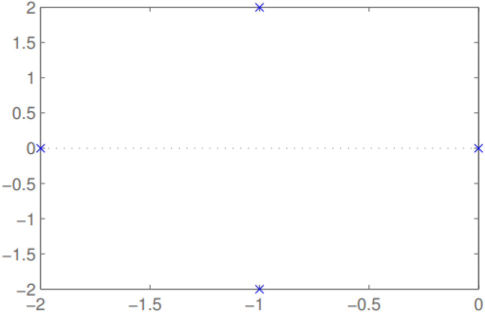
Asymptotes:

- Pole at origin -> zero at s = -1.
- Pole at origin -> -∞.

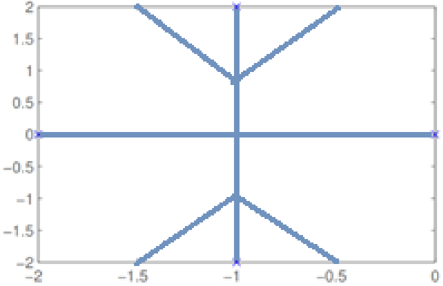
Root Locus Construction Exercise #4

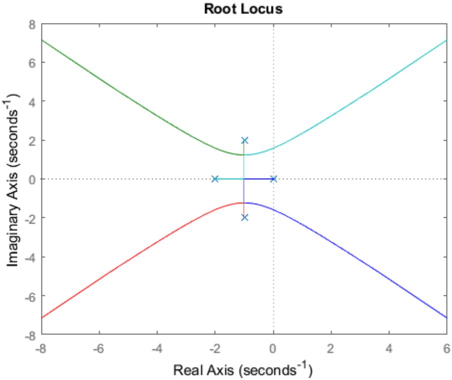
$$G_4(s) = \frac{1}{s(s+2)[(s+1)^2+4]}$$

- No zero.
- A pole at origin.
- A real pole at s = -2.
- A pair of complex poles s = -1 ± j2.



Root Locus Construction Exercise #4





Asymptotes:

- Pole at origin -> +∞ (-j∞).
- Pole at s = -2 -> +∞ (+j∞).
- Complex pole at s = -1 + j2 -> -∞ (+j∞).
- Complex pole at s = -1 - j2 -> -∞ (-j∞).

Refining the Root Locus

- There are many features of the root locus as yet undetermined.
  - Where do pole pairs break away / break into the real axis?
  - Where do pole pairs cross the imaginary axis?
  - At what angles do pole pairs depart from the open-loop poles and enter open-loop zeros?
- We have two options for determining these factors:
  - Use MATLAB (or some other tools).
  - Apply some more rules.

Pole Break-away / Break-in Angles

- We often encounter systems where poles break away from the real axis into the complex plane or, conversely, merge back into the plane.
- The angle at which the poles leave the plane is given by:

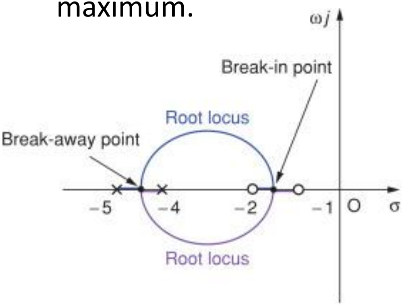
$$\angle\theta = \frac{180^\circ}{n}$$

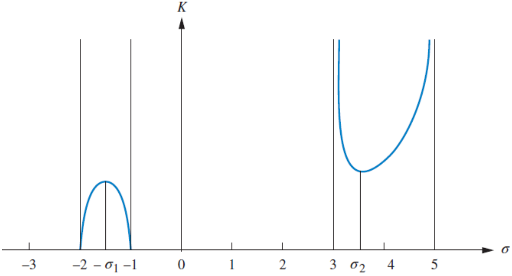
Where: n is the number of poles breaking away/in.

- Typically, n = 2, so the angle of departure/arrival from the real axis is 90°.

Pole Break-away Locations

- We know that a breakaway occurs as the gain is increasing.
- Therefore, the breakaway point has the highest gain of any point where the locus is on the real axis.
- In cases where there are multiple segments on the real axis, it might not be the highest gain globally, but it will be a local maximum.





