XMUT315 Control Systems Engineering

Note 10a: Introduction to Bode Plots

Topic

- Frequency response methods.
- Foundation of frequency response.
- Bode plots.
- Bode plots and transfer functions.
- Examples of Bode plots.
- Resonance response.
- Right-hand plane roots.
- Non-minimum phase.

1. Introduction to Frequency Methods

There are some control system analysis methods in control system engineering that are based on the frequency response. One of these methods is the Bode plots. The other methods are root locus diagram, Nyquist diagram, and Nichols chart that we will cover later in the course.

1.1. Frequency Response Methods

Frequency response methods are a set of graphical techniques that focus on how the gain and phase of a system change with frequency. Frequency responses can be used when we have a good mathematical model for a system (a transfer function). Frequency response methods can also be used even when we do not have a good model of the system (plant) that we are trying to stabilise. We just use experimental data in place of a model.

1.2. Frequency Response Overview

We will cover in this course, the detailed construction of:

- Bode plots (plots of gain and phase vs. frequency).
- Root locus diagrams (plot of real vs imaginary parts of transfer function).
- Nyquist plots (plot of gain vs. phase).
- If time permits, we will also look at the Nichols plot, which can be considered an alternate form of the Nyquist plot.

For each plot type, we will discuss:

- How to assess system stability.
- How to determine the closed-loop characteristics of a control loop.
- How to design compensators (or controllers).

2. Foundations of the Frequency Response

Consider a system described by the transfer function:

$$G(s) = \frac{N(s)}{D(s)}$$

Where: $N(s)$ and $D(s)$ are polynomials such that $G(s)$ is proper.

2.1. Adding Input Signal

Let us apply a sinusoidal signal $r(t)$ at the input of the system and determine the corresponding output signal $y(t)$.

$$r(t) = A\cos(\omega t)u(t)$$

Applying Laplace transform to the equation above:

$$R(s) = A\left(\frac{s}{s^2 + \omega^2}\right)$$

The Laplace transform of the resulting output signal is then:

$$Y(s) = G(s)R(s) = A\left(\frac{s}{s^2 + \omega^2}\right)G(s)$$

2.2. Partial Fraction

If we were to expand this using partial fractions we would obtain:

$$Y(s) = \frac{c}{s - j\omega} + \frac{c^*}{s + j\omega} + \frac{V(s)}{D(s)}$$

Where: the first two terms arise from the sinusoidal excitation and the last term arises from the poles of the system (as contained in $D(s)$).

The polynomial $V(s)$here arises from the polynomial simplification. We can use the Heaviside method to find c and c^*.

$$c = (s - j\omega)Y(s)|_{s \to j\omega} = A\left[\frac{s(s - j\omega)}{s^2 + \omega^2}\right] G(s)\Big|_{s \to j\omega}$$

The equation becomes:

$$c = A\left[\frac{s(s - j\omega)}{(s + j\omega)(s - j\omega)}\right] G(s)\Big|_{s - j\omega}$$

As a result:

$$c = A\left(\frac{s}{s + j\omega}\right) G(s)\Big|_{s \to j\omega} = A\left(\frac{j\omega}{j\omega + j\omega}\right) G(j\omega)$$

Thus

$$c = \left(\frac{A}{2}\right) G(j\omega) \qquad \text{and} \qquad c^* = \left(\frac{A}{2}\right) G^*(j\omega)$$

Substituting back into the expression for $Y\,(s)$, we obtain:

$$Y(s) = \left(\frac{A}{2}\right)\left(\frac{G(j\omega)}{s - j\omega}\right) + \left(\frac{A}{2}\right)\left(\frac{G^*(j\omega)}{s + j\omega}\right) + \frac{V(s)}{D(s)}$$

2.3. Inverse Laplace Transform

We now take the inverse Laplace transform to move back into the time domain.

$$y(t) = \left(\frac{A}{2}\right) G(j\omega)e^{j\omega t} + \left(\frac{A}{2}\right) G^*(j\omega)e^{-j\omega t} + \mathcal{L}^{-1}\left(\frac{V(s)}{D(s)}\right)$$

Where: the term $\mathcal{L}^{-1}\left(\frac{V(s)}{D(s)}\right) = y_{tr}(t)$ is a transient arising from the poles of the system.

