



# Stability Analysis

XMUT315 Control Systems Engineering

# Topics

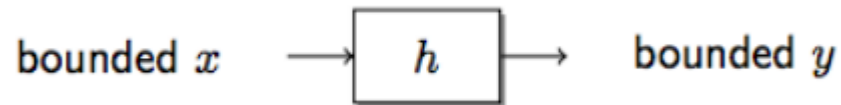
- Stability and system responses.
- Stability of the systems and stability analysis.
- Methods of stability analysis.
- Routh-Hurwitz criterion.
- Construction of the criterion.
- Special cases of the Routh-Hurwitz criterion:
  - Zero in a single column.
  - Zeros in a row.

# Stability

- A linear system is where the principles of superposition do apply (e.g. no saturation or hysteresis effects)
- A time-invariant system is one whose characteristics do not vary with respect to time (e.g. no ageing)
- In LTI systems, we often approximate systems over a specific range or time period.
- We want to build up a relationship between the response of the system and stability:
  - if input is bounded and output ( $c(t)$ ) does not approach  $\infty$  as  $t$  approaches  $\infty$  (e.g. natural response is not approaching  $\infty$ ).
  - if input is unbounded, we cannot conclude stability.

# Stability

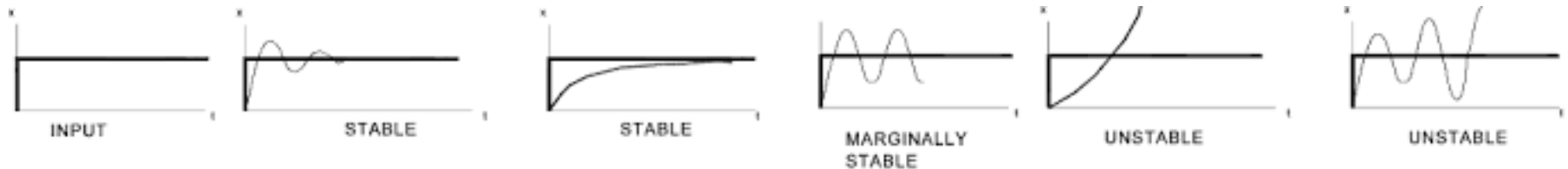
- A system is stable if every bounded input yields a bounded output, or it is bounded-input bounded-output (BIBO).



- We want to build up a relationship between the total response and instability:
  - If the input is bounded, but the output ( $c(t)$ ) is unbounded, the system is unstable.
  - If input is unbounded, we can not conclude instability.
- A system is unstable if any bounded input yields an unbounded output.

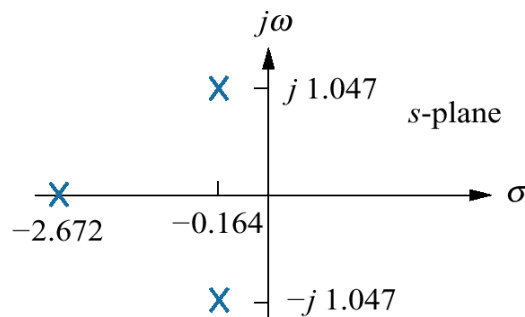
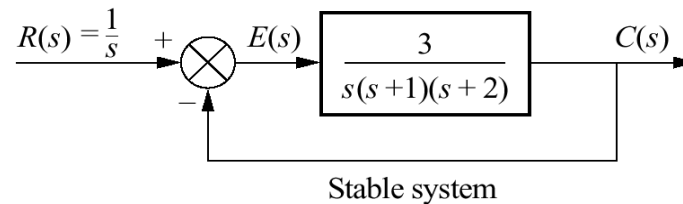
# Stability and System Response

- A linear, time-invariant system is *stable* if the natural response approaches zero as time approaches infinity.
- A linear, time-invariant system is *unstable* if the natural response grows without bound as time approaches infinity.
- A linear, time-invariant system is *marginally stable* if the natural response neither decays nor grows but remains constant or oscillates as time approaches infinity.

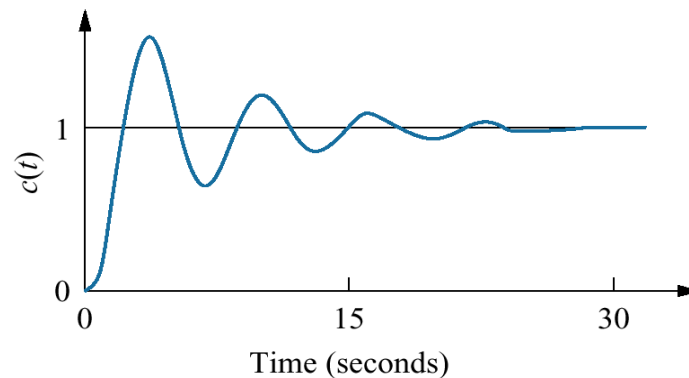


# Stability and System Response

- **Stable**, a linear, time-invariant system is stable if the natural response approaches zero as time approaches infinity

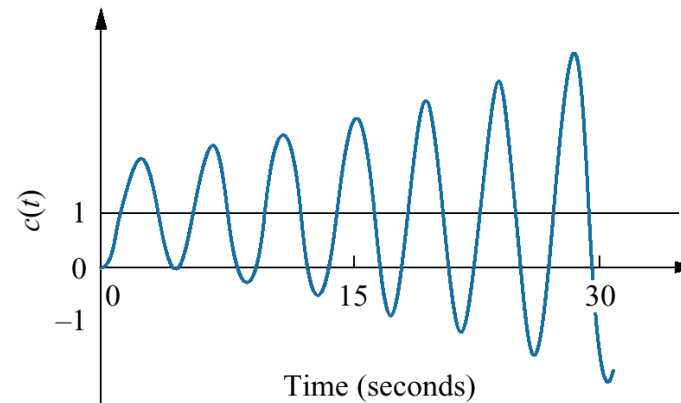
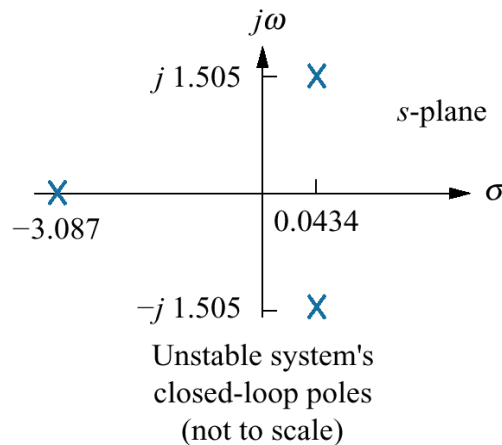
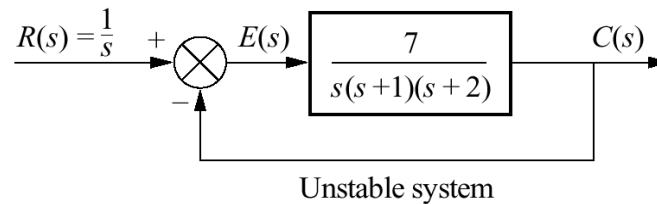


Stable system's  
closed-loop poles  
(not to scale)



# Stability and System Response

- **Unstable**, if the natural response grows without bound as time approaches infinity.



# System Damping (Indicator of Stability)

- Consider a control system described as:

$$as^2 + bs + c$$

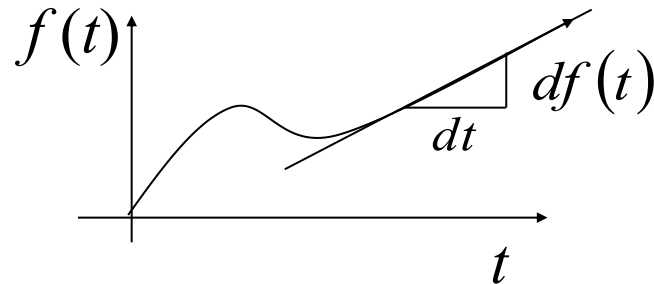
- Find roots of a quadratic equation:

$$\text{Root}_{1,2} = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

Equation	Roots	Transient Response
$b^2 - 4ac > 0$	Real, different	Overdamped
$b^2 - 4ac = 0$	Real, same	Critically damped
$b^2 - 4ac < 0$	Complex, different	Underdamped
$b = 0$	Complex, same	Undamped

# 's' Variable

- 's' Laplace operator or Laplace transform variable.
- 's' can be considered as a differentiator ( $df(t)/dt$ ).
- 's' can be considered as a gradient.



