VICTORIA UNIVERSITY OF WELLINGTON
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Physical System Modelling

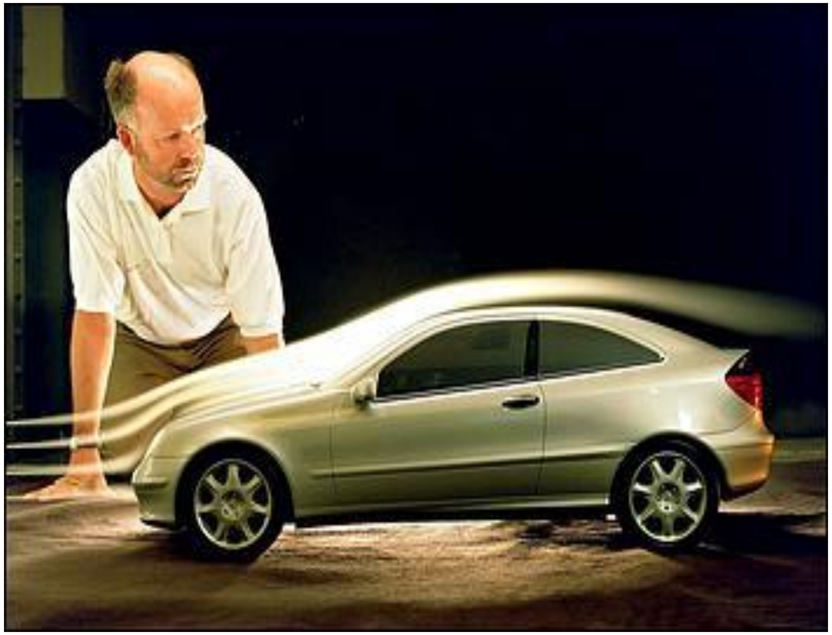
XMUT315 Control System Engineering

Topics

- Modelling physical systems.
- Lumped parameters models.
- LTI models.
- Linearisation.
- Modelling aspects and process.
- Modelling mechanical systems.
- Modelling electrical systems.
- Modelling electromechanical systems.

How to Model Physical Systems

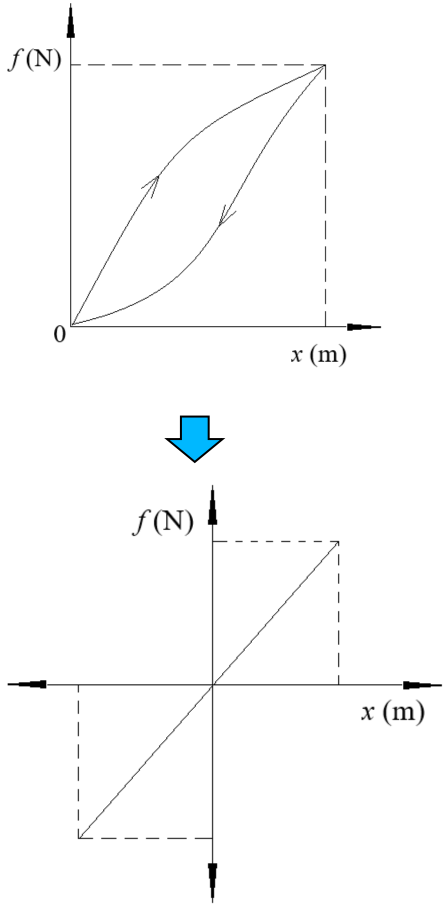
- Scaled physical model: proportional to the actual model.
- Mathematical model: described as function and variable in mathematical equation.



- Numerical model: represented as a set of numbers to describe system characteristic and behaviour.

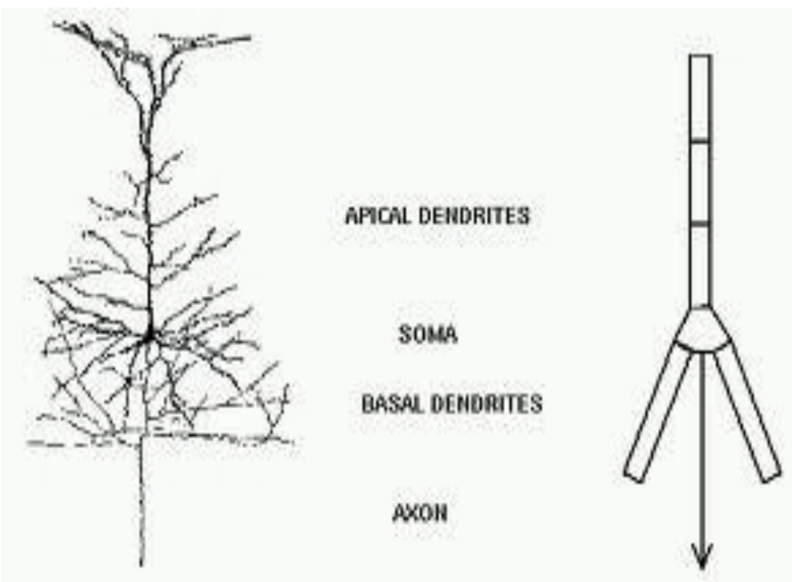
Modelling of Physical Systems

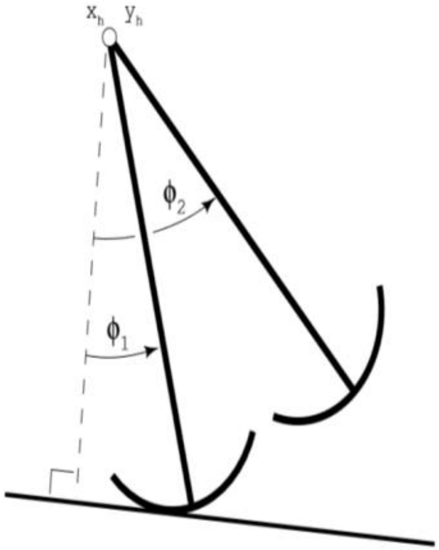
- Develop mathematical models, i.e. ordinary differential equations that describe the relationship between input and output characteristics of a system.
- These equations can then be used to forecast the behaviour of the system under specific conditions.
- All systems can normally be approximated and modelled by one of several models, e.g. mechanical, electrical, thermal or fluid.
- We also find that we can translate a system from one model to another to facilitate the modelling.



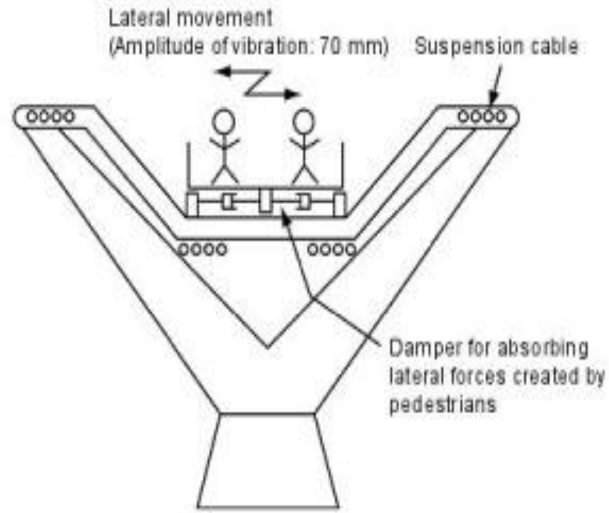
Lumped Parameter Models

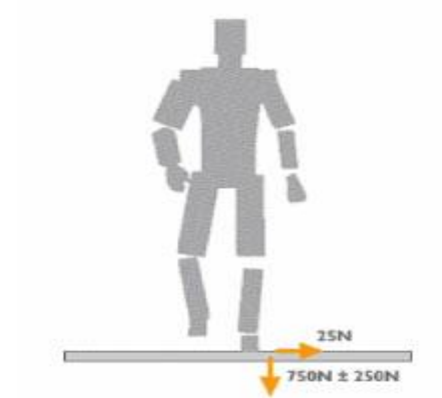
- Use standard laws of physics and break a system down into several building blocks.
- Each of the parameters (property or function) is considered independently.





Lumped Parameter Models

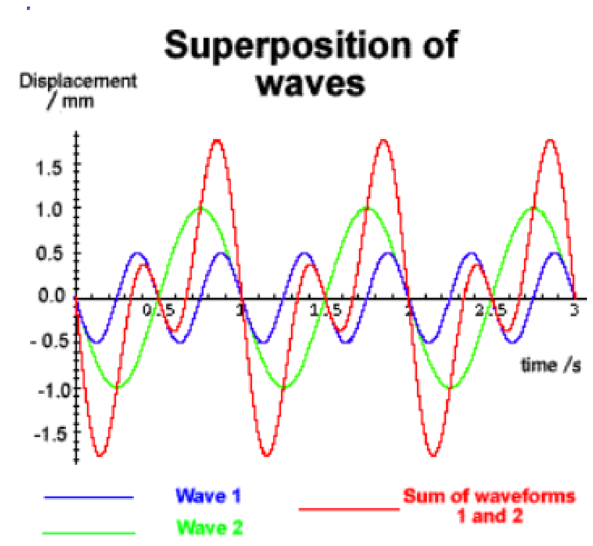




Millennium wobbly bridge: http://www.youtube.com/watch?v=eAXVa__XWZ8

Linear Time Invariant Models

- Assume the property of linearity for these models.
- A linear system will possess two properties:
  - Superposition.
  - Homogeneity.
- Allows us to use standard mathematical operations to simplify our models.