If the system is stable, then the transient will (eventually) decay to zero and we will be left with just the first two terms. After the decay of the transient, we have:

$$\begin{aligned} y(t) &= \left(\frac{A}{2}\right) G(j\omega)e^{j\omega t} + \left(\frac{A}{2}\right) G^*(j\omega)e^{-j\omega t} \\ &= \left(\frac{A}{2}\right)\left(|G(j\omega)|e^{j\theta}e^{j\omega t} + |G^*(\omega t)|e^{-j\theta}e^{-j\omega t}\right) \\ &= \left(\frac{A}{2}\right)\left(|G(j\omega)|e^{j(\omega t + \theta)} + |G^*(\omega t)|e^{-j(\omega t + \theta)}\right) \end{aligned}$$

Applying exponential form to trigonometry identity:

$$y(t) = \left(\frac{A}{2}\right)|G(j\omega)| 2\cos(\omega t + \theta) = A|G(j\omega)|\cos(\omega t + \theta) \quad \text{where: } \theta = \angle G(j\omega)$$

If we compare this with the input signal $r(t)$, we can see that the effect of the system is therefore to multiply the gain of the input signal by $|G(j\omega)|$ and phase shift it by $\angle G(j\omega)$

3. Bode Plots

A Bode plot is a pair of plots showing the variation of gain (in dB) and phase (normally in degrees) against the logarithm of the frequency.

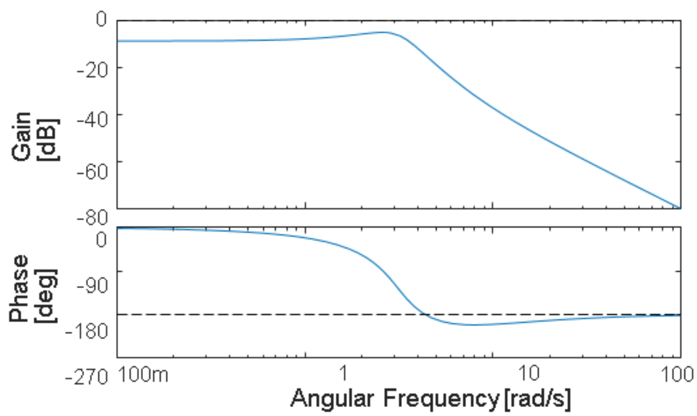


Figure 1: Bode plots of a control system (gain and phase plots)

We can use either angular or linear frequency on the x-axis, although angular frequency (in rad/s) is usually more convenient.

3.1. Gain and Phase Response

We wish to determine the gain and phase response as a function of ω for a system having transfer function $G(s)$. We can find this by plotting $|G(j\omega)|$ and $\angle G(j\omega)$ as we vary ω.

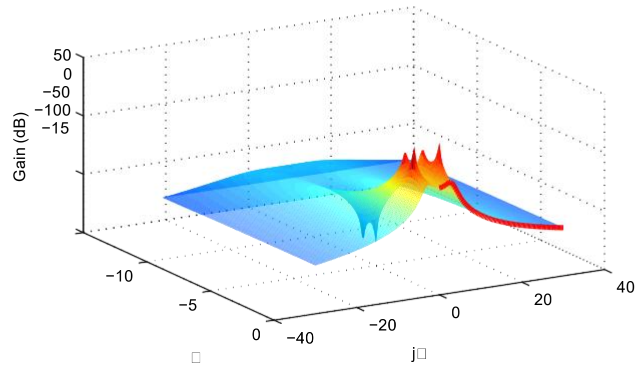


Figure 2: Gain and phase response of a control system

3.2. Bode Approximations

If we need to plot an accurate frequency response, then we could solve the gain and phase responses at many points. Alternatively, we could find explicit expressions for $|G(j\omega)|$ and $\angle G(j\omega)$. Tools like Matlab make plotting an accurate Bode plot very easy – `bode(tf([num],[den]))` and `bode(zpk([z],[p],[k]))` produce Bode plots. Read the Matlab help file for details and options.

However, much of the time we only need an approximate frequency response for control design. In fact, in many cases, the approximations are easier to work with than the accurate curves would be. Bode developed a set of straight-line approximations to the real response curves. Using these approximations makes it simple to plot a frequency response by hand.

3.3. Bode Plots - The Big Picture

Bode plots are intended to be quick and easy to draw:

- They produce reasonably accurate gain responses and phase responses that are adequate for many purposes.
- The approximations become less accurate for systems containing lightly damped oscillatory modes.

The basic idea of the Bode plot is to break a transfer function into smaller simple parts, each of which has a known Bode plot. We will build the Bode plot of an arbitrarily complex transfer function by adding the constituent plots graphically.

3.4. Dividing A Transfer Function into Its Parts

For example, consider a system having a transfer function:

$$G(s) = \frac{K(s+z_1)(s+z_2)}{s(s^3+d_2s^2+d_1s+d_0)} \qquad \text{for} \quad K \in R$$

We may be able to break this down into a set of simpler elemental transfer functions, say:

$$G(s) = K(s+z_1)(s+z_2)\left(\frac{1}{s}\right)\left(\frac{1}{s+p_1}\right)\left(\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}\right)$$

Usually, a transfer function is broken down into these terms:

- Poles and zeros at dc: s^n for $n \in Z$
- Simple poles and zeros: $(s+a)^{\pm 1}$
- Complex pairs of poles and zeros: $(s+2\zeta\omega s+\omega_n^2\ \ )^{\pm 1}$