- A variable at an instance is a number:
  - Numbers can be real or imaginary.

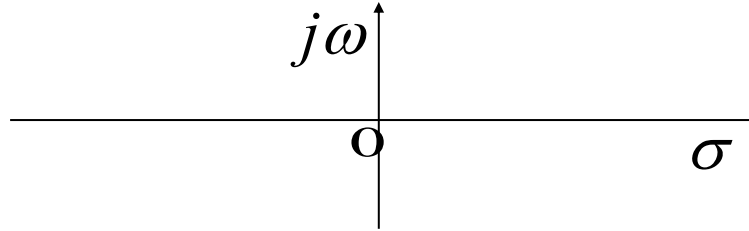
$$s = a \quad \text{or} \quad s = a + bj$$

- Usually given in control systems as:

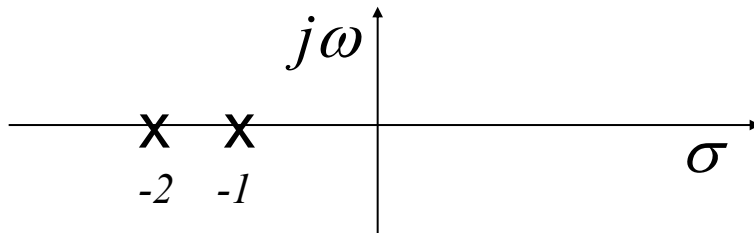
$$s = \sigma \quad \text{or} \quad s = \sigma + j\omega$$

# 's' Domain

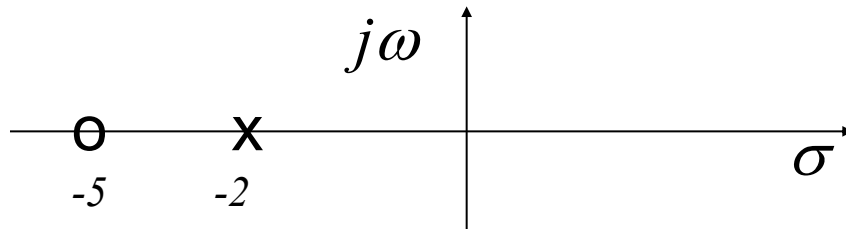
- We can plot the  $s$  variable on a  $s$ -domain diagram:



- Consider the systems given below:



*Poles (x) cause the system to be infinite in the  $s$ -domain*



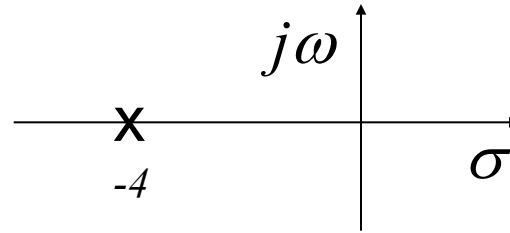
*Zeros (o) cause the system to be zero in the  $s$ -domain*

# System Response

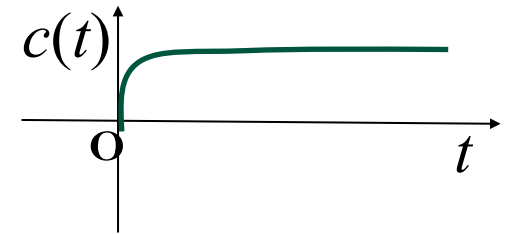
## Transfer Function

$$G_1(s) = \frac{4}{s + 4}$$

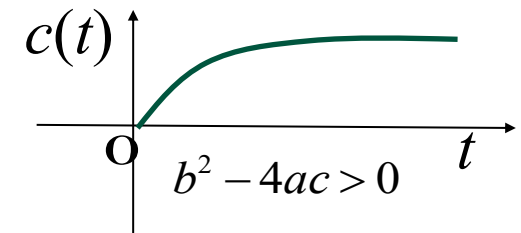
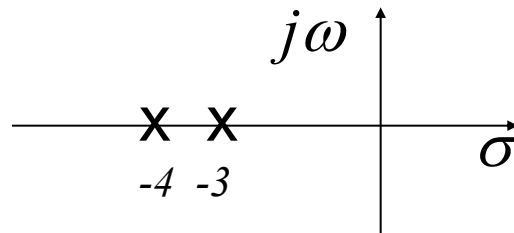
## s-Domain



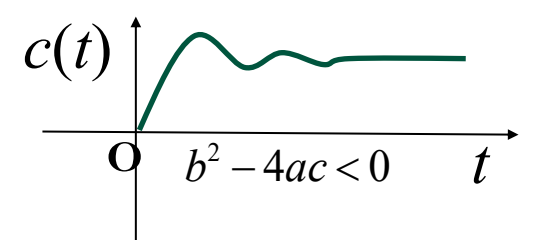
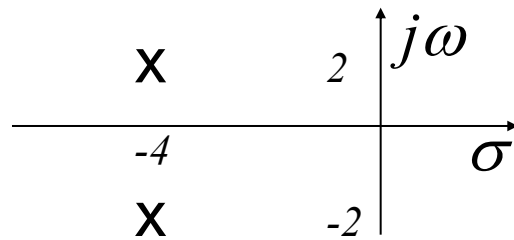
## Time Response



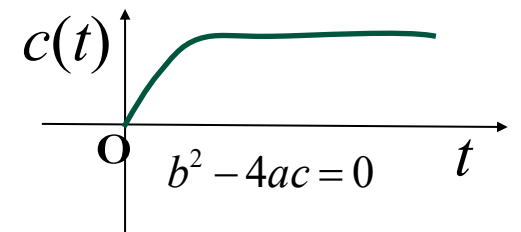
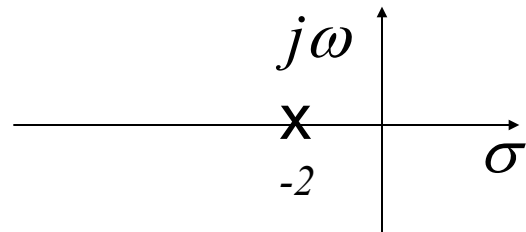
$$G_2(s) = \frac{12}{s^2 + 7s + 12}$$



$$G_3(s) = \frac{100}{s^2 + 8s + 100}$$



$$G_4(s) = \frac{4}{s^2 + 4s + 4}$$



# Example of Stability of Systems

Consider the following two control systems:

i. System 1:

$$G_1(s) = \frac{1}{(s + 1)(s + 2)}$$

ii. System 2:

$$G_2(s) = \frac{s + 5}{s + 2}$$

What happens when:

- a. Root  $s = -1$ ? [4 marks]
- b. Root  $s = -2$ ? [4 marks]
- c. Root  $s = -5$ ? [4 marks]

# Zeros (0) and Poles ( $\infty$ )

What happens when the following conditions exist:

a. When  $s = -1$ ?

$$G_1(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{(1-1)(2-1)} = \infty$$

$$G_2(s) = \frac{s+5}{s+2} = \frac{5-1}{2-1} = \frac{4}{1} = 1$$

Notice that the output of system 1,  $G_1(s)$  is  $\infty$  when  $s = -1$  that indicates its response is a growing. The system is unstable.

On the other hand, the output of system 2,  $G_2(s)$  is 1 when  $s = -1$  that indicates its response settles at 1. The system is stable.

