VICTORIA UNIVERSITY OF
WELLINGTON
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Steady-State Analysis

XMUT315 Control Systems Engineering

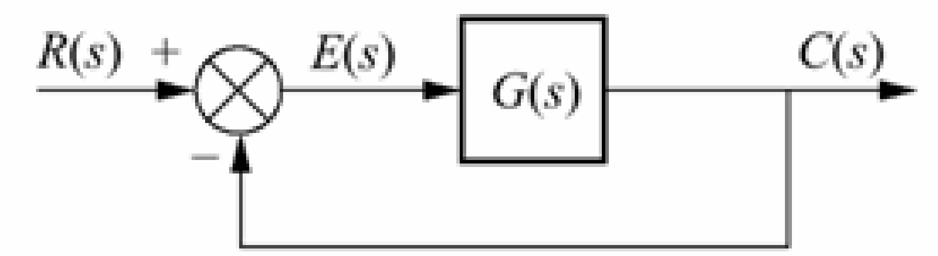
Topics

- Introduction to steady-state error.
- Derivation of steady-state error.
- Steady-state analysis of step input.
- Steady-state analysis of ramp input.
- Steady-state analysis of parabola input.
- Steady-state error of other types of systems.
- Sensitivity of system parameters towards steady-state errors.

Error in Control Systems

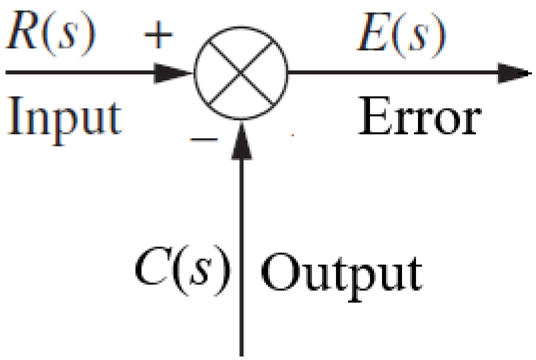
- For the following feedback control system, the system error $e(t)$ for a feedback control system is given by the difference between the demanded output $r(t)$ and the actual output $c(t)$.

$$e(t) = r(t) - c(t)$$



Steady-State Error

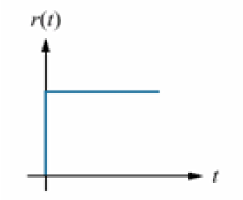
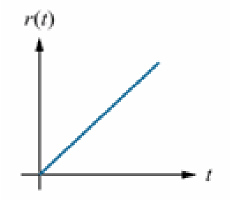
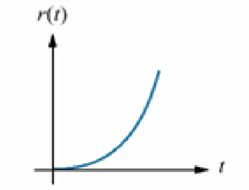
- The steady-state error is then defined as the difference between demanded and actual output when $t \to \infty$.



- In this course, the steady-state error is also defined for specific test inputs (e.g. there are other types of input tests available in control system engineering, e.g. sinusoidal, square wave, etc.):
    - Step input.
    - Ramp input.
    - Parabola input.

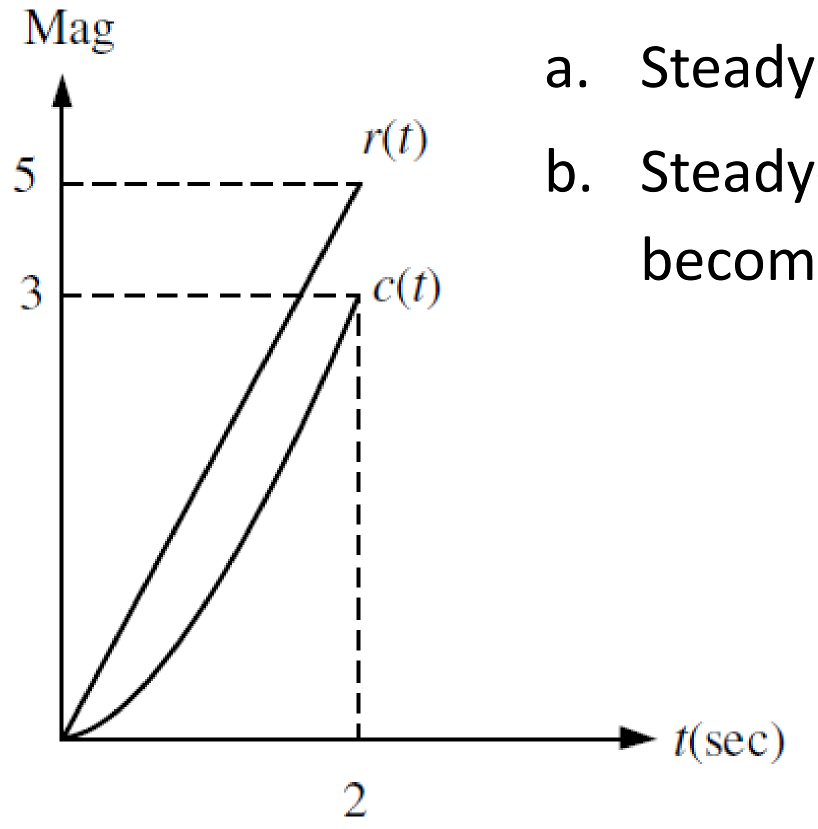
Test Inputs for Steady-State Error Analysis

- For steady-state analysis, test inputs: step, ramp and parabola.

Name	**Waveform**	**Physical Interpretation**	**Time Function**	**Laplace Transform**
Step		Constant position	1	$\frac{1}{s}$
Ramp		Constant velocity	t	$\frac{1}{s^2}$
Parabola		Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Example of Steady-State Error

Figure given below shows the ramp input $r(t)$ and the output $c(t)$ of a system. Assuming the output's steady state can be approximated by a ramp, find:



a. Steady-state error. [2 marks]

b. Steady-state error if the input becomes $r(t) = tu(t)$. [2 marks]

Example of Steady-State Error

a. From the figure, the steady-state error of the system is:

$$e(\infty) = r(\infty) - c(\infty) = 5 - 3 = 2$$

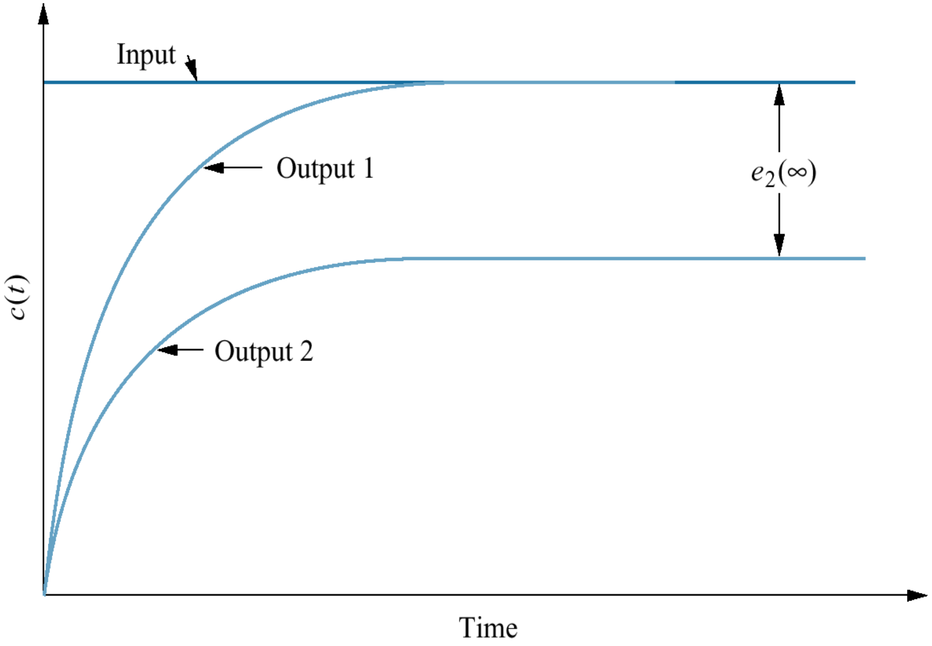
b. Since the system is linear, and because the original input was $r(t) = 2.5tu(t)$, the new steady-state error is:

$$e(\infty) = \frac{2}{2.5} = 0.8$$

Steady-State Error of Step Input

For a step input, compare the time response of different systems:

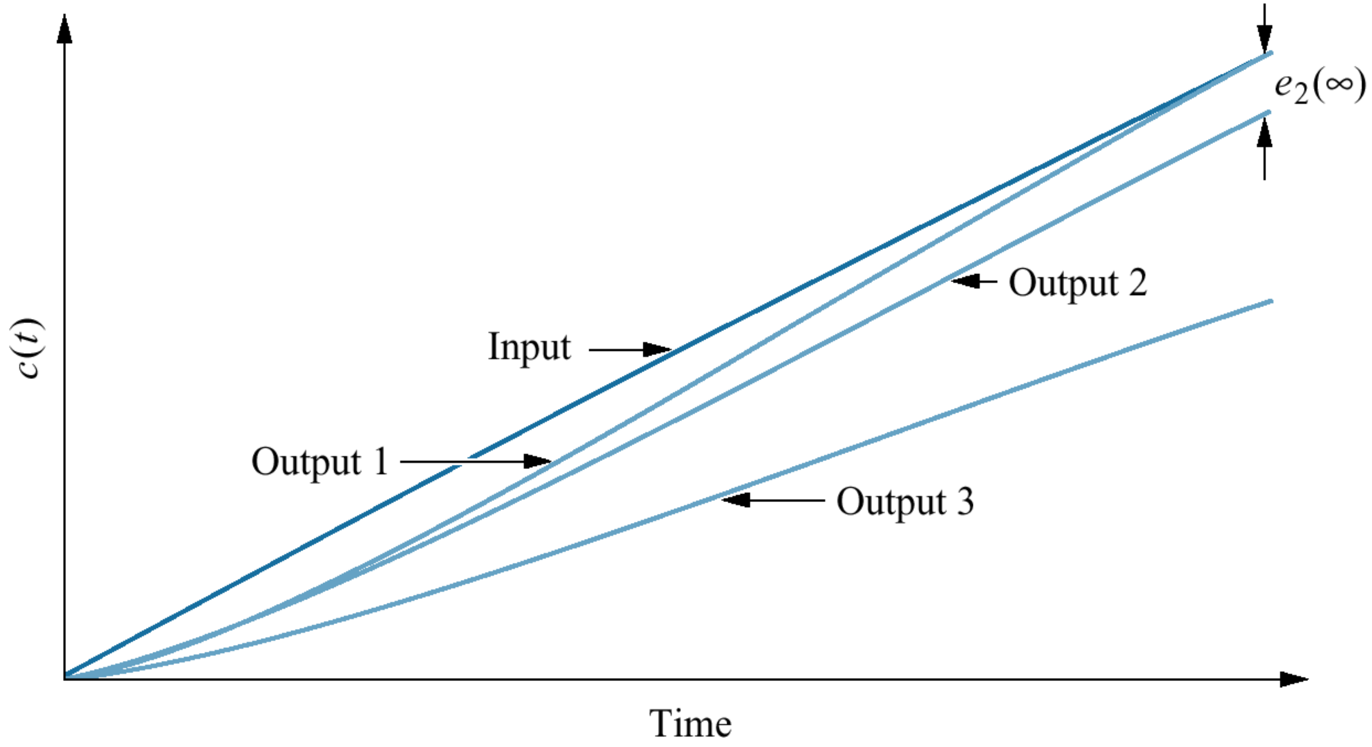
- Output 1: $e_1(\infty) = 0$ because Output 1 is equal to Input at $t = \infty$, and the steady-state error is thus zero.
- Output 2: $e_2(\infty) \neq 0$ because Output 2 is NOT equal to Input at $t = \infty$ and the steady-state error is thus non-zero.



Steady-State Error of Ramp Input

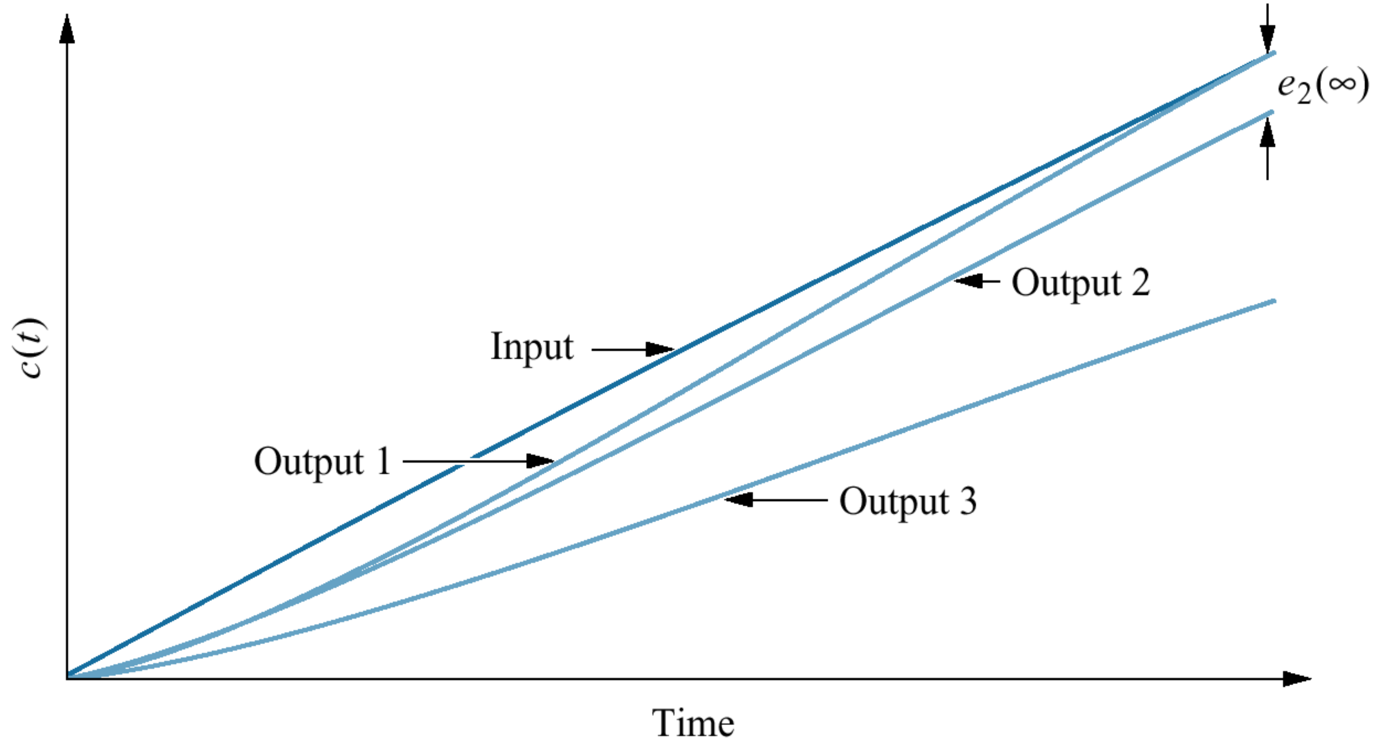
For a ramp input:

- Output 1: $e_1(\infty) = 0$ because Output 1 = Input at $t = \infty$ and the steady-state error is thus zero.

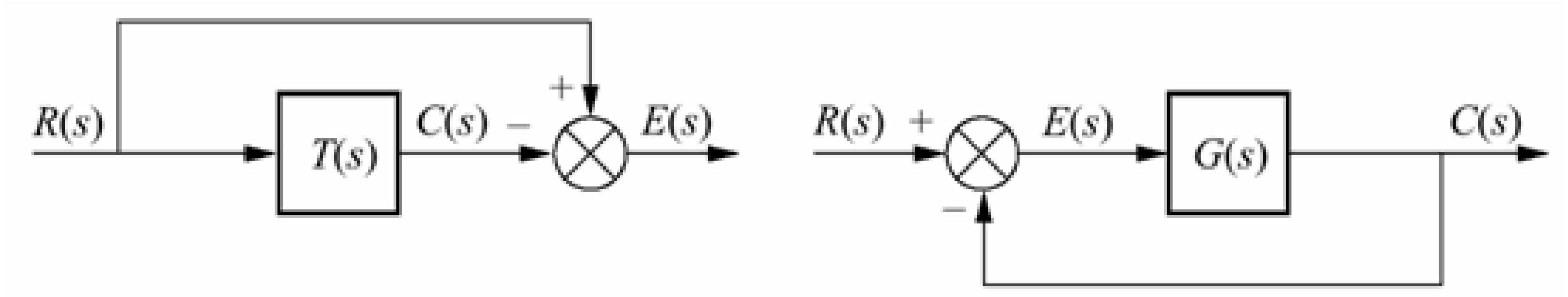


Steady-State Error of Ramp Input

- Output 2: Although the response has the same slope as the ramp input, $e_2(\infty) \neq 0$ because there will be a finite error at $t = \infty$ and the steady-state error is thus non-zero.
- Output 3: $e_3(\infty) = \infty$ because the error will increase with time as the response has a different slope than the ramp input.



General Closed Loop (Unit Feedback System)



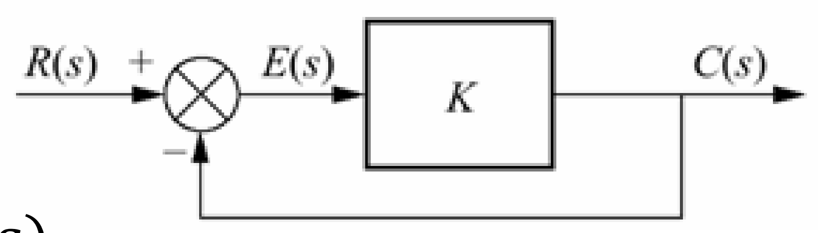
- The system error (in both cases) is then given as by the definition as:

$$E(s) = R(s) - C(s)$$

- We will now derive expressions for the steady-state error in unit feedback systems and then expand to non-unity feedback.

Sources of Steady-State Error

- Consider steady-state errors due to system configuration. System with a pure gain element.