Linear Time Invariant Models

- Assume system is time-invariant.
- Constants stay constants in the time-scales of our model.
- Proportionality between variables does not change.

  Note: Our shock absorbers do not wear out in our car suspension model!

Linearisation

- Linearisation: Intuitively we could do it with order reduction, tangent, or Taylor series:

Order reduction method:

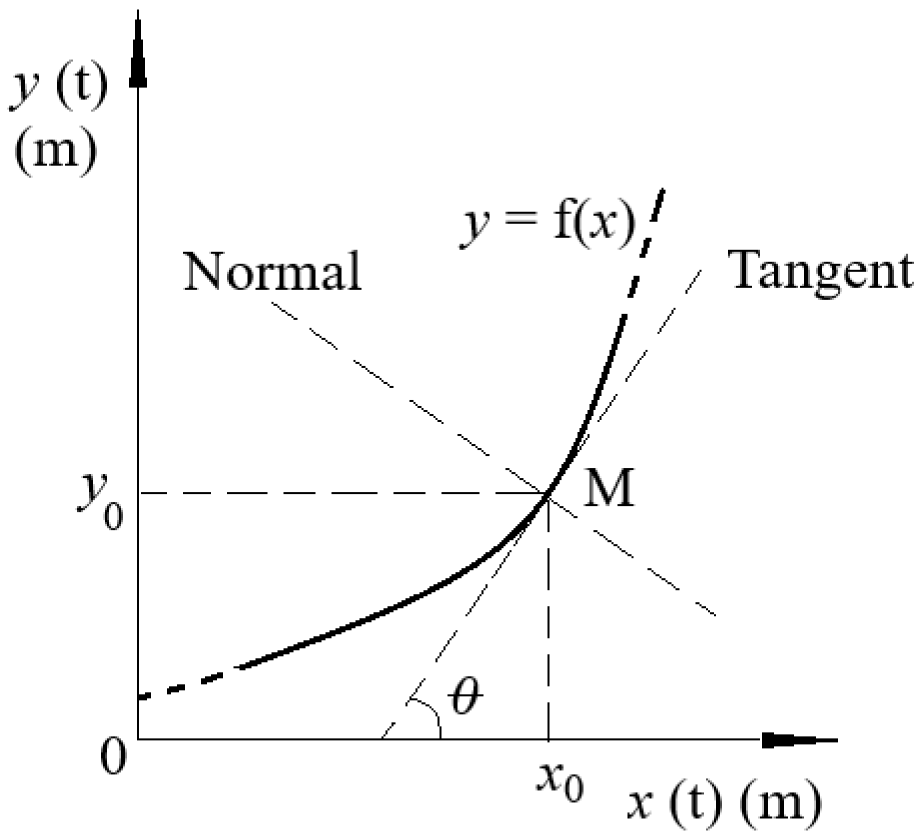
- A finite number of terms will give an approximation of the function e.g. the first two terms will give a linear approximation.

$$y = f(x_0) + \left[\frac{dy}{dx}\right]_{x_0} (x - x_0) + \left[\frac{d^2y}{dx^2}\right]_{x_0} \frac{(x - x_0)}{2!} + \ldots$$

Linearisation

Tangent method:

- Using a linear function evaluated at a given point (i.e. tangent of the curve) instead of higher order function.



Linearisation

Taylor series method:

- Suppose that we know y is a function of x, and we know the values of y and y' when $x = a$, that is $y(a)$ and $y'(a)$ are known.
- We can use $y(a)$ and $y'(a)$ to determine a linear polynomial which approximates to $y(x)$. Let this polynomial be:

$$p_1(x) = c_0 + c_1 x$$

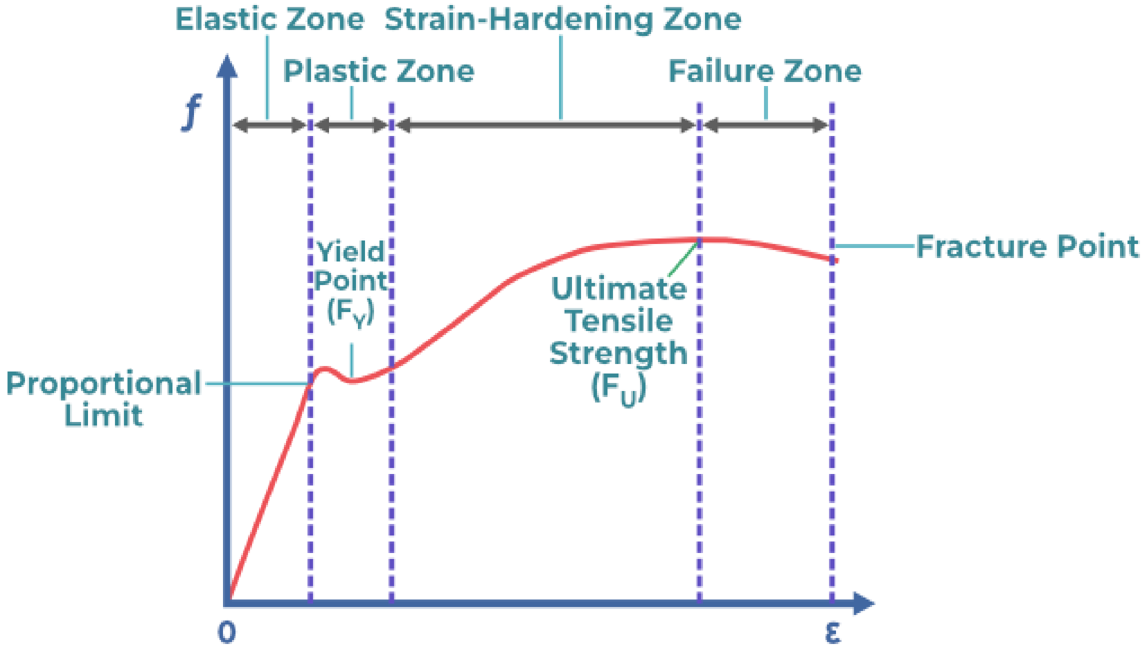
- Thus

$$p_1(x) = y(a) + y'(a)(x - a)$$

- The $p_1(x)$ is the first-order Taylor polynomial generated by y at $x = a$.

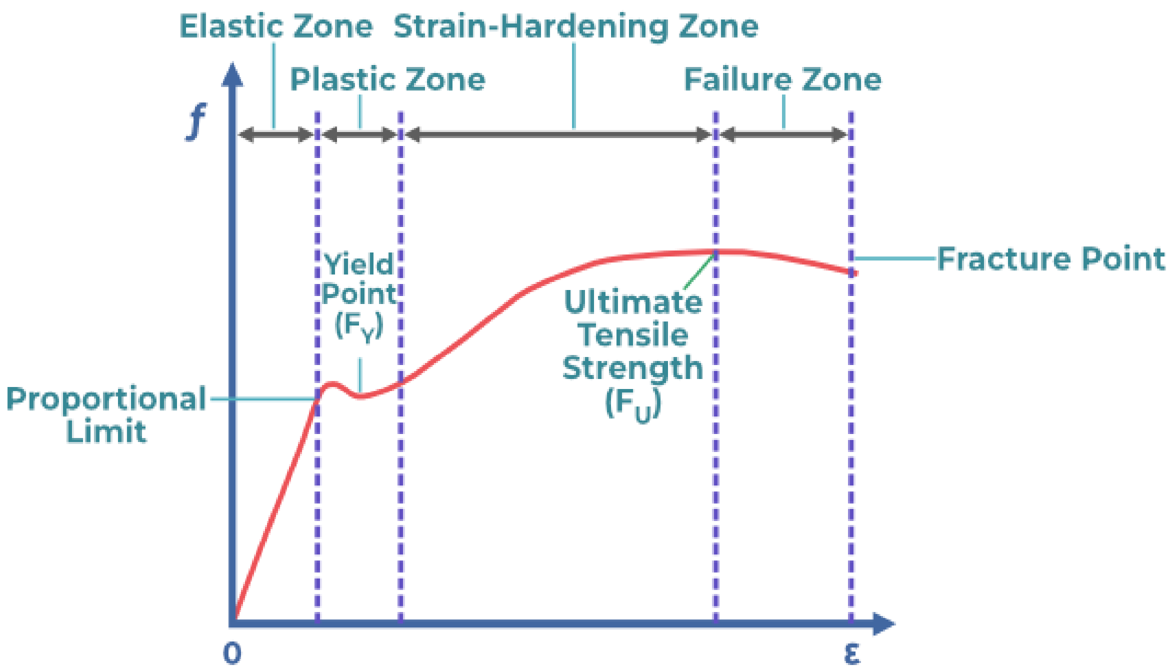
Example of Linearisation 1

- Consider the force acting in a spring during the plastic zone condition that can be described as a third-order function of extension (ε): $f(\varepsilon) = 2\varepsilon + 5\varepsilon^3$.