3.5. Gain Response of An Arbitrary TF

Let's derive an equation for the gain and phase of an arbitrary transfer function $G(s)$, where:

$$G(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_k)}{(s+p_1)(s+p_2)\dots(s+p_n)}$$

That is, the transfer function has k zeros at $-zi \in\ C$ and n poles at $s=-pi \in\ C$. The z_i and p_i are not necessarily distinct. The gain of $G(s)$ at $s=j\omega$ is:

$$|G(j\omega)| = \frac{|K||j\omega+z_1||j\omega+z_2|\dots|j\omega+z_k|}{|j\omega+p_1||j\omega+p_2|\dots|j\omega+p_n|}$$

Converting to dB, we write this as:

$$\begin{aligned}20\log|G(j\omega)| = 20\log|K| + 20\log|j\omega+z_1| + \cdots + 20\log|j\omega+z_k| \\ - 20\log|j\omega+p_1| - \cdots - 20\log|j\omega+p_n|\end{aligned}$$

To find the gain response of our overall function in dB we can find the gain responses arising from each pole and zero separately and then add them. This is why we use a dB scale, because otherwise, we would have to go to the bother of multiplying the individual responses. We shall see shortly that adding the plots graphically is trivial.

3.6. Phase Response of An Arbitrary TF

Recall that a complex number $a = bc/de$ will have a phase given by:

$$\angle a = \angle b\ + \angle c\ - \angle d\ - \angle e$$

Similarly, our transfer function:

$$G(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_k)}{(s+p_1)(s+p_2)\dots(s+p_n)}$$

This will have a phase response of:

$$\angle G(s) = \angle K +\ \angle(s+z_1) +\ \angle(s+z_2) + \cdots +\ \angle(z_k) -\ \angle(s+p_1) -\ \angle(s+p_2) - \cdots -\ \angle(s+p_n)$$

Remember that $K \in R$, so $\angle K = 0$ if $K > 0$, or $180°$ if $K < 0$. If we calculate the phase responses for our family of prototype pole/zero combinations, we will be able to add them to determine the overall phase response of an arbitrary transfer function.

3.7. Transfer Functions for Bode Plots

It is easier to draw a Bode plot if we rearrange the transfer functions so that each term has unity gain at dc. Normally we write a transfer function as:

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_k)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

But we will find it easier for Bode plotting if we first place it in the equivalent form:

$$G(s) = \frac{K\left(1+\frac{s}{z_1}\right)\left(1+\frac{s}{z_2}\right)\ldots\left(1+\frac{s}{z_k}\right)}{\left(1+\frac{s}{p_1}\right)\left(1+\frac{s}{p_2}\right)\ldots\left(1+\frac{s}{p_n}\right)}$$

Converting to this form is accomplished by dividing through by the constant in each term and adjusting the overall gain to compensate.

Example for Tutorial 1 - Transfer Function Modification

Convert the transfer function given below into a suitable modified form for Bode plots.

$$G(s) = \frac{s+10}{s(s+2)(s^2+3s+9)}$$

a. Calculate the form manually. [4 marks]

b. Use simulation in MATLAB. [5 marks]

Answer

a. Calculate manually, the transfer function of the system is modified as follows:

$$G(s) = \frac{s+10}{s(s+2)(s^2+3s+9)} = \frac{10s+2\left(\frac{s}{10}+1\right)}{s\times 2\left(\frac{s}{2}+1\right)\times 9\left(\frac{s^2}{3^2}+\frac{s}{3}+1\right)} = \frac{5}{9}\left[\frac{\left(\frac{s}{10}+1\right)}{s\left(\frac{s}{2}+1\right)\left(\left(\frac{s}{3}\right)^2+\frac{s}{3}+1\right)}\right]$$

Note the change in the constant (dc) gain term. Notice also the form change for the complex pair of poles, from:

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

Thus, the equation above becomes:

$$\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1$$

b. MATLAB does not distinguish internally between the two forms of the transfer function that we have discussed. However, you can specify which form MATLAB uses to present the transfer function. This is useful to convert between the two.

```
>> G = zpk(-6,[-1+j -1-j -2],2)

          2 (s+6)                  % This is the default, with
   --------------------            % DisplayFormat = 'roots'
   (s+2) (s^2 + 2s + 2)

>> G.DisplayFormat = 'frequency';G

                  3 (1+s/6)
   -------------------------------------------
   (1+s/2) (1 +1.414(s/1.414) +(s/1.414)^2)
```

4. Forms and Plots of Bode Plots

In this section, we will look into a number of forms and plots of Bode plots. We look into forms such as: $Ks^{\pm n}$, $\left(\frac{s}{a}+1\right)^{\pm 1}$, $\left(1+\frac{s}{a}\right)^{\pm 2}$, and $\left[\left(\frac{2}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1\right]^{\pm 1}$.