- System output: $C(s) = KE(s)$
- The steady-state error can then never be equal to zero, nor will the output of the system be zero.
- There will thus always be a steady-state error present.
- If C_{ss} is the steady-state value of the output and e_{ss} is the steady-state value of the error, then:

$$c_{ss}(t) = Ke_{ss}(t)$$

- For a unity feedback system, error will diminish as K increases.

Steady-State Error in Terms of G(s)

- For the system:

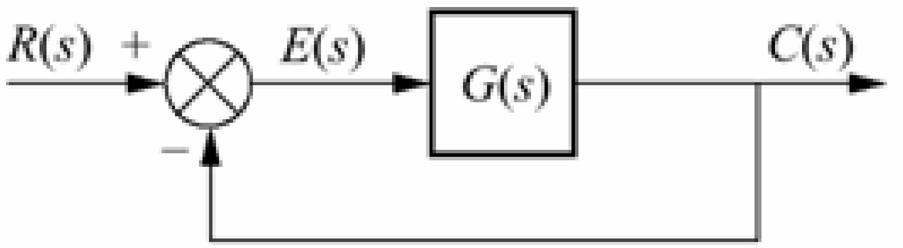
$$E(s) = R(s) - C(s) \text{ and } C(s) = E(s)G(s)$$

- Thus:

$$E(s) = R(s) - E(s)G(s)$$

- Rearrange, so that:

$$E(s) = \frac{R(s)}{1 + G(s)}$$



- From the final value theorem:

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}$$

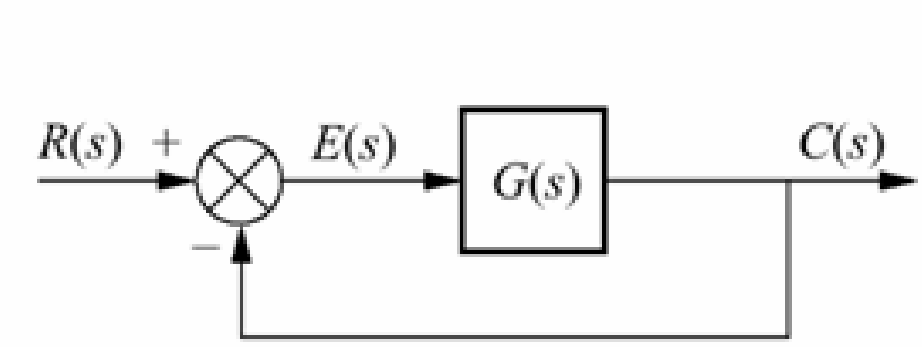
- Above equation will thus allow us to calculate the steady-state error given a particular input $R(s)$.

Example of S/S Error of Closed-Loop Systems

Determine the steady-state error of the unity feedback system as shown below if the plant $G(s)$ is given a step input $(1/s)$:

[4 marks]

$$G(s) = \frac{2}{s(s+2)}$$



Example of S/S Error of Closed-Loop Systems

- The steady-state error of the unity feedback system is determined from:

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

- Entering the transfer-function equation of the plant to the equation above, it becomes:

$$e(\infty) = \lim_{s\to 0} \frac{s(1/s)}{1+\left[\frac{2}{s(s+2)}\right]} = \lim_{s\to 0} \frac{s(s+2)}{s(s+2)+2} = 0$$

Steady-State Error of Step Input

Step Input: With $R(s) = 1/s$, we have:

$$e(\infty) = e_{step}(\infty) = \lim_{s\to 0} \frac{s(1/s)}{1 + G(s)} = \frac{1}{1 + \lim_{s\to 0} G(s)}$$

- For zero steady-state error, we need:

$$\lim_{s\to 0} G(s) = \infty$$

- To satisfy the above equation, $G(s)$ must have the form:

$$G(s) = \frac{(s + z_1)(s + z_2)\ldots}{s^n(s + p_1)(s + p_2)\ldots}$$

- The $G(s) \to \infty$ in the limit $s \to 0$, as the denominator will become zero.
- To have a zero steady-state error, we must have at least one pole at the origin so that $n \geq 1$.

Steady-State Error of Step Input

- The term s in the denominator of the equation for $G(s)$ represents an integrating element in the feedforward path.
- Division by s in the frequency domain represents integration in the time domain.
- At least one integrating element must be present in the forward path in order to ensure a zero steady-state error.
- If there are no integrations, then $n = 0$ and

$$\lim_{s\to 0} G(s) = \frac{z_1 z_2 \cdots}{p_1 p_2 \cdots}$$

- This will be finite and will thus produce a finite steady-state error.
- In order to have a zero steady-state error for a step input, we thus need at least one integrating element in the forward path.

Steady-State Error of Ramp Input

Ramp Input: For a ramp input, we have $r(t) = tu(t)$, where $r(t) = t$ for $t > 0$ and $r(t) = 0$ elsewhere.

- With $R(s) = 1/s^2$ we have:

$$e(\infty) = \lim_{s\to 0} \frac{s(1/s^2)}{1 + G(s)} = \lim_{s\to 0} \frac{1}{s + sG(s)} = \frac{1}{\lim\limits_{s\to 0} sG(s)}$$

- In order to have zero steady-state error, we need:

$$\lim_{s\to 0} sG_0(s) = \infty$$

- For this condition, we need poles at origin, $n \geq 2$, i.e. we need at least two integrators in the open-loop transfer function.

Steady-State Error of Ramp Input

- If there is one integrator ($n = 1$):

$$\lim_{s \to 0} sG_0(s) = \frac{sKz_1z_2 \dots}{sp_1p_2 \dots} = \text{finite}$$

- This will lead to a finite steady-state error.

- If there are no integrators ($n = 0$):

$$\lim_{s \to 0} sG_0(s) = \frac{sKz_1z_2 \dots}{p_1p_2 \dots} = 0$$

- So that, we have an infinite steady-state error.

Steady-State Error of Parabolic Input

Parabolic Input: For a parabolic input, we have:

$$r(t) = 0.5t^2$$

- Thus, $R(s) = 1/s^3$, the steady-state error is then:

$$e(\infty) = \lim_{s\to 0} \frac{s(1/s^3)}{1 + G(s)} = \lim_{s\to 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{\lim_{s\to 0} s^2 G(s)}$$

- In order to have zero steady-state error, we need:

$$\lim_{s\to 0} s^2 G_0(s) = \infty$$

- We will thus require three integrators in the open-loop transfer function ($n \geq 3$).
- If $n = 2$, there will be a finite steady-state error and for $n < 2$ there will be an infinite steady-state error.

Summary of Steady-State Errors

Expressions for the steady-state error (for unity feedback) to different inputs:

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Where:

$$e_{step}(\infty) = \frac{1}{1+\lim_{s\to 0} G(s)}$$

$$e_{ramp}(\infty) = \frac{1}{1+\lim_{s\to 0} sG(s)}$$

$$e_{parabola}(\infty) = \frac{1}{1+\lim_{s\to 0} s^2G(s)}$$

Summary of Steady-State Errors

For a zero steady-state error, we need at least:

- one integrator in the transfer function for a step input.

$$e(\infty) = \frac{1}{1 + \lim_{s\to 0}\left[\frac{as^n + bs^{n-1} + \cdots}{s(as^n + bs^{n-1} + \cdots)}\right]}$$

- two integrators in the transfer function for a ramp input.

$$e(\infty) = \frac{1}{1 + \lim_{s\to 0} s\left[\frac{as^n + bs^{n-1} + \cdots}{s^2(as^n + bs^{n-1} + \cdots)}\right]}$$

- three integrators in the transfer function for a parabola input.

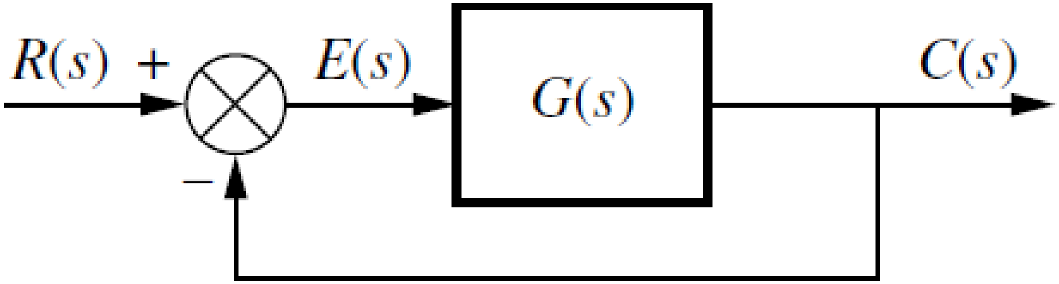
$$e(\infty) = \frac{1}{1 + \lim_{s\to 0} s^2\left[\frac{as^n + bs^{n-1} + \cdots}{s^3(as^n + bs^{n-1} + \cdots)}\right]}$$

Example of Steady-State Errors & Inputs

For the unity feedback system shown in the figure below, where:

$$G(s) = \frac{450(s+8)(s+12)(s+15)}{s(s+38)(s^2+2s+28)}$$

Find the steady-state errors for the following test inputs: $25u(t)$, $37tu(t)$, and $47t^2u(t)$. [6 marks]