- At its operating point at $\varepsilon = 1$, this function can be approximated as: $f(\varepsilon) \cong -10 + 17\varepsilon$.

Example of Linearisation 1



- Model the spring force as $f(\varepsilon) = -10 + 17\varepsilon$ around the point $\varepsilon = 1.0$ and the linearised spring forced constant would be given by:

$$\frac{df}{d\varepsilon} = 17 \text{ N/m}$$

Example of Linearisation 2

Find a linear approximation to a function $y(t) = t^2$ near $t = 3$ using Taylor series.

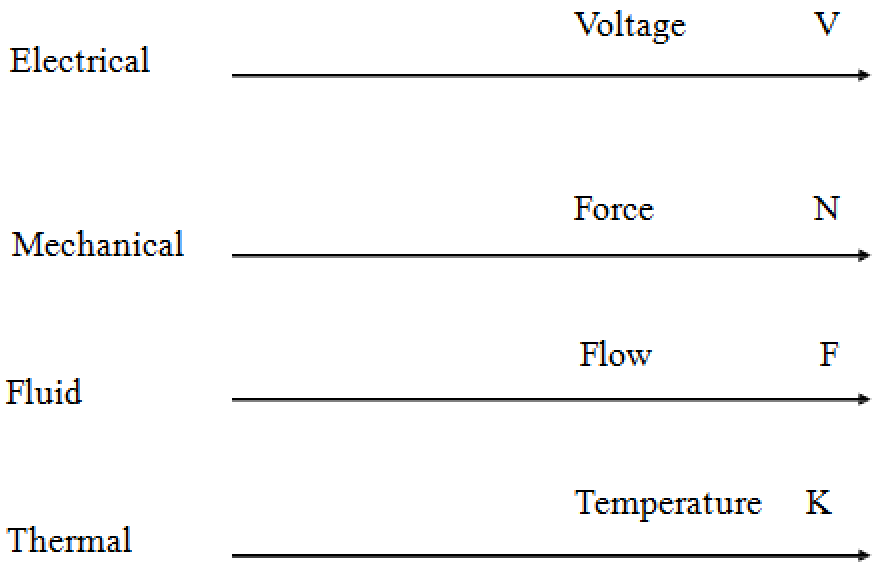
- We require the equation of the tangent to $y = t^2$ at $t = 3$, that is the first-order Taylor polynomial about $t = 3$.
- Note that $y(3) = 9$ and $y'(3) = 6$.

$$p_1(t) = y(a) + y'(a)(t - a) = y(3) + y'(3)(t - 3)$$
$$= 9 + 6(t - 3) = 6t - 9$$

- At $t = 3$, $p_1(t)$ and $y(t)$ have an identical value.
- Near to $t = 3$, $p_1(t)$ and $y(t)$ have similar values, for example $p_1(2.8) = 7.8$, and on the other hand $y(2.8) = 7.84$.

Signals

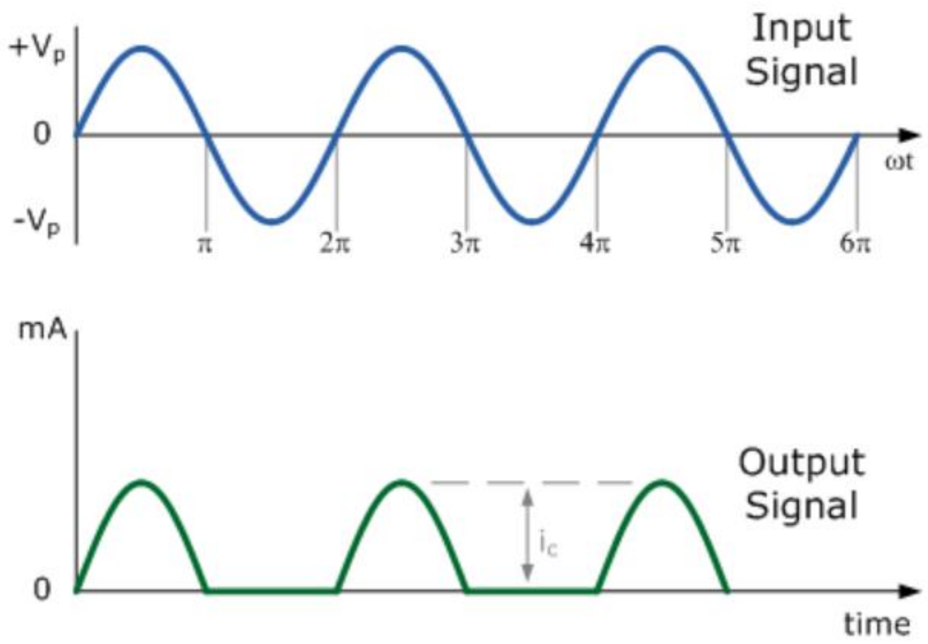
- Components are connected together by signals.
- Signals have many different forms.
- Signals must also have direction and name.



- Signals continue until interrupted.
- Signals and components are considered ideal.
- We add other signals and components to alter the signals.

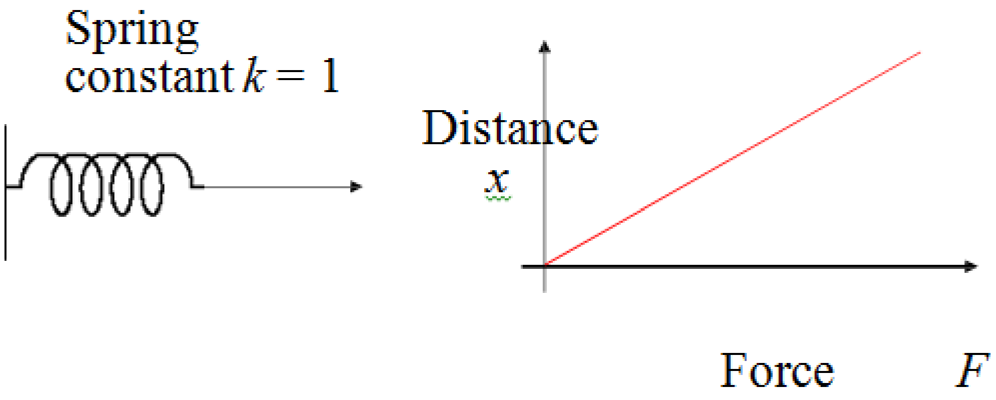
Signals

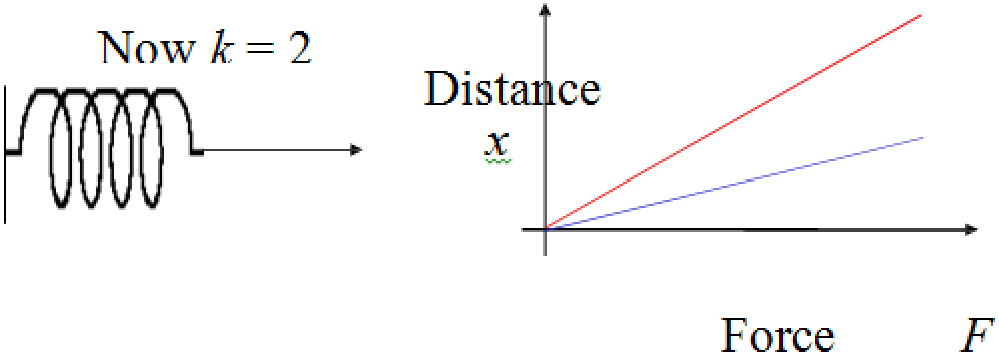
- We wish to know how the output signal varies with an input signal for a fixed (invariant) system.
- We may plot two signals against each other invariant of time (system relationship).