4.1. Systems with Form Ks^n

The most basic form of Bode plots is Ks^n where the pole is located at the origin (0, 0) in the s-plane.

4.1.1. Transfer Functions of Form Ks^n

The gain of a transfer function $G(s) = Ks^n$ is given by:

$$|G(j\omega)| = K\omega^n$$

So, the gain in dB is:

$$|G(j\omega)| = 20\log(K\omega)^n = 20\log K + 20n\log\omega$$

Thus, the transfer gain response is a straight line with a slope of $20n$ dB/decade and is equal to $20\log K$ at $\omega = 1$.

$$G(j\omega) = K(j\omega)^n = j^n K\omega^n$$

Thus

$$\angle G(j\omega) = \angle j^n = 90n° \quad \text{if} \quad K > 0$$

The phase of $G(s) = Ks^n$ is constant at $90n$ degrees.

4.1.2. The Plot of a TF with Form Ks^n

For example, an integrator (which has $G(s) = 1/s = s^{-1}$) has a gain response of $1/\omega$ and a constant phase of −90°.

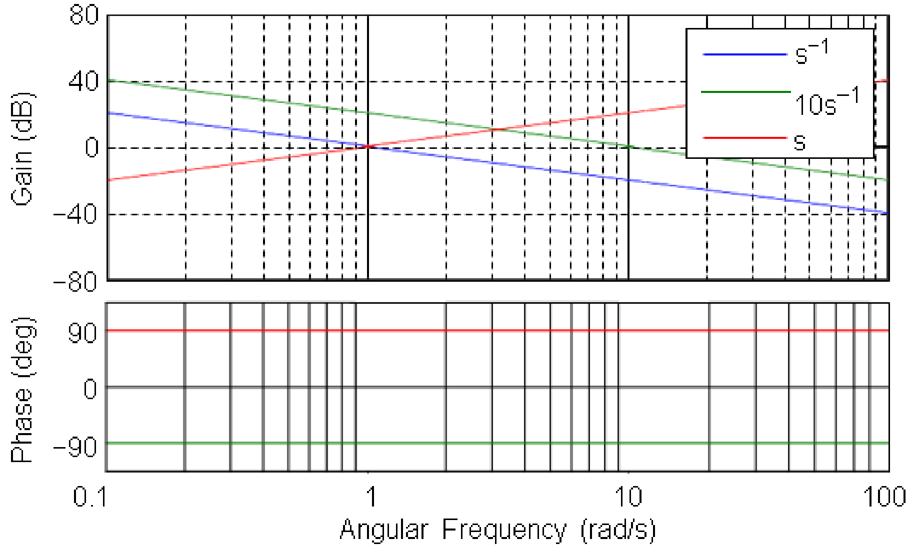


Figure 3: Bode plot of a TF with form Ks^n

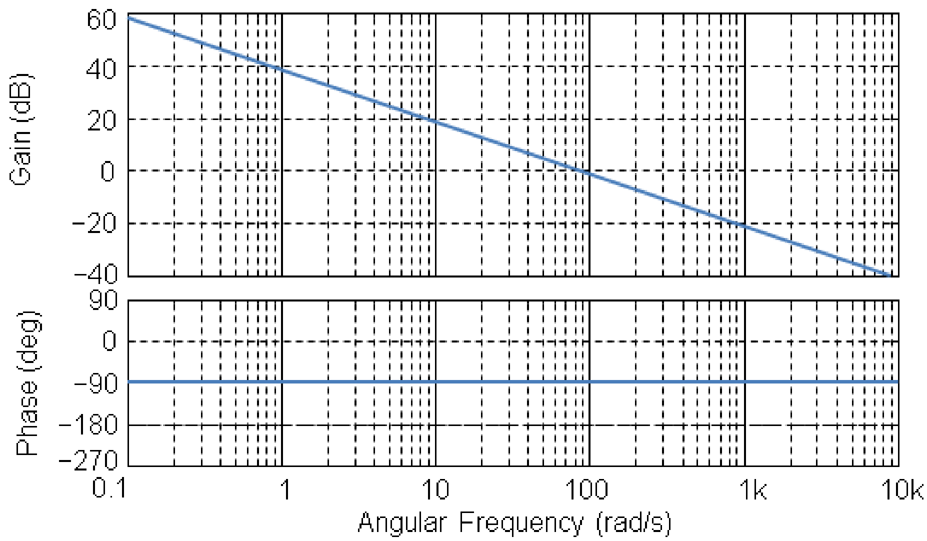
Example for Tutorial 2 – System with Pole at Origin

Given a first-order system with a zero at origin and gain of 100, sketch its Bode plots: [5 marks]

$$G(s) = \frac{100}{s}$$

Answer

The following figure shows the Bode plots of system $G(s) = 100/s$



Note:

Gain of the system is:

$$|G(j\omega) = 100/j\omega| = 20\log(100) - 20\log(\omega) = 40 - 20\log(\omega)\ \text{dB/dec}$$