Example of Steady-State Errors & Inputs

- The steady-state error of the system is calculated from:

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Where:

$$G(s) = \frac{450(s+8)(s+12)(s+15)}{s(s+38)(s^2+2s+28)}$$

- For step input, $25u(t)$, $R(s) = 25/s$. Thus, the steady-state error of the system is:

$$e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Example of Steady-State Errors & Inputs

$$= \lim_{s\to 0} \frac{s(25/s)}{1 + \frac{450(s+8)(s+12)(s+15)}{s(s+38)(s^2+2s+28)}}$$

$$= 0$$

- For ramp input, $37tu(t)$, $R(s) = 37/s^2$. Thus, the steady-state error of the system is:

$$e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

$$= \lim_{s\to 0} \frac{s\left(\frac{37}{s^2}\right)}{1 + \frac{450(s+8)(s+12)(s+15)}{s(s+38)(s^2+2s+28)}}$$

Example of Steady-State Errors & Inputs

$$= \frac{37}{\frac{450(8)(12)(15)}{(38)(28)}} = 6.075 \times 10^{-2}$$

- For parabolic input, $47t^2u(t)$, $R(s) = 47/s^3$. Thus, the steady-state error of the system is:

$$e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}$$

$$= \lim_{s\to 0} \frac{s(47/s^3)}{1 + \frac{450(s+8)(s+12)(s+15)}{s(s+38)(s^2+2s+28)}}$$

$$= \infty$$

Static Error Constants

Static-error constant and system type:

- The term in the denominator of the definition of the steady-state error for each input type is taken to limit the steady-state error.
- These are then called the static-error constants and are defined as follows:
    - Position constant: $K_p = \lim_{s \to 0} G(s)$
    - Velocity constant: $K_v = \lim_{s \to 0} sG(s)$
    - Acceleration constant: $K_a = \lim_{s \to 0} s^2 G(s)$
- These constants depend on the form of $G(s)$ and will determine the value of the steady-state error.
- Error decreases as the value of the static-error constant increases.

Static Error Constants

Static position error constant (K_p):

- It is associated with step input signal applied to a closed-loop system. For a given step input signal:

$$R(s) = A/s \qquad (\text{Eq. }1)$$

- Steady-state error is given as:

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)H(s)} \qquad (\text{Eq. }2)$$

- Put equations (1) into (2):

$$e_{ss} = \lim_{s\to 0} \frac{s(A/s)}{1 + G(s)H(s)} = \frac{A}{1 + \lim_{s\to 0} G(s)H(s)} = \frac{A}{1 + K_p}$$

Where: $K_p = \lim_{s\to 0} G(s)H(s)$

Static Error Constants

Static velocity error constant (K_v)

- It is associated with ramp input signal applied to a closed loop system. The ramp input signal is:

$$R(s) = A/s^2 \qquad (\mathrm{Eq.}\,3)$$

- Steady-state error is given as:

$$e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)H(s)} \qquad (\mathrm{Eq.}\,4)$$

- Put equations (3) into (4):

$$e_{ss} = \lim_{s\to 0} \frac{s(A/s^2)}{1 + G(s)H(s)} = \frac{A}{\lim\limits_{s\to 0}(1)s + \lim\limits_{s\to 0} sG(s)H(s)} = \frac{A}{K_v}$$

Where: $K_v = \lim\limits_{s\to 0} sG(s)H(s)$

Static Error Constants

Static acceleration error constant (K_a)

- It is associated with parabolic input signal applied to a closed loop system. The parabolic input signal is:

$$R(s) = A/s^3 \qquad (\mathrm{Eq.\,5})$$

- Steady-state error is given as:

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)H(s)} \qquad (\mathrm{Eq.\,6})$$

- Put equations (5) into (6):

$$e_{ss} = \lim_{s \to 0} \frac{s(A/s^3)}{1 + G(s)H(s)} = \frac{A}{\lim\limits_{s \to 0}(1)s^2 + \lim\limits_{s \to 0} s^2 G(s)H(s)} = \frac{A}{K_a}$$

Where: $K_a = \lim\limits_{s \to 0} s^2 G(s)H(s)$

Example of Static Error Constants

For a system that has the open-loop transfer function as given below.

$$G(s) = \frac{20(s+1)}{s(s+2)(s+5)}$$

a. Determine the position, velocity and acceleration error constants (i.e. K_p, K_v, and K_a) and steady-state errors. [12 marks]

b. Comment on influence of the input on the tracking of the output of the system. [2 marks]

Example of Static Error Constants

a. The steady-state error constants and steady-state errors for the given system are:

- Step input:

$$K_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{20(s+1)}{s(s+2)(s+5)} = \frac{20(1)}{(0)(2)(5)} = \infty$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

- Ramp input:

$$K_v = \lim_{s\to 0} sG(s) = \lim_{s\to 0} \frac{(s)20(s+1)}{s(s+2)(s+5)} = \frac{(20)(1)}{(2)(5)} = 2$$

$$e_{ss} = \frac{1}{K_v} = \frac{1}{2} = 0.5$$

Example of Static Error Constants

- Parabolic input:

$$K_a = \lim_{s\to 0} s^2 G(s) = \lim_{s\to 0} \frac{(s^2)20(s+1)}{s(s+2)(s+5)} = \frac{(0)(20)(1)}{(2)(5)}$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty$$

b. Since the open-loop transfer function of this system has one integrator, the output of the closed-loop system can perfectly track only the unit step.

System Type

- The system type is taken to be the number of integrations in the feed-forward path.
- The value of n in s^n of the denominator. This value of n (the system type) then determines the steady-state error of a unit feedback system for a particular type of input.
- In general, the system transfer function can be written as:

$$G(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^n\prod_{k=1}^{Q}(s+p_k)}$$

Where: $\prod$ denotes a multiplication of factors.

- The index 'n' denotes the system type number (if n = 0, the system type is 0; if n = 1, the system type is 1, etc.)

Steady-State Error Constant & System Type

- The relationships between types of inputs, steady-state error constants and system types are summarised as in the following table:

Input	**Steady-state error formula**	**Type 0**		**Type 1**		**Type 2**	
		Static error constant	**Error**	**Static error constant**	**Error**	**Static error constant**	**Error**
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $1/2t^2u(t)$	$\frac{1}{K_\mathrm{a}}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

Example of Steady-State Errors & System Type

Consider the second-order system whose open-loop transfer function is given below.

$$G(s) = \frac{(s+3)}{(s+1)(s+2)}$$

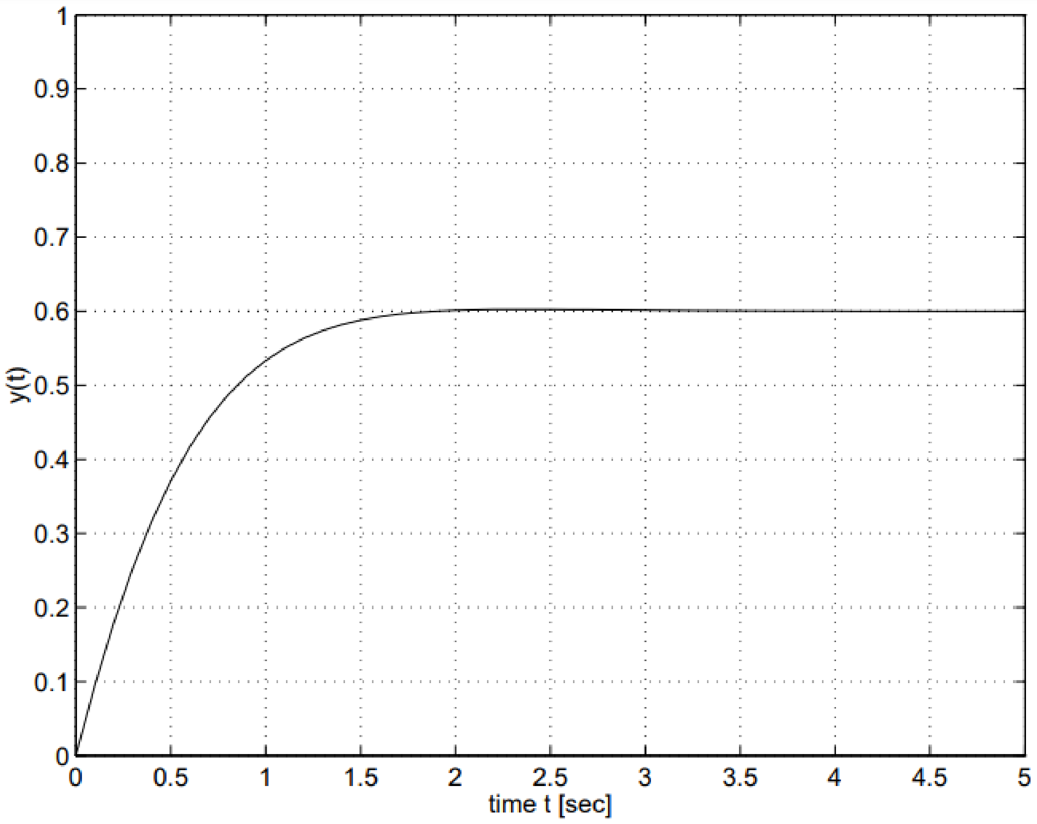
a. Sketch the time response of the system. [5 marks]

b. Calculate the position error constant (K_p) and steady-state error of the system toward unit-step input. [6 marks]

c. What type of system is the system? Can you eliminate the steady-state error of this system? [4 marks]

Example of Steady-State Errors & System Type

a. The unit-step response of the given system. Notice the steady-state output is equal to 0.6, and hence, the steady-state error is 0.4.