Constants

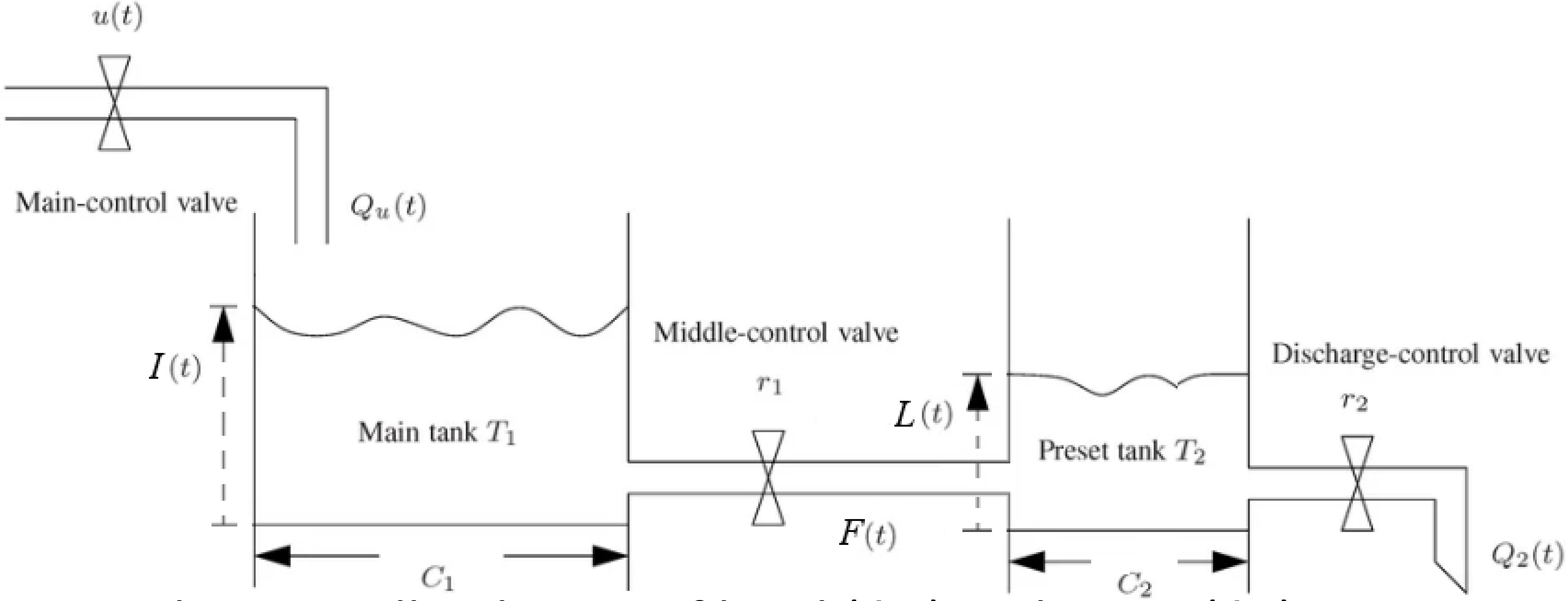
- System constants are time invariant for the given system.
- We now consider a different system as the spring has been changed.
- However, the analysis stays the same.





Differentiation

- Alternative method: level of water, L, changes because of the flow of liquid.



- Mathematically, change of level (ΔL) with time (Δt) is calculated from:

$$\frac{\Delta L}{\Delta t} = \frac{dL(t)}{dt}$$

Differentiation

- In fact, the change in L is proportional to flow, $F(t)$ and inversely proportional to cross-sectional area of the connecting pipe, C:

$$\frac{dL(t)}{dt} = \left[\frac{1}{C}\right] F(t) \qquad \text{and} \qquad F(t) = \frac{I - L(t)}{R}$$

Where: I is height of larger tank, L is height of smaller tank, and R is radius of pipe.

- The flow is related to difference in levels, thus:

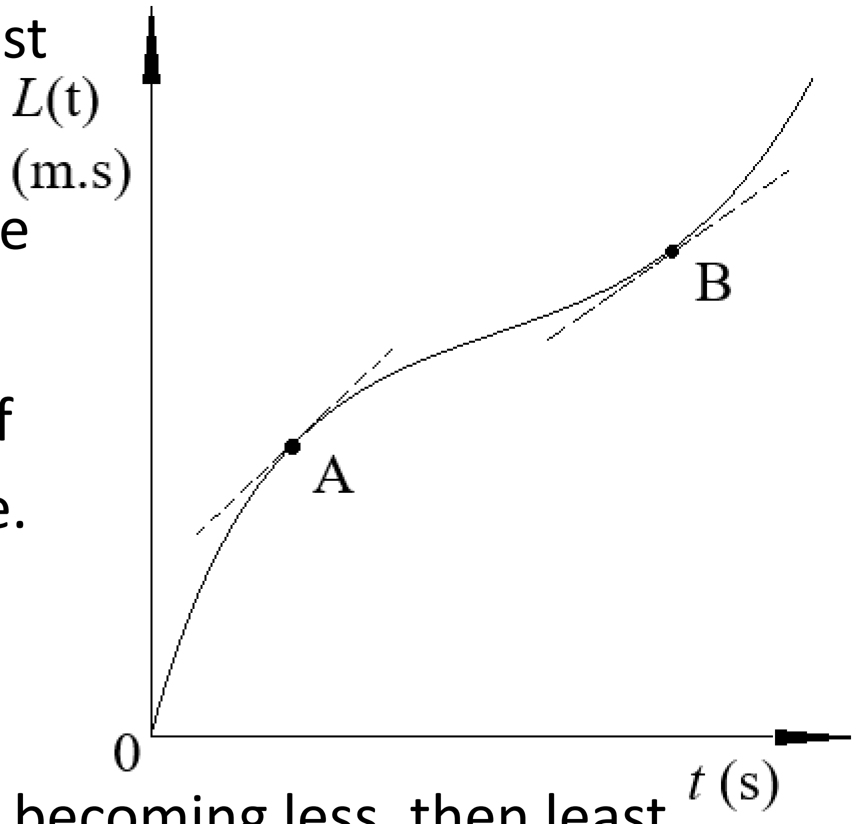
$$\frac{dL(t)}{dt} = \left[\frac{1}{C}\right]\left[\frac{I - L(t)}{R}\right] = \frac{I - L(t)}{CR}$$

Where: CR is the time constant, T.

- Note: the above case is a differential equation. It has the differential of $L(t)$ being a function including $L(t)$.

Differentiation: Slopes

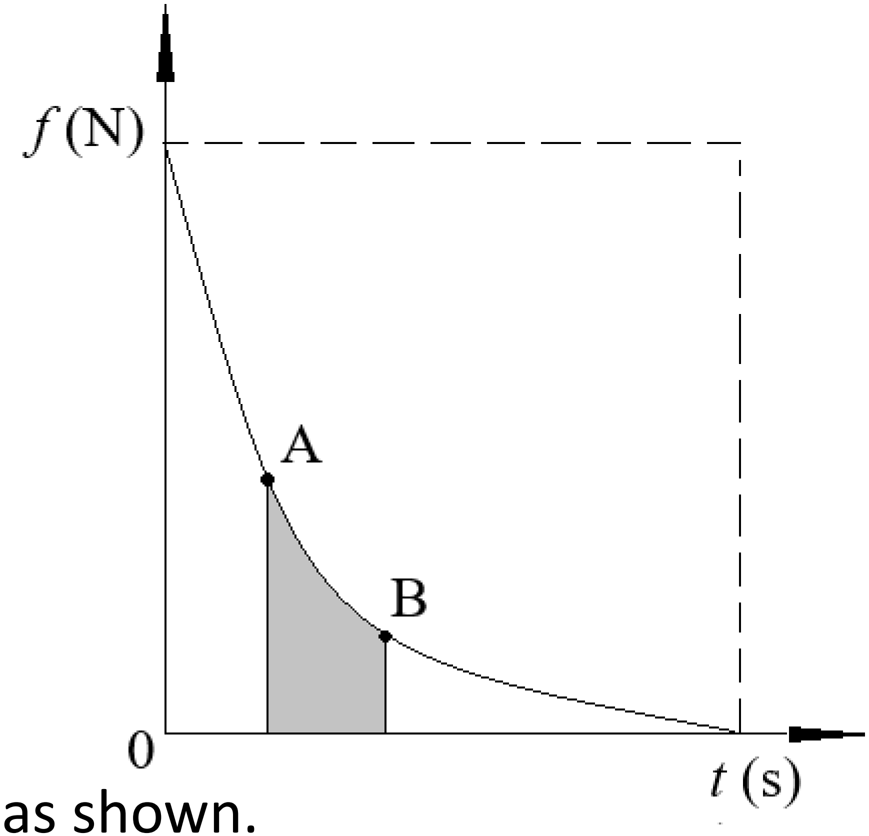
- Consider the graph of $L(t)$ against time.
- At any instant of time, we can see value of $L(t)$.
- The change in $L(t)$ is the slope of the graph, which varies with time.
- Initially steep (high value), then less, then less.



- But the flow is initially high, then becoming less, then least.
- Thus, slope of $L(t)$ is like $F(t)$, but slope is change of $L(t)$.
- In fact, $F(t)$ is proportional to derivate of $L(t)$ with time.

Integration: Area

- The reverse process is integration.
- Graphical interpretation: area under a graph.
- Consider the flow graph: the area at different times is shown.



- After a short time, the area is as shown.
- Later, the area has grown, but by less, etc.