# Zeros (0) and Poles ( $\infty$ )

b. When  $s = -2$ ?

$$G_1(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{(-1)(0)} = \infty$$

$$G_2(s) = \frac{s+5}{s+2} = \frac{3}{0} = \infty$$

Both systems 1 and 2 are growing. Both systems are unstable.

c. When  $s = -5$ ?

$$G_1(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{(-4)(-3)} = \frac{1}{12}$$

$$G_2(s) = \frac{s+5}{s+2} = \frac{0}{-3} = 0$$

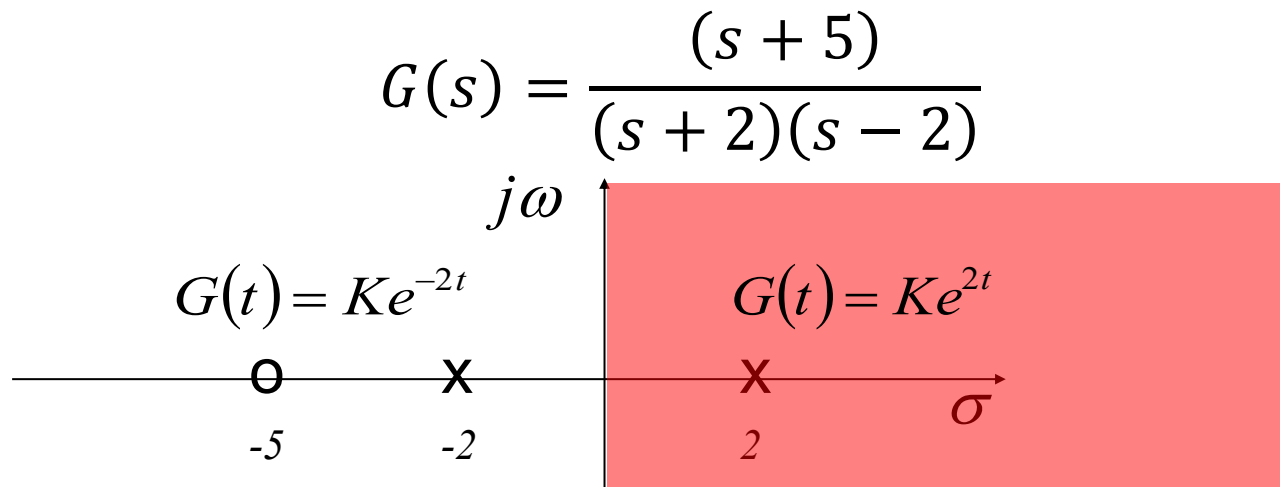
Both systems settle to  $1/12$  and  $0$ . Both systems are stable.

# Time Domain

- We are worried if the system is infinite or 0 in the TIME domain.
- System response often contains an exponential component:

$$G(t) = ke^{at}$$

- When  $a > 0$ , the output will reach infinity! We MUST avoid the right half of the plane poles.



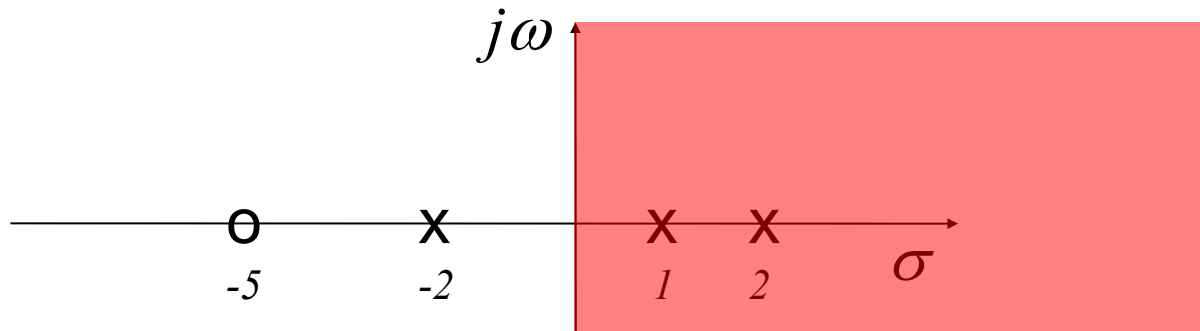
*All poles must be in the left-half plane for stability*

*Any poles in the right-half plane will cause the system to be UNSTABLE*

# Stability

- If the poles of a system's transfer-function equation are all on the left half of the  $s$ -plane, then that system is stable.

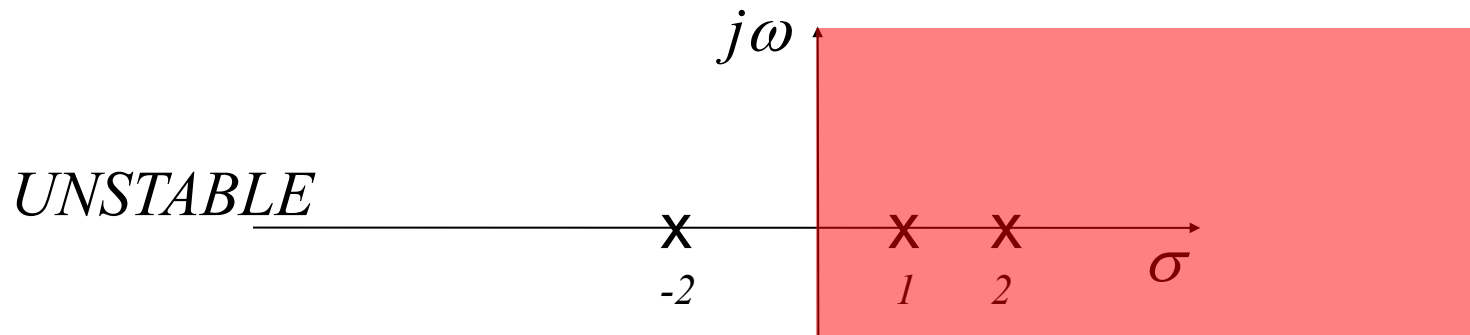
$$G(s) = \frac{(s + 5)}{(s + 2)(s - 1)(s - 2)}$$



# Stability

- It is only the poles of a system transfer function which are important as far as the stability is concerned (non-cancelling zeros can be ignored).

$$G(s) = \frac{1}{(s + 2)(s - 1)(s - 2)}$$



# Stability

- The poles of a system are the polynomial roots obtained when the system denominator is equated with 0.

$$G(s) = \frac{(s + 5)}{(s + 2)(s - 1)(s - 2)}$$

- The system denominator is known as the *characteristic polynomial*.

$$G(s) = \frac{(s + 5)}{(s + 2)(s - 1)(s - 2)}$$

- The system denominator equated to zero is the *characteristic equation*.

$$(s + 2)(s - 1)(s - 2)$$

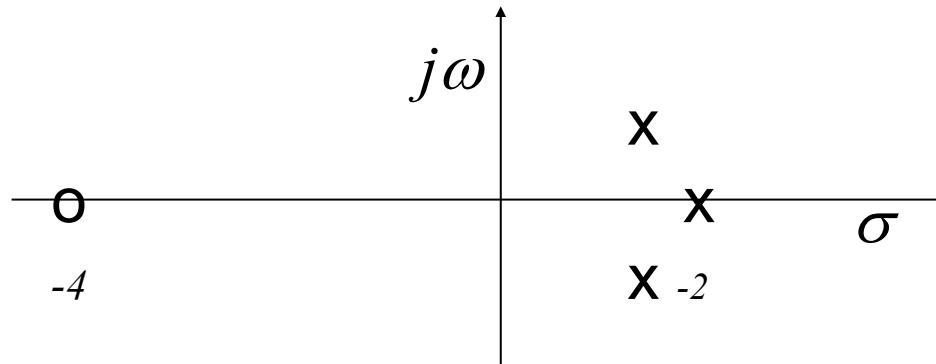
- Open-loop stable may/may not be closed-loop stable
- Open-loop unstable, may/may not be closed-loop unstable!

# Unstable System Response

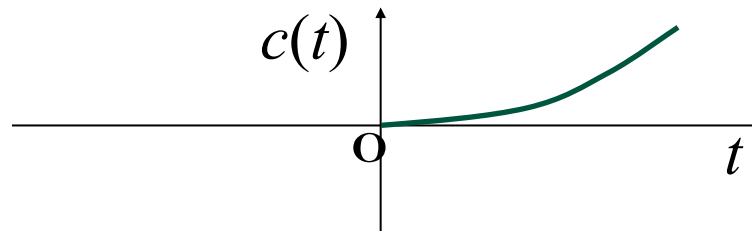
- Transfer function:

$$G(s) = \frac{(s + 4)}{s^3 - 4s^2 + 21s - 34}$$

- s-Domain

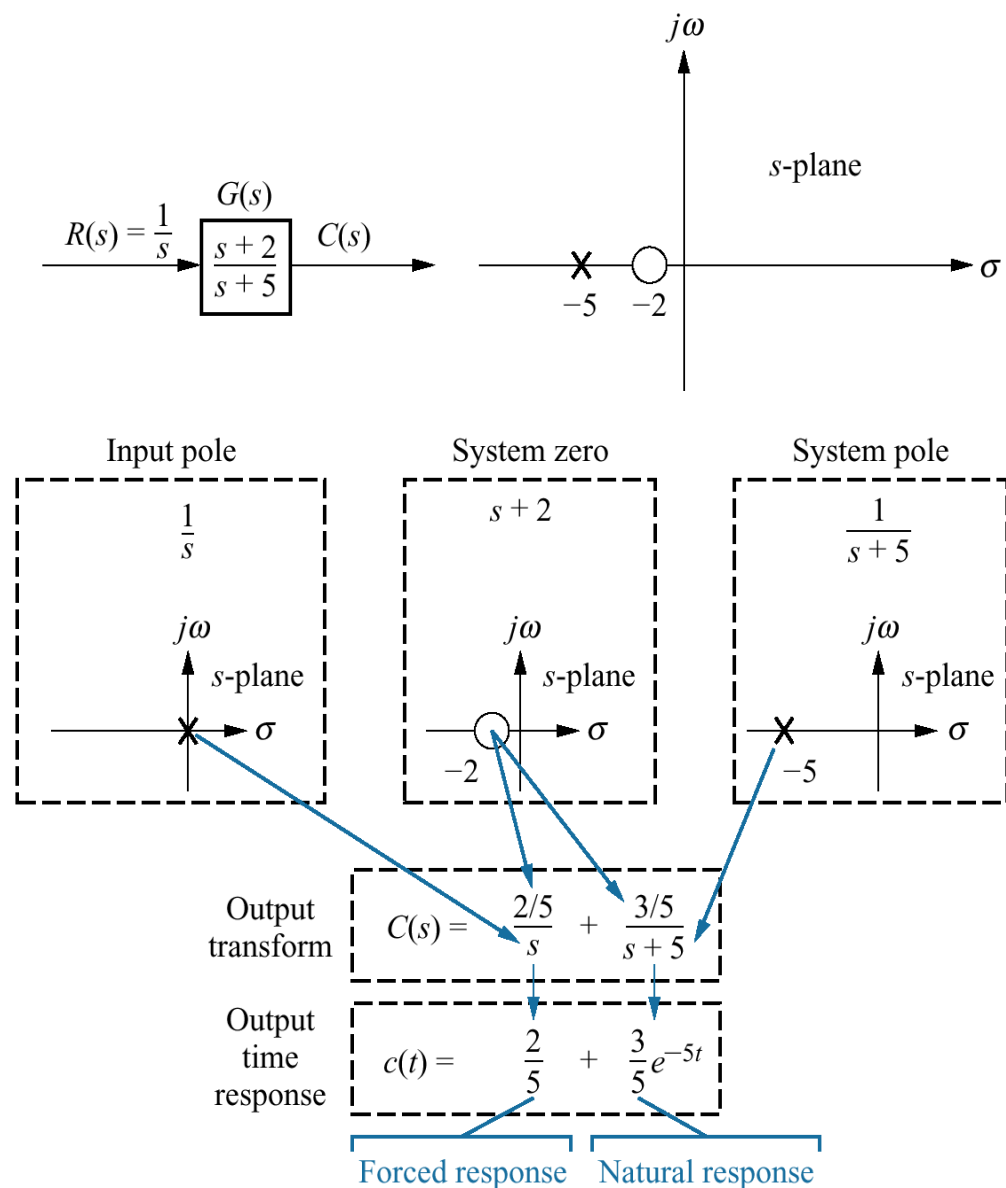


- Time response

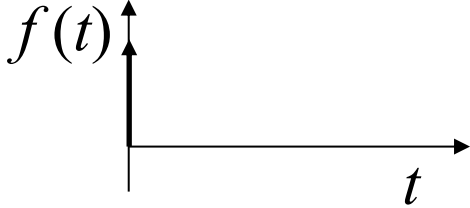
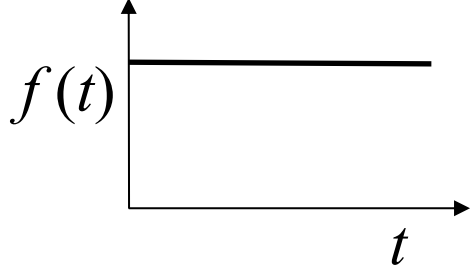
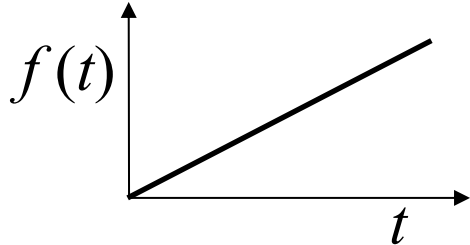
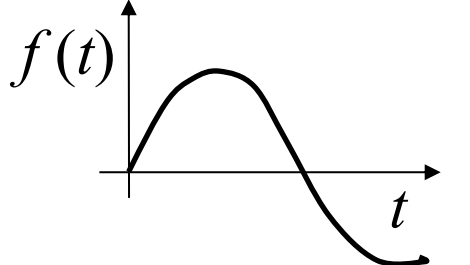


# Forced and Natural Responses

- Natural response: output due to the response of the system itself, without external input.
- Forced response: output due to intentional input, external to the system.
- Stability of the system mainly depends on the natural response of the system.

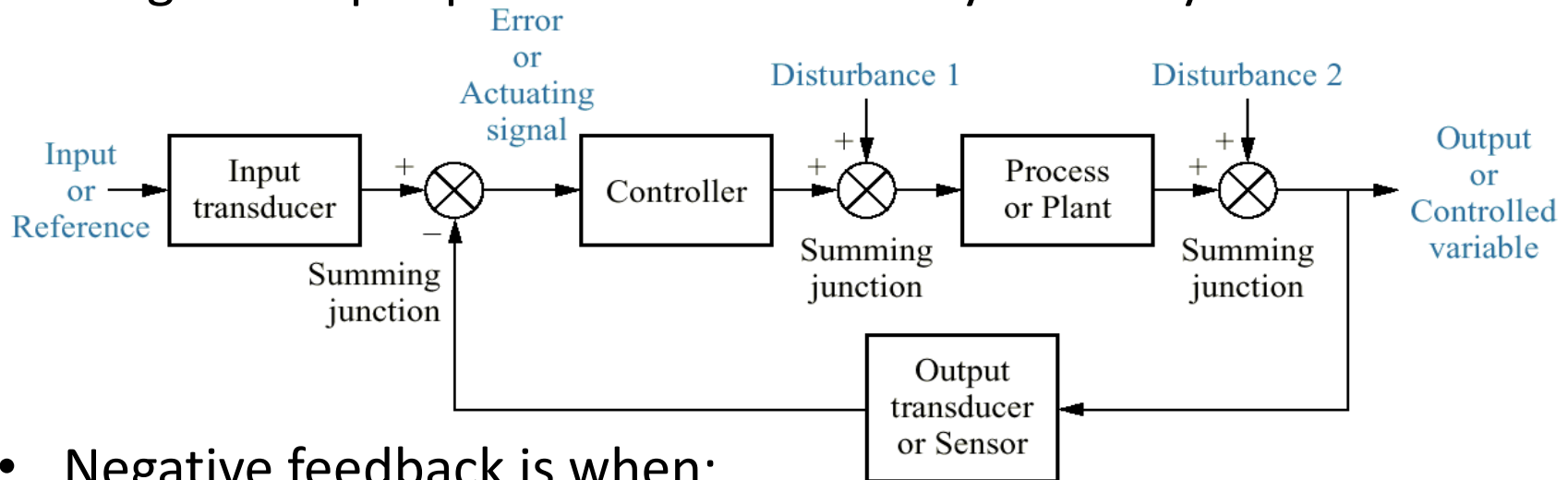


# Forced Response

Input	Function	Sketch	s-domain
Impulse	$\delta(t)$		1
Step	$u(t)$		$\frac{1}{s}$
Ramp	$tu(t)$		$\frac{1}{s^2}$
Sinusoid	$\sin \omega t$		$\frac{\omega}{s^2 + \omega^2}$

# Improvements to Stability of System

- Negative feedback can reduce the effects of disturbances and changes to input parameters -> stability of the system.



- Negative feedback is when:

$$|closed\ loop\ gain| < |open\ loop\ gain|$$

- This reduces the steady-state error by making the output closer to the input.
- To improve the stability of the system further, we could add a controller or compensator.

# Improvements to Stability of System

- Controllers or compensators change the natural response of the system.
- They adjust the poles of the system.
- They help achieve the desired output from a given input.
- Controllers or compensators can be mechanical, natural or electrical (used in industry).
- Three main types of controller or compensator:
  - Gain (proportional).
  - Lead or lag (lead, lag, and lead-lag).
  - PID (proportional, integral, derivative, or any of their combinations).