Phase shift of the system is:

$$\angle(G(j\omega) = 100/j\omega) = -90°$$

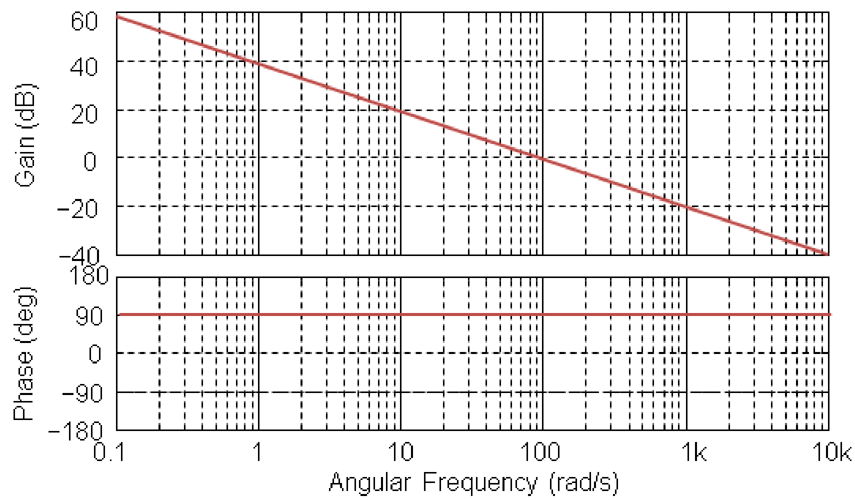
Example for Tutorial 3 – System with Pole and Negative Gain

Given a first-order system with a negative gain of 100 and a pole at the origin as in the following transfer function, sketch its Bode plots. [5 marks]

$$G(s) = -\frac{100}{s}$$

Answer

The following figure shows the Bode plots of system $G(s) = -100/s$.



Notice the difference between positive gain and negative gain the Bode plot. It is difficult to distinguish which gain is positive and which one is negative.

$$|G(j\omega) = -100/j\omega| = 20\log(100) - 20\log(\omega) = 40 - 20\log(\omega) \text{ dB/dec}$$

To cope with $K < 0$, you just need to account for the extra 180° phase shift associated with the negative gain.

$$\angle(G(j\omega) = -100/j\omega) = +90°$$

4.2. Systems with Form $(s/a + 1)^{-1}$

Consider a transfer function of form $(s) = \frac{1}{(s/a)+1}$, with $a > 0$.

This is a system with a single pole at $s = -a$ (a low pass filter).

- At low frequencies ($\omega \ll a$) we find $|G(j\omega)| = 1$ (or 0 dB).
- At high frequencies ($\omega \gg a$) we have $|G(j\omega)| = \frac{1}{(\omega/a)} = \frac{a}{\omega}$

The response in these two frequency regimes forms low-and high-frequency asymptotes. The low-frequency asymptote is a straight line with a zero slope and unity gain. On a dB scale, the high-frequency asymptote is given by:

$$|G(j\omega)| = 20\ \log a - 20\ \log \omega$$

This is therefore a straight line with a slope of −20 dB/decade.

Example for Tutorial 4 – System with Pole

Consider the example of a first-order system with pole at -100 as shown in the following transfer function as shown below, perform the following tasks:

$$G(s) = \frac{1}{\left(\frac{s}{100}\right) + 1}$$

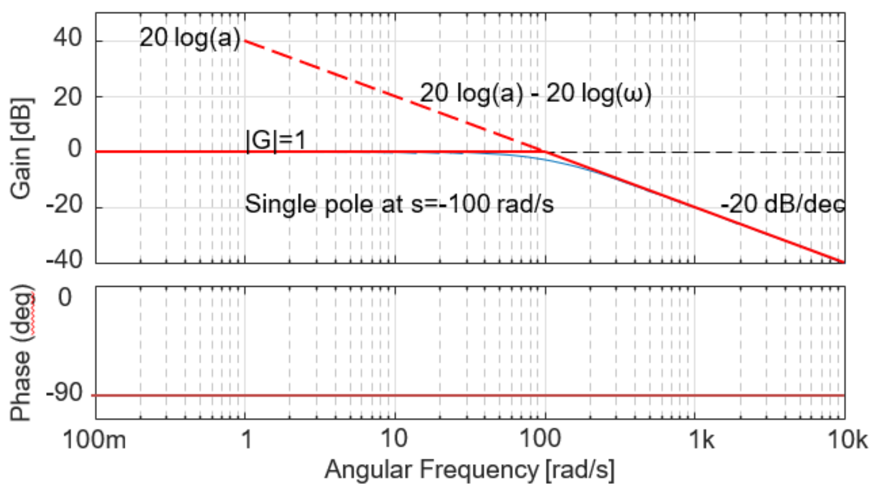
a. Sketch its Bode plots. [5 marks]

b. Describe the frequency response of the system. [6 marks]

c. Sketch its Bode plots. [5 marks]

Answer

a. Notice that the high and low-frequency asymptotes form a reasonable approximation of the real response. The following figure shows the Bode plots of system $G(s) = \frac{1}{(s/100)+1}$



b. At low frequency, the transfer function $G(s) = 1$. It, therefore, has a phase of 0° in this region. At high frequency $G(s) \approx \frac{a}{j\omega} = -j\left(\frac{a}{\omega}\right)$.

