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Laplace Transforms

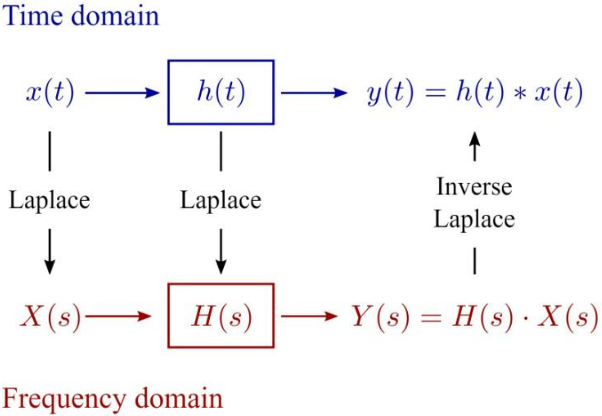
XMUT315 Control System Engineering

Topics

- Differential equations and Laplace transforms.
- Transfer functions, poles and modes, and zeros.
- Modal decomposition and expansion method.
- Cover up (Heaviside) method.
- Complex factors.
- Repeated factors.
- Partial fractions.
- S-plane and final value theorem.

Solving DEs with the Laplace Transform

- The Laplace transform is useful because it allows us to convert linear, constant-coefficient differential equations into algebraic equations.



Solving DEs with the Laplace Transform

- This results from the differentiation in time property of the Laplace transform.

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0^-)$$

$$\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0^-) - y'(0^-)$$

$$\ldots$$

$$\mathcal{L}\left\{y^{(n)}(t)\right\} = s^n\mathcal{L}\{y(t)\} - s^{(n-1)}y(0^-) \ldots y^{(n-1)}(0^-)$$

- Recall that $y(0^-)$,$y^j(0^-)$ and so forth are initial conditions.

- For an n-th order DE, we need to know the initial values of the first n derivatives to solve a differential equation uniquely using the Laplace transform.

Example of Solving DEs

Find $y(t)$ in a system described by the differential equation:

$$y''(t) + 4y'(t) + 3y(t) = 0$$

With initial conditions:

$$y(0) = 3, y'(0) = 1$$

[10 marks]

Example of Solving DEs

- We start by taking the Laplace transform of the entire differential equation.
- Using the differentiation in time formula, we can write the transforms of each of the derivatives of y.

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$
$$= sY(s) - 3$$

- And

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$
$$= s^2Y(s) - 3s - 1$$

Example of Solving DEs

- We can therefore write the complete Laplace transform.

$$(s^2Y(s) - 3s - 1) + 4(sY(s) - 3) + 3Y(s) = 0$$

$$(s^2 + 4s + 3)Y(s) = 3s + 13$$

- Rearrange the equation and factorise roots:

$$Y(s) = \frac{3s + 13}{(s^2 + 4s + 3)} = \frac{3s + 13}{(s + 1)(s + 3)}$$

- Apply partial fraction expansion to simplify the form of the equation:

$$Y(s) = \frac{5}{(s + 1)} + \frac{-2}{(s + 3)}$$

Example of Solving DEs

- We have solved our DE by Laplace transforming it, solving an algebraic equation.
- Then, using inverse Laplace transform, transform the equation back to the time domain (see table of Laplace transform).

$$y(t) = (5e^{-t} - 2e^{-3t})u(t)$$

The Transfer Function

- When characterising a system, we are interested in what the system does to an arbitrary input signal.
- We typically assume that any initial transients have been given time to die away, which is equivalent to assuming zero initial conditions.

$$y''(t) + y'(t) + y(t) = x(t)$$

- Take Laplace transform of the equation

$$s^2Y(s) + sY(s) + Y(s) = X(s)$$

- Rearrange the equation

$$(s^2 + s + 1)Y(s) = X(s)$$

The Transfer Function

- Rearrange the equation in terms of ratio of the parameters that we are interested e.g. $Y(s)$ and $X(s)$.

$$G(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + s + 1}$$

- This is the so-called transfer function (TF) -> it tells us what the system does to an arbitrary input, $X(s)$.