Pole Break-in Locations

- Similarly, around a break-in point, the locus first touches the real axis at some value of gain.
- The locus then continues on the real axis as the gain increases. The break-in point will therefore correspond to a local minimum of gain.
- We know that the root locus satisfies the characteristic equation:

$$1 + KG = 0$$

- Thus, everywhere along the root locus:

$$K = -\frac{1}{G(s)}$$

- Along the real axis, we know that s is real, so we can write $s = \sigma$.
- Hence, $K = -G(\sigma)$ on the real axis.

Pole Break-in Locations

- For a given system, the open-loop transfer-function equation is:

$$G(s) = \frac{(s+z_1)(s+z_2)\ldots(s+z_k)}{(s+p_1)(s+p_2)\ldots(s+p_k)} = \frac{n_1 s^k + n_2 s^{k-1} .. + n_k}{d_1 s^m + d_2 s^{m-1} .. + d_m}$$

- Thus

$$K = -\frac{1}{G(s)} = \frac{d_1 s^m + d_2 s^{m-1} .. + d_m}{n_1 s^k + n_2 s^{k-1} .. + n_k} = \frac{P(s)}{Z(s)}$$

- Knowing the form of equation is $u/v = (u'v - uv')/v^2$, the first derivative of the equation above is:

$$\frac{dK}{ds} = \frac{\frac{dP(s)}{ds} Z(s) - P(s) \frac{dZ(s)}{ds}}{[D(s)]^2}$$

Pole Break-away / Break-in without Differentiation

- There is also an easier method without using differentiation, though it is a less obvious technique.
- Using this methos, the *breakaway and break-in points,* $s = \sigma_{1,2}$*,* satisfy the relationship*:*

$$\sum_{i=1}^{Z} \frac{1}{\sigma_b - z_i} = \sum_{j=1}^{P} \frac{1}{\sigma_b - p_j}$$

Where: p_i and z_i are the pole and zero values of $G_C G_P$, where we have Z total zeros and P total poles.

Example of Pole Break-away / Break-in Location

Given a control system with the following transfer-function equation, it is connected in series with a proportional controller with a gain of K:

$$G(s) = \frac{(s-3)(s-5)}{(s+1)(s+2)}$$

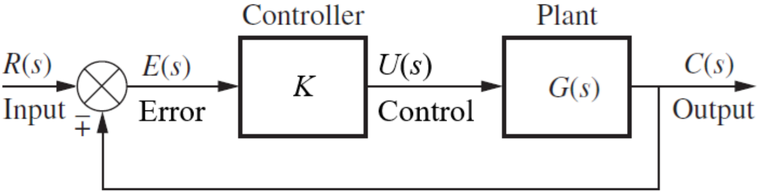
a. Determine the expression for the closed-loop system for determining value of K in terms of value of σ. [4 marks]

b. Simulate in MATLAB the break-away and break-in points of the system. [5 marks]

c. Determine the break-away and break-in points of the system using differentiation method. [4 marks]

Example of Pole Break-away / Break-in Location

d. Determine the break-away and break-in points of the system without using the differentiation method. Comment on the difference using this method. [4 marks]



Example of Pole Break-away / Break-in Location

Answer

a. Along the root locus, $KG = -1$.

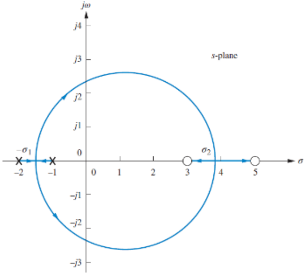
$$\frac{K(s-3)(s-5)}{(s+1)(s+2)} = -1$$

Considering gains only on the real axis -> substitute $s = \sigma$.

$$\frac{K(\sigma-3)(\sigma-5)}{(\sigma+1)(\sigma+2)} = -1$$

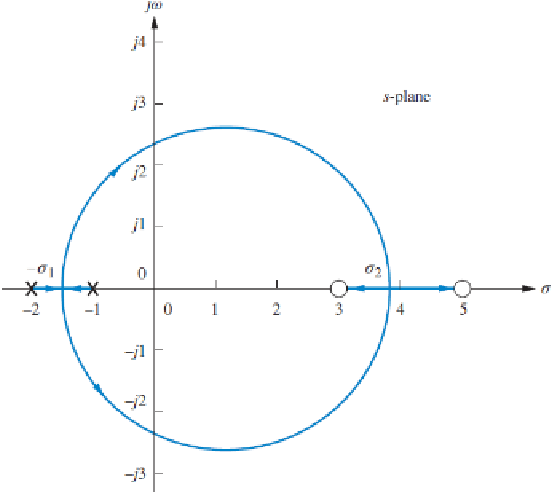
We can solve for K and obtain:

$$K = -\frac{\sigma^2 + 3\sigma + 2}{\sigma^2 - 8\sigma + 15}$$



Example of Pole Break-away / Break-in Location

b. The following figure shows the results of MATLAB simulation of breakaway and break-in points of the system.