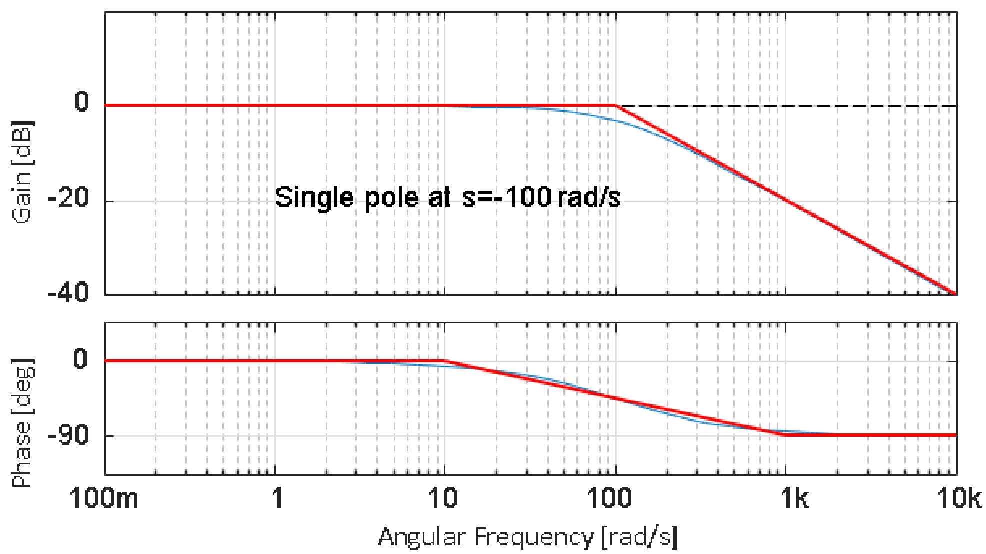
The high-frequency asymptote thus has a fixed value of −90°. At the breakpoint ($\omega = a$) we have $G(s) = \frac{1}{1+j} = \frac{1-j}{2}$ which has a phase of −45°.

The normal approximation for the phase response is to draw a straight line at 0° up to a frequency a factor of ten below the break point, a straight line with a phase of −90° beyond ten times the breakpoint and then join the two asymptotes with a straight line.

c. Again, for the example of the following transfer function:

$$G(s) = \frac{1}{\left(\frac{s}{100}\right) + 1}$$

The following figure shows the Bode plots of system $G(s) = \frac{1}{(s/100)+1}$



Example for Tutorial 5 – System with Pole

For a first-order system with a real pole at -20 and gain of 7 as the following transfer function:

$$G(s) = \frac{7}{s + 20}$$

a. Sketch the Bode plots of the system [5 marks]

b. Simulate the Bode plots of the system in MATLAB. [5 marks]

Answer

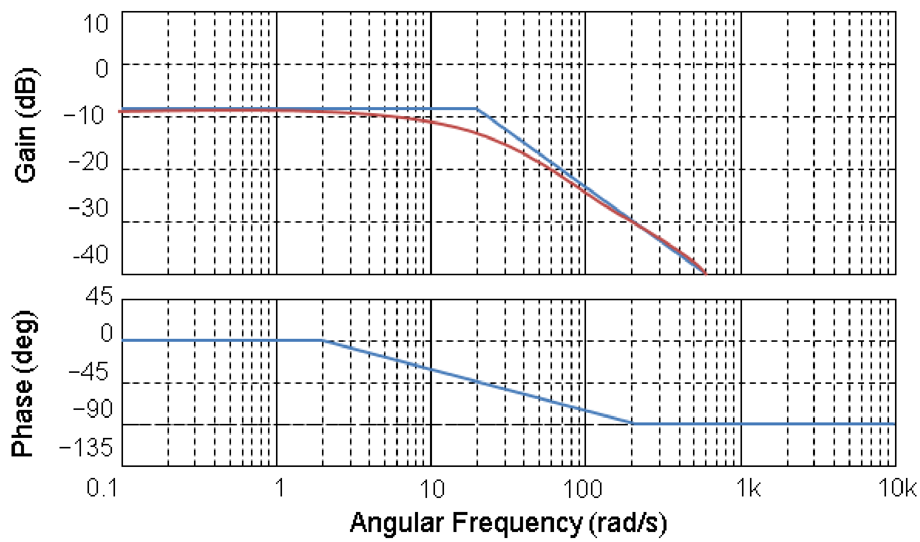
a. Let us begin by putting the transfer function into a form suitable for Bode plotting.

