Time-Response Analysis

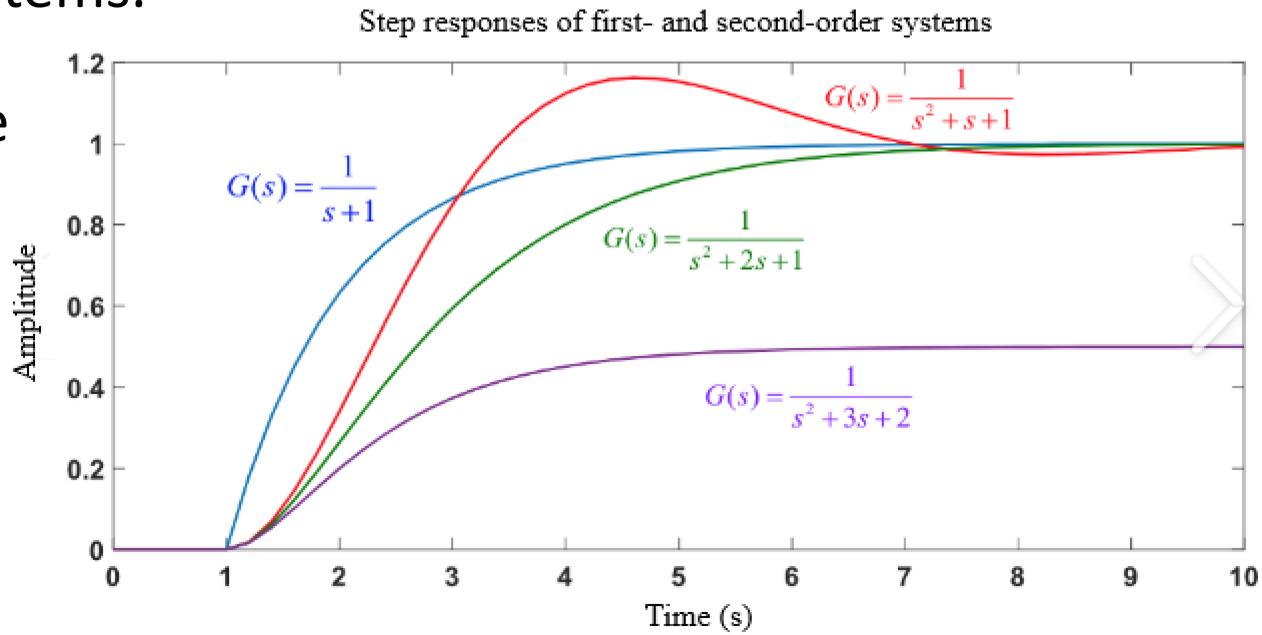
XMUT315 Control Systems Engineering

Topics

- First-order time responses analysis (e.g. time constant, rise time, and settling time).
- Second-order time responses analysis (e.g. damping ratio, rise time, settling time, time-to-peak, percentage overshoot, and steady-state error).
- Damping of the systems.
- Second-order system time responses.
- Trends in second-order system time responses.

Time Responses

- It is the time response of a system to an input that sets the criteria for our control systems.
- Many quantitative criteria have been defined to characterise the response of a system.



- The given diagrams show the time response of a first-order system (left) and a second-order system (right).
- The time response of a higher-order system could be approximated as a first- or second-order system.

Time Response of First-Order System

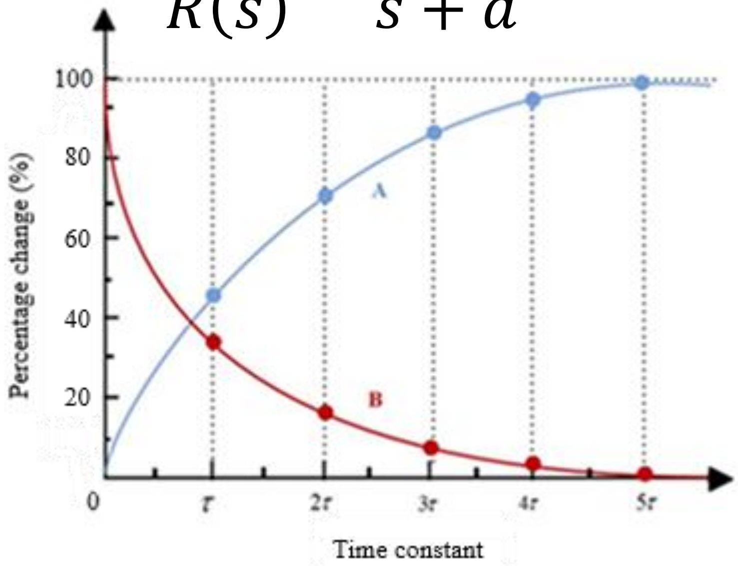
- A first-order system may be written as:

$$\frac{C(s)}{R(s)} = \frac{k}{s\tau + 1} \quad \text{or} \quad \frac{C(s)}{R(s)} = \frac{a}{s + a}$$

Where: $k = 1$ and $\tau = 1/a$

- Rearrange the equation:

$$C(s) = R(s)\left(\frac{a}{s + a}\right)$$



- The time constant and system gain of a first-order system are useful in its analysis.
- But other criteria describe the time response more accurately to an engineer.

Time Response of First-Order System

- For a unit step e.g. $R(s) = 1/s$:

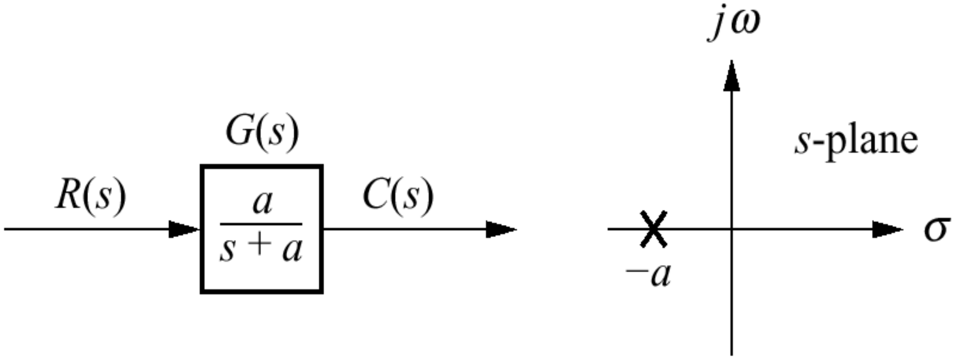
$$C(s) = R(s)\left(\frac{a}{s+a}\right) = \frac{1}{s}\left(\frac{a}{s+a}\right)$$

- Apply partial fraction:

$$C(s) = \frac{1}{s} - \frac{1}{s+a}$$

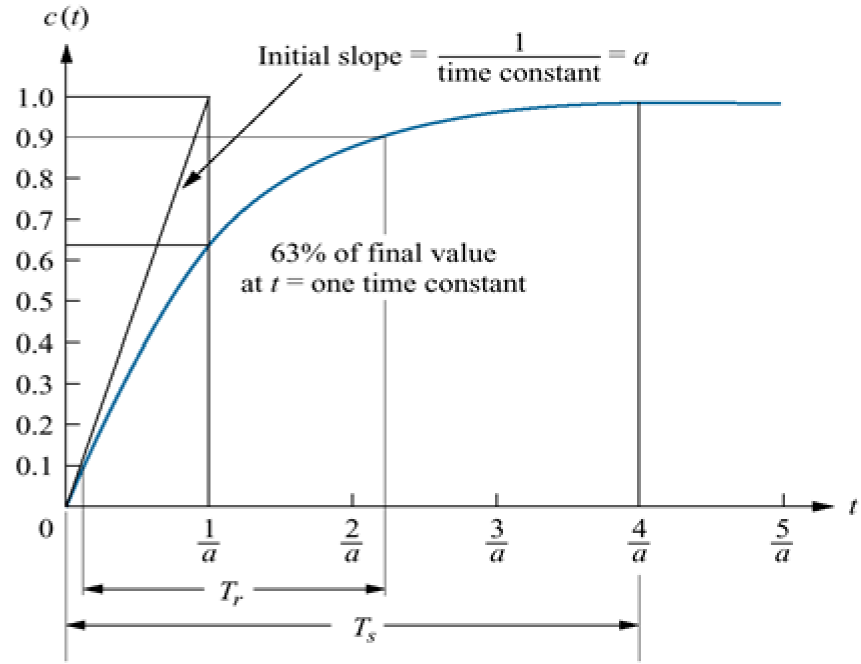
- By using Laplace table, we obtain the standard response:

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$



Time Response of First-Order System

- The time response for a first-order system depends on the gain and time constant of the system.
- Generally, the time response of a first-order system is exponential.
- Changing the gain or time constant only changes the steady-state value and time.
- Typical parameters:
  - Time constant (τ).
  - Rise time (T_r).
  - Settling time (T_s).

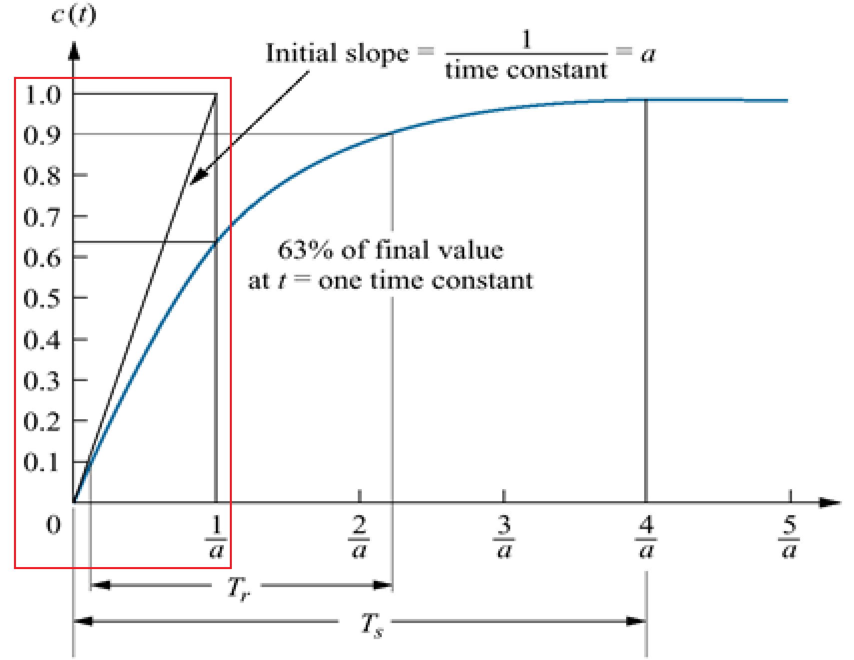


Time Constant of First-Order System

- For a first-order system response to a unit step, a time constant (τ) is defined as the time for the step response to rise to 63% of its final value.

