VICTORIA UNIVERSITY OF WELLINGTON
TE HERENGA WAKA

Introduction to Nyquist Diagram

XMUT315 Control System Engineering

Topics

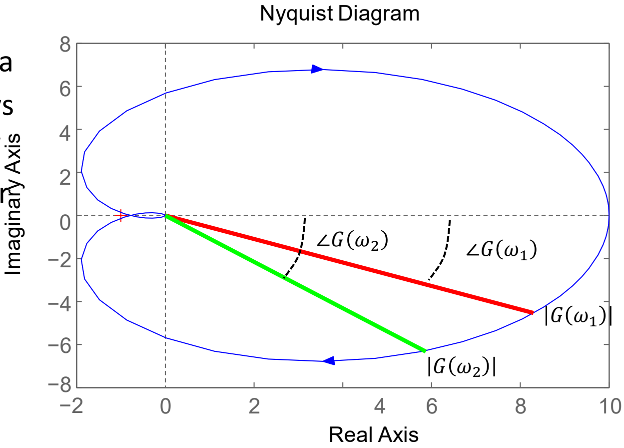
- Fundamentals of Nyquist diagram.
- From Bode to Nyquist diagram.
- Examples of Bode to Nyquist diagram.
- From Pole-Zero to Nyquist diagram.
- Examples of Pole-Zero to Nyquist diagram.

Nyquist Diagram

- We have been using the Bode and Root Locus plots of an open-loop system to determine stability.
- However, as we discussed earlier, this method is only reliable for simple cases.
- The Nyquist diagram provides a simple, universal method for assessing the stability of SISO systems.
- It works for simple systems that are manageable with a Bode plot, but also for more complicated systems.
- The root locus and Routh-Hurwitz techniques also provide methods for determining stability, but the Nyquist diagram has the advantage of being applicable when you do not have a mathematical description of your system (you can use it on experimental data).

The Nyquist Diagram

- A Nyquist diagram is a plot of the real part vs the imaginary part of an open-loop transfer function.
- Equivalently, you can think of it as a polar plot of a transfer function.



Frequency ω	**Magnitude $\lvert G(\omega)\rvert$**	**Phase $\angle G(\omega)$**
ω_1	$\lvert G(\omega_1)\rvert$	$\angle G(\omega_1)$
ω_2	$\lvert G(\omega_2)\rvert$	$\angle G(\omega_2)$

The Nyquist Diagram

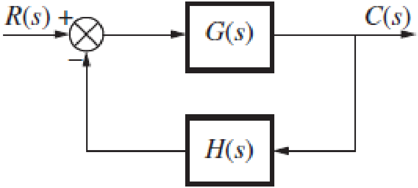
- The graph's axes are linear (not logarithmic), which makes the plot awkward for visualising the entire behaviour of a system that has high-gain regions.
- The Nyquist diagram is specialised for considering system stability, as it focuses on the low-gain region (i.e. the region near unity gain).
- We will examine three ways to construct a Nyquist diagram:
  - Based on a given root locus diagram.
  - Using a Bode plot or frequency response.
  - Directly on the diagram.

Stability from Nyquist Diagram

- Recall that a system is stable if and only if it has no poles in the right half of the s-plane.
- We seek a method that will tell us whether a system will be stable, once we enclose it in a feedback loop - we want to know about the stability of a closed-loop system.
- The Nyquist plot is a graphical technique that enables us to determine whether a system has closed-loop poles in the right half of the s-plane by examining its open-loop poles.
- In fact, it does more - it counts the number of closed-loop poles in the right-half plane.

From Root Locus to Nyquist

- For the given system, the Nyquist criterion can tell us how many closed-loop poles are in the right half-plane.



- Four important concepts that will be used during the derivation of criteria:

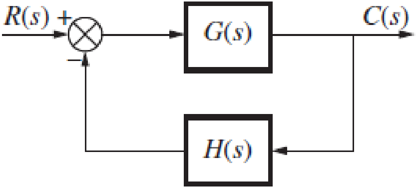
  a. The relationship between the poles of $1 + G(s)H(s)$ and the poles of $G(s)H(s)$.

  b. The relationship between the zeros of $1 + G(s)H(s)$ and the poles of the closed-loop transfer function, $T(s)$.

  c. The concept of mapping points.

  d. The concept of mapping contours.