$$G(s) = \frac{7}{s + 20} = \frac{7}{20\left(\frac{s}{20} + 1\right)} = \frac{7}{20}\left(\frac{1}{1 + \frac{s}{20}}\right)$$

Note that:

$$20 \log\ (7/10) = -9.1\ \text{dB} \approx -10\ \text{dB}$$

The following figure shows the Bode plots of a system $G(s) = \frac{7}{s+20}$



b. We can check our work using MATLAB. Be careful using the `zpk()` function – check that you have the right dc gain and put the pole at −20 rad/s not +20 rad/s.

```
>> G = zpk([],[-20],7)
>> bode(G)
```

You can also use the `tf()` function instead. The following will both work:

```
>> G = tf(7,[1,20])
>> G = tf(7/20,[1/20,1])
```

Though MATLAB is very useful for control design, it can be error prone. One important reason to understand how to draw a Bode plot by hand is that it allows you to recognize errors when using computer-based tools.

Most of the errors are due to algorithms used in MATLAB and accuracy of the simulation results. It is also possible errors are due to extreme points e.g. infinite results obtained in the simulation.

4.3. Systems with Form $\left(\frac{s}{a}+1\right)$

Now, consider the case of a single zero at $s=-a$, where $a>0$. The low-frequency asymptote arising from a zero is the same as that for a pole (a straight line at 0 dB). However, for a zero, the high-frequency asymptote is given by:

$$|G(j\omega)| = -20\log a + 20\log\omega$$

The high-frequency asymptote is therefore a straight line with a slope of +20 dB/decade. The phase response is also the opposite of that produced by a pole. At high frequencies, we have $G(j\omega) \approx j\omega$, leading to a phase shift of +90°. As we might expect the phase is +45° at the breakpoint.

Example for Tutorial 6 – System with Zero

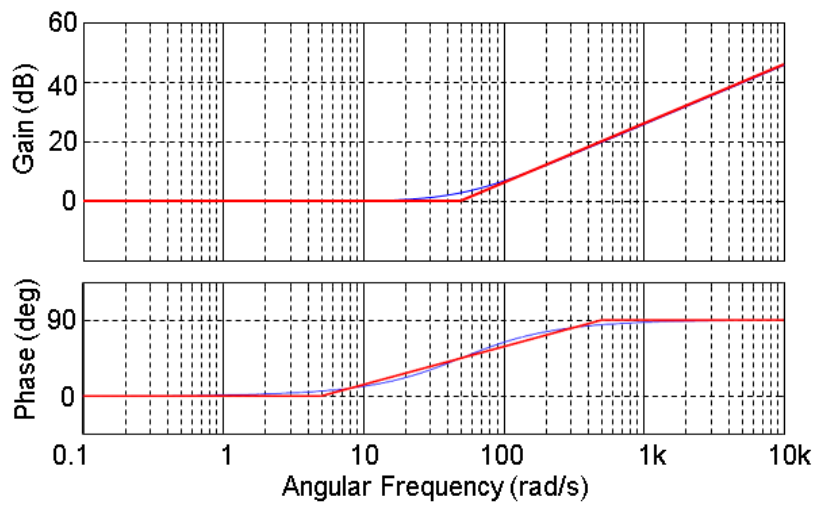
For the following first-order system with a real zero at -50 as given below, sketch its Bode plots:

[5 marks]

$$G(s) = (s + 50)$$

Answer

For a system with a zero at s = −50. The following figure shows the Bode plots of a system with a zero at s = −50.



4.4. Systems with Repeated Roots

As a transition to complex pairs of poles/zeros, consider the case of a transfer function with a double pole:

$$G(s) = \frac{1}{\left(1+\frac{s}{a}\right)^2} = \left(\frac{1}{1+\frac{s}{a}}\right)\left(\frac{1}{1+\frac{s}{a}}\right) \qquad \text{for} \quad a > 0$$

We know that the gain and phase responses are the sum of the two parts. So, we will have a response that falls off at −40 dB/decade beyond the breakpoint and moves from 0° to −180° in phase (over the same frequency range that a single pole TF would take to move 90°).

Notice that the presence of two poles means that the gain at $s = a$ is 6 dB down from the dc value. The plots for a repeated zero are opposite, with a slope of 40 dB/decade and a phase that moves from 0° to 180°.