Example of The Transfer Function

Find the transfer function of the following differential equation:

$$y''(t) + 4y'(t) + 3y(t) = 0$$

[4 marks]

Example of The Transfer Function

- Take Laplace transform of the equation

$$s^2Y(s) + \ 4sY(s) + 3Y(s) = X(s)$$

- Gather all coefficients of $Y(s)$ to the left and the rest of other coefficients to the right.

$$(s^2+4s + 3)Y(s) = X(s)$$

- Form the equation of $Y(s)/X(s)$:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 4s + 3}$$

Poles and Modes

- The poles of the transfer function are important (e.g. the values of s that make the denominator of the TF zero), as they allow us to find the modes of the system.
- The modes are simply the characteristic responses that the system will exhibit when excited by a signal, or by initial conditions.

$$G(s) = \frac{1}{(s+a)(s+b)}$$

- Poles at $s = -a$ and $s = -b$. The modes will be e^{-at} and e^{-bt}

$$y(t) = [Ae^{-at} + Be^{-bt}]u(t)$$

- Where: A and B depend on the input and the initial conditions.

Example of Poles and Modes

Find the mode of the characteristic response of the system as given below. [4 marks]

$$G(s) = \frac{1}{s^2 + 4s + 3}$$

Example of Poles and Modes

- For the given system, factorise its transfer function equation as shown below:

$$G(s) = \frac{1}{s^2 + 4s + 3} = \frac{1}{(s+1)(s+3)}$$

- With the given system, the poles at $s = -1$ and $s = -3$.
- The modes will be e^{-t} and e^{-3t} which are exponential responses.

$$g(t) = [Ae^{-t} + Be^{-3t}]u(t)$$

Where: A and B depend on the input and the initial conditions.

Zeros

- In general, we can also have a polynomial of s in the numerator of the transfer function.
- The values of s that make the numerator zero are called *zeros* of the transfer function.
- The system will exhibit no output when driven by a signal having these values of s.
- The zeros do not produce modes, but they play an important role in setting the relative magnitude of the various modes.
- In particular, a pole that has an s value that is close to that of a zero will have a “small” mode.

Modal Decomposition

- We use partial fraction expansion to simplify expressions in the *s*-domain.
- We must begin with a proper rational polynomial, otherwise the form of the system modes will be obscured.

Not a rational polynomial:

$$Y_1(s) = \frac{\frac{1}{s} + 1}{(s+2)(s+3)}$$

Not strictly proper:

$$Y_2(s) = \frac{s^2}{(s+2)(s+3)}$$

Example of Modal Decomposition

Decompose the following functions into its simplest forms.

[4 marks]

- System 1:

$$Y_1(s) = \frac{(1/s) + 1}{(s+2)(s+3)}$$

- System 2:

$$Y_2(s) = \frac{s^2}{(s+2)(s+3)}$$

Example of Modal Decomposition

- If the transfer function of the system is not a rational polynomial, perform modal decomposition of the equation as shown below:

$$Y(s) = \frac{(1/s) + 1}{(s+2)(s+3)}$$

$$= \left(\frac{s}{s}\right)\frac{(1/s + 1)}{(s+2)(s+3)}$$

$$= \frac{(s+1)}{s(s+2)(s+3)}$$

Example of Modal Decomposition

- If the transfer function of the system is not strictly proper, we can simplify the equation as below:

$$Y(s) = \frac{s^2}{(s+2)(s+3)}$$

$$= \frac{s^2+5s+6}{s^2+5s+6} - \frac{5s+6}{s^2+5s+6}$$

$$= 1 - \frac{5s+6}{(s+2)(s+3)}$$

Modal Expansion

- We then write our expression as the sum of a set of appropriate terms, each of which corresponds to a particular mode and has an unknown amplitude.

- We write the denominator as a combination of three types of poles:
  - Simple real poles: $(s + a) \Leftrightarrow Ae^{-at}$
  - Complex pole pairs: $(s + a)^2 + \omega_d^2 \Leftrightarrow Be^{-at}\cos(\omega_d t + \phi)$

    or

    Control systems: $s^2 + 2\zeta\omega_n s + \omega_n^2 \Leftrightarrow Be^{-\zeta t}\cos(\omega_n t + \phi)$
  - Repeated real poles: $(s + a)^n \Leftrightarrow C_n t^{n-1} e^{-at} + \ \ldots$

Modal Expansion

- You only need to use partial fraction expansion when you need to write an equation for the output of a system.
- If you need to know the amplitudes of the modes, then use partial fractions.
- If you don’t need the amplitudes, just wish to find out the mode of the system, then stop!