From Root Locus to Nyquist



- Given the transfer functions of feedback control system:

$$G(s) = \frac{N_G}{D_G} \qquad \text{and} \qquad H(s) = \frac{N_H}{D_H}$$

- Thus

$$G(s)H(s) = \frac{N_G N_H}{D_G D_H}$$

From Root Locus to Nyquist

- The characteristic equation of the transfer function is:

$$1 + G(s)H(s) = 1 + \frac{N_G N_H}{D_G D_H} = \frac{D_G D_H + N_G N_H}{D_G D_H}$$

- As a result, the transfer function of closed loop system is:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_G D_H}{D_G D_H + N_G N_H}$$

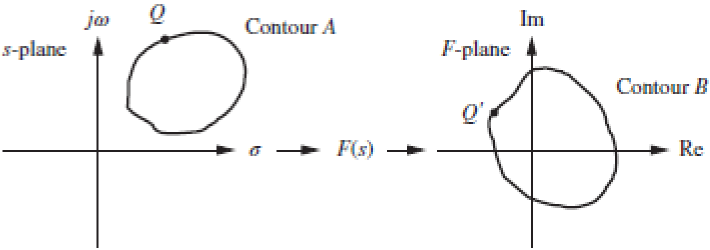
- We conclude that:

1. Poles of $1 + G(s)H(s)$ are the same as the poles of $G(s)H(s)$, the open-loop system.
2. Zeros of $1 + G(s)H(s)$ are the same as the poles of $T(s)$, the closed-loop system.

From Root Locus to Nyquist

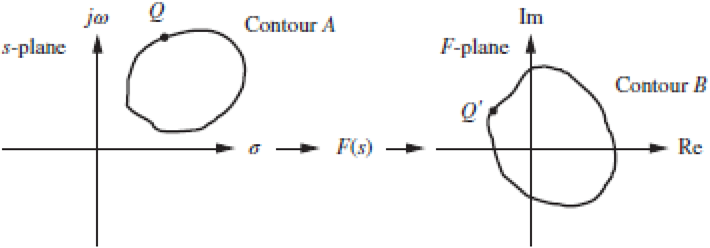
- Finally, we discuss the concept of mapping contours.
- Consider the collection of points, called a contour, shown in the figure below as contour A.
- Also, assume that:

$$F(s) = \frac{(s - z_1)(s - z_2)\ldots}{(s - p_1)(s - p_2)\ldots}$$



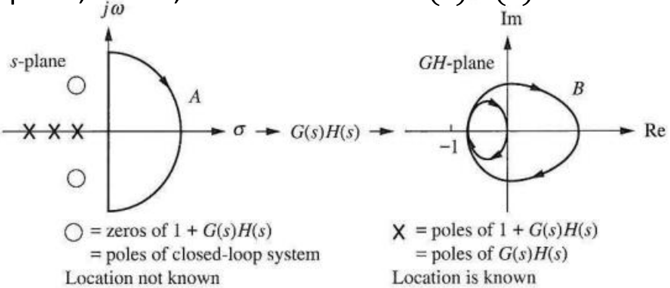
From Root Locus to Nyquist

- Contour A can be mapped through $F(s)$ into contour B by substituting each point of contour A into the function $F(s)$ and plotting the resulting complex numbers.
- For example, point Q in the figure above maps into point Q' through the function $F(s)$.



From Root Locus to Nyquist

- Figure below shows a contour A that does not enclose closed-loop poles, that is, the zeros of $1 + G(s)H(s)$.

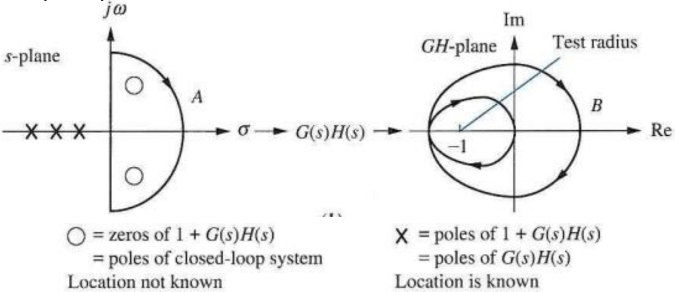


- The contour thus maps through $G(s)H(s)$ into a Nyquist diagram that does not encircle the test point $(-1, 0)$. Hence, $P = 0$; $N = 0$, and $Z = P - N = 0$.

- Since Z is the number of closed-loop poles inside contour A, which encircles the right half-plane, this system has no right-half-plane poles and is stable.

From Root Locus to Nyquist

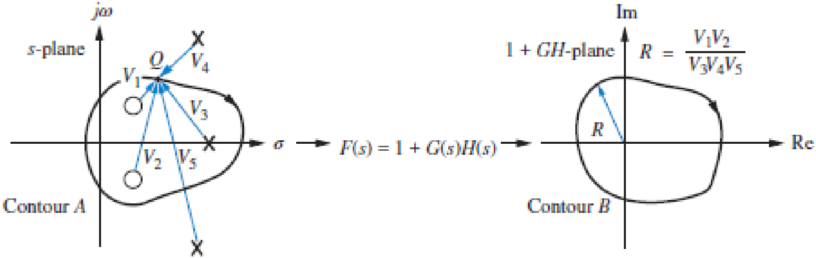
- Figure below shows a contour A that, while it does not enclose open-loop poles, does generate two clockwise encirclements of the test point (-1,0).