- It is derived from the time for e^{-at} to decay to 37% of its final value.

$$e^{-at}\Big|_{t=1/a} = e^{-1} = 0.37$$



- So

$$c(t)\Big|_{t=1/a} = 1 - e^{-at}\Big|_{t=1/a}$$
$$= 1 - 0.37 = 0.63$$

Rise Time of First-Order System

- For a first-order system response to a unit step, rise time (T_r) is defined as the time for the response to go from 0.1 to 0.9 of its final value:

For $c(t)$ = 0.1, $c(t) = \left(1 - e^{-aT_{(0.1)}}\right) = 0.1 \rightarrow T_{(0.1)} = 0.11/a$

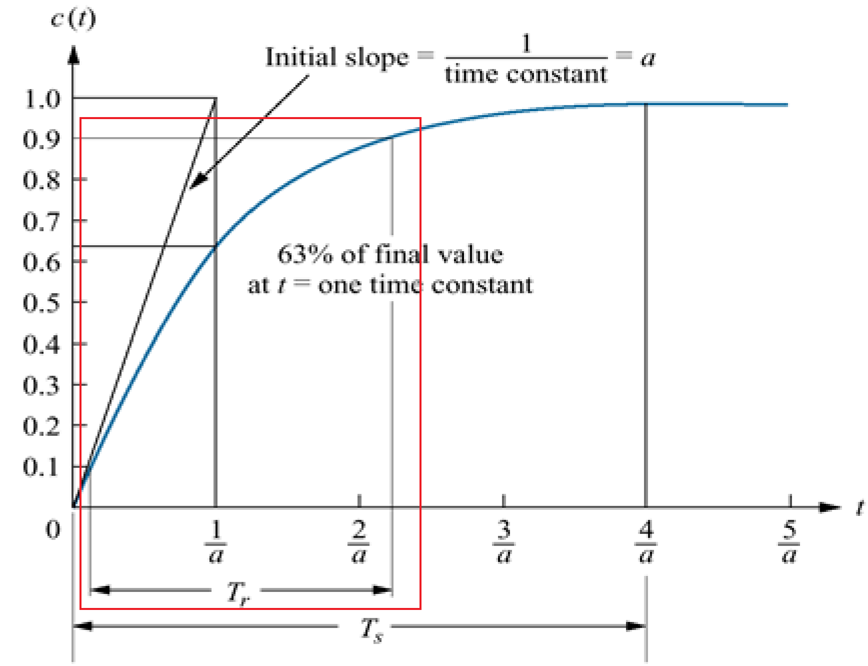
For c(t) = 0.9, $c(t) = \left(1 - e^{-aT_{(0.9)}}\right) = 0.9 \rightarrow T_{(0.9)} = 2.31/a$

- So, the rise time is:

$$T_r = T_{(0.9)} - T_{(0.1)}$$

$$= \frac{2.31}{a} - \frac{0.11}{a}$$

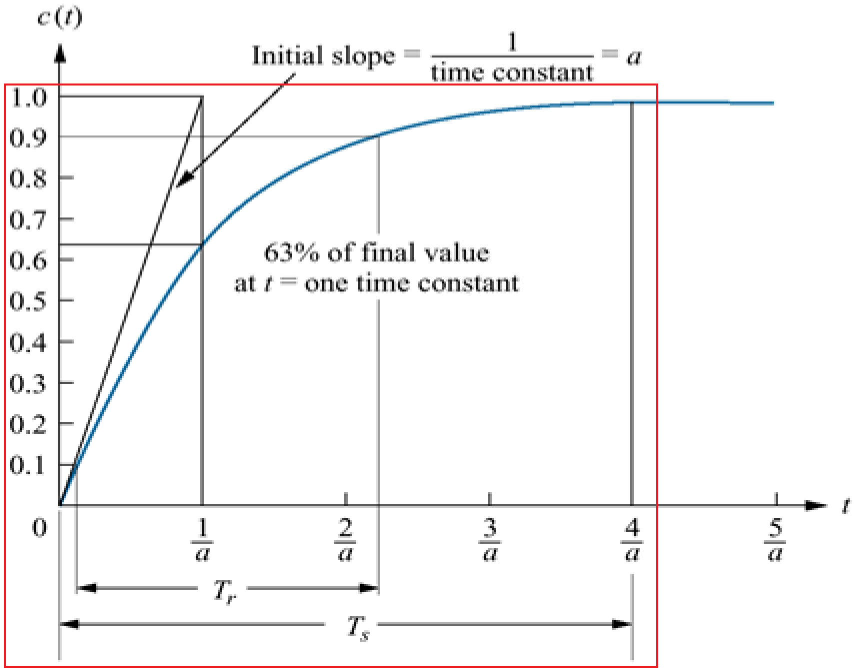
$$= \frac{2.2}{a}$$



Settling Time of First-Order System

- For a first-order system response to a unit step, settling time (T_s) is defined as the time for the response to reach a steady-state level.
- For first order systems, it is typically calculated as 4 times the time constant (τ) of the system.

$$T_s = 4\tau = \frac{4}{a}$$



Settling Time of First-Order System

- It could also be determined as the time taken by the system to stay within 2% of its final value (typically 2% is the standard).
- For a first-order system, since $\tau = 1/a$, the response of the system is calculated from:

$$c(t) = 1 - e^{at} = 1 - e^{-t/\tau}$$

- Rearrange the equation above:

$$e^{-t/\tau} = 1 - c(t)$$

- To calculate the 2% settling time:

$$e^{-T_s/\tau} = 1 - 0.98 = 0.02$$

- Thus:

$$T_s = -\tau \ln 0.02 = 3.9\tau \approx 4\tau$$

Example of Time Response of First-Order System

For a first-order system given as the transfer function below, calculate the following time-domain parameters of the system:

$$\frac{C(s)}{R(s)} = \frac{2.5}{s+3}$$

a. The time constant (τ). [2 marks]

b. The rise time (T_r). [2 marks]

c. The settling time (T_s). [2 marks]

Example of Time Response of First-Order System

a. The time constant (τ) is calculated from:

$$\tau = \frac{1}{a} = \frac{1}{3} = 0.33 \text{ s}$$

b. The rise time (T_r) is calculated from:

$$T_r = \frac{2.2}{a} = \frac{2.2}{3} = 0.733 \text{ s}$$

c. The settling time (T_s) is calculated from:

$$T_s = 4\tau = 4(0.33) = 1.32 \text{ s}$$

Time Response of Second Order System

- Consider a second-order system with the following transfer-function equation:

$$\frac{C(s)}{R(s)} = \frac{k}{a's^2 + b's + c'}$$

- To work out the natural frequency and damping characteristics, convert the transfer function equation of the system to a monic polynomial form with unity in front of the leading coefficient (s^2 term) and k such that:

$$\frac{C(s)}{R(s)} = \frac{b}{s^2 + as + b}$$

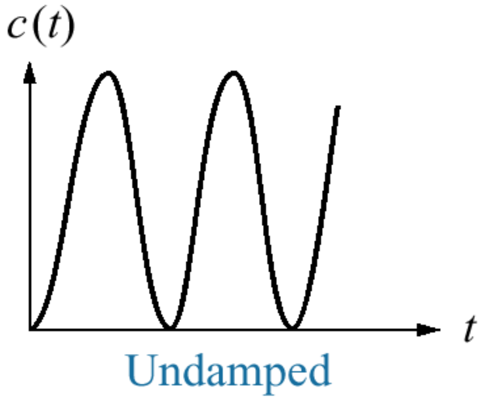
- Note that the constants a', b', and c' are not equivalent to a and b.

Natural Frequency

- Natural frequency is when there is no damping in the system (a = 0 in this case).

- We obtain:

$$\frac{C(s)}{R(s)} = \frac{b}{s^2 + b}$$



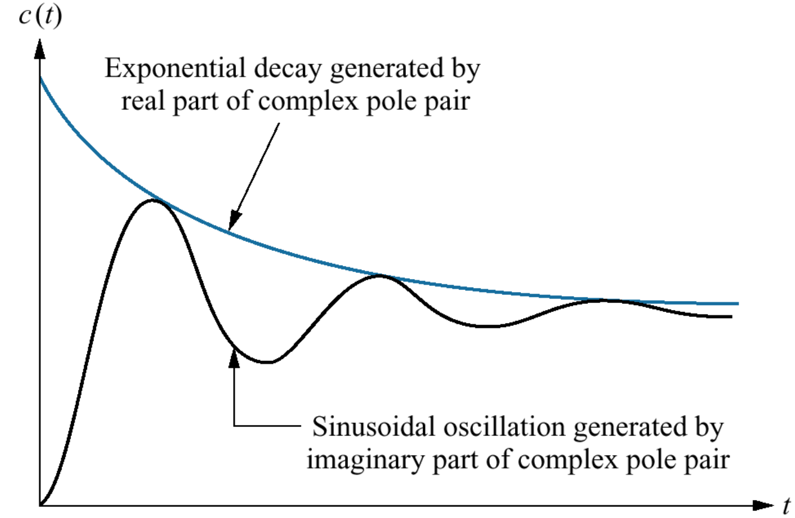
- With poles:

$$s_{1,2} = \pm j\sqrt{b}$$

- We know that: $\omega = \sqrt{b} = \omega_n$ as a complex number.
- So, the frequency of oscillation: $s = \sigma + j\omega$, which is termed the natural frequency.

Exponential Decay Frequency

- Exponential decay frequency is when the exponential function shapes up the sinusoidal oscillation function of the system response.



- Considering an underdamped second-order system:

$$\frac{C(s)}{R(s)} = \frac{b}{s^2 + as + b}$$

- The real part of $s = \sigma + j\omega,$ is $\sigma = -a/2$, where $|\sigma|$ is termed the exponential decay frequency.

Damping Ratio

- Damping ratio (ζ) is defined as a measure describing how oscillations in a system decay after a disturbance.
- It is equated as the ratio of the exponential decay frequency to the natural frequency.

$$\zeta = \frac{|\sigma|}{\omega_n} = \frac{(a/2)}{\omega_n}$$

Where:

- $|\sigma|$ is the exponential decay frequency.
- ω_n is the natural frequency.