# Stability Analysis

- Various types of stability analysis in control systems:
  - Analytical (i.e. requiring a model of the physical system and maths to solve differential equations – mostly approximation for complex systems).
  - Experimentation (i.e. running a set of trials and observation or measurement).
- Approaches used in the course:
  - Routh-Hurwitz criterion – mathematical process.
  - Nyquist plot – graphical tool.
  - Nichol's chart – graphical tool.
  - Bode plot – graphical tool.
  - Root locus diagram – graphical tool.

# Routh-Hurwitz Criterion

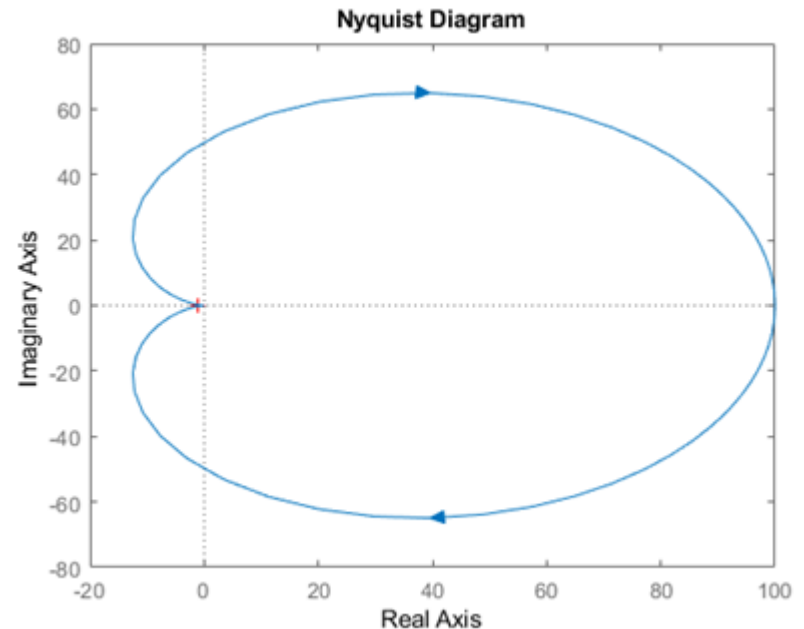
- Analyse the stability of a system through mathematical analysis of the characteristic equation.

$$G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s^1 + a_0s^0}$$

$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
$s^1$	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
$s^0$	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

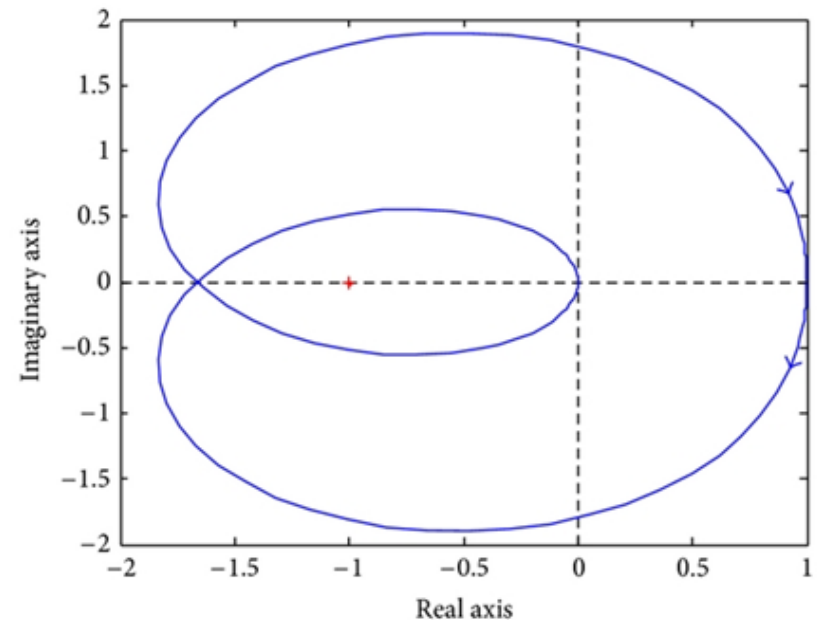
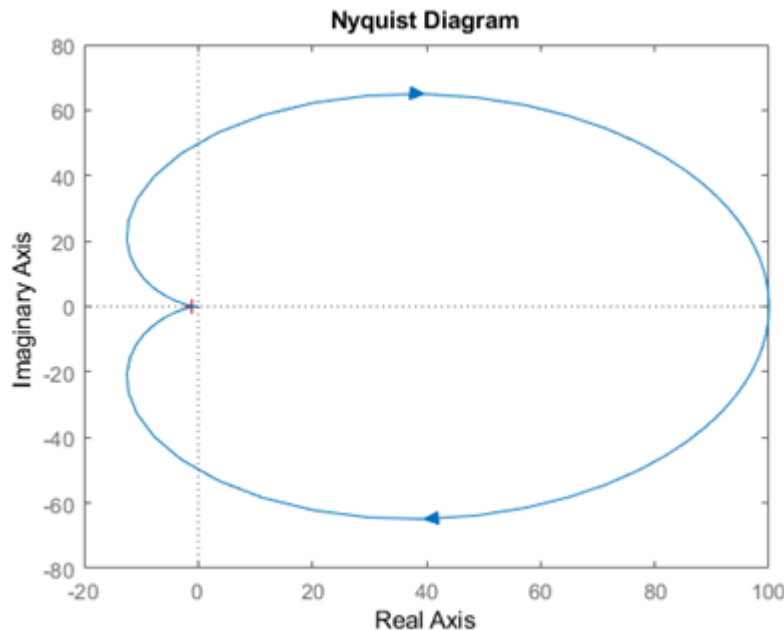
# Nyquist Plot

- Analyse the stability of a system through the use of a polar plot of the system equation.
- In Cartesian coordinates, the real part of the transfer function is plotted on the x-axis.
- The imaginary part is plotted on the y-axis. The frequency is swept as a parameter, resulting in a plot per frequency.
- Alternatively, in polar coordinates, the gain of the transfer function is plotted as the radial coordinate, while the phase of the transfer function is plotted as the angular coordinate.



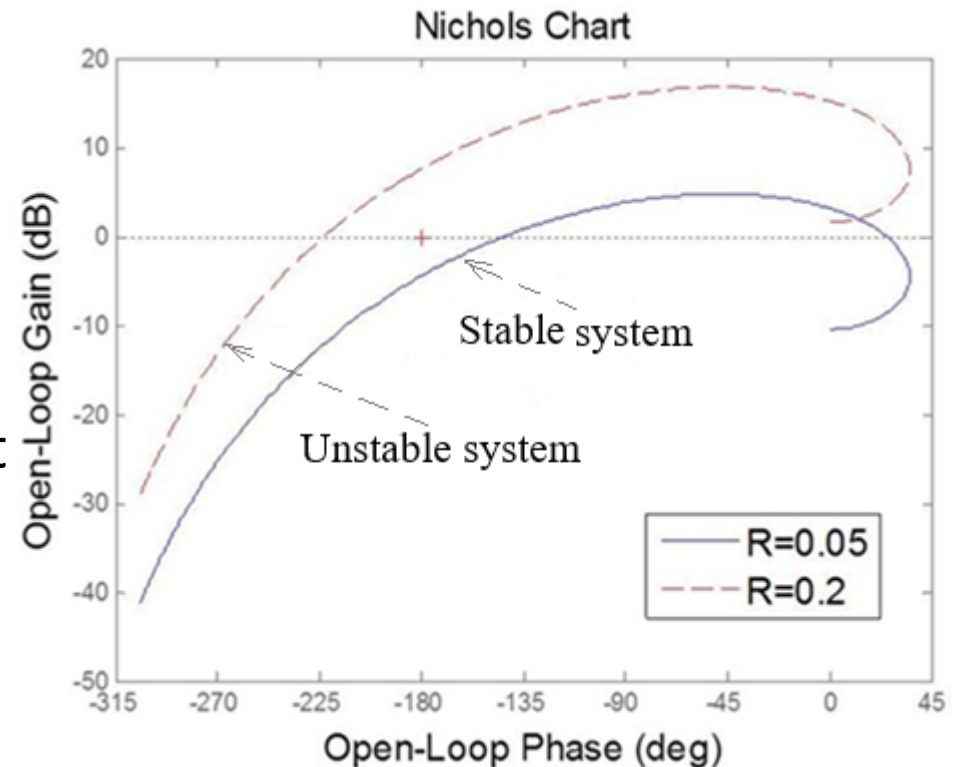
# Nyquist Plot

- System is stable if the plot does not encircle the unity gain point  $(-1, 0)$  in the Nyquist plot.
- The system below on the left is deemed to be stable as the plot is not encircling the Nyquist stability node  $(-1, 0)$ .
- The system on the right is not stable as the plot encircles the Nyquist stability node  $(-1, 0)$ .