Integration: Area

- Consider the height of water in the tank.
- Thus, $L(t)$ like area under $F(t)$:

$$L(t) \propto \int F(t)\, dt$$

- $L(t)$ is proportional to integral of $F(t)$ with time.
- In fact, for this system, we have:

$$\frac{dL(t)}{dt} = \left(\frac{1}{C}\right) F(t) \qquad \text{and} \qquad L(t) = \frac{1}{C}\int F(t)\, dt$$

- The flow, $F(t)$ is differential of $L(t)$ and $L(t)$ is integral of $F(t)$.

Integration: Area

- Differentiation and integration are opposites (note: here they are used to model a water systems).
- It can also model electronic circuits, mechanical systems, motors, etc.
- In fact, the differential equation has the same form, and hence the same exponential response as that for many systems.
- Note: there are analogies between water systems and electronics: pipe like a resistor, tank like a capacitor.
- Also, for thermals, walls have thermal resistance, rooms have thermal capacity.

Mechanical Components

- We know that distance $(x(t))$ is related to velocity $(v(t))$ is related to acceleration $(a(t))$ through differentiation.
- Displacement:

$$\text{Distance} = x(t)$$

- Velocity:

$$v(t) = \frac{dx(t)}{dt}$$

- Acceleration:

$$a(t) = \frac{dv}{dt} = \frac{d\left(\frac{dx(t)}{dt}\right)}{dt} = \frac{d^2x(t)}{dt^2}$$

Mechanical Components

- It gets messy writing d/dt all the time!
- Therefore, we use Laplace transform and will write in term of 's' instead.
- Displacement

$$\text{Distance} = X(s)$$

- Velocity

$$V(s) = sX(s)$$

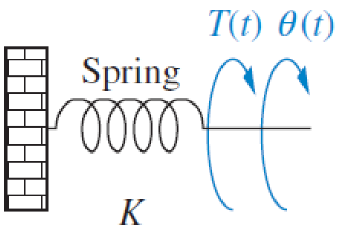
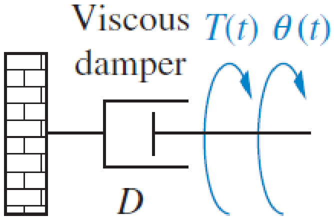
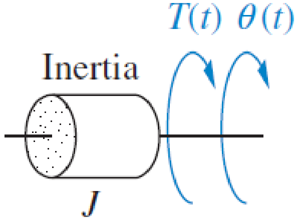
- Acceleration

$$A(s) = sV(s) = s^2X(s)$$

- Note: both time and frequency domain are transformed with respect to the variable.

Mechanical Components

- Standard mechanical components for modelling physical system.

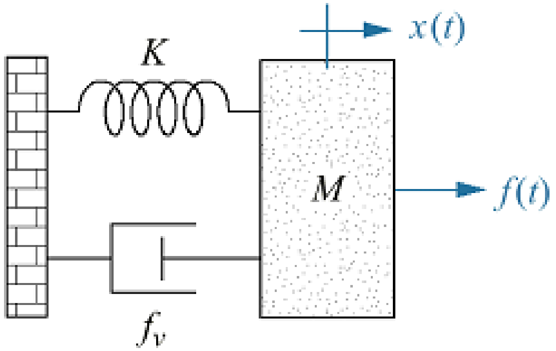
Component	Torque-angular velocity	Torque-angular displacement	Impedence $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Example of Mechanical System Modelling

For the mechanical system below with mass, spring and damper:

- We assume the mass is displaced by $x(t)$ toward the right.
- Note that taking into consideration the zero initial condition, just like the spring, the damper will also oppose the force.
- Thus, only the applied force points to the right.
- All other forces impede the motion and act to oppose it e.g. the spring, damper, and the force due to acceleration point to the left.

Determine the transfer function equation of the system in the time domain. [6 marks]



Example of Mechanical System Modelling

- Write the differential equation of motion using Newton's second law to sum to zero of all the forces shown on the system:

$$m\left(\frac{d^2x(t)}{dt^2}\right) + f_v\left(\frac{dx(t)}{dt}\right) + kx(t) = f(t)$$

- Taking the Laplace transform, assuming zero initial conditions, the equation above becomes:

$$ms^2X(s) + f_v sX(s) + kX(s) = F(s)$$

- As a result, the transfer function equation of the given mechanical system is:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + f_v s + k}$$

Electrical Components

- We know that to find the reactance of electrical devices such inductor and capacitor requires integration and differentiation respectively:
- Voltage across resistor:

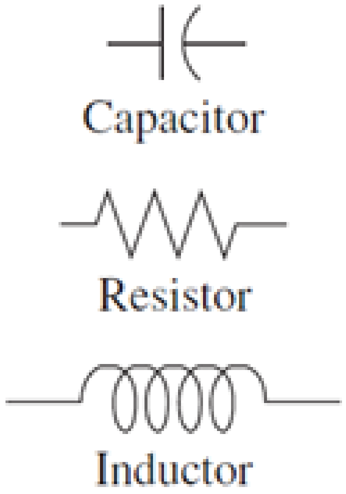
$$v_R(t) = Ri(t)$$

- Voltage across capacitor:

$$v_C(t) = \frac{1}{C}\int_0^t i(t)$$

- Voltage across inductor:

$$v_L(t) = L\left[\frac{di(t)}{dt}\right]$$



Electrical Components

Applying the Laplace transform, we have the following:

- Voltage across resistor:

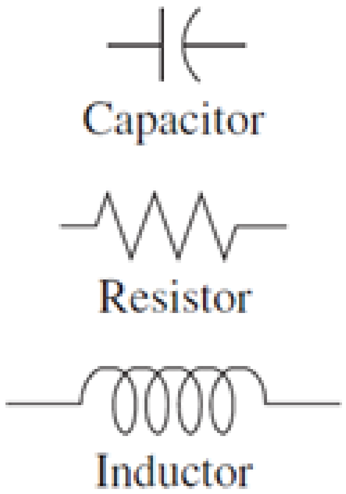
$$V_R(s) = Ri(s)$$

- Voltage across capacitor:

$$V_C(s) = \left(\frac{1}{sC}\right) i(s)$$

- Voltage across inductor:

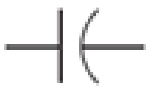
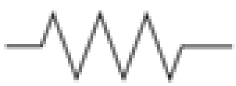
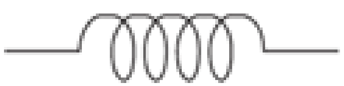
$$V_L(s) = sLi(s)$$



Note: both with respect to the variable.

Electrical Components

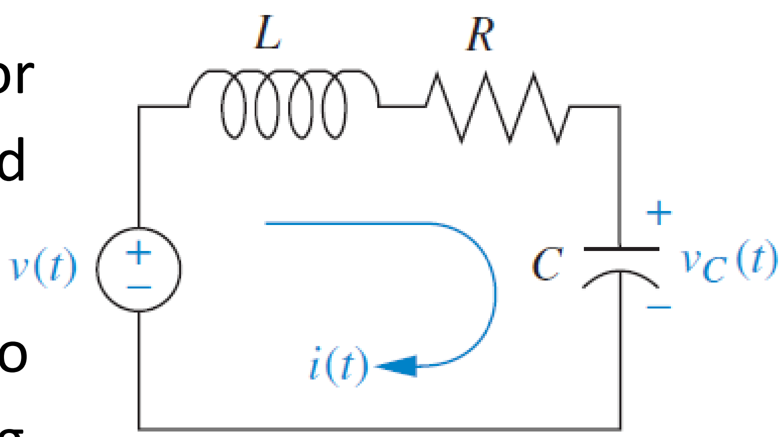
- Standard electrical components for modelling a physical system.

Component	Voltage-current	Current-voltage	Impedance $Z(s) = V(s)/I(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$\frac{1}{Cs}$
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	R
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau$	Ls

Example of Electrical System Modelling

For the electrical system as shown below:

- It is a series RLC circuit.
- Assume in this case that capacitor voltage as the output and applied voltage as the input.
- Assume zero initial conditions (no prior conditions before modelling existed).