Natural Frequency and Damping Ratio

- Consider a second-order system with the following transfer-function equation:

$$\frac{C(s)}{R(s)} = k\left(\frac{c}{as^2 + bs + c}\right)$$

- Written as a standardised equation for a second-order system in terms of damping ratio and natural frequency:

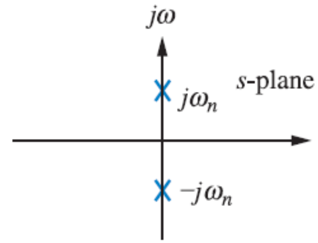
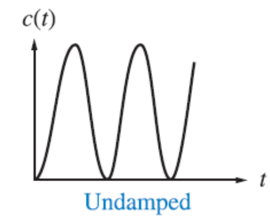
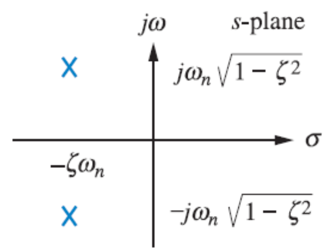
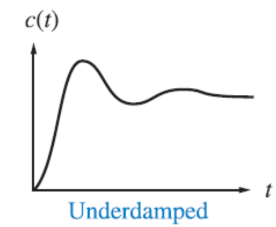
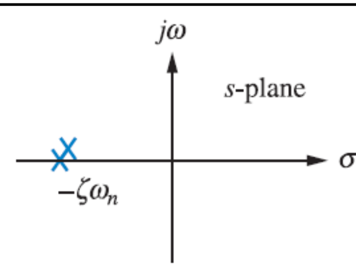
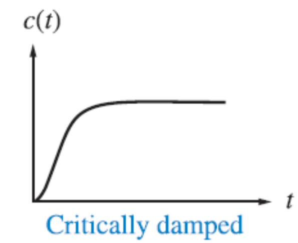
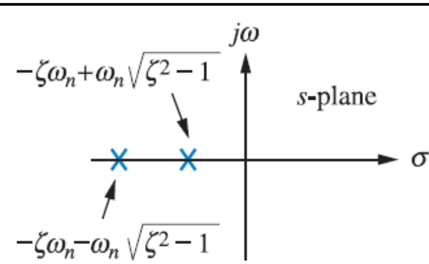
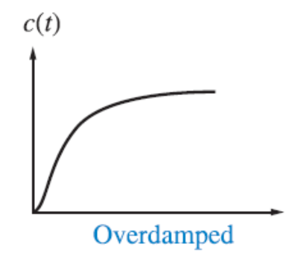
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Consider the roots of the characteristic equation:

$$s_{1,2} = \zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

- The transient response of the second-order systems depends on based on location of poles (ω_n) and the damping ratio (ζ).

The ζ, ω_n and Step Response of System

Damping (ζ)	Roots (ω_n)	Poles in the S-plane	Response of System
$\zeta = 0$	$\pm j\omega_n$		
$0 < \zeta < 1$	$-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$		
$\zeta = 1$	$\zeta\omega_n$		
$\zeta > 1$	$-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$		

Determining ζ and ω_n of Second Order System

- This system is with unity in front of the s^2 term and k such that:

$$\frac{C(s)}{R(s)} = \frac{b}{s^2 + as + b} \qquad (Eq.1)$$

- Equating the transfer function of the system with the standardised equation for a second-order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n\zeta s + \omega_n^2} \qquad (Eq.2)$$

- Equating equation (1) with equation (2), the natural frequency is:

$$\omega_n^2 = b \qquad (Eq.3)$$

Determining ζ and ω_n of Second Order System

- Thus, the natural frequency is:

$$\omega_n = \sqrt{b} \qquad (Eq. 4)$$

- The damping ratio of the second order system is:

$$2\omega_n \zeta = a \qquad (Eq. 5)$$

- Substituting equation (4) into equation (5), the damping ratio (ζ) is:

$$\zeta = \frac{a}{2\sqrt{b}}$$

Example of Second Order Time Response

For the following second-order system, determine the following time-domain parameters of the system:

$$G(s) = \frac{4}{3s^2 + 6s + 9}$$

a. The poles and zeros of the system. [6 marks]

b. Natural frequency (ω_n). [4 marks]

c. Damping ratio (ζ). [4 marks]

Example of Second Order Time Response

a. Convert the transfer function equation of the system into a monic polynomial first.

$$G(s) = \frac{4}{3s^2 + 15s + 9} = \frac{4}{(3)(3)}\left(\frac{3}{s^2 + 5s + 3}\right)$$

Notice that 4/[(3)(3)] term is becoming the gain of the system.

To determine the poles and zeros of the system, we use the standard equation for determining the roots of the second-order equation.

$$s_{1,2} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Example of Second Order Time Response

Poles and zeros of the system are determined from:

$$s_{1,2} = -\frac{(5)}{2(1)} \pm \frac{\sqrt{(5)^2 - 4(1)(3)}}{2(1)} = -2.5 \pm \frac{\sqrt{13}}{2}$$

The poles and zeros of the system are $-2.5 + \sqrt{13}/2$ and $-2.5 - \sqrt{13}/2$.

b. To calculate the natural frequency of the system, we use the standardised equation for the second order system.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{3}{s^2 + 5s + 3}$$

Example of Second Order Time Response

Natural frequency (ω_n) of the system is calculated from:

$$\omega_n^2 = 3$$

Thus, the natural frequency of the system is $\sqrt{3}$.

c. Knowing the natural frequency of the system (ω_n) from part (b), the damping ratio (ζ) of the system is calculated from:

$$2\zeta\omega_n = 5$$

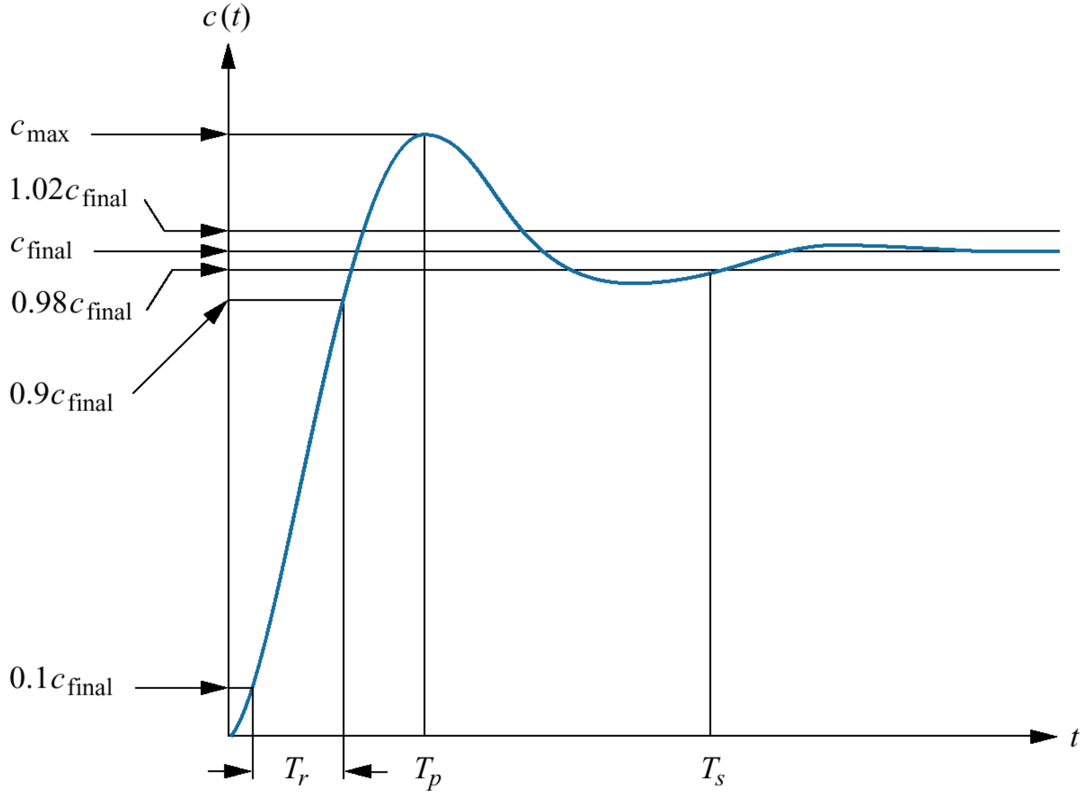
Thus

$$\zeta = \frac{5}{2\omega_n} = \frac{5}{2\sqrt{3}} = \frac{5}{6}\sqrt{3}$$

As a result, the damping ratio of the system is $(5/6)\sqrt{3}$.

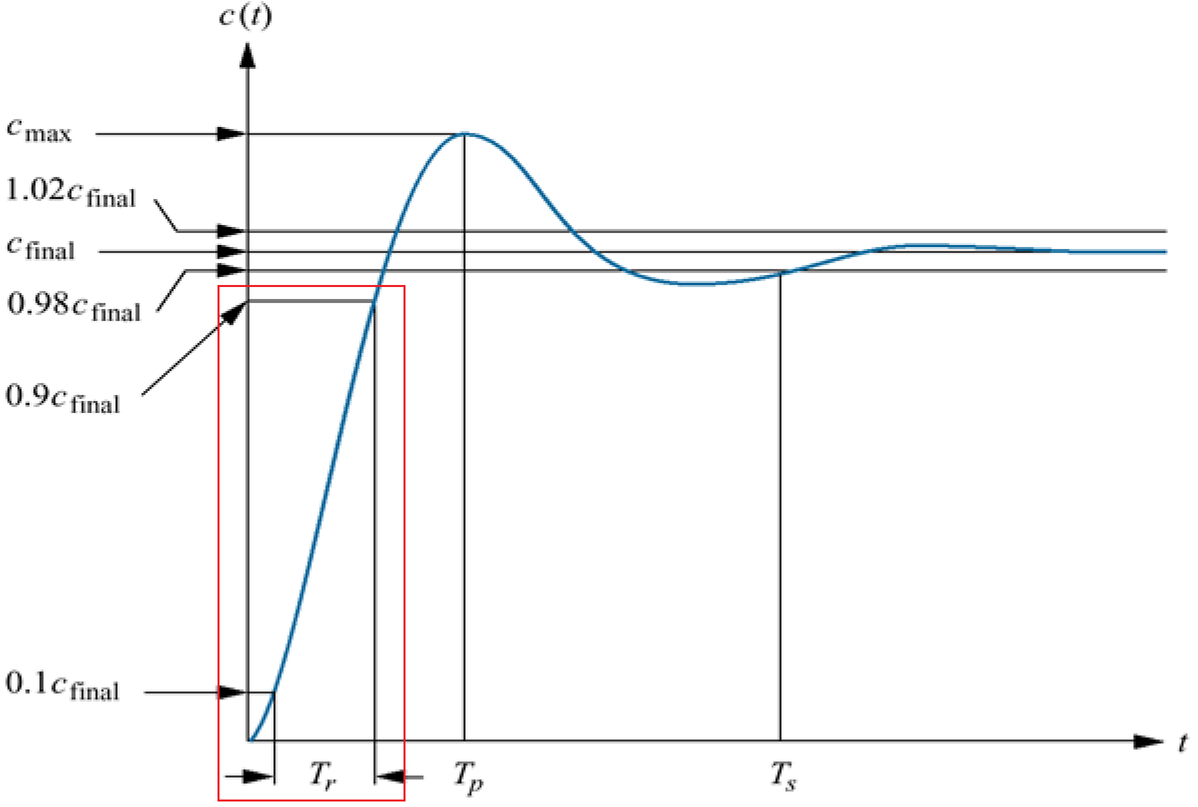
Time Response of Second-Order System

- Second-order underdamped response specifications.
- We can not use equations for a first-order system.
- Consider:
  - Rise time (T_r).
  - Settling time (T_s).
- But also:
  - Time-to-peak (T_p).
  - Percentage overshoot ($\%OS$).
  - Steady-state error ($e(\infty)$).