Example of Pole Break-away / Break-in Location

c. Find local extrema in the K value by finding the derivative of the expression for K and setting it to zero.

$$K = -\frac{\sigma^2 + 3\sigma + 2}{\sigma^2 - 8\sigma + 15}$$

Differentiate the equation above ($d(u/v) = (u'v - uv')/v^2$):

$$\frac{dK}{d\sigma} = \frac{(2\sigma + 3)(\sigma^2 - 8\sigma + 15) - (\sigma^2 + 3\sigma + 2)(2\sigma - 8)}{(\sigma^2 - 8\sigma + 15)^2}$$

$$= -\frac{11\sigma^2 + 26\sigma + 61}{(\sigma^2 - 8\sigma + 15)^2}$$

Solve $11\sigma^2 + 26\sigma + 61 = 0$ using the quadratic equation to find the critical values of σ.

Example of Pole Break-away / Break-in Location

Knowing $\sigma_{1,2}$ = -1.45, 3.82. Thus, the breakaway point is at s = -1.45 and the break-in point is at s = 3.82.

d. Without the differentiation method, the *breakaway and break-in points,* $s = \sigma_{1,2}$, satisfy the relationship:

$$\sum_{i=1}^{Z} \frac{1}{\sigma_b - z_i} = \sum_{j=1}^{P} \frac{1}{\sigma_b - p_j}$$

Where: p_i and z_i are the pole and zero values of $G_C G_P$, where we have Z total zeros and P total poles.

Example of Pole Break-away / Break-in Location

For our example, we get:

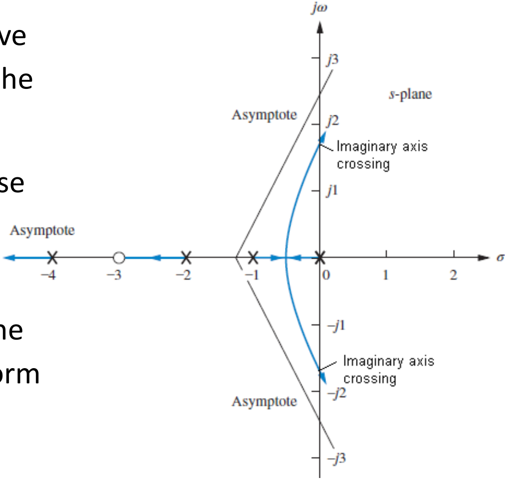
$$\frac{1}{\sigma_b - 3} + \frac{1}{\sigma_b - 5} = \frac{1}{\sigma_b + 1} + \frac{1}{\sigma_b + 2}$$

After some algebra, we get $11\sigma^2 + 26\sigma + 61 = 0$ as before.

We can complete the problem as earlier.

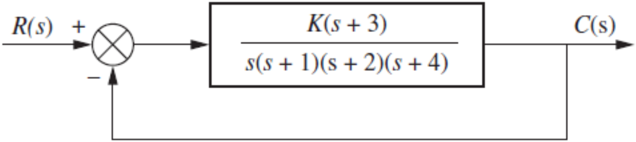
Imaginary Axis Crossings

- The gain K at which the root locus crosses the imaginary axis is the value at which the system becomes unstable.
- Stable systems cannot have poles in the right-half of the s-plane.
- There is no easy way to use the root locus to find this point.
- If you need to calculate the crossing point, then perform a Routh-Hurwitz test.



Example of Imaginary Axis Crossings

For the system given below, find the frequency and gain, K, for which the root locus crosses the imaginary axis. For what range of K is the system stable?