- Thus, $P = 0$; $N = 2$, and the system is unstable; it has two closed-loop poles in the right half-plane, since $Z = P - N = 2$.

- The two closed-loop poles are shown inside contour A in the figure above as zeros of $1 + G(s)H(s)$.

From Root Locus to Nyquist



Thus, the number of counterclockwise rotations of contour B about the origin is:

$$N = P - Z$$

Where:

P - Number of poles of $1 + G(s)H(s)$ inside contour A.

Z - Number of zeros of $1 + G(s)H(s)$ inside contour A.

From Root Locus to Nyquist

Nyquist stability criterion is as follows:

If a contour, A, that encircles the entire right half-plane is mapped through $\mathrm{G}(\mathrm{s})\mathrm{H}(\mathrm{s})$, then the number of closed-loop poles, Z, in the right half-plane equals the number of open-loop poles, P, that are in the right half-plane minus the number of counterclockwise revolutions, N, around the test point (1,0) of the mapping.

That is, the number of zero is:

$$Z = P - N$$

The mapping is called the Nyquist diagram, or Nyquist plot, of $G(s)H(s)$.

From Root Locus to Nyquist Example

Given a first-order system with its transfer function equation:

$$G(s) = \frac{1}{s + 0.1}$$

a. Derive the real and imaginary equations needed for sketching the Nyquist diagram. [4 marks]

b. Using equations derived in part (a), calculate the points required for sketching the Nyquist diagram. [4 marks]

c. Sketch the Nyquist diagram of the system. [6 marks]

d. Simulate the root locus diagram of the system in MATLAB. By determining and obtaining values of the points in the diagram required for sketching Nyquist diagram, convert the root locus diagram to Nyquist diagram. [12 marks]

From Root Locus to Nyquist Example

Answer

a. Substituting $s = j\omega$, the transfer function equation of the system becomes:

$$G(j\omega) = \frac{1}{j\omega + 0.1} = \left(\frac{1}{j\omega + 0.1}\right)\left(\frac{j\omega - 0.1}{j\omega - 0.1}\right) = -\left(\frac{j\omega - 0.1}{\omega^2 + 0.01}\right)$$

For sketching the Nyquist diagram, we need the following equations for determining the points in the Nyquist diagram:

The real part and imaginary part of the complex equation:

$$\mathrm{Re}\{G(j\omega)\} = \frac{0.1}{\omega^2 + 0.01} \qquad \mathrm{Im}\{G(j\omega)\} = -\left(\frac{j\omega}{\omega^2 + 0.01}\right)$$

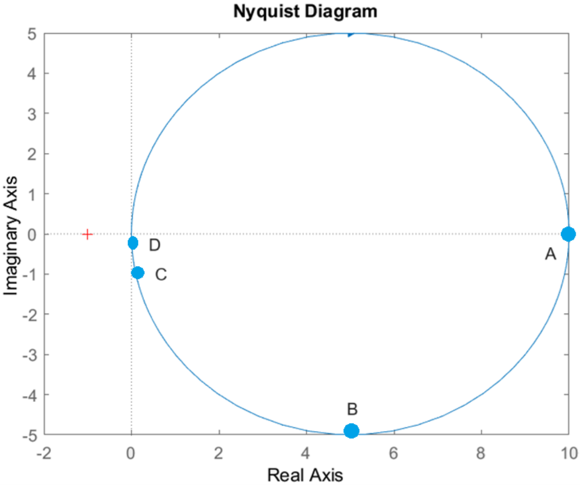
From Root Locus to Nyquist Example

b. Choose the frequencies from 0 to +∞ along the y-axis (imaginary axis) e.g. A = 0 rad/s, B = 0.1 rad/s, C = 1 rad/s, and D = 10 rad/s. The points for sketching the Nyquist diagram are calculated and tabulated in the following table.

Points	ω	$\mathbf{Re}\{G(j\omega)\}$	$\mathbf{Im}\{G(j\omega)\}$
A	0	$\frac{0.1}{(0)^2+0.01} = 10$	$-\left(\frac{j(0)}{(0)^2+0.01}\right) = 0$
B	0.1	$\frac{0.1}{(0.1)^2+0.01} = 5$	$-\left(\frac{j(0.1)}{(0.1)^2+0.01}\right) = -5$
C	1	$\frac{0.1}{(1)^2+0.01} = 0.099$	$-\left(\frac{j(1)}{(1)^2+0.01}\right) = -0.99$
D	10	$\frac{0.1}{(10)^2+0.01} = 0.001$	$-\left(\frac{j(10)}{(10)^2+0.01}\right) = -0.099$