Rise Time of Second-Order System

- Like in the first-order system, rise time (T_r) is defined as the time for the response to go from 0.1 to 0.9 of its final value.
- We could use the formulae for the rise time of the first-order response – but this is not very accurate.
- To simplify the maths required, we consider the rise time for a second-order system as the time response from 0 to its final value.



Rise Time of Second-Order System

- For an underdamped second-order system, its time response is:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_d t + \phi)$$

Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

- The magnitude of the output signal at rise time, T_r is approximated to 1 for easy calculation e.g. $c(t) = 1$.

$$1 - \left(\frac{e^{-\zeta\omega_n T_r}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_d T_r + \phi) = 1$$

Rise Time of Second-Order System

- Thus, equating both sides:

$$\left(\frac{e^{-\zeta\omega_n T_r}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_d T_r + \phi) = 0$$

- Knowing that $\sin^{-1}(0) = n\pi$, then the above given equation becomes:

$$\omega_d T_r + \phi = n\pi$$

- The rise time is calculated from ($n = 1$):

$$T_r = \frac{\pi - \phi}{\omega_d}$$

Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

Rise Time of Second-Order System

- As $\omega_d = \omega_n\sqrt{1-\zeta^2}$, the formula for calculating rise time (T_r) of the second-order system as a function of damping ratio (ζ) and natural frequency (ω_n) is:

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$$

Where: $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad (\text{i.e. convert to rad/s})$

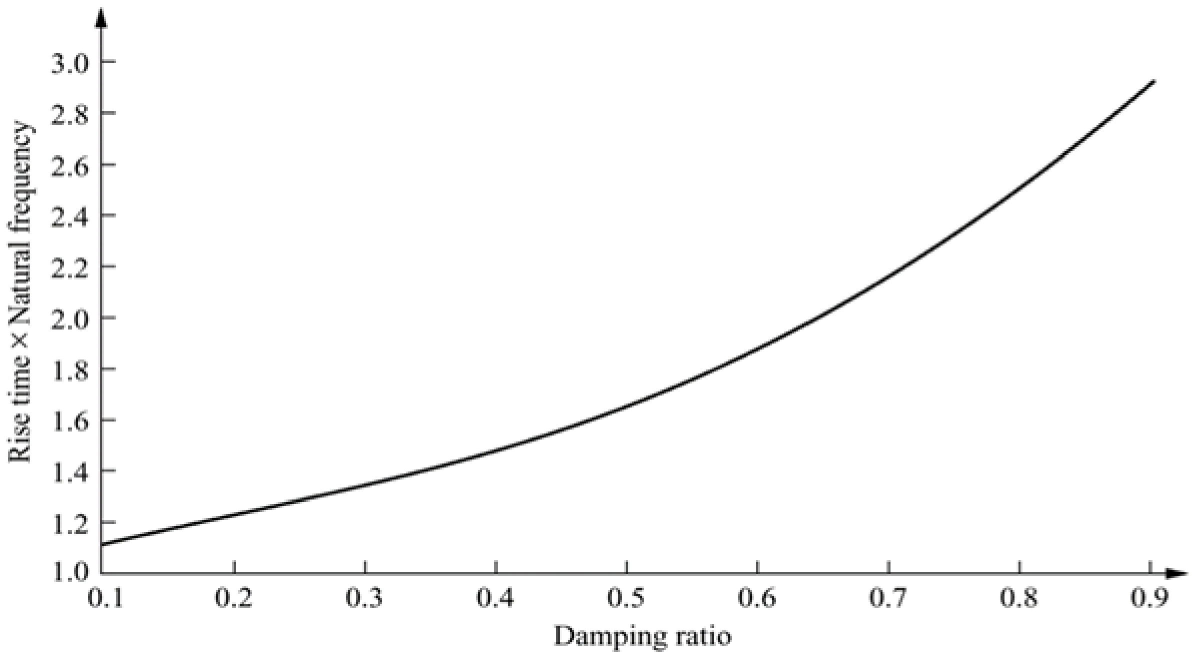
- Alternative formulae for calculating rise time (T_r) of the second-order system:

$$T_r = \frac{(1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)}{\omega_n}$$

Rise Time of Second-Order System

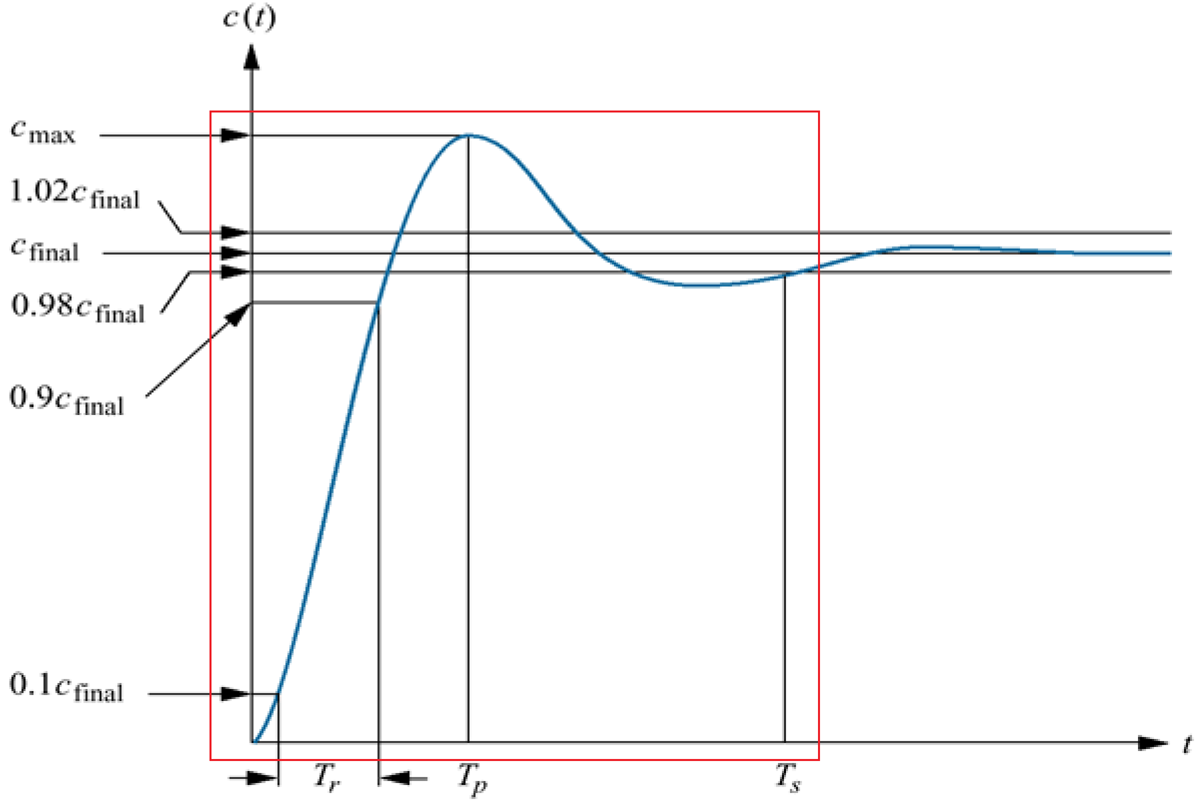
- We can also use the graph of normalised rise time vs. damping ratio for a second-order underdamped response as shown below.
- This is quicker and without a maths process to analyse.

Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883



Settling Time of Second-Order System

- Settling time (T_s) is defined as the time for the response to reach and stay close to its final steady-state value.
- There are several settling time standards in control system engineering, e.g. 0.1%, 0.5%, 1%, 2%, 5 %, etc., of the final value.



Settling Time of Second-Order System

- For an underdamped second-order system, its time response:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_d t + \phi)$$

Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

- Thus, to find the settling time, $\sin(\omega_d t + \phi) = 1$:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n T_s}}{\sqrt{1-\zeta^2}}\right)$$

- Rearrange the equation above:

$$\left(\frac{e^{-\zeta\omega_n T_s}}{\sqrt{1-\zeta^2}}\right) = 1 - c(t)$$

Settling Time of Second-Order System

- For 2% settling time standard, therefore $1 - c(t) = 0.02$:

$$\left(\frac{e^{-\zeta\omega_n T_s}}{\sqrt{1-\zeta^2}}\right) = 0.02$$

- For an underdamped second-order system, ζ lies between 0 and 1. As a result, neglect the denominator for easy calculation.

$$e^{-\zeta\omega_n T_s} = 0.02$$

- Taking natural log on both sides:

$$-\zeta\omega_n T_s = \ln 0.02$$

- So, the settling time of the system for 2% standard is:

$$T_s = \frac{3.9}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n}$$

Settling Time of Second-Order System

- For 5% settling time standard, therefore $1 - c(t) = 0.05$:

$$\left(\frac{e^{-\zeta\omega_n T_s}}{\sqrt{1-\zeta^2}}\right) = 0.05$$

- For underdamped second order system:

$$e^{-\zeta\omega_n T_s} = 0.05$$

- Taking natural log on both sides:

$$-\zeta\omega_n T_s = \ln 0.05$$

- So, the settling time for 5% standard is:

$$T_s = \frac{2.9957}{\zeta\omega_n} \approx \frac{3}{\zeta\omega_n}$$

Time-to-Peak of Second-Order System

- Time-to-peak (T_p), it is the time required to reach the first, or maximum, peak.
- For an underdamped second-order system, its time response:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_d t + \phi)$$

Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

- As per definition at the peak time, the response curve reaches to its maximum value. Hence at that point,

$$\frac{dc(t)}{dt} = 0$$

Time-to-Peak of Second-Order System

- Now, substitute $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and perform partial differentiation (i.e. $uv' + u'v$) of the equation above:

$$\frac{dc(t)}{dt} = \left(\frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\right)\omega_n\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}T_p + \phi\right)$$

$$+\left[\frac{(-\zeta\omega_n)e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\right]\sin\left(\omega_n\sqrt{1-\zeta^2}T_p + \phi\right)$$

- Assign the $dc(t)/dt$ to zero.

$$\left(\frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\right)\omega_n\sqrt{1-\zeta^2}\cos\left(\omega_n\sqrt{1-\zeta^2}T_p + \phi\right)$$

$$+\left[\frac{(-\zeta\omega_n)e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\right]\sin\left(\omega_n\sqrt{1-\zeta^2}T_p + \phi\right) = 0$$

Time-to-Peak of Second-Order System

- Rearranging and equating both sides:

$$\tan\left(\omega_n\sqrt{1-\zeta^2}T_p+\phi\right)=\frac{\sqrt{1-\zeta^2}}{\zeta}=\tan\phi$$