Example of Imaginary Axis Crossings

Answer

- The closed-loop transfer function for the system is:

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K}$$

- The Routh table is as shown below.

s^4	1	14	$3K$
s^3	7	$8 + K$	
s^2	$90 - K$	$21K$	
s^1	$\dfrac{-K^2 - 65K + 720}{90 - K}$		
s^0	$21K$		

Example of Imaginary Axis Crossings

- A complete row of zeros yields the possibility for imaginary axis roots.
- For positive values of gain, those for which the root locus is plotted, only the s^1 row can yield a row of zeros.
- Thus,

$$-K^2 - 65K + 720 = 0$$

- From this equation K is evaluated as:

$$K = \frac{-65 \pm \sqrt{(-65)^2 - 4(-1)(720)}}{2(-1)} = 9.65$$

Example of Imaginary Axis Crossings

- Forming the even polynomial by using the s^2 row with K = 9.65, we obtain:

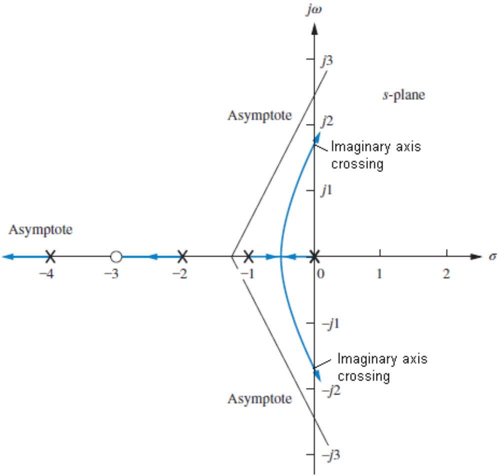
$$(90-K)s^2+21K=(90-9.65)s^2+21(9.65)$$

$$=80.35s^2+202.65$$

- Solving the equation, the s is found to be equal to $\pm j1.59$.

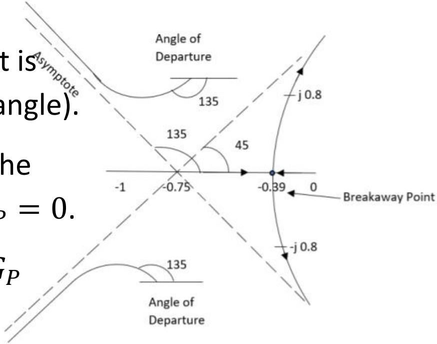
Example of Imaginary Axis Crossings

- Thus, the root locus crosses the $j\omega$-axis at $\pm j1.59$ at a gain K of 9.65.



- Also, we could conclude that the system is stable for $0 \le K < 9.65$.

Angles of Departure/Arrival at Complex Roots



- The angles when locus leaving or entering the complex roots (e.g. it is NOT the same as the asymptote angle).
- All points on a root locus satisfy the characteristic equation, $1 + G_C G_P = 0$.

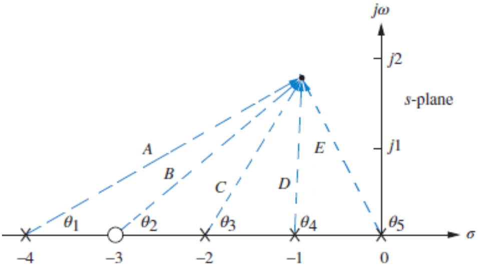
$$0 = 1 + G_C G_P \quad \Rightarrow \quad -1 = G_C G_P$$

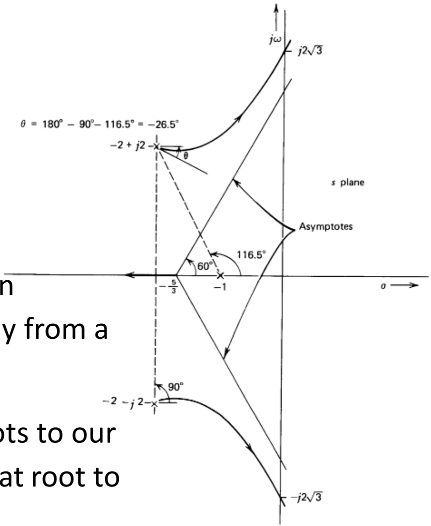
- Thus:

$$|G_C G_P| = 1 \qquad \text{and} \qquad \angle G_C G_P = -(2k+1)180^\circ$$

- Angle (phase) of $G_C G_P$ is the sum of the phases of the poles and zeros that make up $G_C G_P$.
- To be on the root locus, the sum of the angles to the closed loop roots is always -180°.