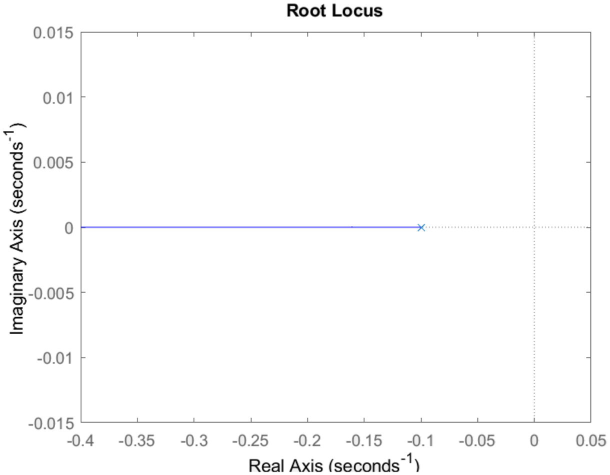
From Root Locus to Nyquist Example

c. Based on the points listed in the table in part (b), the following diagram shows the sketched Nyquist diagram.



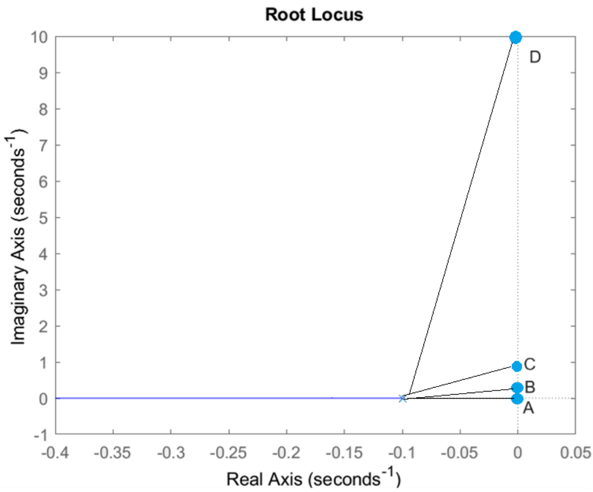
From Root Locus to Nyquist Example

d. The result of root locus diagram simulation of the system in MATLAB is shown in the figure below.



From Root Locus to Nyquist Example

Choose the frequencies from 0 to +∞ along the y-axis (imaginary axis) e.g. A = 0j, B = 0.1j, C = 1j, and D = 10j.



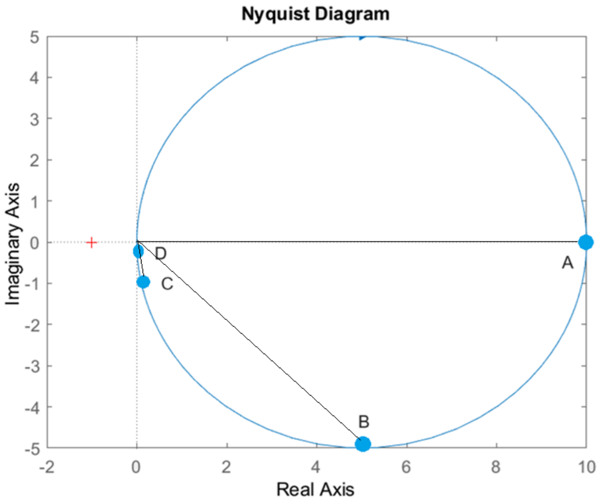
From Root Locus to Nyquist Example

Calculate the magnitudes and angles formed by the zero at (-1, 0) with the chosen points in the diagram.

Points	$\boldsymbol{\omega}$	$\vert \mathbf{G}(j\boldsymbol{\omega})\vert$	$\angle \mathbf{G}(j\boldsymbol{\omega})$
A	0	$\frac{1}{\sqrt{(0.1)^2+(0)^2}}=10$	$-\tan(0/0.1)=0°$
B	0.1	$\frac{1}{\sqrt{(0.1)^2+(0.1)^2}}=7.07$	$-\tan\left(\frac{0.1}{0.1}\right)=-45°$
C	1	$\frac{1}{\sqrt{(0.1)^2+(1)^2}}=0.99$	$-\tan\left(\frac{1}{0.1}\right)=-84.29°$
D	10	$\frac{1}{\sqrt{(0.1)^2+(10)^2}}=0.01$	$-\tan(10/0.1)=-89.43°$

From Root Locus to Nyquist Example

Sketch the Nyquist diagram based on the magnitudes and angles obtained above.

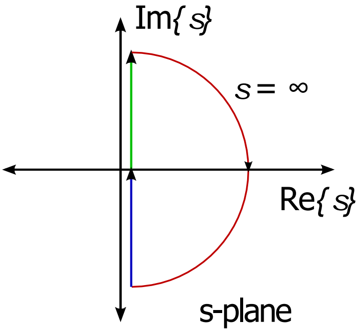


From Bode to Nyquist

- The curve on a Nyquist diagram can be determined by choosing gain-phase points from a Bode plot at multiple frequencies.
- Thus, the Nyquist diagram is the locus traced out by the transfer function as we vary frequency.
- Frequency thus varies along the Nyquist curve, but not in a regular way. There is no way to use a Nyquist diagram to determine the frequency at which something happens.
- To construct the Nyquist diagram, choose the points where something "interesting" happens on the Bode plot and transfer them to the Nyquist diagram.
- Join the points with sensible curves. This produces a plot of gain vs. phase as the frequency varies from 0 to ∞.