Example of Steady-State Errors & System Type

b. The position-error constant for this system is:

$$K_p = \lim_{s \to 0} \frac{(s+3)}{(s+1)(s+2)} = 1.5$$

So, the corresponding steady-state error of the system is:

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+1.5} = 0.4$$

The unit-step response of the system is presented in the figure in part (a), from which it can be clearly seen that the steady-state output is equal to 0.6.

Hence, the steady-state error is equal to:

$$e(\infty) = 1 - 0.6 = 0.4$$

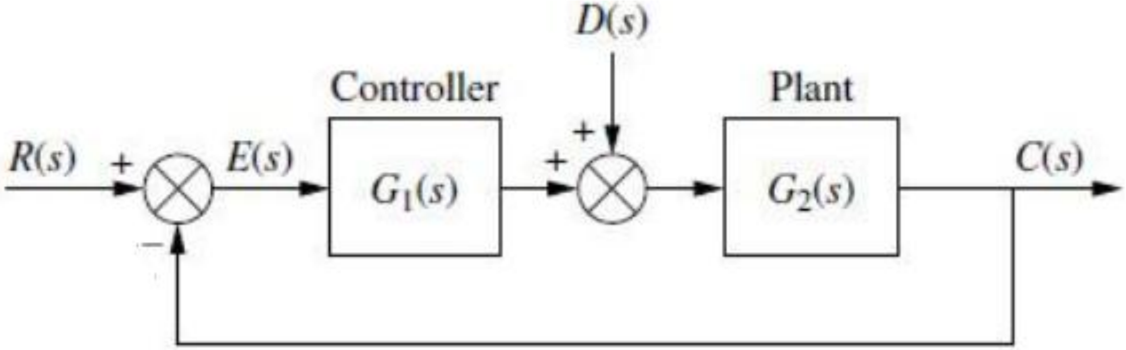
Example of Steady-State Errors & System Type

c. The system is a Type 0 system as it does not have any integral.

   The steady-state error of the system can be eliminated by introducing an integral into the system.

Steady-State Error for Disturbances

- Feedback control systems are often used to compensate for disturbances or unwanted inputs that enter a system.



- For a feedback control system with a disturbance, $D(s)$, injected between the controller and the plant, the transform of the output is:

$$C(s) = R(s) - E(s)$$

- Thus

$$C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s)$$

Steady-State Error for Disturbances

- The equation for deriving steady-state error is:

$$E(s) = \frac{1}{1+G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1+G_1(s)G_2(s)}D(s) \quad (\text{Eq. } 7)$$

- The first part is relating $E(s)$ to $R(s)$ and the second term relating $E(s)$ to $D(s)$.
- Apply final value theorem to find steady-state value of the error:

$$e(\infty) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} \frac{s}{1+G_1(s)G_2(s)}R(s) - \lim_{s\to 0} \frac{sG_2(s)}{1+G_1(s)G_2(s)}D(s)$$

Steady-State Error for Disturbances

- Equation for the steady-state error for disturbance is:

$$e(\infty) = e_R(\infty) + e_D(\infty)$$

Where:

$$e_R(\infty) = \lim_{s \to 0} \frac{s}{1 + G_1(s)G_2(s)} R(s)$$

And

$$e_D(\infty) = \lim_{s \to 0} \frac{sG_2(s)}{1 + G_1(s)G_2(s)} D(s)$$

- The first term $e_R(\infty)$ is the steady-state error due to $R(s)$ and the second term $e_D(\infty)$ is the steady-state error due to disturbance $D(s)$.

Steady-State Error for Disturbances

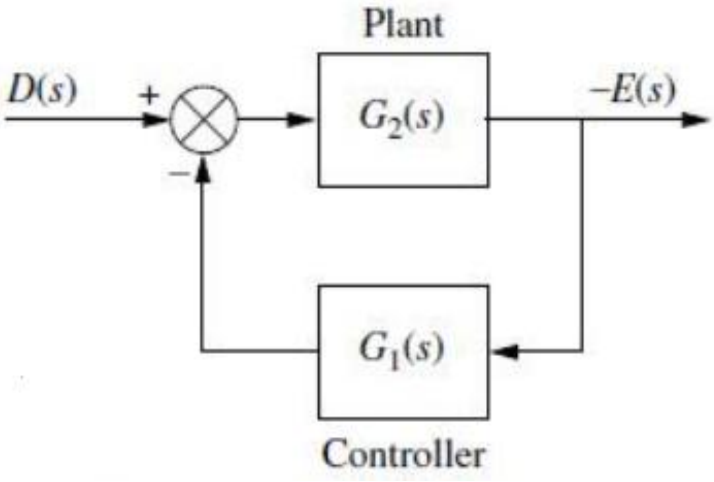
- Assume a step disturbance $D(s) = 1/s$.
- Substitute this value of step disturbance into the second term of equation (7), $e_D(\infty)$, the steady-state error due to a step disturbance is:

$$e_D(\infty) = -\frac{1}{\lim\limits_{s\to 0}\dfrac{1}{G_2(s)} + \lim\limits_{s\to 0} G_1(s)}$$

- The steady-state error produced by a step disturbance can be reduced by increasing the dc gain of $G_1(s)$ or decreasing the dc gain of $G_2(s)$.

Steady-State Error for Disturbances

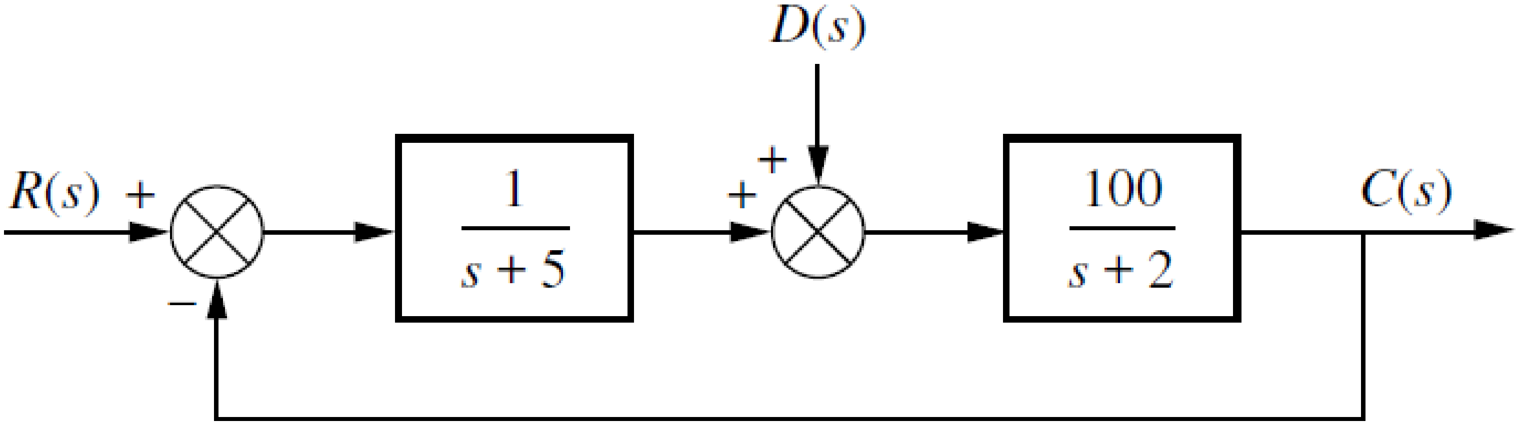
- If we want to minimise the steady-state value of $E(s)$, (the output), we must increase the DC gain of $G_1(s)$ so that a lower $E(s)$ be fed back to match the steady-state value of $D(s)$ or decrease the dc value of $G_2(s)$.
- This yields a smaller value of $e(\infty)$, as predicted by the feedback formula.



Example of Steady-State Error for Disturbances

Find the total steady-state error due to a unit step input and a unit step disturbance in the system of the figure below.