Angles of Departure/Arrival at Complex Roots



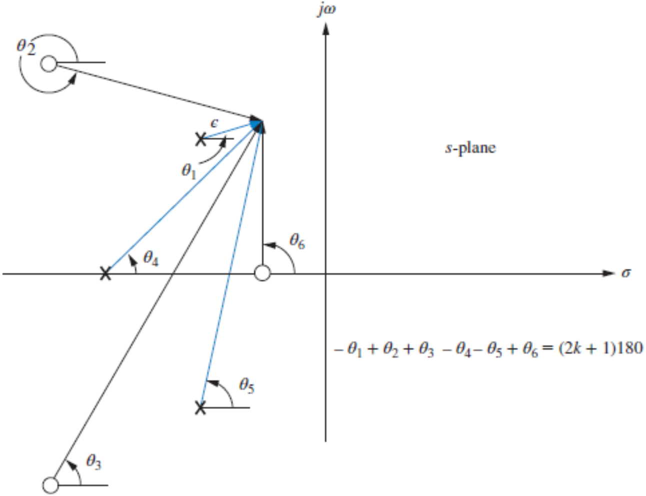


- Assume that we are on a point an arbitrarily small distance, ∈, away from a system root.
- Angle from each of the other roots to our test point is the same as from that root to the root of interest.
- Calculate these other angles directly e.g. we can find the angle from ∈ to the root of departure or arrival.

Angles of Departure/Arrival at Complex Roots

- Angle of departure at complex roots:

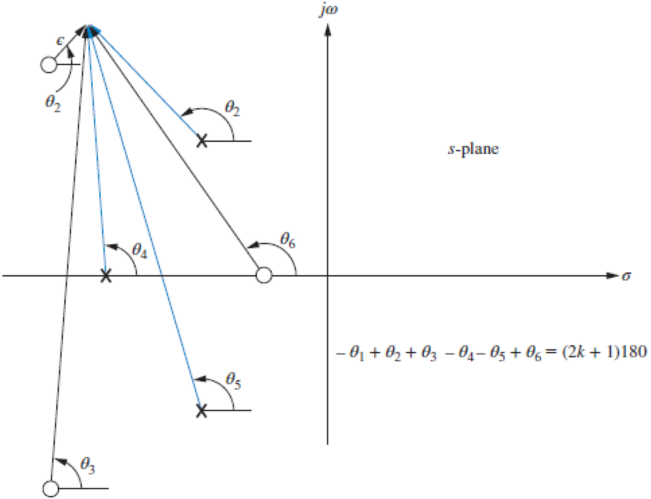
$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)180°$$



Angles of Departure/Arrival at Complex Roots

- Angle of arrival at complex roots:

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)180°$$



Example of Angles of Departure

Consider a system with a transfer-function equation:

$$G(s) = \frac{s+2}{(s+3)(s^2+2s+2)}$$

a. Determine the poles and zeros of the open-loop system.

[2 marks]

b. Sketch the angle of departure of the system. [6 marks]

c. Calculate the angle of departure from the pole at s = -1 + j.

[6 marks]

Example of Angles of Departure

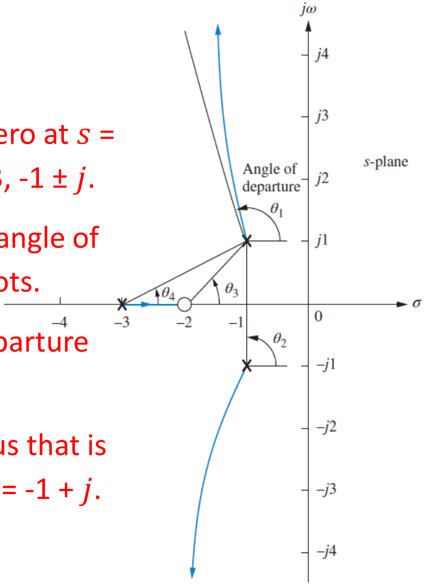
Answer

a. This system has an open-loop zero at s = -2 and open-loop poles at s = -3, -1 ± j.

b. The following figure shows the angle of departure from the complex roots.

c. Let us calculate the angle of departure from the pole at s = -1 + j.

   Assume a point on the root locus that is a small distance $\in$ away from s = -1 + j.



Example of Angles of Departure

- Equate all angles of the poles and zeros to -180°:

$$-180° = -\theta_1 - \theta_2 + \theta_3 - \theta_4$$

$$= -\theta_1 - 90° + 45° - \tan^{-1}\left(\frac{1}{2}\right)$$

- Thus, rearrange the above equation:

$$\theta_1 = 180° - 90° + 45° - 26° = 109°$$

- This is the required angle of departure from the open loop pole at s = -1 + j.
- Notice that zeros cause phase lead (so, we add their angle), while poles cause phase lag (so, we subtract their angle).