Determine the time-domain expression for the output voltage over the input voltage for the given circuit. [12 marks]

Example of Electrical System Modelling

- Summing the voltages around the loop, assuming zero initial conditions, yields the integral-differential equation for this network as:

$$L\left[\frac{di(t)}{dt}\right] + Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$

- Changing variables from current to charge using $i(t) = dq(t)/dt$ yields:

$$L\left[\frac{d^2q(t)}{dt^2}\right] + R\left[\frac{dq(t)}{dt}\right] + \left(\frac{1}{C}\right)q(t) = v(t)$$

- From the voltage-charge relationship for a capacitor:

$$q(t) = Cv_c(t)$$

Example of Electrical System Modelling

- Substituting Eq. (2) into Eq. (1) yields:

$$C\left[\frac{d^2v_c(t)}{dt^2}\right] + RC\left[\frac{dv_c(t)}{dt}\right] + v_c(t) = v(t)$$

- Taking the Laplace transform assuming zero initial conditions, rearranging terms, and simplifying yields:

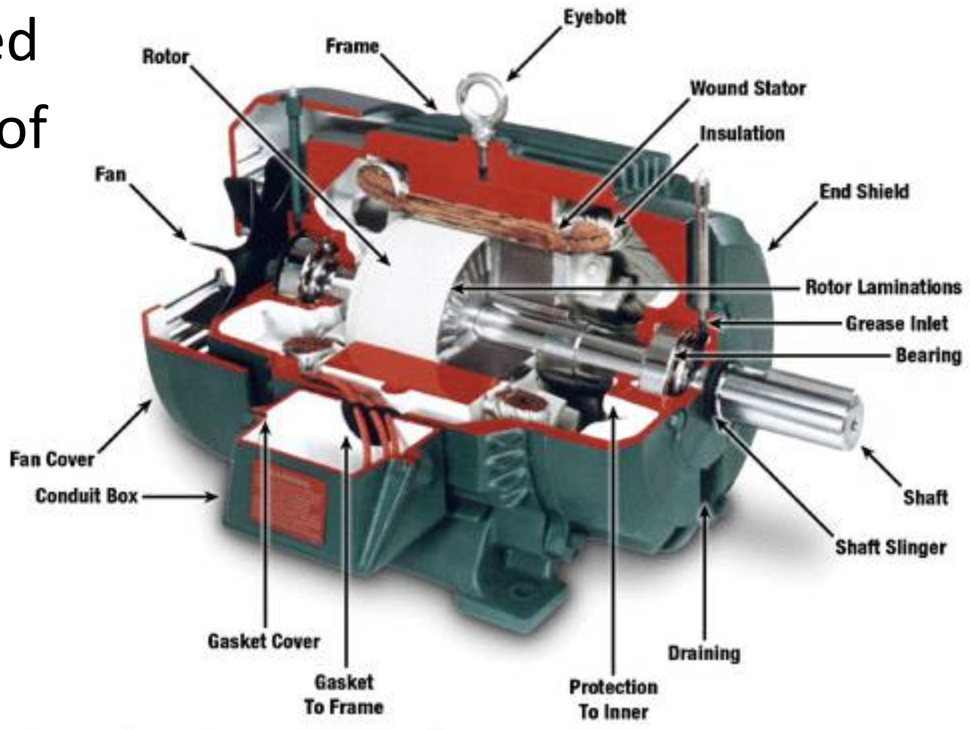
$$(LCs^2 + RCs + 1)V_c(s) = V(s)$$

- Solving for the transfer function, $V_C(s)/V(s)$, we obtain:

$$\frac{V_C(s)}{V(s)} = \frac{\left(\frac{1}{LC}\right)}{s^2 + \left(\frac{R}{L}\right)s + 1/LC}$$

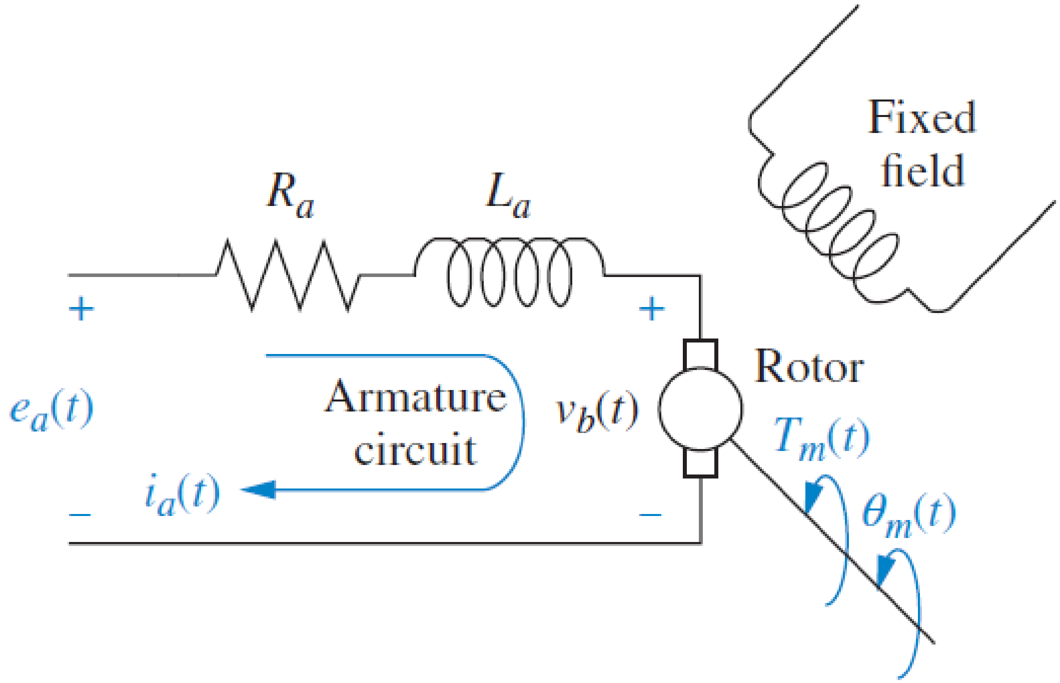
Modelling Electromechanical System

- Modelling of electromechanical system could be performed using the standard components of the mechanical and electrical systems.
- DC motor is commonly used to illustrate the modelling of the electromechanical systems.
- Modelling process divided into electrical system, mechanical system and electromechanical system.



Modelling Electrical System

- The modelling components of a DC motor are illustrated as shown in the figure below.



- Typically, there are two windings in the DC motor e.g. armature winding and (fixed) field excitation winding.

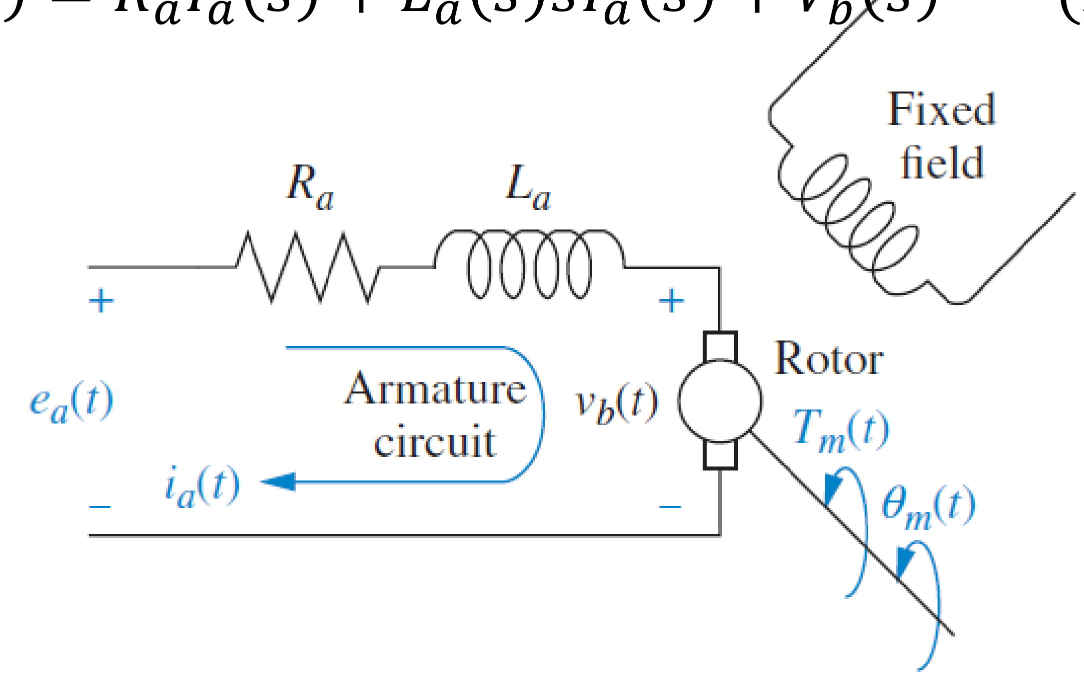
Modelling Electrical System

- Applying KVL in armature winding, the equation for the circuit is:

$$e_a(t) = R_a i_a(t) + L_a(t)\left[\frac{di(t)}{dt}\right] + v_b(t)$$

- Using Laplace transform

$$E_a(s) = R_a I_a(s) + L_a(s) s I_a(s) + V_b(s) \qquad (Eq. 1)$$



Modelling Electrical System

- Where K_b is the back EMF constant and $d\theta_m(t)/dt = \omega_m(t)$, for a given motor, the back EMF of the motor is:

$$v_b(t) = K_b\left[\frac{d\theta_m(t)}{dt}\right] \quad \text{so} \quad V_b(s) = K_b s\theta_m(s) \qquad (Eq.2)$$

- The torque developed by motor is proportional to armature current: $T_m(t) = K_t i_a(t)$ where K_t is a motor torque constant, thus, it is determined from:

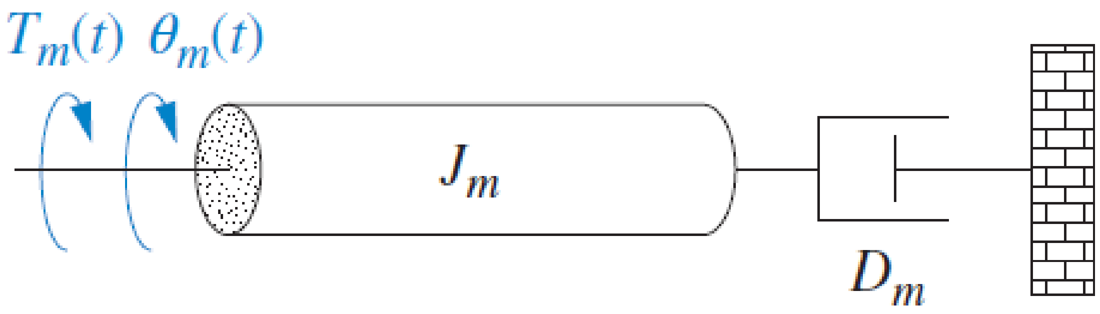
$$T_m(s) = K_t I_a(s) \qquad (Eq.3)$$

- Thus

$$I_a(s) = \left(\frac{1}{K_t}\right)T_m(s) \qquad (Eq.4)$$

Modelling Mechanical System

- The following figure shows the equivalent mechanical loading that typically connected to a DC motor.



Where:

- J_m is the equivalent inertia of the motor (e.g. the inertia of the armature and load).
- D_m is the vicious damping (e.g. both the vicious damping of the armature and load).

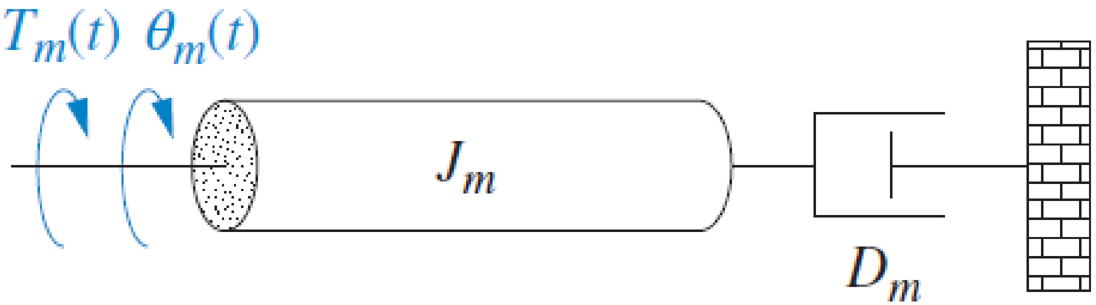
Modelling Mechanical System

- The torque of the DC motor is calculated from:

$$T_m(t) = J_m\left[\frac{d^2\theta_m(t)}{dt^2}\right] + D_m\left[\frac{d\theta_m(t)}{dt}\right]$$

- Thus

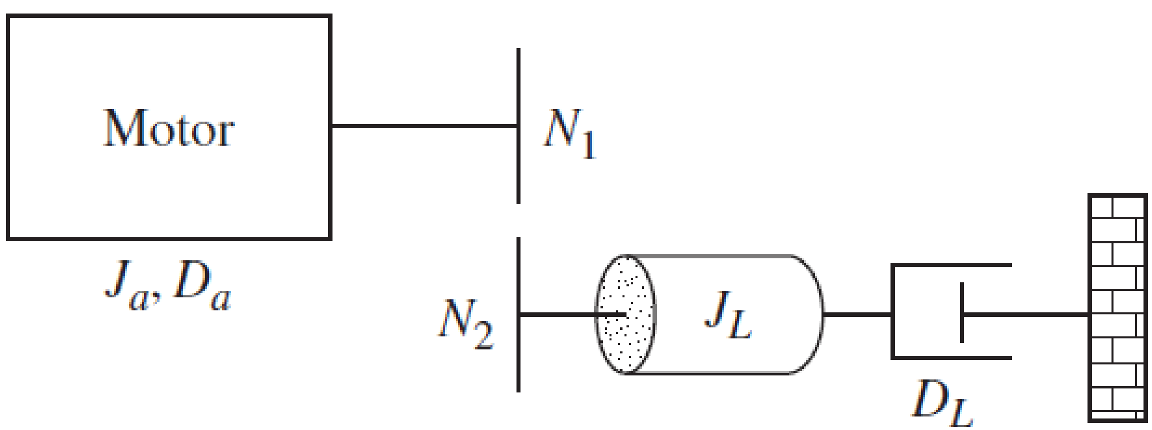
$$T_m(s) = (J_m s^2 + D_m s)\theta_m(s) \qquad (Eq.5)$$



Modelling Mechanical System

For a DC motor connected with a mechanical load as given in the figure below, modelling components of the system are:

- Motor is used to drive a mechanical load (J_L) pushing a damper (D_L).
- Motor has inertia (J_a) and damping factors (D_a).
- Gear ratios of the DC motor (N_1) and mechanical load (N_2).



Modelling Mechanical System

- Knowing inertia (J_m) and damping factor (D_m) of the motor are related through:

$$J_m = J_a + J_L \left(\frac{N_1}{N_2}\right)^2$$

And

$$D_m = D_a + D_L \left(\frac{N_1}{N_2}\right)^2$$

- Substituting eqs. (2) and (4) into eq. (1), with $L_a = 0$, yields:

$$\left(\frac{R_a}{K_t}\right) T_m(s) + K_b s\theta_m(s) = E_a(s)$$

Modelling Mechanical System

- As $s\theta_m(s) = d\theta_m(t)/dt = \omega_m(t)$, applying the inverse Laplace transform, we get:

$$\left(\frac{R_a}{K_t}\right)T_m(t) + K_b\omega_m(t) = e_a(t)$$

- Rearrange the equation, the equation above becomes:

$$T_m(t) = -\left(\frac{K_bK_t}{R_a}\right)\omega_m(t) + \left(\frac{K_t}{R_a}\right)e_a(t)$$

- When the equation above is plotted, it becomes a straight-line graph, T_m vs. ω_m, as shown in the figure below.

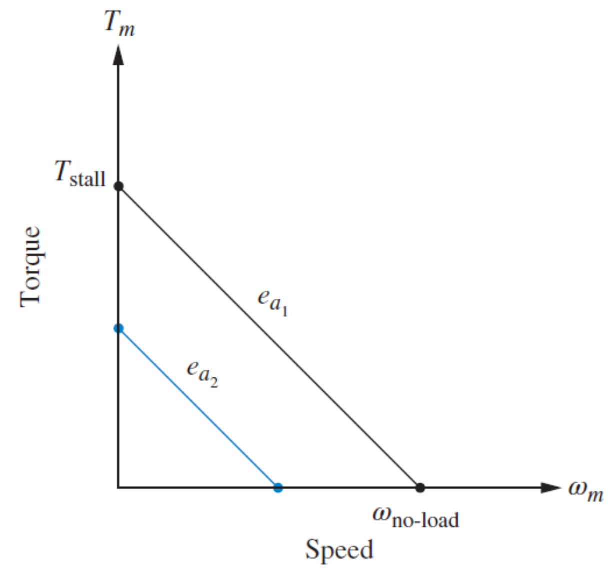
Modelling Mechanical System

- The torques-speed curve diagram, with the armature voltage is set at e_{a_1}, the DC motor is set at the extreme conditions.
  - Stalling state when $\omega_m = 0$ (motor stops and max current).
  - No-load state when $T_m = 0$ (max speed with no load).
- The intercepts in the diagram = the extreme conditions.
- Stall torque, T_{stall}:

$$T_{\text{stall}} = \left(\frac{K_t}{R_a}\right) e_a(t)$$

- No load speed, $\omega_{\text{no-load}}$:

$$\omega_{\text{no-load}} = \frac{e_a(t)}{K_b}$$



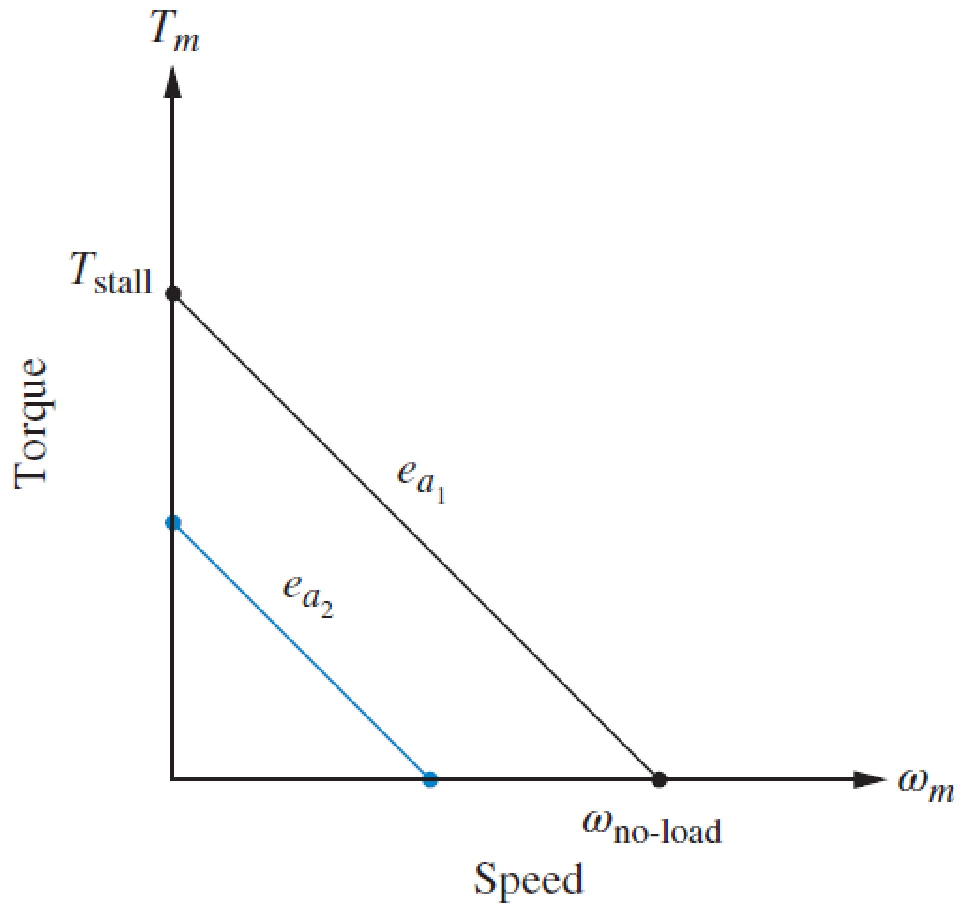
Modelling Mechanical System

- We could obtain the electrical constants, K_t/R_a and K_b from the torques-speed curve diagram given above.

$$\frac{K_t}{R_a} = \frac{T_{\text{stall}}}{e_a(t)}$$

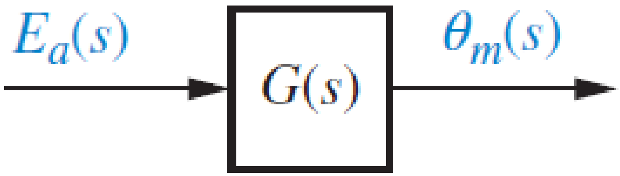
- And

$$K_b = \frac{e_a(t)}{\omega_{\text{no-load}}}$$



Modelling Electromechanical System

- The DC motor is typically illustrated as a block diagram ($G(s)$) with armature voltage, $E_a(s)$ as input and angular speed of the motor, $\theta_m(s)$ as output.



- Substitute equations (4) and (2) into equation (1), the equation becomes:

$$\frac{(R_a + L_a s)T_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s) \qquad (Eq.6)$$

Modelling Electromechanical System

- Substitute equation (5) into equation (6), it is:

$$\frac{(R_a + L_a s)(J_m s^2 + D_m s)\theta_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$$

- As $L_a \ll R_a$, common for a DC motor, the equation becomes:

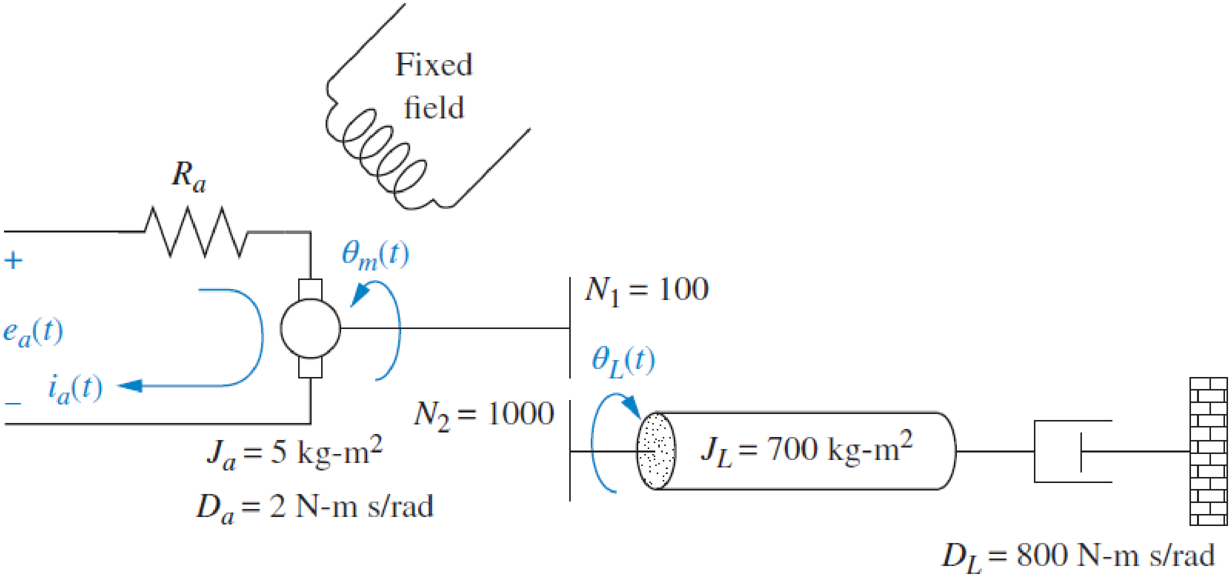
$$\left[\frac{R_a}{L_a}(J_m s + D_m) + K_b\right] s\theta_m(s) = E_a(s)$$

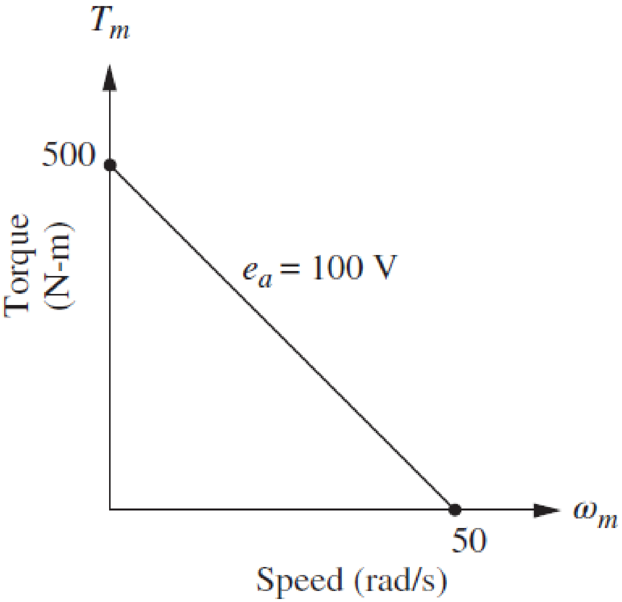
- Rearrange the equation to a ratio of $\theta_m(s)/E_a(s)$, it is:

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]}$$

Example of Electromechanical System Modelling

Given the DC motor connected to a mechanical load as shown in part (a) in the figure below used as an example of electromechanical system and torque-speed curve shown in part (b), find the transfer function, $\theta_L(s)/E_a(s)$. [20 marks]





Example of Electromechanical System Modelling

- Begin by finding the mechanical constants, J_m and D_m.
- From the equation given below, the total inertia at the armature of the motor is:

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12 \qquad (Eq.1)$$

- The total damping at the armature of the motor is:

$$D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10 \qquad (Eq.2)$$

Example of Electromechanical System Modelling

- Now, we will find the electrical constants, $K_t = R_a$ and K_b.
- From the torque-speed curve of the part (b) in the figure,

$$T_{stall} = 500 \qquad (Eq.3)$$

$$\omega_{\text{no-load}} = 50 \qquad (Eq.4)$$

$$e_a(t) = 100 \qquad (Eq.5)$$

- Hence, the electrical constants are:

$$\frac{K_t}{R_a} = \frac{T_{stall}}{e_a(t)} = \frac{500}{100} = 5 \qquad (Eq.6)$$

- And

$$K_b = \frac{e_a(t)}{\omega_{no-load}} = \frac{100}{50} = 2 \qquad (Eq.7)$$

Example of Electromechanical System Modelling

- Substituting Eqs. (1), (2), (6), and (7) into the equation below.

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{K_t}{(R_a J_m)}}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]}$$

$$= \frac{(5/12)}{s\left\{s + \frac{1}{12}[10 + (5)(2)]\right\}} = \frac{0.417}{s(s + 1.667)}$$

- To find $\theta_L(s)/E_a(s)$, we use the gear ratio, $N_1/N_2 = 1/10$, and find:

$$\frac{\theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s + 1.667)}$$

Example of Electromechanical System Modelling

- Or, as shown as a block diagram in the figure below.