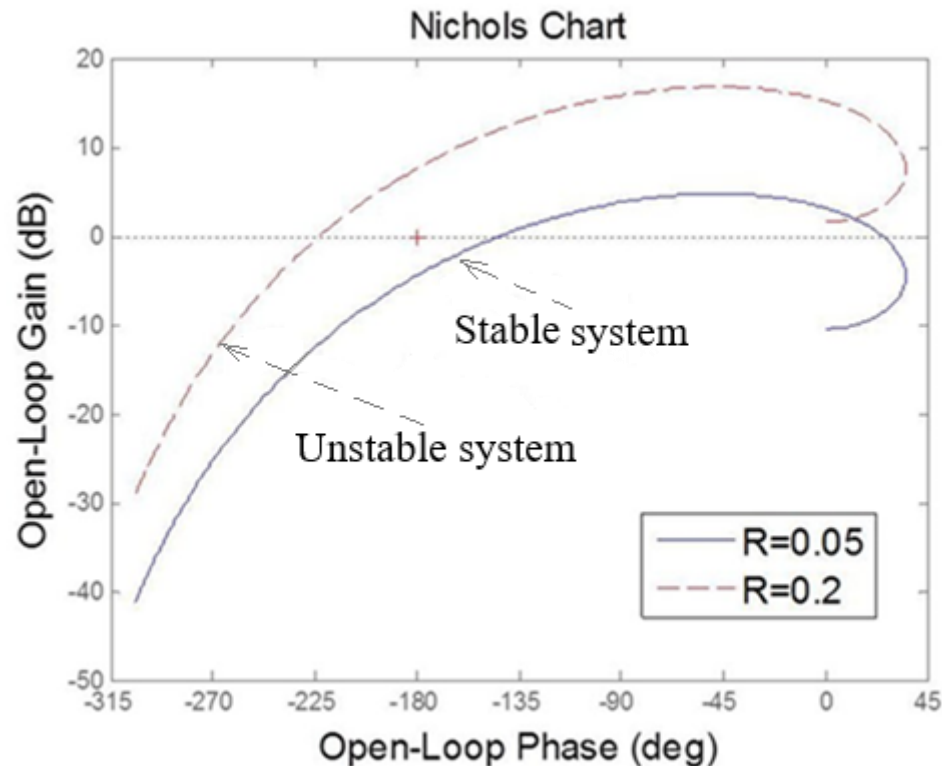
# Nichols Chart

- Analyse the stability of a system through the use of the gain and phase plot of the system.
- A Nichols chart displays the magnitude (in dB) plotted against the phase (in degrees) of the system response.
- It is used further to analyse the stability of a system beyond the Nyquist plot.
- System is considered to be stable if the value of gain at  $180^\circ$  in the Nichols chart is positive.



# Nichols Chart

- The system below is deemed to be stable as the curve of the plot is at the gain  $> 0$  dB whenever phase =  $180^\circ$  in the Nichols chart.
- Contrast with the unstable system when its gain  $< 0$  dB at  $180^\circ$ .



# Routh-Hurwitz Criterion

This stability analysis technique is based on the work of Edward Routh, whose algorithm was proposed in 1876, and Adolf Hurwitz, who independently proposed a contribution in 1895.



Edward Routh



Adolf Hurwitz

# Routh-Hurwitz Criterion

- Any system that has closed-loop poles in the right-half of the  $s$ -plane will be unstable.
- The Routh Hurwitz table is a simple method that allows us to determine the number of poles in the left- and right-halves of the  $s$ -plane and the number of poles on the imaginary axis.
- It tells us nothing about *where* those poles are, so it is not generally sufficient to design a control system.
- It is just a specialised tool to tell us about system stability.

# Routh-Hurwitz Criterion

- Using this method, we can tell how many closed-loop system poles are in the left-half plane, in the right-half plane, and on the  $j\omega$ -axis.
- We can not tell where, but only how many are in each plane, determining the system's stability.
- The method requires two steps:
  - Generate a data table called a Routh table.
  - Interpret the Routh table to tell how many closed-loop system poles are in the left half-plane, the right half-plane, and on the  $j\omega$  -axis.

$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$	$-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
$s^1$	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
$s^0$	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

# Construction of the Routh Array

- Consider a *closed-loop* transfer function having form:

$$G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s^1 + a_0s^0}$$

- When considering stability, we are interested only in the poles of  $G(s)$ , therefore we examine the denominator polynomial of  $G(s)$ .
- To form the Routh array, start by writing powers of  $s$  down the left-hand side and then fill in the coefficients of the denominator polynomial.

$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$	$-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
$s^1$	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
$s^0$	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

# Construction of the Routh Array

- Note that in the first row, we start with the coefficient of the highest power of  $s$  and then write every second coefficient.
- In the second row, we write in the coefficients that were not included in the top row.
- Include a zero at the end of row two if necessary to fill out the array.
- To complete the array, we fill out successive rows as a function of the two rows above.
- See the table for the required pattern.

$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
$s^1$	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
$s^0$	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

# Filling in the Array

- The Routh table:

$$G(s) = \frac{N(s)}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s^1 + a_0s^0}$$

$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
$s^1$	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
$s^0$	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

# Filling in the Array

- The Routh criterion states that the number of closed-loop poles in the right half of the s-plane is equal to the number of sign changes in the first column of the Routh table.
- For stability, we therefore require zero sign changes in the first column.
- Any change of sign indicates that the system is unstable.

$s^4$	$a_4$	$a_2$	$a_0$	+/-
$s^3$	$a_3$	$a_1$	0	+/-
$s^2$	$b_1$	$b_2$	0	+/-
$s^1$	$c_1$	0	0	+/-
$s^0$	$d_1$	0	0	+/-

# Interpreting a Routh Table

- If the closed-loop transfer function has all poles in the left-half of the s-plane, the system is stable.
- Thus, a system is stable if there are no sign changes in the first column of the Routh table.

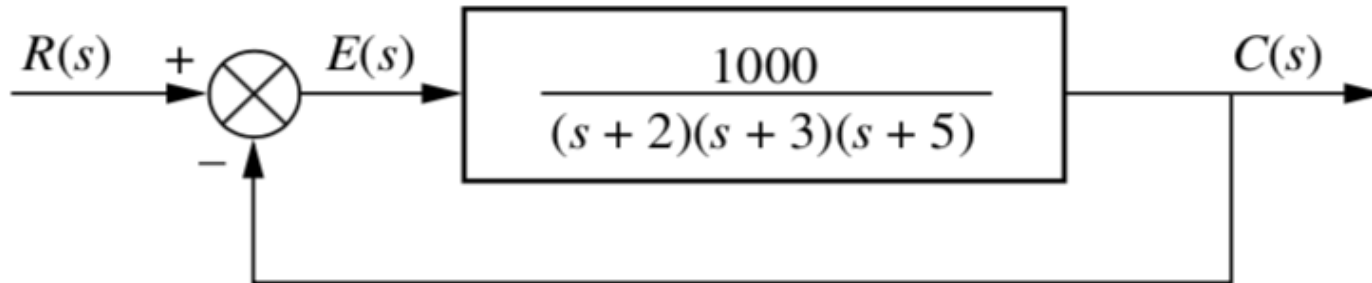
<b>Sign change</b>	<b>Interpretation</b>
0	No right-hand side pole
1	1 right-hand side pole
2	2 right-hand side poles
...	
N	Nth right-hand side poles

# Example of Routh Array Construction

- Consider a system with a transfer function of:

$$G(s) = \frac{1000}{(s + 2)(s + 3)(s + 5)}$$

- Determine whether this system is stable when closed in a unity gain feedback loop. [14 marks]

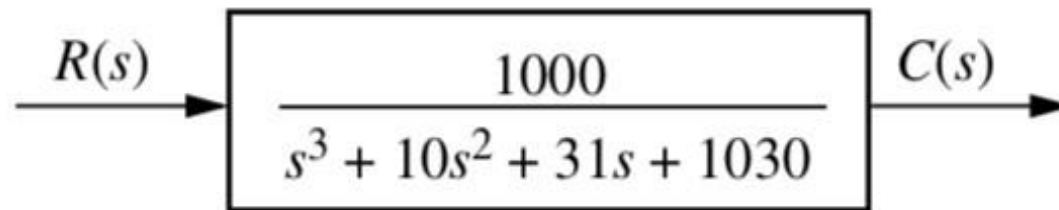


# Find Closed Loop Transfer Function

- We first need to find the closed-loop transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{1000}{s^3 + 10s^2 + 31s + 1030}$$

- Notice that we have completely expanded the denominator rather than leaving it as the product of poles as we normally do.