From Bode to Nyquist

- Drawing a Nyquist diagram is slightly more complex than plotting the gain vs phase curve for $f = 0 \rightarrow \infty$.
- Nyquist's criterion (see later) actually requires that we evaluate the transfer function as we traverse a clockwise path that completely encloses the right half of the s-plane.
- Evaluating the transfer function for $f = 0 \rightarrow \infty$ corresponds to traversing the upper straight part of the semicircle shown in the diagram.
- As the pole-zero diagram must be symmetric we know that the Nyquist diagram must be symmetric in the section from $f = -\infty \rightarrow 0$.



From Bode to Nyquist

- To draw the complete Nyquist diagram, we need to add to the plot that you produced by transferring data from the Bode plot.
- The section from -∞ to 0 is straightforward, as it is just the mirror image of the 0 to ∞ section that you have already drawn.
- Most transfer functions that you will meet have the degree of the denominator larger than the numerator (they are strictly proper).
- A consequence of this is that the gain is infinitesimally small at infinite frequency.
- Thus, the response for the circular part of the contour from $\omega = +\infty$ to $\omega = -\infty$ is always zero.
- The whole sweep is therefore mapped to the origin of the Nyquist diagram.

From Bode to Nyquist Example

Consider a first-order system with the transfer function:

$$G(s) = \frac{100}{s + 10}$$

a. Simulate the Bode plots of the system the in MATLAB.

[5 marks]

b. Determine the gain and phase of the frequency response of the system from the Bode plots for ω = 1, 10, and 100 rad/s.

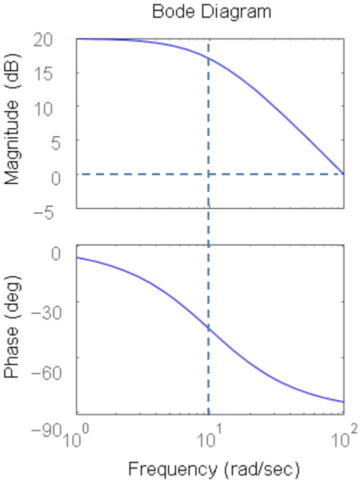
[6 marks]

c. Based on the results obtained in part (b), construct Nyquist diagram of the system. [5 marks]

From Bode to Nyquist Example

- Consider the first-order system with the transfer function:

$$G(s) = \frac{100}{s+10}$$



ω	$\lvert G(\omega)\rvert$	$\angle G(\omega)$
1	20 dB (10)	0°
10	17 dB (7)	-45°
100	0 dB (1)	−90°

From Bode to Nyquist Example

- Consider a first-order system with the transfer function:

$$G(s) = \frac{100}{s+10}$$

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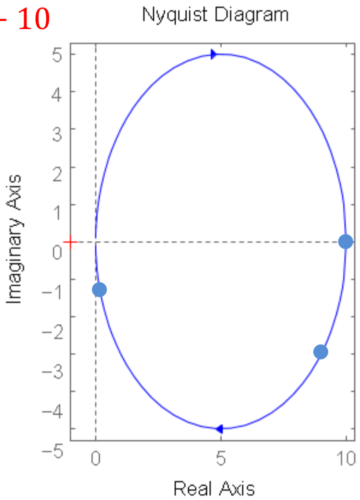


From Bode to Nyquist Example

Consider a second-order system with the transfer function:

$$G(s) = \frac{100}{s^2 + 2s + 26}$$

a. Simulate the Bode plots of the system the in MATLAB.

[5 marks]

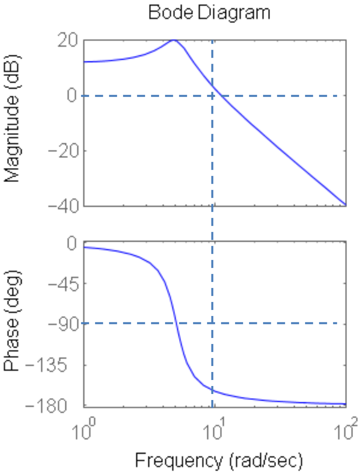
b. Determine the gain and phase of the frequency response of the system from the Bode plots for ω = 1, 5, 10, 50, and 100 rad/s. [10 marks]

c. Based on the results obtained in part (b), construct Nyquist diagram of the system. [5 marks]