[8 marks]



Example of Steady-State Error for Disturbances

- From the given block diagram of the system, the equation for the steady-state error of the system is:

$$e(\infty) = \lim_{s\to 0} \frac{sR(s) - sD(s)G_2(s)}{1 + G_1(s)G_2(s)}$$

Where:

$$G_1(s) = \frac{1}{s+5} \qquad \text{and} \qquad G_2(s) = \frac{100}{s+2}$$

- From the problem statement, the input signal is:

$$R(s) = D(s) = \frac{1}{s}$$

Example of Steady-State Error for Disturbances

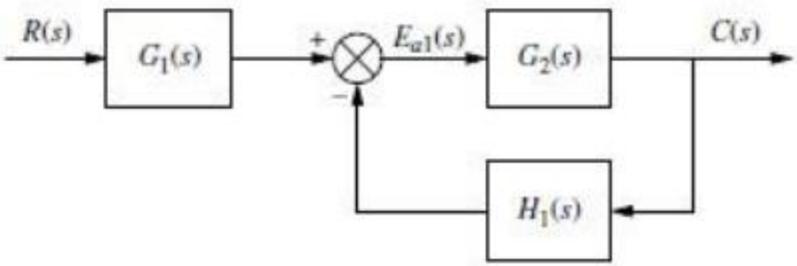
- Hence, the steady-state error of the system is:

$$e(\infty) = \lim_{s\to 0} \frac{s\left(\frac{1}{s}\right) - s\left(\frac{1}{s}\right)\left(\frac{100}{s+2}\right)}{1 + \left(\frac{1}{s+5}\right)\left(\frac{100}{s+2}\right)}$$

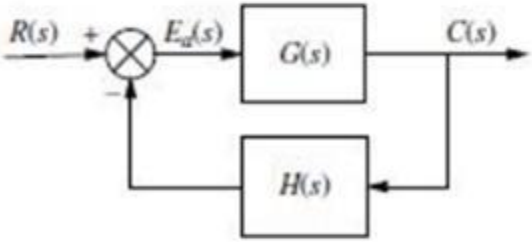
$$= \lim_{s\to 0} \frac{1 - \left(\frac{100}{s+2}\right)}{1 + \left(\frac{1}{s+5}\right)\left(\frac{100}{s+2}\right)} = -\frac{49}{11}$$

Steady-State Error for Non-Unity Feedback

- A general feedback system, showing the input transducer, $G_1(s)$, controller and plant, $G_2(s)$, and feedback, $H_1(s)$, is shown in Figure (a).
- Pushing the input transducer to the right past the summing junction yields the general non-unity feedback system shown in Figure (b), where: $G(s) = G_1(s)G_2(s)$ and $H(s) = H_1(s)/G_1(s)$.



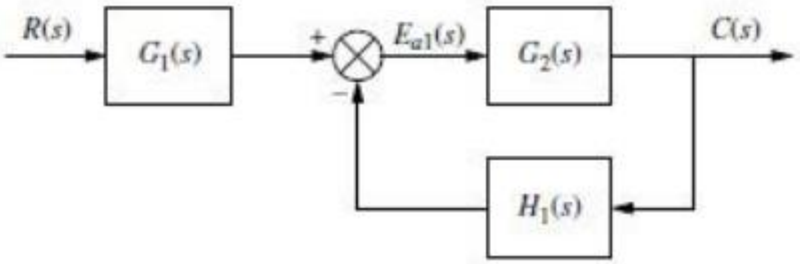
(a)



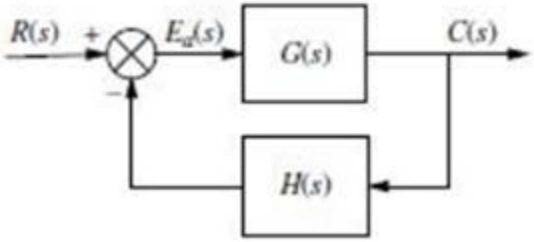
(b)

Steady-State Error for Non-Unity Feedback

- Unlike a unity feedback system, where $H(s) = 1$, the error in non-unity feedback is not the difference between the input and the output.
- For this case we call the signal at the output of the summing junction the actuating signal, $E_a(s)$.
- If $r(t)$ and $c(t)$ have the same units, we can find the steady-state error, $e(\infty) = r(\infty) - c(\infty)$.



(a)



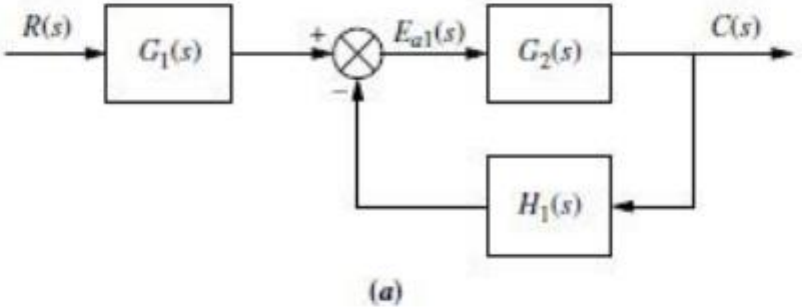
(b)

Steady-State Error for Non-Unity Feedback

- To find out the steady-state value of the actuating signal, $E_{a1}(s)$, in figure (a), there is no restriction that the input and output units be the same, since we are finding the steady-state difference between signals at the summing junction, which do have the same units.
- The steady-state actuating signal for Figure (a) is:

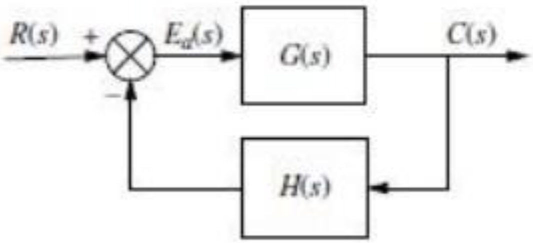
$$e_{a1}(\infty) = \lim_{s\to 0} \frac{sR(s)G_1(s)}{1 + G_2(s)H_1(s)}$$

- The first step is to show explicitly $E(s) = R(s) - C(s)$ on the block diagram.

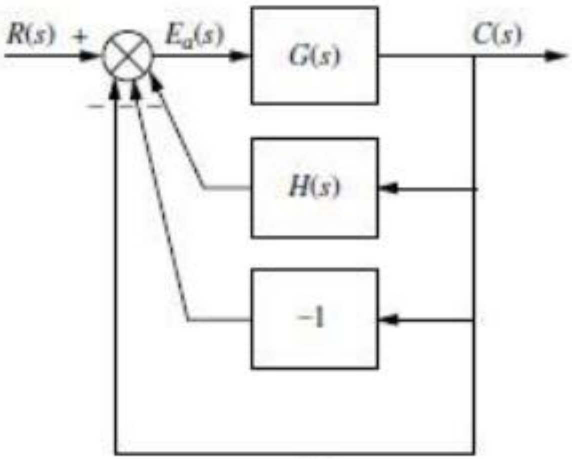


Steady-State Error for Non-Unity Feedback

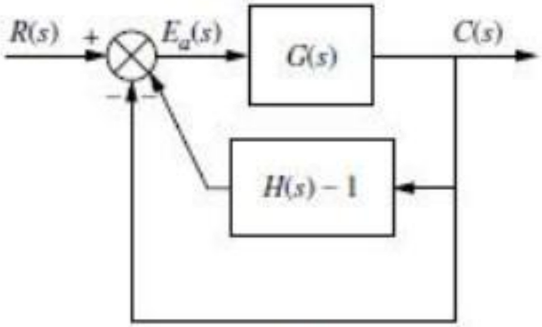
- Then, we form an equivalent unity feedback system from a general non-unity feedback system as illustrated below.



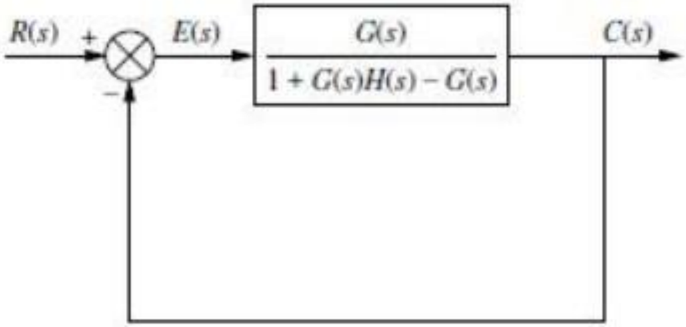
(b)



(c)



(d)



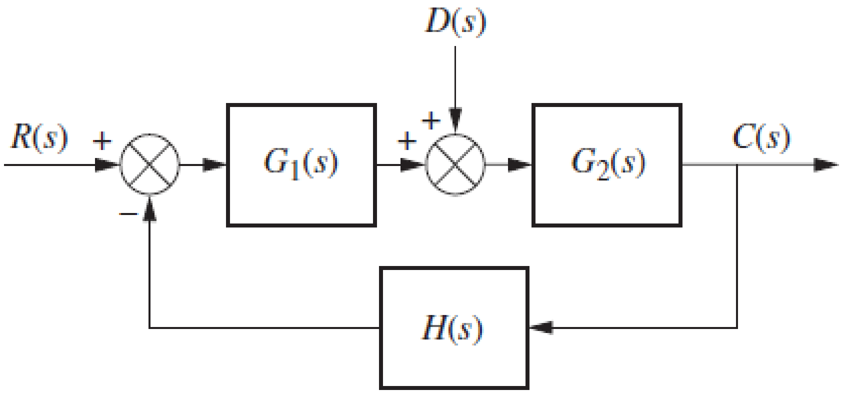
(e)

Steady-State Error for Non-Unity Feedback

- Take the non-unity feedback control system shown in Figure (b) and form a unity feedback system by adding and subtracting unity feedback paths, as shown in Figure (c). This step requires that input and output units be the same.
- Next combine $H(s)$ with the negative unity feedback, as shown in Figure (d).
- Finally, combine the feedback system consisting of $G(s)$ and $[H(s) - 1]$, leaving an equivalent forward path and a unity feedback, as shown in Figure (e).
- Notice that the final figure shows $E(s) = R(s) - C(s)$ explicitly.