- The characteristic equation of the system is:

$$s^3 + 10s^2 + 31s + 1030$$

# Scaling in the Rows in the Table

- Now write the powers of  $s$  down the left side and fill in the coefficients from the denominator polynomial of the closed-loop transfer function.

$s^3$	1	31
$s^2$	10	1030
$s^1$		
$s^0$		

- We are allowed to divide any row of the array by a *positive* number if it simplifies the calculation.
- We will divide the second row by ten.

$s^3$	1	31
$s^2$	$10/10 = 1$	$1030/10 = 103$
$s^1$		
$s^0$		

# Filling in the Array

- We see that the first column changes from positive to negative after the second row and then from negative to positive after the third row.
- We therefore have two sign changes.
- The Routh criterion tells us that the transfer function has two poles in the right half of the s-plane.

$s^3$	1	31	0
$s^2$	1	103	0
$s^1$	$\frac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	$\frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
$s^0$	$\frac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

# Interpreting a Routh Table

- Simply stated, the Routh-Hurwitz criterion declares that the number of roots of the polynomial that are in the right-hand plane is equal to the number of sign changes in the first column.

$s^3$	1	31	0	+
$s^2$	1	103	0	+
$s^1$	$\frac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	$\frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	-
$s^0$	$\frac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	+

# Special Cases of the Routh Array

- A couple of special situations can arise when constructing a Routh array.
- While constructing the array, we might find that we end up with:
  - A zero in the first column.
  - A row of zeros.
- Either of these occurrences will “break” the procedure outlines above.
- However, there are simple tricks to deal with such situations - consult any control text for further information.

# Special Cases of the Routh Array

- A zero in the first column.

$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 = (\mathbf{0})$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
$s^1$	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$
$s^0$	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

- A row of zeros.

$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 = (\mathbf{0})$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 = (\mathbf{0})$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = (\mathbf{0})$
$s^1$	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$
$s^0$	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

# Routh-Hurwitz Special Cases

Zero in the first row/column:

- In this case, the zero is replaced with epsilon ( $\epsilon$ ).

$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$	$-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 = (0)$	$-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
$s^1$	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 = \infty$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$
$s^0$	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

- Evaluate the sign change in the first column. Interpret the stability of the system.

$s^4$	$a_4$	$a_2$	$a_0$	+/-
$s^3$	$a_3$	$a_1$	0	+/-
$s^2$	$b_1 = (\epsilon)$	$-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$	+/-
$s^1$	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	+/-
$s^0$	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	+/-

# Example of Routh-Hurwitz Special Cases 1

Given the following transfer function of a control system, we perform stability analysis of the system using the Routh-Hurwitz method. [6 marks]

$$T(s) = \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$$

- The characteristic equation of the system is:

$$2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1$$

- Create Routh table based on the equation given above.

# Example of Routh-Hurwitz Special Cases 1

- This example shows an example of zero in the first row or column.
- In this case, zero is replaced with epsilon ( $\epsilon$ ) and will tend to zero.

$s^5$	2	2	2
$s^4$	3	3	1
$s^3$	$\frac{-\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}}{3} = 0$	$\frac{-\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}}{3} = 4/3$	$\frac{-\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}}{3} = 4/3$
$s^2$	$\frac{-\begin{vmatrix} 3 & 3 \\ 0 & 4/3 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} 2 & 1 \\ 0 & 4/3 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} 2 & 1 \\ 0 & 4/3 \end{vmatrix}}{0} = \infty$
$s^1$	$\frac{-\begin{vmatrix} 0 & 4/3 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$	$\frac{-\begin{vmatrix} 0 & 4/3 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$	$\frac{-\begin{vmatrix} 0 & 4/3 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$
$s^0$	$\frac{-\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$

# Example of Routh-Hurwitz Special Cases 1

- There are two sign changes in the first row of the Routh table.
- System has two unstable poles located on the right-hand side of the s-plane.

$s^5$	2	2	2	+
$s^4$	3	3	1	+
$s^3$	$(\epsilon)$	$4/3$	0	+
$s^2$	$\frac{3\epsilon - 4}{\epsilon}$	1	0	-
$s^1$	$\frac{12\epsilon - 16 - 3\epsilon^2}{9\epsilon - 12}$	0	0	+
$s^0$	1	0	0	+

# Routh-Hurwitz Special Cases

Entire row of zeros:

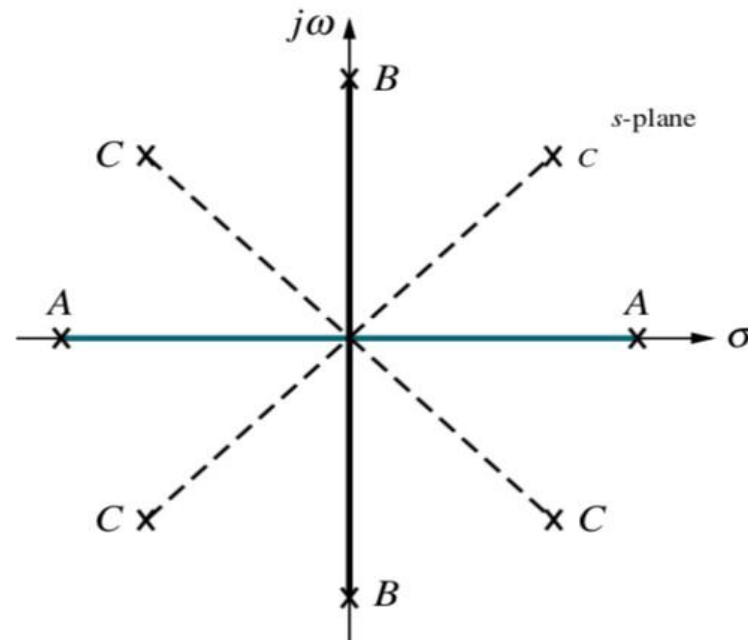
$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 = (0)$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 = (0)$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = (0)$
$s^1$	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$
$s^0$	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

- There is an even polynomial that is a factor of the original polynomial.
- This case has to be handled differently from the case with a zero in the first column of a row.

# Routh-Hurwitz Special Cases

Entire row of zeros in the s-plane:

- Real and symmetrical about the origin (A).
- Imaginary and symmetrical about the origin (B).
- Quadrantal and symmetrical about the origin (C).



- A: Real and symmetrical about the origin —
- B: Imaginary and symmetrical about the origin —
- C: Quadrantal and symmetrical about the origin - - -

# Routh-Hurwitz Special Cases

- The row of zeros tells us of the existence of an even polynomial whose roots are symmetric about the origin.
- Some of these roots could be on the  $j\omega$ -axis.
- On the other hand, since  $j\omega$  roots are symmetric about the origin, if we do not have a row of zeros, we cannot possibly have  $j\omega$  roots.
- Another characteristic of the Routh table for the case in question is that the row previous to the row of zeros contains the even polynomial that is a factor of the original polynomial.
- Finally, everything from the row containing the even polynomial down to the end of the Routh table is a test of only the even polynomial.

# Routh-Hurwitz Special Cases

- For the entire row of zeros, in the Routh-Hurwitz table, go up a row above the row with all zero coefficients.

$s^4$	$a_4$	$a_2$	$a_0$
$s^3$	$a_3$	$a_1$	0
$s^2$	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 = (0)$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 = (0)$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = (0)$
$s^1$	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$
$s^0$	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

- Create an equation that is made up of the coefficients of the given row.

$$P = a_3s^3 + a_1s^1 + 0$$

# Routh-Hurwitz Special Cases

- Differentiate the respective equation concerning the 's' function.

$$\frac{dP}{ds} = 3a_3s^2 + a_1$$

- Replace the row with all zero coefficients with the coefficients of the resulting function after the differentiation.