Simple Real Poles

- For real poles:

$$Y(s) = \frac{n(s)}{(s+a)d(s)} = Y_1(s) + \frac{A}{s+a}$$

- Thus

$$y(t) = Y_1(t) + Ae^{-at}u(t)$$

- We can solve this system with simple real poles using ordinary partial-fraction expansion method or using Heaviside or cover-up method.

Example of Simple Real Poles

Find the time domain equation of a system described by the transfer function equation below using the ordinary partial fraction expansion method. [8 marks]

$$Y(s) = \frac{s+1}{s^3 + s^2 - 6s}$$

Example of Simple Real Poles

- For the given system, factorise the transfer function equation.

$$Y(s) = \frac{s+1}{s^3 + s^2 - 6s}$$

$$= \frac{s+1}{s(s^2 + s - 6)}$$

$$= \frac{s+1}{s(s-2)(s+3)}$$

- Then, perform partial fraction expansion.

$$Y(s) = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

Example of Simple Real Poles

- The first method we will use is multiplying out the new form of the equation and equating it with the original form.

$$\frac{s+1}{s^3+s^2-6s} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

$$= \frac{A(s-2)(s+3) + Bs(s+3) + Cs(s-2)}{s^3+s^2-6s}$$

$$= \frac{(A+B+C)s^2 + (A+3B-2C)s - 6A}{s^3+s^2-6s}$$

- The denominators of these expressions are identical, so the numerators must be equivalent.

Example of Simple Real Poles

- We therefore equate coefficients of the various powers of s in the numerator polynomials of the two sides.

$$s^2:\ 0 = A + B + C$$

$$s^1:\ 1 = A + 3B - 2C$$

$$s^0:\ 1 = -6A$$

- From the last of these equations, we know that $A = -1/6$.
- Substituting into the other two equations we find:

$$B + C = \frac{1}{6} \qquad \text{and} \qquad 3B - 2C = \frac{7}{6}$$

- Solving these two equations simultaneously we find:

$$B = \frac{3}{10}, C = -\frac{2}{15}$$

Example of Simple Real Poles

- So,

$$Y(s) = \frac{s+1}{s(s^2+s-6)} = -\frac{1}{6}\left(\frac{1}{s}\right) + \frac{3}{10}\left(\frac{1}{s-2}\right) - \frac{2}{15}\left(\frac{1}{s+3}\right)$$

- Taking the inverse Laplace transform (from the table), we therefore find:

$$y(t) = \left[-\frac{1}{6} + \frac{3}{10}e^{2t} - \frac{2}{15}e^{-3t}\right]u(t)$$

The Heaviside or Cover-up Method

- There is a quicker method for finding the partial fraction expansion, known as the *Heaviside* or *cover-up* method.
- "Cover" the term in the denominator for which you are trying to find the coefficient and then calculate the value of the remaining fraction at the value that would cause the covered term to be zero.
- Say for example that you have the following function to be decomposed into partial fractions:

$$\frac{x-7}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

The Heaviside or Cover-up Method

- On the left-hand side, we mentally remove (or cover up with a finger) the factor $x - 1$ associated with A, and substitute $x = 1$ into what's left; this gives A:

$$\left.\frac{x-7}{x+2}\right|_{x=1} = \frac{1-7}{1+2} = -2 = A$$

- Similarly, B is found by covering up the factor $x + 2$ on the left and substituting $x = -2$ into what's left. This gives:

$$\left.\frac{x-7}{x-1}\right|_{x=-2} = \frac{-2-7}{-2-1} = 3 = B$$

- Thus, the partial fraction of the function is:

$$\frac{x-7}{(x-1)(x+2)} = \frac{-2}{x-1} + \frac{3}{x+2}$$

Example of The Heaviside or Cover-up Method

Find the time domain equation of a control system given as the following transfer function equation below using the cover-up method. [5 marks]

$$Y(s) = \frac{s+1}{s^3 + s^2 - 6s}$$

Example of The Heaviside or Cover-up Method

- For the given system, factorise the transfer function equation.

$$\frac{s+1}{s^3+s^2-6s} = \frac{s+1}{s(s-2)(s+3)}$$

- Then, perform partial fraction expansion using cover-up method.