Steady-State Error for Non-Unity Feedback

- Let us look at the general system of the figure below which has both a disturbance and non-unity feedback.



- We will derive a general equation for the steady-state error and then determine the parameters of the system in order to drive the error to zero for step inputs and step disturbances.

Steady-State Error for Non-Unity Feedback

- The steady-state error for this system, $e(\infty) = r(\infty) - c(\infty)$, is:

$$e(\infty) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0} \left\{ \left[1 - \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} \right] R(s) - \left[\frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \right] D(s) \right\}$$

Steady-State Error for Non-Unity Feedback

- Now limiting the discussion to step inputs and step disturbances, where $R(s) = D(s) = 1/s$, the above equation becomes:

$$e(\infty) = \lim_{s\to 0} sE(s)$$

$$= \left[1 - \frac{\lim_{s\to 0} G_1(s)G_2(s)}{1 + \lim_{s\to 0} G_1(s)G_2(s)H(s)}\right]$$

$$- \left[\frac{\lim_{s\to 0} G_2(s)}{1 + \lim_{s\to 0} G_1(s)G_2(s)H(s)}\right]$$

Steady-State Error for Non-Unity Feedback

- For zero error,

$$\frac{\lim_{s\to 0} G_1(s)G_2(s)}{1+\lim_{s\to 0} G_1(s)G_2(s)H(s)} = 1$$

- And

$$\frac{\lim_{s\to 0} G_2(s)}{1+\lim_{s\to 0} G_1(s)G_2(s)H(s)} = 0$$

- The two equations above can always be satisfied if:

  (1) the system is stable,

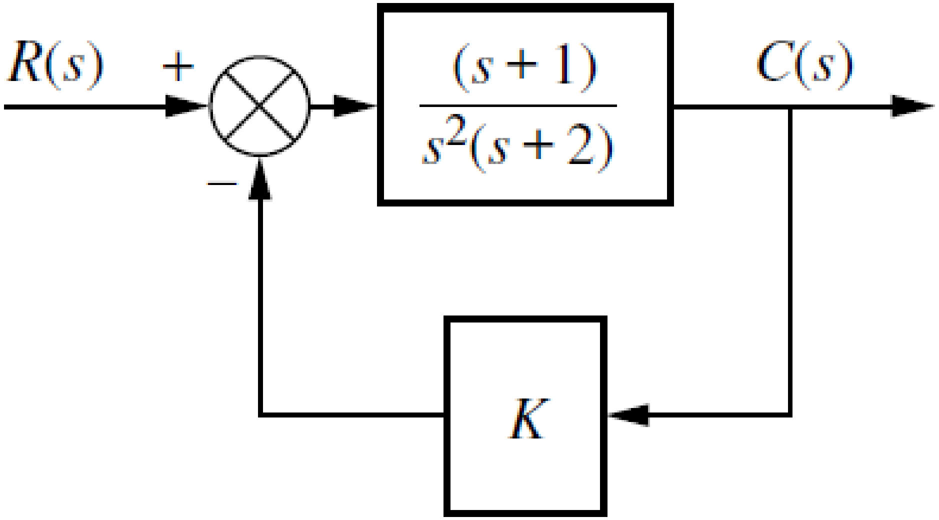
  (2) $G_1(s)$ is a Type 1 system,

  (3) $G_2(s)$ is a Type 0 system, and

  (4) $H(s)$ is a Type 0 system with a dc gain of unity.

Example for Steady-State Non-Unity Feedback

Given the non-unity feedback system as shown in the figure given below, find the following:

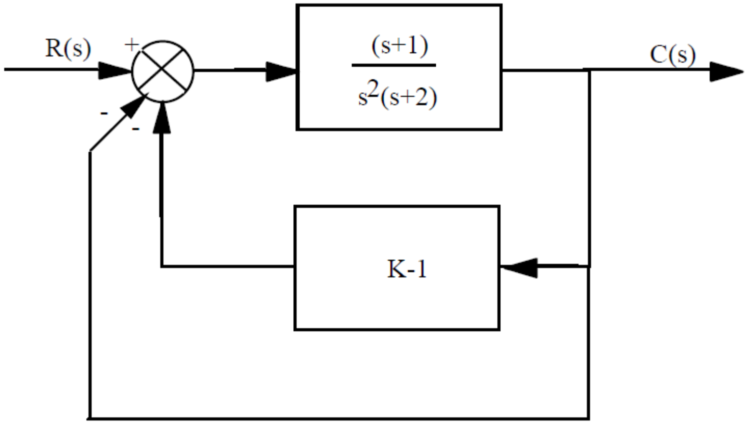


a. The system type. [4 marks]

b. The value of K to yield 0.1% error in the steady state. [14 marks]

Example for Steady-State Non-Unity Feedback

a. Produce a unity-feedback system of the system as shown in the figure below.



Thus, the unity-feedback system of the system is:

$$G_e(s) = \frac{\frac{(s+1)}{s^2(s+2)}}{1+\frac{(s+1)(K-1)}{s^2(s+2)}} = \frac{s+1}{s^3+2s^2+(K-1)s+(K-1)}$$

As shown above, the system is Type 0.

Example for Steady-State Non-Unity Feedback

b. Since the system is Type 0, the appropriate static error constant is K_p. Thus, the steady-state error due to the step input is:

$$e_{step}(\infty) = 0.001 = \frac{1}{1+K_p}$$

Therefore,

$$K_p = 999 = \frac{1}{K-1}$$

Hence, K = 1.001001.

Example for Steady-State Non-Unity Feedback

Check stability: Using original block diagram, the closed-loop transfer function of the system is:

$$T(s) = \frac{\dfrac{(s+1)}{s^2(s+2)}}{1 + \dfrac{K(s+1)}{s^2(s+2)}} = \frac{s+1}{s^3 + 2s^2 + Ks + K}$$

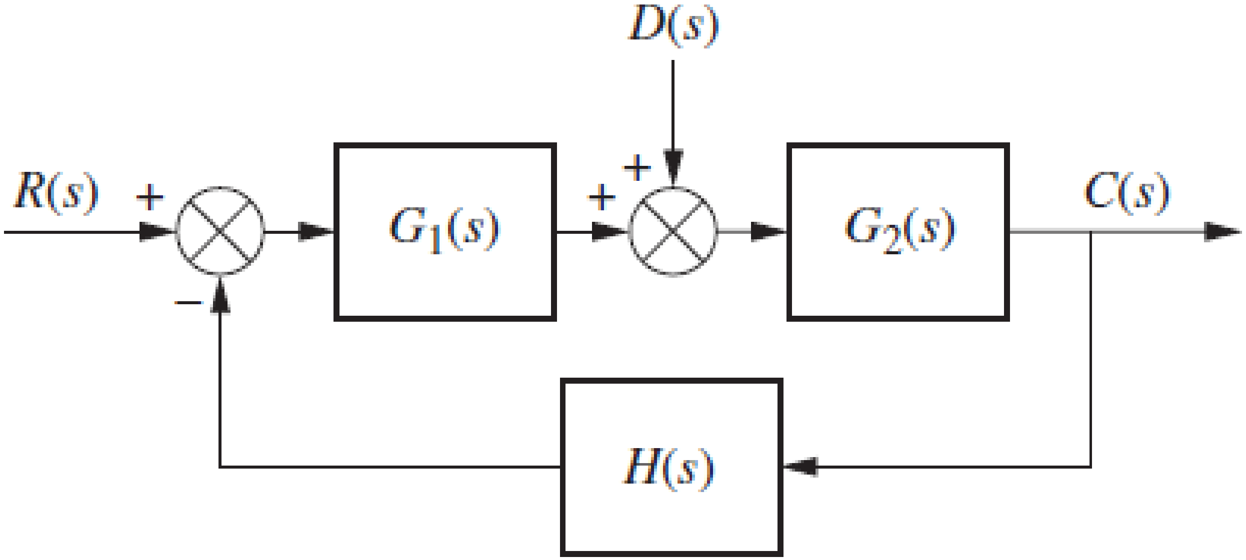
Making a Routh table:

s^3	1	K
s^2	2	K
s^1	$\frac{K}{2}$	0
s^0	K	0

Therefore, system is stable and steady-state error calculations are valid.

Non-Unity Feedback Steady-State & Disturbance

- Let us look at the general system of the figure below which has both a disturbance and non-unity feedback.



- Derive a general equation for the steady-state error and then determine the parameters of the system in order to drive the error to zero for step inputs and step disturbances.