From Bode to Nyquist Example

- Consider a second-order system with the transfer function:

$$G(s) = \frac{100}{s^2 + 2s + 26}$$



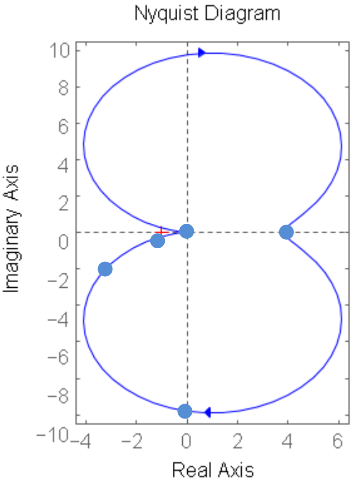
ω	$\lvert G(\omega)\rvert$	$\angle G(\omega)$
1	12 dB (4)	5°
5	20 dB (10)	−90°
10	5 dB (1.77)	-165°
50	-30 dB (0.03)	-175°
100	-40 dB (0.01)	-180°

From Bode to Nyquist Example

- Consider a second-order system with the transfer function:

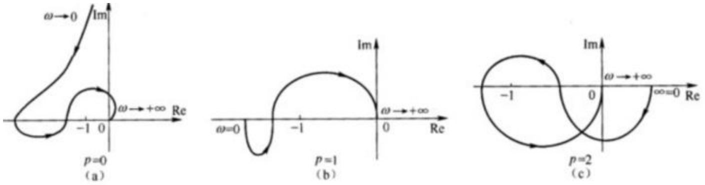
$$G(s) = \frac{100}{s^2 + 2s + 26}$$

ω	$\lvert G(\omega) \rvert$	$\angle G(\omega)$
1	12 dB (4)	0°
5	20 dB (10)	-90°
10	5 dB (1.77)	−165°
50	-30 dB (0.03)	-175°
100	-40 dB (0.01)	180°



Nyquist Diagram Analysis

- A system is stable if there is no encirclement of the curve around the test point (-1,0) in the Nyquist criterion.



- The number of encirclement correspond to unstable pole and zero in the closed-loop system:

$$N = P - Z$$

- More precise stability condition of the system is determined through gain and phase margin in the Nyquist diagram (e.g. we will discuss about this topic later).

Nyquist Diagram Analysis Example

Apply Nyquist criterion to determine the stability of the following feedback systems:

a. System (i) [5 marks]

$$G_1(s) = \frac{s+20}{(s+2)(s+7)(s+50)}$$

b. System (ii) [5 marks]

$$G_2(s) = \frac{s+3}{(s+2)(s^2+2s+25)}$$

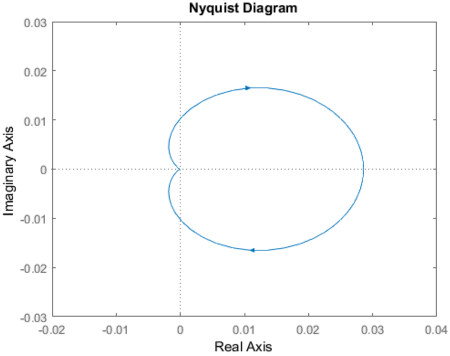
c. System (iii) [5 marks]

$$G_3(s) = \frac{500(s-2)}{(s+2)(s+7)(s+50)}$$

Nyquist Diagram Analysis Example

For system (i):

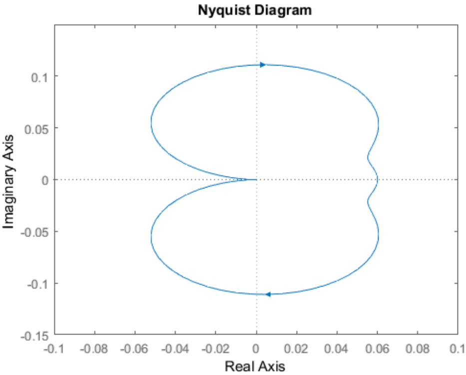
- We have $P = 0$ (open loop stable system).
- The Nyquist diagram does not enclose (-1, j0), ($N = 0$)
- Thus, $Z = P - N = 0$.
- Systems (i) is stable since there are no closed loop poles in the right half plane.



Nyquist Diagram Analysis Example

For system (ii):

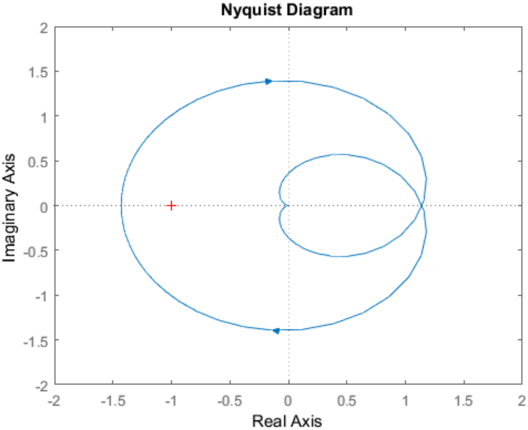
- We have $P = 0$ (open loop stable system).
- The Nyquist diagram does not enclose (-1, j0), ($N = 0$)
- Thus, $Z = P - N = 0$.
- Systems (ii) is stable since there are no closed loop poles in the right half plane.