$$Y(s) = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

- The coefficients A, B, and C in the equation above are calculated from:

$$A = \left.\frac{s+1}{(s-2)(s+3)}\right|_{s=0} = -\frac{1}{6}$$

Example of The Heaviside or Cover-up Method

$$B = \left.\frac{s+1}{s\,(s+3)}\right|_{s=2} = \frac{3}{2(2+3)} = \frac{3}{10}$$

$$C = \left.\frac{s+1}{s(s-2)}\right|_{s=-3} = \frac{-2}{-3(-3-2)} = -\frac{2}{15}$$

- Thus

$$\frac{s+1}{s^3+s^2-6s} = -\frac{1}{6s} + \frac{3}{10(s-2)} - \frac{2}{15(s+3)}$$

Unique Complex Factors

- For complex pole pair:

$$Y(s) = \frac{n(s)}{[(s+a)^2+\omega^2]d(s)} = Y_1(s) + \frac{As+B}{(s+a)^2+\omega^2}$$

- Thus

$$y(t) = y_1(t) + [Ae^{-at}\cos(\omega t+\phi)]u(t)$$

Where: ϕ depends on A and B.

- The coefficients of complex factors must be found by the cross-multiplication method. Find the residuals of other factors first.

Example of Unique Complex Factors

For a given system described by the following transfer function equation with a pair of complex factors, find its time-domain equation. [8 marks]

$$Y(s) = \frac{1}{s(s^2 + s + 1)}$$

Example of Unique Complex Factors

- The coefficients of complex factors must be found by the cross-multiplication method.
- Find the residuals of other factors first.

$$Y(s) = \frac{1}{s(s^2 + s + 1)}$$

$$= \frac{1}{s} + \frac{A_2 s + A_3}{s^2 + s + 1}$$

$$= \frac{(s^2 + s + 1) + A_2 s^2 + A_3 s}{s(s^2 + s + 1)}$$

$$= \frac{(A_2 + 1)s^2 + (A_3 + 1)s + 1}{s(s^2 + s + 1)}$$

Example of Unique Complex Factors

- We equate the coefficients of the powers of s in the numerators of the two sides.

$$s^2\colon\ 0 = A_2 + 1 \Longrightarrow A_2 = -1$$

$$s^1\colon\ 0 = A_3 + 1 \Longrightarrow A_3 = -1$$

$$s^0\colon\ 1 = 1$$

- Thus

$$Y(s) = \frac{1}{s} + \frac{-s-1}{s^2+s+1} = \frac{1}{s} - \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

Example of Unique Complex Factors

The equation becomes:

$$Y(s) = \frac{1}{s} - \frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{\frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \frac{1}{s} - \frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

- Taking inverse Laplace transform:

$$y(t) = \left[1 - e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]u(t)$$

Repeated Real Factors

- Repeated Real poles:

$$Y(s) = \frac{n(s)}{(s+a)^k d(s)}$$

- Or

$$Y(s) = Y_1(s) + \frac{A_k}{(s+a)^k} + \frac{A_{k-1}}{(s+a)^{k-1}} + \cdots + \frac{A_0}{s+a}$$

- Thus

$$y(t) = y_1(t) + [A_k t^{k-1} e^{-at} + A_{k-1} t^{k-2} e^{-at} + \cdots + A_0 e^{-at}] u(t)$$

- Repeated poles lead to a set of partial fractions, with decreasing multiplicity of the pole.

Example of Repeated Real Factors

Find the time-domain equation of a system expressed as the following transfer function equation with a pair of repeated poles. [5 marks]

$$Y(s) = \frac{3s + 8}{(s + 2)^2}$$

Example of Repeated Real Factors

- With the given transfer function equation, factorise and perform partial fraction expansion.

$$Y(s) = \frac{3s+8}{(s+2)^2} = \frac{A_2}{(s+2)^2} + \frac{A_1}{s+2}$$

- The coefficients A_1 and A_2 are found from:

$$A_2 = \lim_{s\to -2} \frac{(s+2)^2(3s+8)}{(s+2)^2} = 3s+8\Big|_{s\to -2} = 2$$

- (Just the Heaviside technique)

$$A_1 = \lim_{s\to -2} \frac{d}{ds}\frac{(s+2)^2(3s+8)}{(s+2)^2} = \lim_{s\to -2} \frac{d}{ds}(3s+8)$$