Non-Unity Feedback Steady-State & Disturbance

- The steady-state error for this system, $e(\infty) = r(\infty) - c(\infty)$, is:

$$e(\infty) = \lim_{s\to 0} sE(s)$$

$$= \lim_{s\to 0}\left\{\left[1 - \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}\right]R(s) - \left[\frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}\right]D(s)\right\}$$

- Now limiting the discussion to step inputs and step disturbances, where $R(s) = D(s) = 1/s$, the above equation becomes:

$$e(\infty) = \lim_{s\to 0} sE(s)$$

$$= \left[1 - \frac{\lim_{s\to 0} G_1(s)G_2(s)}{1 + \lim_{s\to 0} G_1(s)G_2(s)H(s)}\right] - \left[\frac{\lim_{s\to 0} G_2(s)}{1 + \lim_{s\to 0} G_1(s)G_2(s)H(s)}\right]$$

Non-Unity Feedback Steady-State & Disturbance

- For zero error,

$$\frac{\lim_{s\to 0} G_1(s)G_2(s)}{1+\lim_{s\to 0} G_1(s)G_2(s)H(s)} = 1$$

- And

$$\frac{\lim_{s\to 0} G_2(s)}{1+\lim_{s\to 0} G_1(s)G_2(s)H(s)} = 0$$

- The two equations above can always be satisfied if:
  1. the system is stable,
  2. $G_1(s)$ is a Type 1 system,
  3. $G_2(s)$ is a Type 0 system, and
  4. $H(s)$ is a Type 0 system with a DC gain of unity.

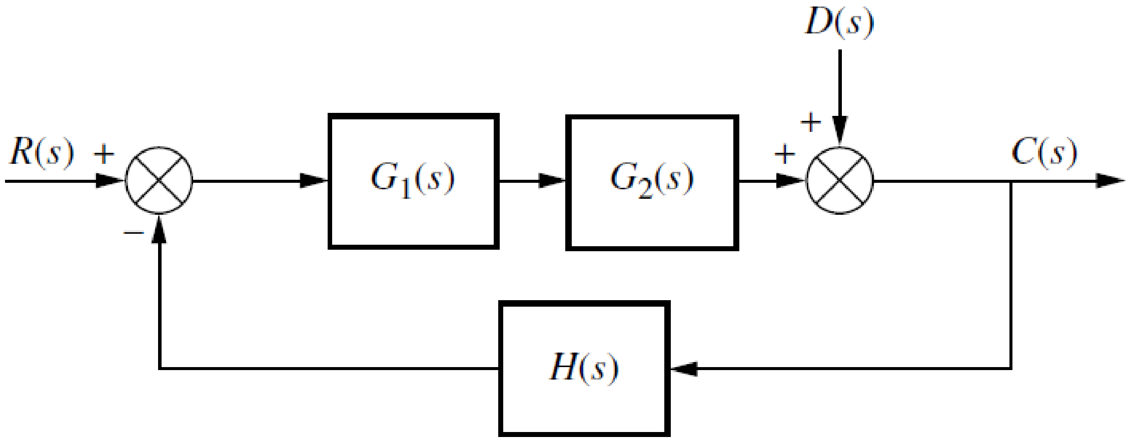
Example of Non-Unity S/S & Disturbance

Given the system shown in the figure below, do the following:

a. Derive the expression for the error, $E(s) = R(s) - C(s)$, in terms of $R(s)$ and $D(s)$. [8 marks]

b. Derive the steady-state error, $e(\infty)$, if $R(s)$ and $D(s)$ are unit step functions. [4 marks]

c. Determine the attributes of $G_1(s)$, $G_2(s)$, and $H(s)$ necessary for the steady-state error to become zero. [2 marks]



Example of Non-Unity S/S & Disturbance

a. The error in the system is calculated from:

$$E(s) = R(s) - C(s)$$

But, considering the disturbance, the output of the system is:

$$C(s) = [R(s) - C(s)H(s)]G_1(s)G_2(s) + D(s)$$

Solving for $C(s)$:

$$C(s) = \frac{R(s)G_1(s)G_2(s)}{1 + G_1(S)G_2(s)H(s)} + \frac{D(s)}{1 + G_1(s)G_2(s)H(s)}$$

Example of Non-Unity S/S & Disturbance

Substituting the above equation into $E(s)$, the equation becomes:

$$E(s) = \left[1 - \frac{G_1(s)G_2(s)}{1 + G_1(S)G_2(s)H(s)}\right]R(s) - \left[\frac{1}{1 + G_1(s)G_2(s)H(s)}\right]D(s)$$

b. For $R(s) = D(s) = 1/s$, the steady-state error of the system is:

$$e(\infty) = \lim_{s\to 0} sE(s)$$

$$= 1 - \frac{\lim_{s\to 0} G_1(s)G_2(s)}{1 + \lim_{s\to 0} G_1(s)G_2(s)H(s)} - \frac{1}{1 + \lim_{s\to 0} G_1(s)G_2(s)H(s)}$$

c. Zero error if $G_1(s)$ and/or $G_2(s)$ is Type 1. Also, $H(s)$ is Type 0 with unity DC gain.

Sensitivity of Parameters on Steady-State

- Sensitivity is the degree to which changes in system parameters affect system transfer functions, and hence performance.
- A system with zero sensitivity (that is, changes in the system parameters do not affect the transfer function) is ideal.
- The greater the sensitivity, the less desirable the effect of a parameter change.

Sensitivity of Parameters on Steady-State

- For example, assume the function of:

$$F = \frac{K}{(K + a)}$$

- If K = 10 and a = 100, then F = 0.091.
- If parameter a triples to 300, then F = 0.032.
- We see that a fractional change in parameter a of $(300 – 100)/100 = 2$ (e.g. 200% change) yields a change in the function F of $(0.032 - 0.091)/0.091 = 0.65$ (e.g. 65% change).
- Thus, the function F has reduced sensitivity to changes in parameter a.

Sensitivity of Parameters on Steady-State

- With feedback, it reduces sensitivity to parameter changes.
- Sensitivity is ratio of the fractional change in the function to the fractional change in the parameter as the fractional change of the parameter approaches zero.
- That is,

$$S_{F:P} = \lim_{\Delta P \to 0} \frac{\text{Fractional change in the function}, F}{\text{Fractional change in the parameter}, P}$$

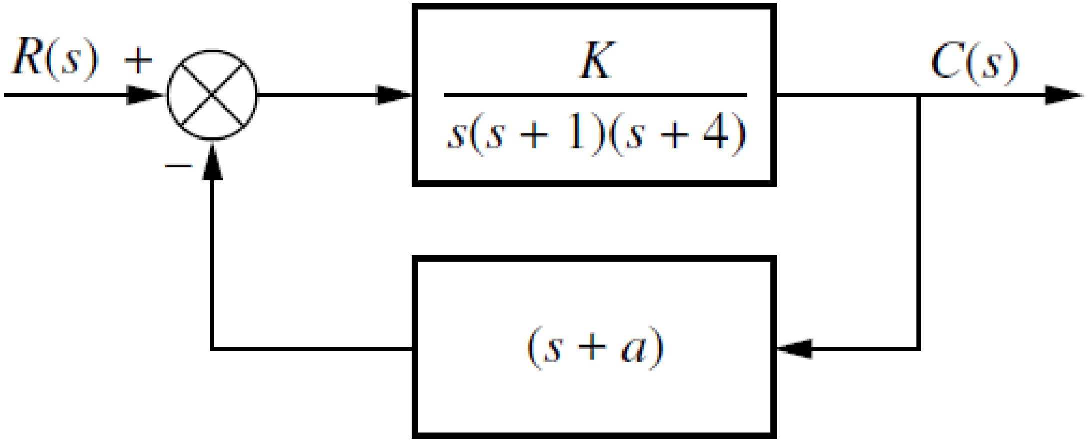
$$= \lim_{\Delta P \to 0} \frac{\Delta F / F}{\Delta P / P} = \lim_{\Delta P \to 0} \frac{P \Delta F}{F \Delta P}$$

- Which reduces to:

$$S_{F:P} = \frac{P}{F}\left(\frac{\delta F}{\delta P}\right)$$

Example of Sensitivity of S/S Parameters

For a system as shown in the figure below, assume it is given a step input.



a. Find the sensitivity of the steady-state error to parameter a. [6 marks]

b. Plot the sensitivity of the system as a function of parameter a. [5 marks]

Example of Sensitivity of S/S Parameters

a. First, find the forward transfer function of an equivalent unity-feedback system.

$$G_e(s) = \frac{\dfrac{K}{s(s+1)(s+4)}}{1 + \dfrac{K(s+a-1)}{s(s+1)(s+4)}}$$

$$= \frac{K}{s^3 + 5s^2 + (K+4)s + K(a-1)}$$

Thus, steady-state error of the system is:

$$e(\infty) = \frac{1}{1+K_p} = \frac{1}{a + \dfrac{K}{K(a-1)}} = \frac{a-1}{a}$$

Example of Sensitivity of S/S Parameters

Finding the sensitivity of $e(\infty)$, it is:

$$S_{e:a} = \frac{a}{e}\left(\frac{\delta e}{\delta a}\right) = \frac{a}{\left(\frac{a}{a-1}\right)}\left[\frac{a-(a-1)}{a^2}\right] = \frac{a-1}{a^2}$$

b. The plot of sensitivity of the system as a function of parameter a is as shown in the figure below.