- The equation above becomes:

$$\left(\omega_n\sqrt{1-\zeta^2}\right)T_p=n\pi$$

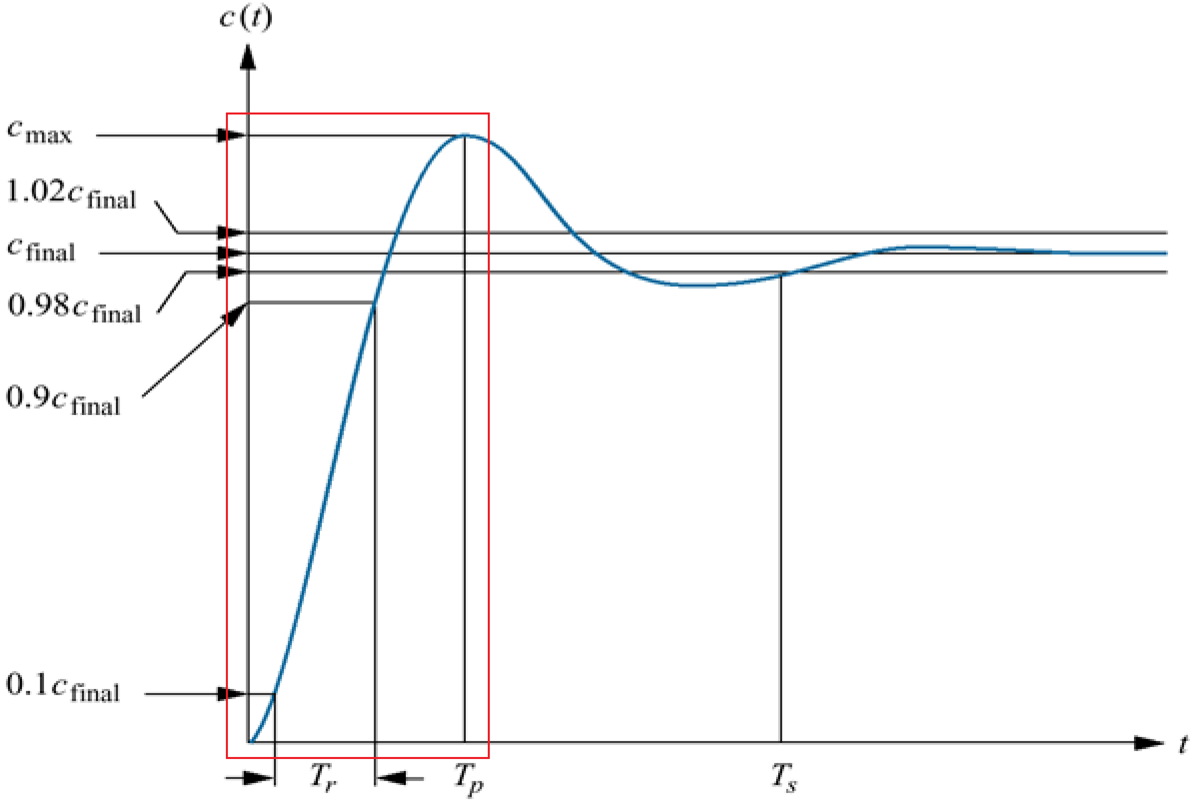
- The time-to-peak (T_p) of the second order system is:

$$T_p=\frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad \text{where: } n=n\text{th peak}$$

- Note: the maximum overshoot of the given second–order system occurs at n = 1.

Time-to-Peak of Second-Order System

- The maximum overshoot of the given second–order system occurs at the first peak.



Percentage Overshoot of Second-Order System

- The percentage overshoot ($\%OS$) is the amount that the waveform overshoots the steady-state or final value compared with the value at the peak time.
- It is typically expressed as a percentage of the steady-state value.
- For an underdamped second-order system, its time response:

$$c(t) = 1 - \left(\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_d t + \phi)$$

Where: $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

- For the percentage overshoot, substitute $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$c(t)_{max} = 1 - \left(\frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_n\sqrt{1-\zeta^2}T_p + \phi)$$

Percentage Overshoot of Second-Order System

- Put the expression of peak time in the expression of output response $c(t)$.

$$c(t)_{max} = 1 - \left[\frac{e^{-\zeta\omega_n\left(\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\right)}}{\sqrt{1-\zeta^2}}\right]\sin\left(\omega_n\sqrt{1-\zeta^2}\left(\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\right)+\phi\right)$$

- Since $\sin(\pi+\phi) = -\sin\phi$

$$c(t)_{max} = 1 - \left(\frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}\right)\sin(\pi+\phi)$$

$$= 1 - \left(\frac{e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}}{\sqrt{1-\zeta^2}}\right)(-\sin\phi)$$

Percentage Overshoot of Second-Order System

- We know that $\sin\phi = \sqrt{1-\zeta^2}$, so:

$$1 + \left(\frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \right) \sin\phi = 1 + \left(\frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \right) \sqrt{1-\zeta^2}$$

$$= 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

- Thus, maximum overshoot is:

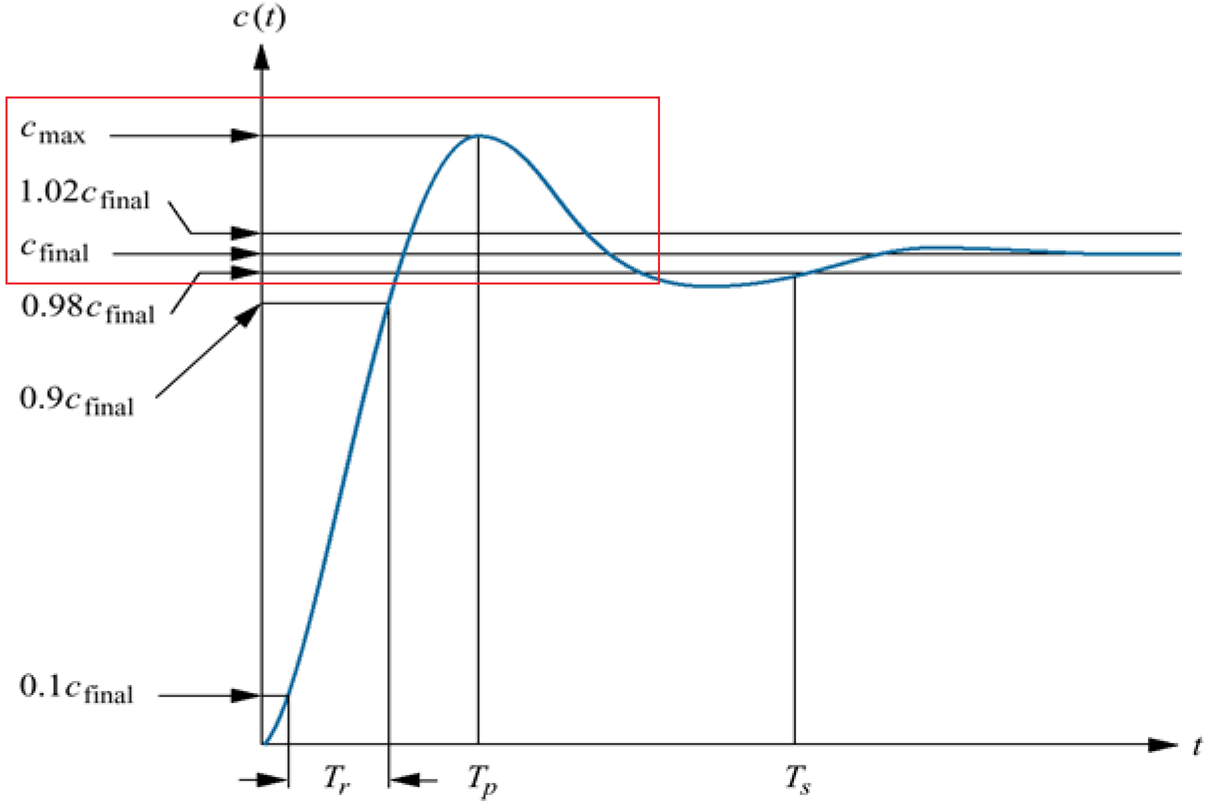
$$M_p = c(t)_{max} - 1$$

$$= 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} - 1 = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Percentage Overshoot of Second-Order System

- Thus, the percentage overshoot ($\%OS$) of the second-order system is:

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100\%$$



Percentage Overshoot and Damping Ratio

- By rearranging the equation for percentage overshoot, we could find ζ from $\%OS$.

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100\%$$

- Equating both sides with natural log.

$$-\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = \ln(\%OS/100)$$

- Rearrange the equation.

$$\zeta\pi = -\sqrt{1-\zeta^2}\,\ln(\%OS/100)$$

- Squaring both sides.

$$(\zeta\pi)^2 = 1-\zeta^2\,[\ln(\%OS/100)]^2$$

Percentage Overshoot and Damping Ratio

- Rearrange the equation:

$$\zeta^2 = \frac{[\ln(\%OS/100)]^2}{(\pi)^2 + [\ln(\%OS/100)]^2}$$

- The relationship between percentage overshoot ($\%OS$) and damping ratio (ζ).

$$\zeta = -\frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + [\ln(\%OS/100)]^2}}$$