Example of Repeated Real Factors

- The equation for coefficient A_1 becomes:

$$A_1 = (3)\Big|_{s \to -2} = 3$$

- Thus, the overall transfer function equation is:

$$Y(s) = \frac{2}{(s+2)^2} + \frac{3}{s+2}$$

- Taking inverse Laplace transform of the transfer function, the equation in the time domain is:

$$y(t) = (2te^{-2t} + 3e^{-2t})u(t)$$

Partial Fractions Summary

- Real poles:

$$Y(s) = \frac{n(s)}{(s+a)d(s)} = Y_1(s) + \frac{A}{s+a}$$

Thus:

$$y(t) = y_1(t) + Ae^{-at}u(t)$$

- Complex pole pair:

$$Y(s) = \frac{n(s)}{[(s+a)^2 + \omega^2]d(s)} = Y_1(s) + \frac{A+B}{(s+a)^2+\omega^2}$$

Thus:

$$y(t) = y_1(t) + [Ae^{-at}\cos(\omega t + \phi)]u(t)$$

Where: ϕ depends on A and B.

Partial Fractions Summary

- Repeated real poles:

$$Y(s) = \frac{n(s)}{(s+a)^k d(s)}$$

$$Y(s) = Y_1(s) + \frac{A_k}{(s+a)^k} + \frac{A_{k-1}}{(s+a)^{k-1}} + \cdots + \frac{A_0}{s+a}$$

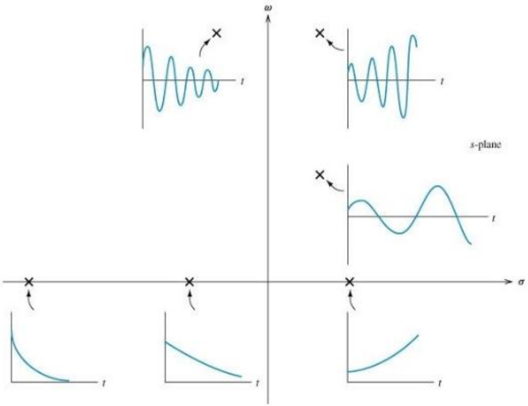
Thus:

$$y(t)$$

$$= y_1(t) + [A_k t^{k-1} e^{-at} + A_{k-1} t^{k-2} e^{-at} + \cdots + A_0 e^{-at}] u(t)$$

The s-plane

- We often don’t care about the precise amplitude of modes but are instead content to talk about the modes themselves.
- Plotting pole-zero diagrams let’s us visualise what is happening.



The s-plane

- Remember that any system having poles only in the left-half side of the s-plane will be stable.
- Its modes will (eventually) decay to zero.
- Conversely, a system having one or more poles in the right half of the *s*-plane will be unstable, and its output will tend to infinity with increasing time.

Example of s-plane

For each of the given control systems below, determine the location of poles and/or zeros in the s-plane and predict its transient response. [9 marks]

a. System 1:

$$Y_1(s) = \frac{s + 20}{s^2 + 101s + 100}$$

a. System 2:

$$Y_2(s) = \frac{0.5s + 2.5}{s^2 + 2s + 10}$$

a. System 3:

$$Y_3(s) = \frac{5s - 500}{s^3 - 3s - 2}$$

Example of s-plane

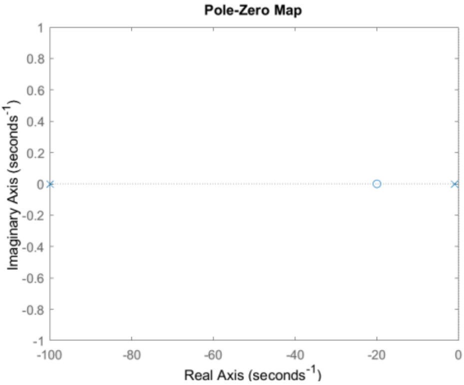
a. System 1

$$Y_1(s) = \frac{(s+20)}{(s+1)(s+100)}$$

The s-plane diagram of the system is as shown in the figure below.

Since all the poles and zero are located at the left-hand side of the diagram, the system is stable.

As the poles are all real, then the transient response of the system is overdamped.