Nyquist Diagram Analysis Example

For system (iii):

- We have P = 0 (open-loop stable system), but N = -1.
- System (iii) is unstable with one closed-loop pole in the right-half plane.

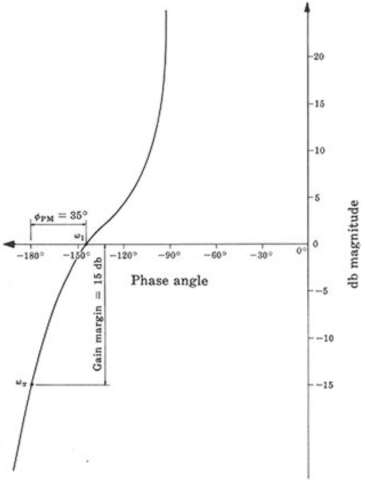


Nichols Chart

- A Nichols chart displays the magnitude (in dB) plotted against the phase (in degrees) of the system response.

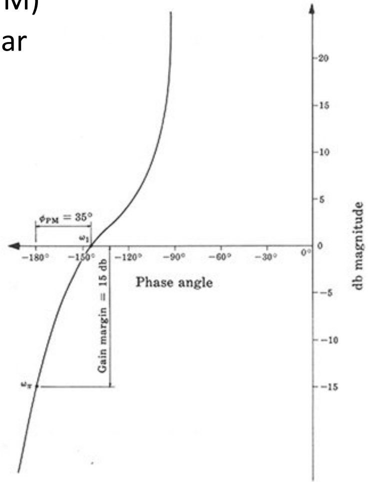
- Nichols charts are useful to analyse open- and closed-loop properties of SISO systems but offer little insight into MIMO control loops.

- Stability of the system is by evaluating the contour around the test point (-180°, 0) - encircles the test point or not



Nichols Chart

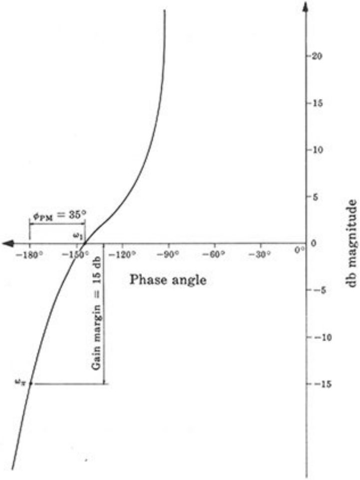
- Gain margin (GM) and phase margin (PM) of the system in the Nichols chart similar to the steps in Nyquist diagram.
- The phase margin is phase difference from the test point to the intersection between the contour with x-axis. The example is GM of 35°.
- The gain margin is gain difference between the gain at the point of the contour at 180° and the x-axis. The example gain margin is 15 dB.



Nyquist Diagram vs. Nichols Chart

Pitfalls of the Nyquist diagram:

- Becomes messy for systems with multiple crossover frequencies.
- Crossover region is imperceptible for systems with large resonant peaks.
- Lacks system composition (superposition) properties of Bode plots.

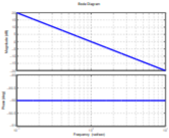
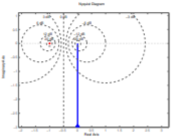
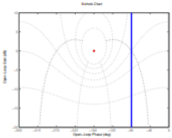
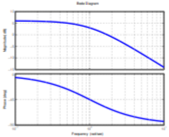
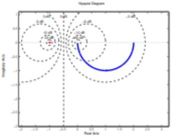
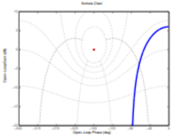
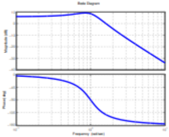
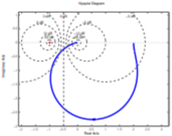
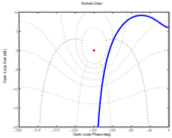
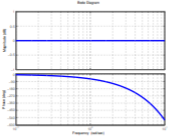
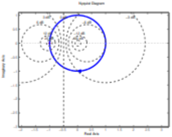
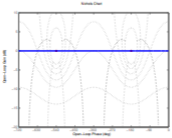


Nyquist Diagram vs. Nichols Chart

Since phase scale is linear rather than polar:

- Nichols chart is typically cleaner than Nyquist diagram especially for systems with large phase lags, like time-delay systems.
- As magnitude scale is in dB, regions with large magnitude don't dominate, hence the crossover region is more visible.
- Also, the consequence of the logarithmic scale of $|\log(j\omega)|$ is that multiplication of systems results in superposition on Nichols chart, almost as easy as on the Bode plots.