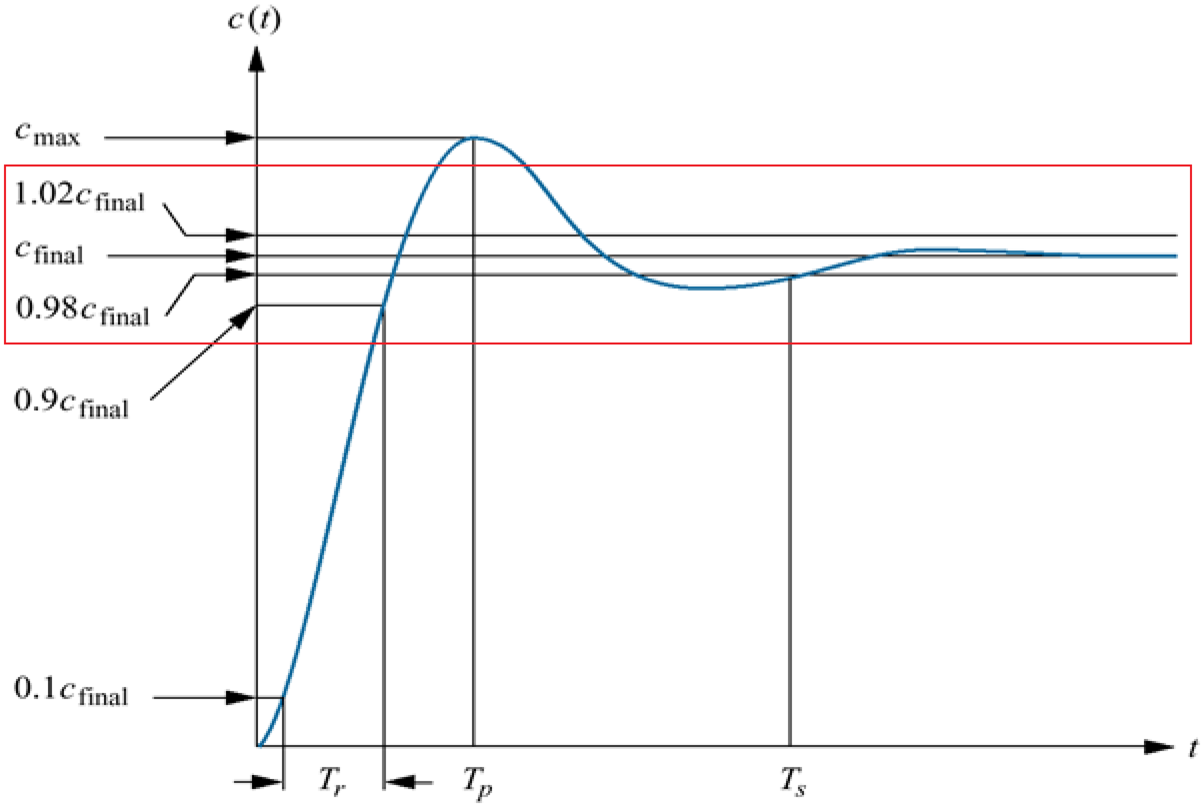
- Selection of the damping ratio is a tradeoff between maximum percentage overshoot ($\%OS$) and time where the peak overshoot occurs (time-to-peak).
- Smaller damping ratio decreases time-to-peak (desirable), but it increases $\%OS$ (undesirable).

Steady-State Error of Second-Order System

- Steady-state error ($e(\infty)$), is the difference between the input ($r(t)$) and output ($c(t)$) for a prescribed test input.

$$e(\infty) = \lim_{t \to \infty} [r(t) - c(t)]$$

- We will look more closely at the steady-state response and steady-state error of the system in the subsequent topic in the course.



Example of Time Response Analysis

For the following second-order system, determine the following time-domain parameters of the system:

$$G(s) = \frac{81}{s^2 + 15s + 81}$$

a. Natural frequency (ω_n). [2 marks]

b. Damping ratio (ζ). [2 marks]

c. Settling time (T_s). [2 marks]

d. Rise time (T_r). [2 marks]

e. Time-to-peak (T_p). [2 marks]

f. Percentage overshoot ($\%OS$). [2 marks]

g. Simulate its transient response in MATLAB, determine the parameters in (a)–(f) and comment on the result. [12 marks]

Example of Time Response Analysis

a. Equating the transfer function equation with the standardised equation for second order system, the natural frequency (ω_n) of the given system is calculated from:

$$G(s) = \frac{81}{s^2 + 15s + 81} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Thus, the natural frequency of the system is as:

$$\omega_n = \sqrt{81} = 9 \text{ rad/s}$$

b. Damping ratio (ζ) is calculated from:

$$\zeta = \frac{15}{2\omega_n} = \frac{15}{2(9)} = 0.833$$

Example of Time Response Analysis

c. Rise time (T_r) is calculated from:

$$T_r = \frac{\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi - \pi(0.664)}{9\sqrt{1-(0.833)^2}} = 0.212 \text{ s}$$

Where:

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = \tan^{-1}\left(\frac{\sqrt{1-(0.833)^2}}{0.833}\right) = 0.664$$

Using alternative equation for rise time, it is:

$$T_r = \frac{(1.76\zeta^3 - 0.417\zeta^2 + 1.039\zeta + 1)}{\omega_n}$$

$$= \frac{(1.76(0.833)^3 - 0.417(0.833)^2 + 1.039(0.833) + 1)}{9}$$

$$= 0.288 \text{ s}$$

Example of Time Response Analysis

d. If the settling time standard is 2%, the settling time (T_s) is calculated from:

$$T_s = \frac{3.9}{\zeta\omega_n} \approx \frac{4}{\zeta\omega_n} = \frac{4}{(0.833)(9)} = 0.533 \text{ s}$$

e. For the first (max) peak (n = 1), the time-to-peak (T_p) is calculated from:

$$T_p = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{9\sqrt{1-(0.833)^2}} = 0.63 \text{ s}$$

Where: n = n-th peak.

f. Percentage overshoot ($\%OS$) is calculated from:

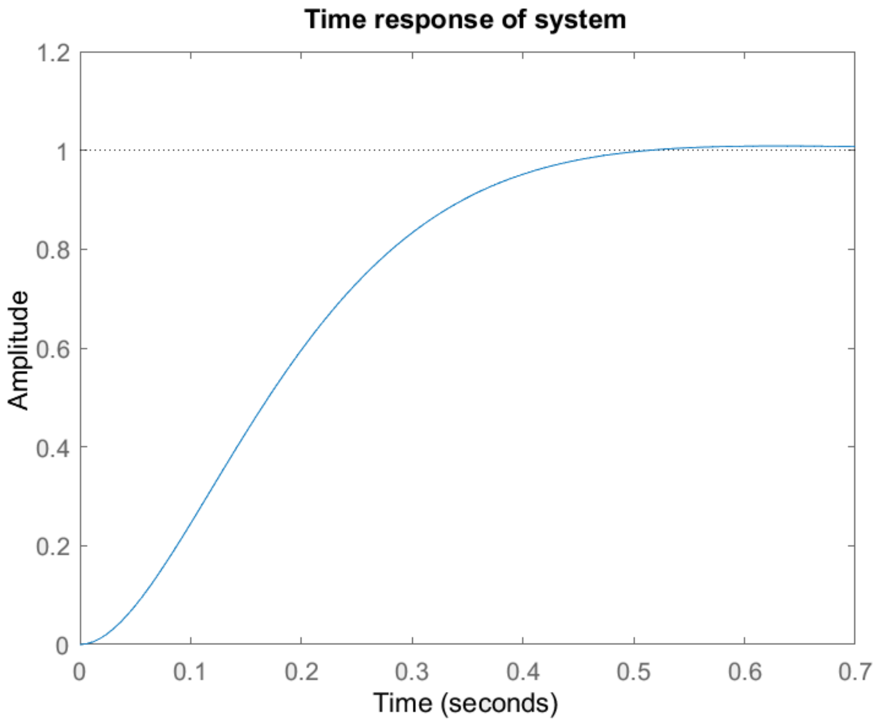
$$\%OS = \left(e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\right)100\% = \left(e^{-\frac{(0.833)\pi}{\sqrt{1-(0.833)^2}}}\right)100\% = 0.88\%$$

Example of Time Response Analysis

g. The results of the simulation in MATLAB are listed below.

It seems that from the plots, the rise time and settling time are found to be 0.3 and 0.5 respectively.

The transient response of the system is underdamped, and it settles down to an amplitude of 1 in the end.

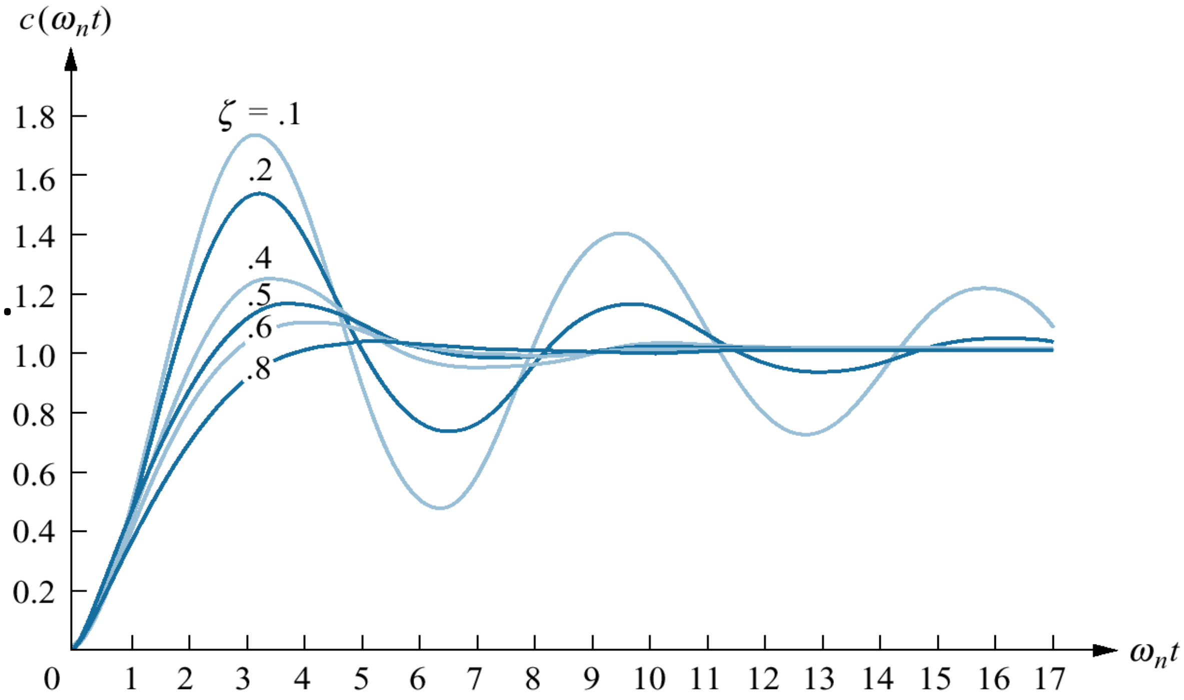


Workspace

Name	Value
den	[1,15,81]
num	81
omegan	9
percent	0.8773
T	*1x1 tf*
Tp	0.6315
Tr	0.2883
Ts	0.5333
zeta	0.8333

Second-Order Step Response

- Step function is typically used for analysing and testing the response of the system.
- For given second-order systems, their step responses are:
  - $\zeta = 0 \rightarrow$ Undamped.
  - $0 < \zeta < 1$ (small ζ) $\rightarrow$ Underdamped.
  - $\zeta = 1 \rightarrow$ Critically damped.
  - $\zeta > 1$ (large ζ) $\rightarrow$ Overdamped.