Example of s-plane

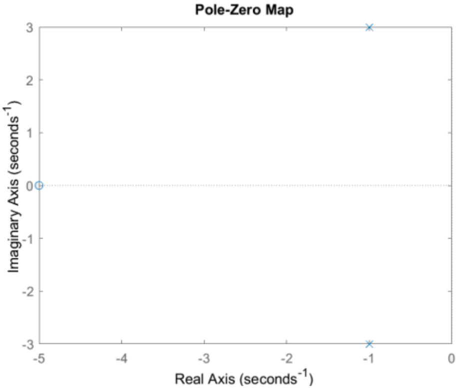
b. System 2

$$Y_2(s) = \frac{0.5(s+5)}{s^2+2s+10} = \frac{0.5(s+5)}{(s+1)^2+(3)^2}$$

The s-plane diagram of the system is as shown in the figure below.

Since all the poles are a pair of complex poles at the left-hand side of the diagram, the system is stable.

Because of these complex poles, the transient response of the system is underdamped.



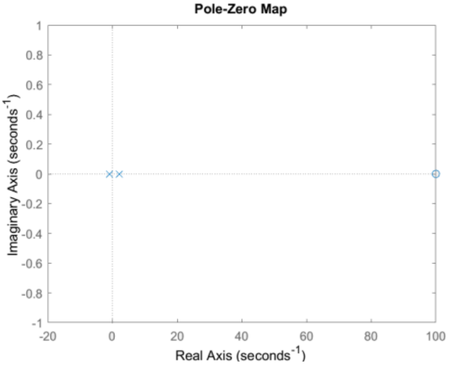
Example of s-plane

c. System 3

$$Y_3(s) = \frac{5(s - 100)}{(s - 2)(s + 1)^2}$$

The s-plane diagram of the system is as shown in the figure below.

Since there is a pole at the right-hand side of the diagram, the system is found to be unstable.



Final Value Theorem

- The final value theorem allows us to calculate the final value that a system's output will take, *without* needing to do partial fraction expansion and inverting the Laplace transform.

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

- We will find this particularly useful in finding the response of a system to a step input, which makes the equation particularly simple.

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sG(s)\frac{1}{s} = \lim_{s \to 0} G(s)$$

- The final theorem is *only* held if the system is stable. Be careful!

Example of Final Value Theorem

Determine the steady-state characteristics of the following control systems given as the following transfer-function equations.

[6 marks]

a. System 1 when it is subjected to a step input ($1/s$):

$$F_1(s) = \frac{s(s+10)}{(s+2)(s+50)}$$

b. System 2 when it is subjected to a ramp input ($1/s^2$):

$$F_2(s) = \frac{10(s+5)}{s(s^2+s+10)}$$

c. System 3 when it is subjected to a parabolic input ($1/s^3$):

$$F_3(s) = \frac{s^2(s+2)}{(s+15)(s+100)}$$

Example of Final Value Theorem

The steady-state characteristics of the following control systems are as outlined below.

a. System 1 (with a step input):

$$\lim_{t\to\infty} f_1(t) = \lim_{s\to 0} sF_1(s)\left(\frac{1}{s}\right)$$

$$= \lim_{s\to 0} s\left[\frac{s(s+10)}{(s+2)(s+50)}\right]\left(\frac{1}{s}\right)$$

$$= \frac{10s}{(2)(50)} = 0$$

Thus, the system settles to 0 at steady-state conditions.

Example of Final Value Theorem

b. System 2 (with a ramp input):

$$\lim_{t\to\infty} f_2(t) = \lim_{s\to 0} sF_2(s)\left(\frac{1}{s^2}\right)$$

$$= \lim_{s\to 0} s\left[\frac{10(s+5)}{s(s^2+s+10)}\right]\left(\frac{1}{s^2}\right)$$

$$= \frac{(10)(5)}{s^2(10)} = \infty$$

The system is becoming unstable at steady-state conditions as the gain is approaching infinity.
This shows that input can have a significant effect on the system's response under steady-state conditions.

Example of Final Value Theorem

c. System 3 (with a parabolic input):

$$\lim_{t\to\infty]} f_3(t) = \lim_{s\to 0} sF_3(s)\left(\frac{1}{s^3}\right)$$

$$= \lim_{s\to 0} s\left[\frac{s^2(s+2)}{(s+15)(s+100)}\right]\left(\frac{1}{s^3}\right)$$

$$= \frac{2}{(15)(100)} = \frac{1}{750}$$

The system settles to a constant, e.g., 1/750, under steady-state conditions, resulting in a large steady-state error.