Nyquist Diagram vs. Nichols Chart

$L(s)$	Bode	Nyquist	Nichols
$\frac{k}{s}$			
$\frac{k}{\tau s+1}$			
$\frac{k\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$			
e^{-sh}			

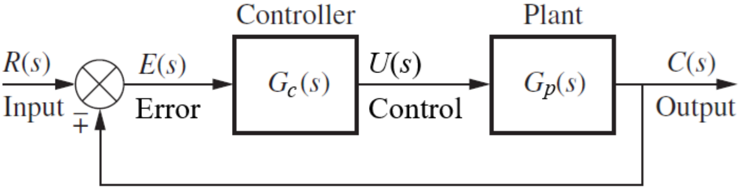
Example of Nichols Chart

- For example, on the following page, we see Bode plots and Nichols charts for an uncompensated system ($G(s)$) and compensated system ($G_P(s)G_C(s)$) with plant is given as:

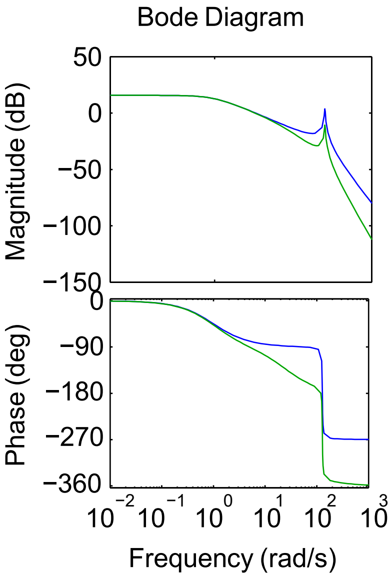
$$G_P(s) = \frac{10^5}{(s+1)(s^2+4s+1.639\times 10^4)}$$

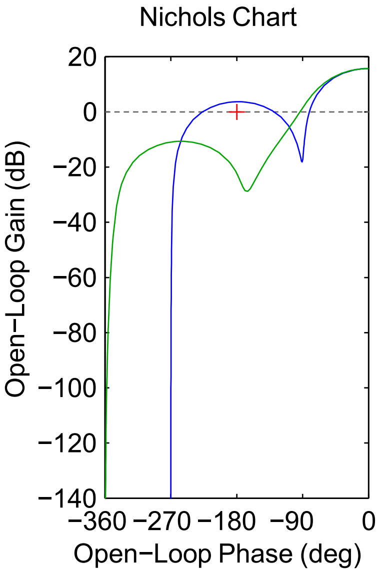
And controller is:

$$G_C(s) = \frac{25}{s+25}$$



Example of Nichols Chart



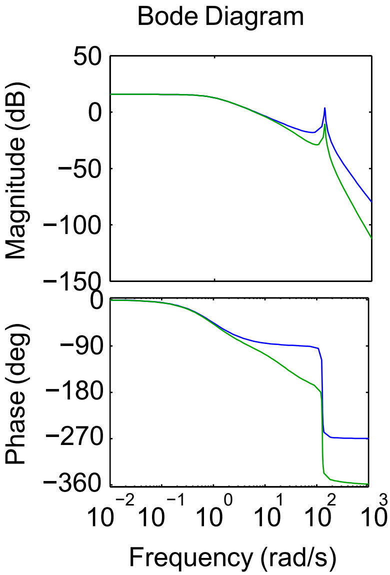


Note: blue line is uncompensated system, and green line is compensated system.

Example of Nichols Chart

a. For the Bode diagram, notice that the gain margin (GM) and phase margin (PM) of the uncompensated system (blue line) are smaller compared with those of the compensated system (green line).

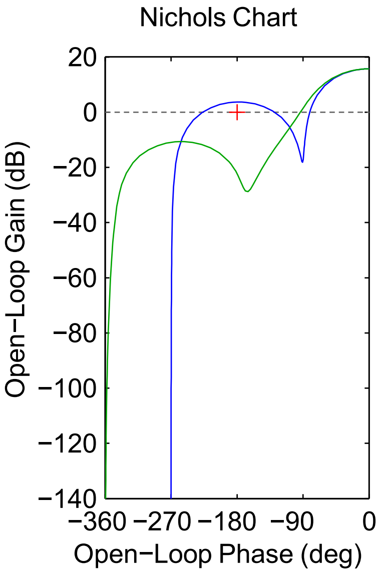
We need to determine the exact values of GM and PM of the system to find out if it is stable or not



Example of Nichols Chart

For the Nichols chart, notice that the contour of the uncompensated system (blue line) encircles the test point (-180°, 0), but the contour of the compensated system is underneath the test point.

From these results, compensated system is stable whereas uncompensated system is unstable.



Example of Nichols Chart

b. From the results of part (b), it seems that Nyquist diagram is easier to use than Bode plots for analysing stability of the system.

We need only to evaluate the contour in the Nyquist diagram whether it is encircling the test point (-180°, 0) to determine if the system is stable or not.

On the other hand, we need to find out the values of GM and PM of the system using Bode plots.