$s^4$	$a_4$	$a_2$	$a_0$	+/-
$s^3$	$a_3$	$a_1$	0	+/-
$s^2$	$b_1 = (3a_3s^2)$	$b_2 = (a_1)$		+/-
$s^1$	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	+/-
$s^0$	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	+/-

- Evaluate the sign change in the first column of the table. Interpret the stability of the system.

# Example of Routh-Hurwitz Special Cases 2

For the transfer function equation of a fifth order system, perform stability analysis using Routh-Hurwitz method.

[10 marks]

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

The characteristic equation of the system is:

$$s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56$$

Create Routh table based on the equation above.

# Example of Routh-Hurwitz Special Cases 2

- The Routh-Hurwitz table of the control system given above is shown in the figure below.

- Row  $s^4$  is multiplied by  $1/7$  for convenience.

- Notice that row with  $s^3$  is all zero.

- The Routh table is summarised as shown below.

$s^5$	1	6	8
$s^4$	$\left(\frac{7}{7}\right) = 1$	$\left(\frac{42}{7}\right) = 6$	$\left(\frac{56}{7}\right) = 8$
$s^3$	$\frac{-\begin{vmatrix} 1 & 6 \\ 1 & 6 \end{vmatrix}}{1} = 0$	$\frac{-\begin{vmatrix} 1 & 8 \\ 1 & 8 \end{vmatrix}}{1} = 0$	$\frac{-\begin{vmatrix} 1 & 8 \\ 1 & 8 \end{vmatrix}}{1} = 0$
$s^2$	$\frac{-\begin{vmatrix} 1 & 6 \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} 1 & 8 \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} 1 & 8 \\ 0 & 0 \end{vmatrix}}{0} = \infty$
$s^1$	$\frac{-\begin{vmatrix} 0 & 0 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$	$\frac{-\begin{vmatrix} 0 & 0 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$	$\frac{-\begin{vmatrix} 0 & 0 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$
$s^0$	$\frac{-\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$

$s^5$	1	6	8
$s^4$	1	6	8
$s^3$	(0)	(0)	(0)
$s^2$	$\infty$	$\infty$	$\infty$
$s^1$	0	0	0
$s^0$	$\infty$	$\infty$	$\infty$

# Example of Routh-Hurwitz Special Cases 2

- In the Routh-Hurwitz table, go up a row above the row with all zero coefficients.
- Create an equation that is made up of the coefficients of the given row.
- At the row of  $s^4$ , the equation below is constructed from the coefficients in this row.

$$P = s^4 + 6s^2 + 8$$

- Differentiate the respected equation about the 's' function.
- The first derivative of the equation above is calculated as follow:

$$\frac{dP}{ds} = 4s^3 + 12s + 0$$

# Example of Routh-Hurwitz Special Cases 2

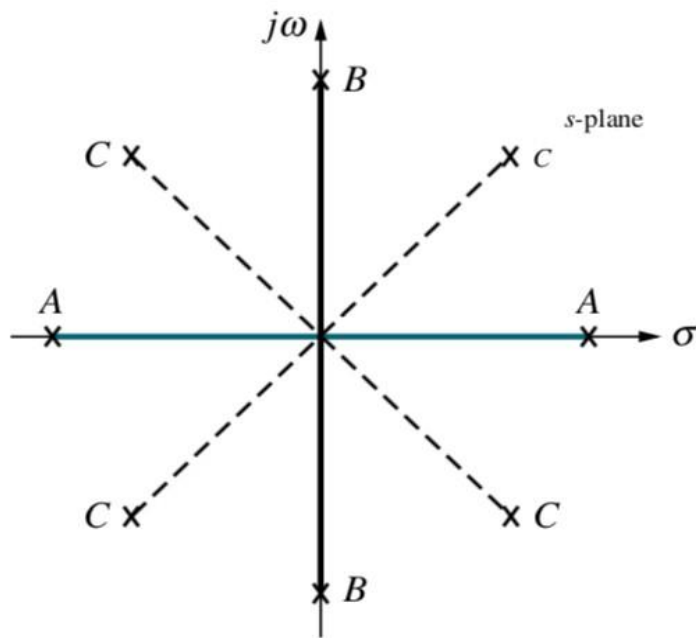
- Replace the row with all zero coefficients with the coefficients of the resulting function after the differentiation.
- As a result, the Routh table becomes as shown in the table below.




$s^5$	1	6	8	+
$s^4$	1	6	8	+
$s^3$	(4)	(12)	(0)	+
$s^2$	3	8	0	+
$s^1$	1/3	0	0	+
$s^0$	8	0	0	+

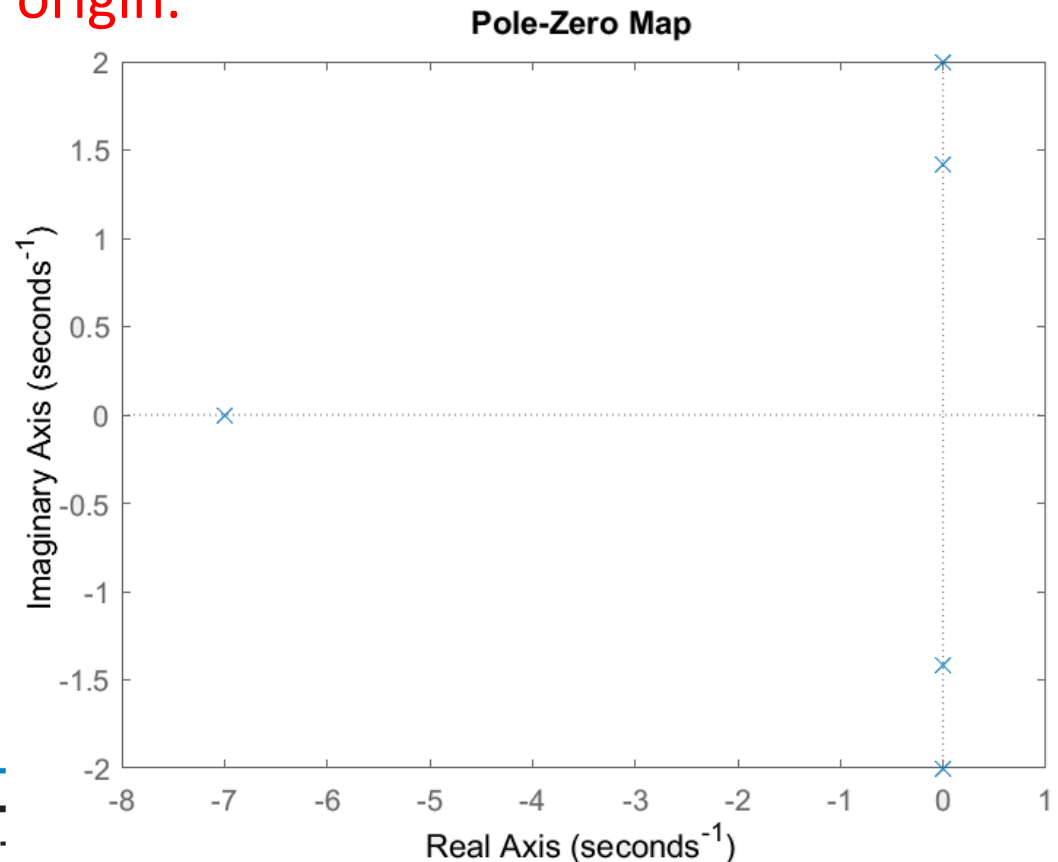
- From the result of the analysis, with all zeros in a given row, it seems that there is no change of sign in the first column of the Routh table.
- As a result, the given system has real and symmetrical poles about the origin.

# Example of Routh-Hurwitz Special Cases 2

- As indicated in the  $s$ -plane diagram of the system, as shown in the figure below, the poles for this system are real and symmetrical about the origin.



- A: Real and symmetrical about the origin 
- B: Imaginary and symmetrical about the origin 
- C: Quadrantal and symmetrical about the origin 



# Example of Routh-Hurwitz Special Cases 2

- Evaluating the even polynomial  $P = s^4 + 6s^2 + 8$ , no change of sign in the first column from row with  $s^4$  to the last row. So, the four poles of the polynomial are located on the y-axis.

- No change of sign from row with  $s^5$  to row with  $s^4$ . So, the remaining fifth pole is located on the left half-side of the s-plane.