Step Response of Second-Order System

- Underdamped second-order step responses are typically generated by complex poles.
- Given a second-order system with a pair of complex poles as shown below.

$$\frac{C(s)}{R(s)} = \frac{k}{(s+a+jb)(s+a-jb)}$$

- For a step input ($R(s) = 1/s$), after implementing partial fraction expansion, the transfer function equation is:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+a+jb)} + \frac{K_3}{(s+a-jb)}$$

Step Response of Second-Order System

- Taking the inverse Laplace transform, the complex pair of roots becomes the co-sinusoidal function.

$$\frac{c(t)}{r(t)} = Ae^{-t}\cos(bt - \phi^\circ)$$

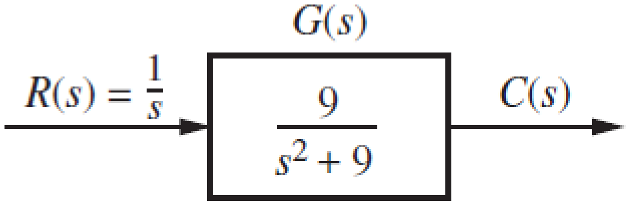
Where:

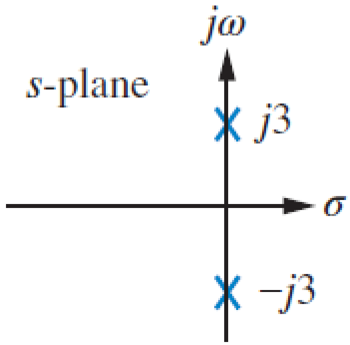
$$\phi = \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}$$

- As a result, the time response of the underdamped second order system is an exponentially decaying sinusoidal oscillation.

Example of Second-Order Step Response 1

For the undamped response of a second-order system given below, determine its roots, step response equation, and simulate its poles and zeros in the s-plane and step response. [12 marks]



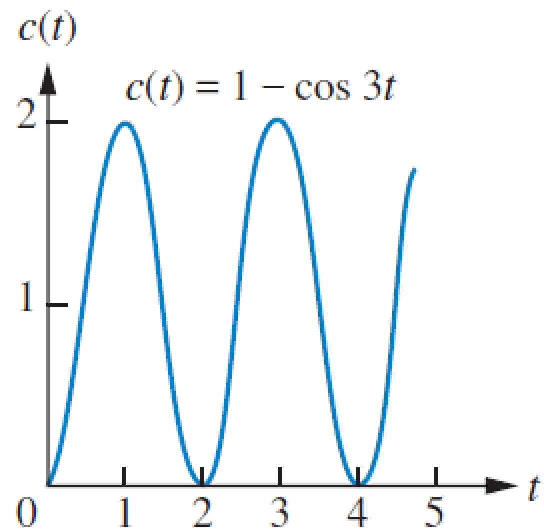


- The transfer function equation of the system is:

$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 9}$$

- Roots are: $s_1 = j3$ and $s_2 = -j3$
- Applying step input $(1/s)$, the response is a constant amplitude sinusoid:

$$c(t) = 1 - \cos 3t$$



Step Response of Second-Order System

- Overdamped second-order step responses are typically generated by real poles.
- Consider a second-order system with real poles as shown below.

$$\frac{C(s)}{R(s)} = \frac{k}{(s+a)(s+b)}$$

- For a step input ($R(s) = 1/s$), after implementing partial fraction expansion, the transfer function equation is:

$$C(s) = \frac{K_1}{s} + \frac{K_2}{(s+a)} + \frac{K_3}{(s+b)}$$

Step Response of Second-Order System

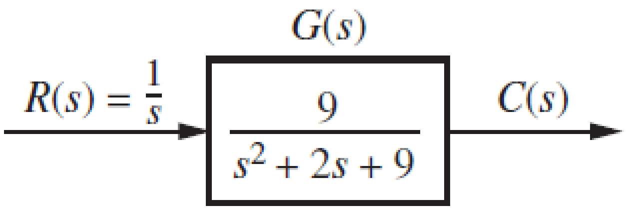
- Taking the inverse Laplace transform, the pole at the origin becomes a constant and both real terms become exponential functions.
- The time domain equation of the system is:

$$c(t) = K_1 + K_2 e^{-at} + K_3 e^{-bt}$$

- As a result, the step response of the system is an exponential increase.

Example of Second-Order Step Response 2

For the underdamped response of a second-order system given below, determine its roots, step response equation, and simulate its poles and zeros in the s-plane and step response. [12 marks]



- The transfer-function equation of the system is:

$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 2s + 9}$$

- Consider the characteristic equation of the system to give roots:

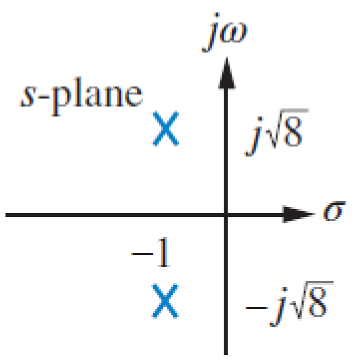
$$s_{1,2} = -1 \pm \sqrt{-8}$$

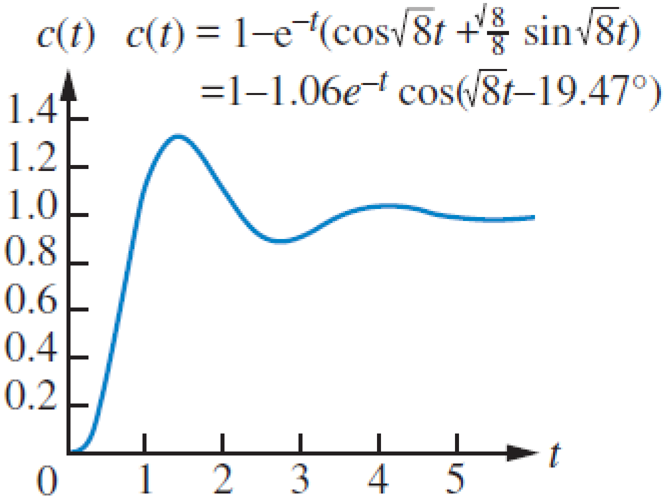
Example of Second-Order Step Response 2

- The real part generates: e^{-t} and the complex part generates: $k\cos\left(\sqrt{8}t-\phi\right)$.
- The response is a decaying sinusoidal oscillation given by:

$$c(t)=1-e^{-t}\left(\cos\sqrt{8}t+\sqrt{8}/8\sin\sqrt{8}t\right)$$
$$=1-1.06e^{-t}\cos\left(\sqrt{8}t-19.47°\right)$$

- This is also called damped oscillation response.





Step Response of Second-Order System

- Undamped second-order step responses are typically generated by complex poles that lie on the y-axis in the s-plane.
- Consider a second-order system with complex poles on y-axis on the s-plane as shown below.

$$\frac{C(s)}{R(s)} = \frac{k}{(s+jb)(s-jb)}$$

- For a step input ($R(s) = 1/s$), after implementing partial fraction expansion, the transfer function equation is:

$$C(s) = \frac{K_1}{s} + \frac{K_1}{(s+jb)} + \frac{K_2}{(s-jb)}$$

Step Response of Second-Order System

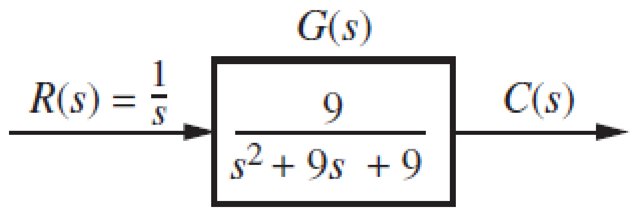
- Taking the inverse Laplace transform, the pole at the origin becomes a constant, and a pair of complex roots become co-sinusoidal function.
- The time-domain equation of the system is:

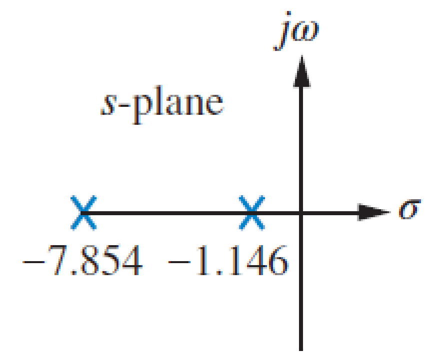
$$c(t) = K_1 - \cos bt$$

- As a result, the step response of the system is a constant amplitude sinusoid.

Example of Second-Order Step Response 3

For the overdamped response of a second-order system given below, determine its roots, step response equation, and simulate its poles and zero in the s-plane and step response. [12 marks]





- The transfer function equation of the system is:

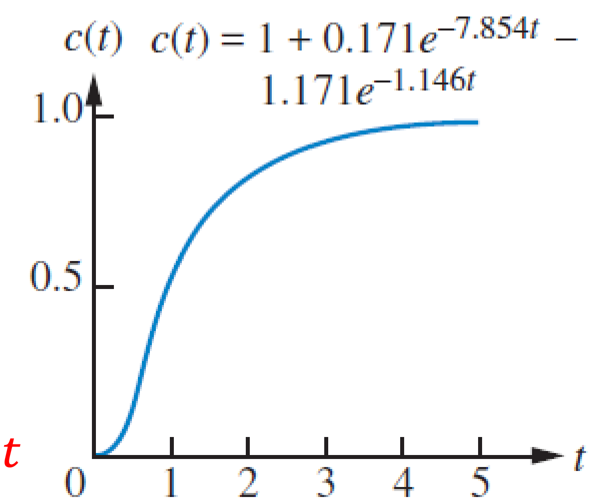
$$\frac{C(s)}{R(s)} = \frac{9}{s^2 + 9s + 9}$$

- Roots are:

$$s_1 = -7.854 \text{ and } s_2 = -1.146$$

- The response is an exponential increase:

$$c(t) = 1 + 0.17e^{-7.854t} - 1.171e^{-1.146t}$$

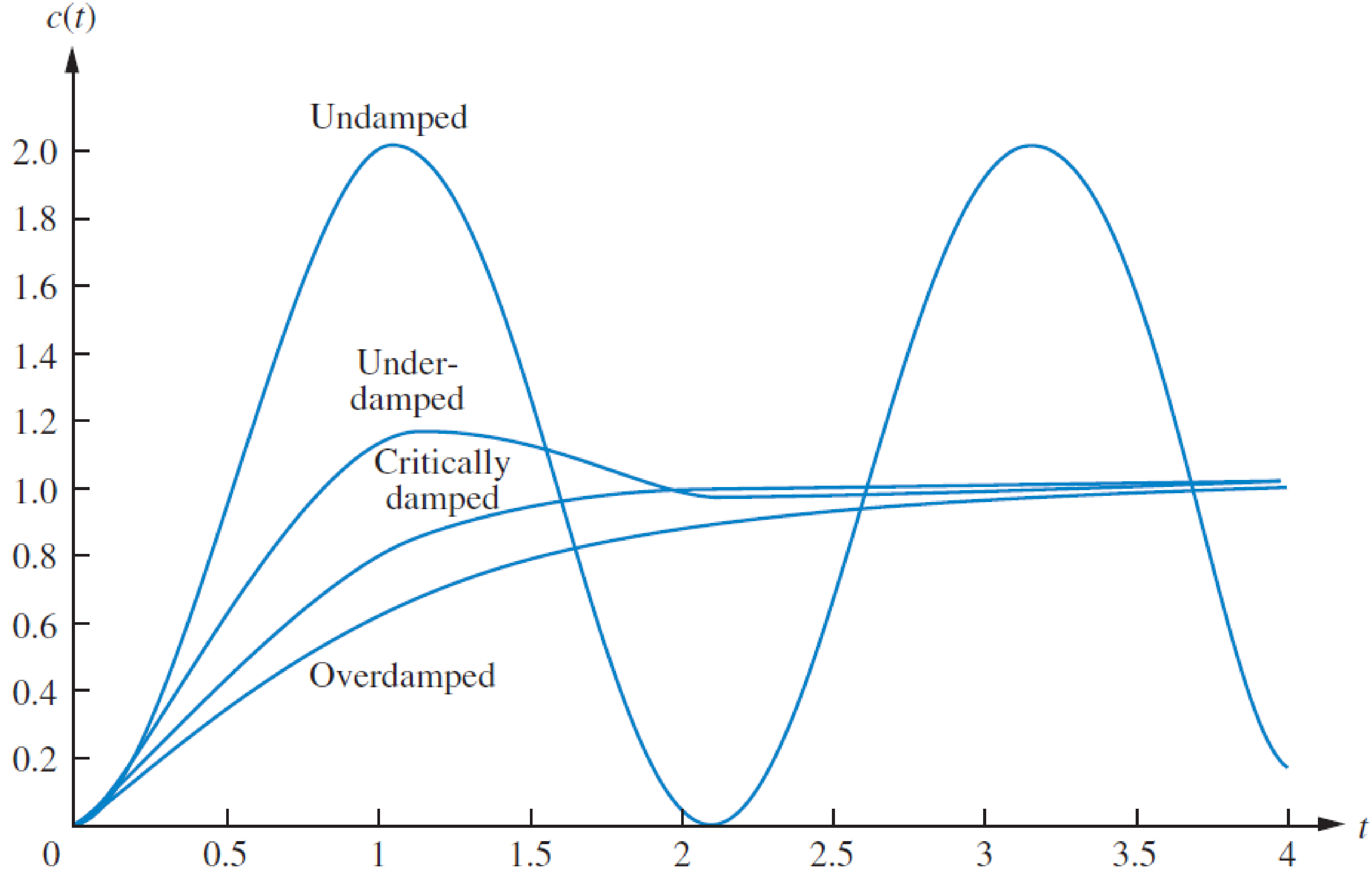


Summary of Second-Order Time Response

- Undamped responses:
  - Poles: $s_{1,2} = \pm j\omega_0$
  - Response: $c(t) = A\cos(\omega_0 t - \phi)$
- Underdamped responses (note: $\omega_d = \omega_n\sqrt{1-\zeta^2}$):
  - Poles: $s_{1,2} = -\sigma_d \pm j\omega_d$
  - Response: $c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$
- Critically damped responses:
  - Poles: $s_{1,2} = -\sigma_0$
  - Response: $c(t) = K_1 e^{-\sigma_0 t} + K_2 t e^{-\sigma_0 t}$
- Overdamped responses:
  - Poles: $s_1 = -\sigma_1, s_2 = -\sigma_2$
  - Response: $c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$

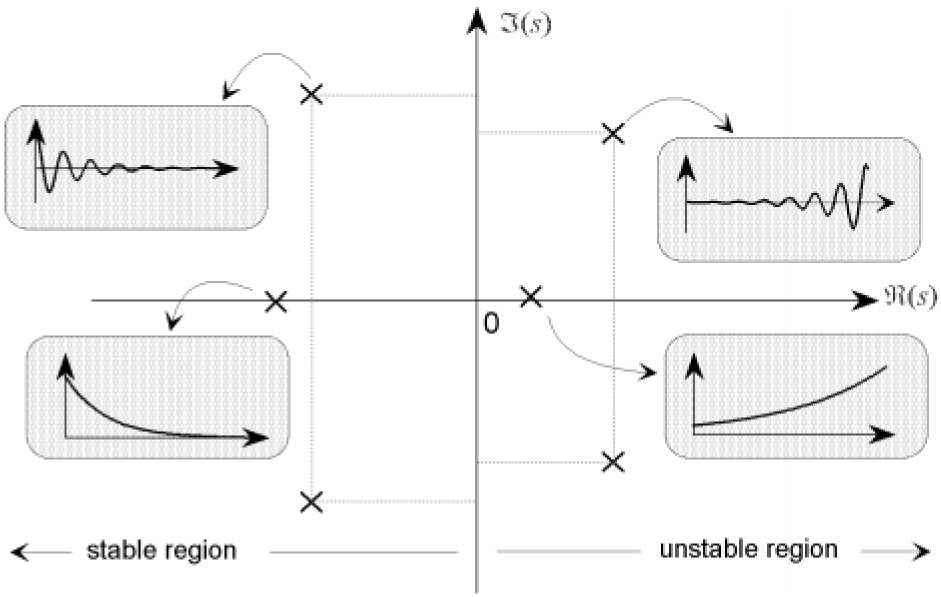
Second-Order Time Response

- Responses of the second-order systems with various damping ratios.



Trends in Second-Order Time Response

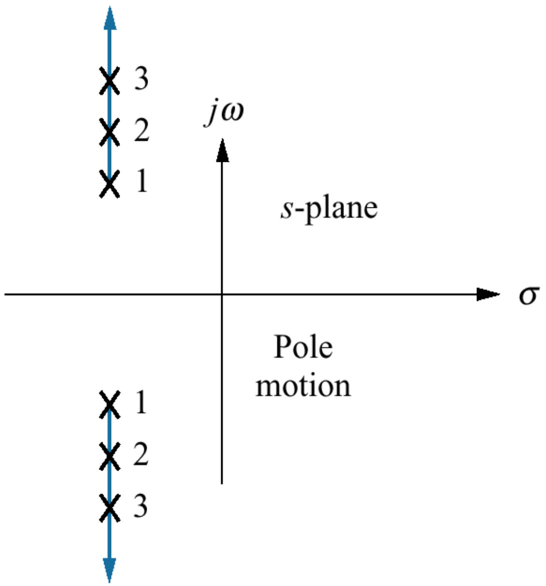
- We evaluate step responses of second-order systems as poles move with:
  - constant real part.
  - constant imaginary part.
  - constant damping ratio.

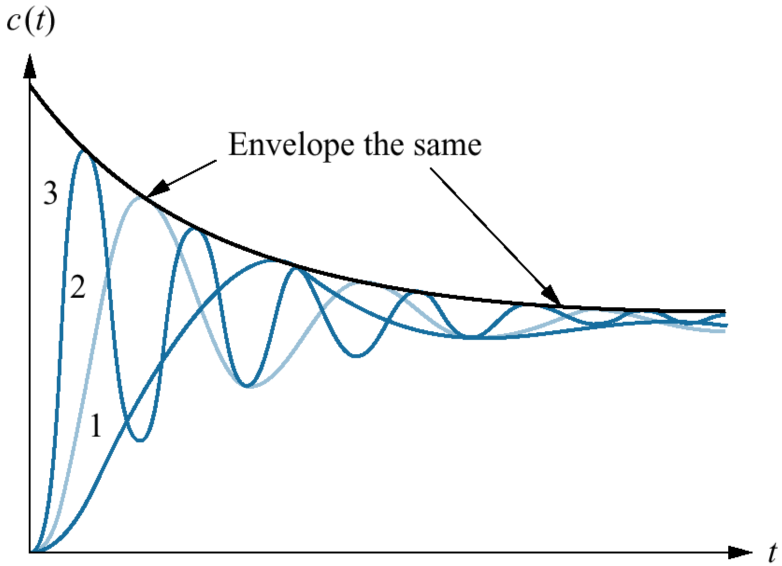


Trend in Second-Order Time Response

Moving in a vertical direction:

- Frequency increases, but the envelope remains the same since the real part of the pole is not changing.
- Constant exponential envelope, even though the sinusoidal response is changing frequency.
- Since all curves fit under the same exponential decay curve, the settling time is virtually the same for all waveforms.
- Overshoot increases, the rise time decreases.

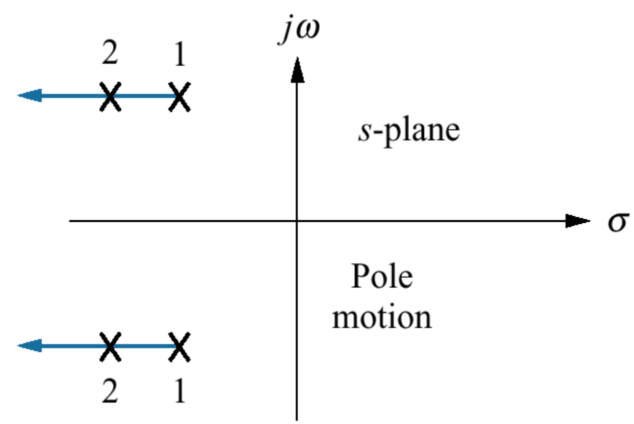


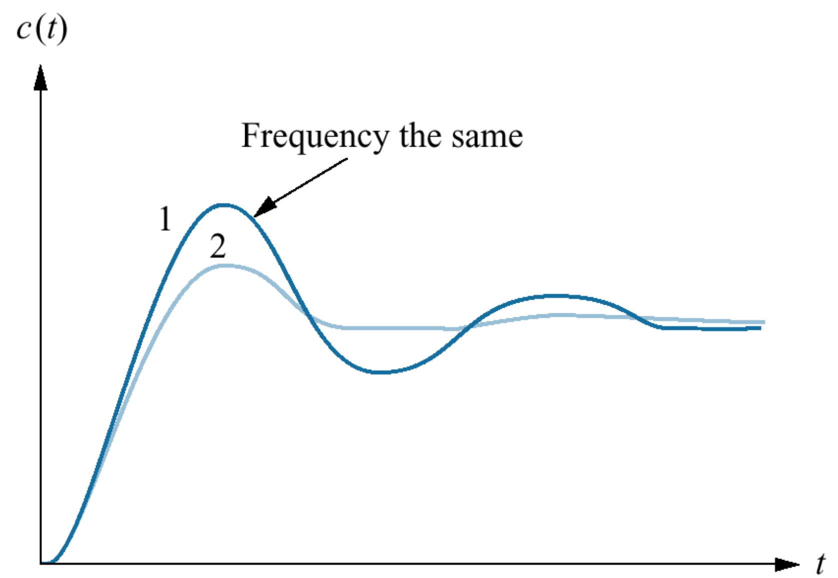


Trend in Second-Order Time Response

Moving the poles to the right or left:

- Imaginary part is now constant.
- Frequency is constant over the range of variation of the real part.
- As the poles move to the left, the response damps out more rapidly while the frequency remains the same.
- Peak time is the same for all waveforms because the imaginary part remains the same.





Trend in Second-Order Time Response

Moving the poles along a constant radial line:

- The percent overshoot remains the same.
- The responses look exactly alike, except for their speed.
- The farther the poles are from the origin, the more rapid the response.