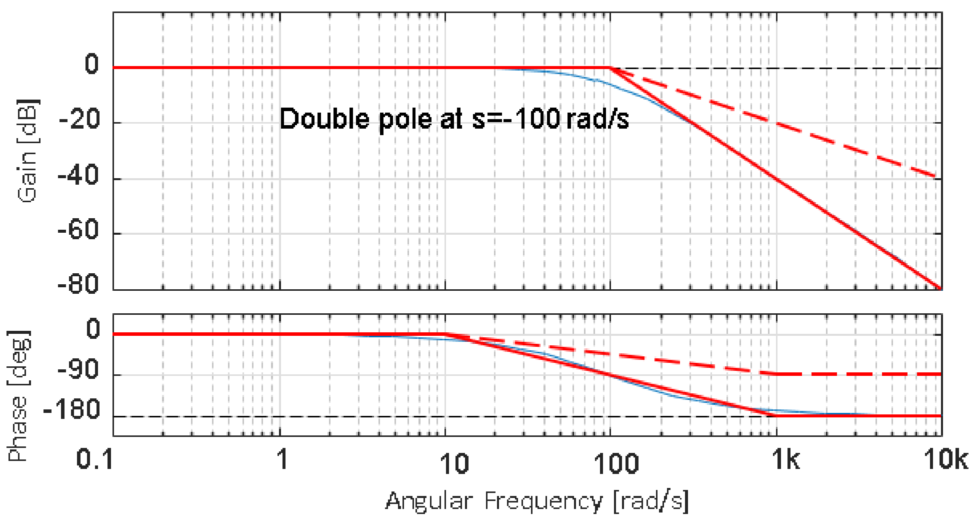
Example for Tutorial 7 – System with Double Poles

For the following second-order system with double poles at -100 as given below, sketch its Bode plots.
[5 marks]

$$G(s) = \frac{1}{(s+100)^2}$$

Answer

The plot of a transfer function with a double pole. The following figure shows the Bode plots of a system with the double pole at s = -100



4.5. Systems with Form $\left[\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1\right]^{\pm 1}$

We might expect that the transfer function produced by a pair of complex poles would look something like that produced by a double pole. As a way from the break point this is true, the gain rolls off at −40 dB/decade at high frequencies and the phase moves from 0° at low frequencies to −180° at high frequencies.

However, when the transfer function is underdamped it leads to some significant deviations in the region of the breakpoint. The smaller the damping the larger the effect. As damping decreases, we get:

- increasing peak in the gain response.
- sharper transition in the phase response.

These effects are shown in the figure.

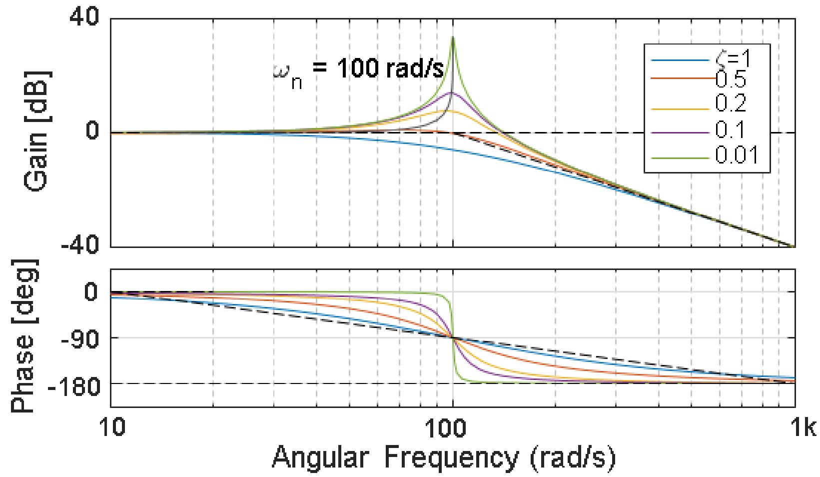


Figure 4: Bode plots of second-order systems with various damping ratios

We represent this family of curves with a straight-line approximation identical to the repeated real pole example above.

Note though that the corner point is at ω_n for the resonance, not at the real part of the pole pair.

If the damping is very low ($\zeta < 0.01$ say), you might prefer to approximate the phase response as a step at the natural frequency.

4.5.1. Corrections for Second-order Systems

To draw an accurate frequency response for a second-order system, it is necessary to make corrections by looking at a previously plotted response.

If you do not happen to have such a response handy, as a rough guide the peak (or trough for zeros) in the gain response has a gain as follows at the breakpoint:

$$M_p = \frac{\sqrt{1}}{2\zeta\sqrt{1-\zeta^2}}$$

For lightly damped systems:

$$M_p \approx \frac{1}{2\zeta} = Q$$

4.5.2. Resonance

This should be familiar, as it is just a description of resonance. The gain of the system becomes large in the vicinity of the resonant frequency. Highly resonant (lightly damped) systems have a more pronounced gain increase at resonance. All systems go through a 180° phase change in the vicinity of a resonance.

4.5.3. Damping and the Resonant Peak

For lightly damped systems we can see that the resonant peak occurs at approximately ω_n. However, the peak in the gain response shifts downwards in frequency as damping increases. However, the passage of the phase response through −90° always occurs at ω_n, which makes this a better feature to search for in experiments.

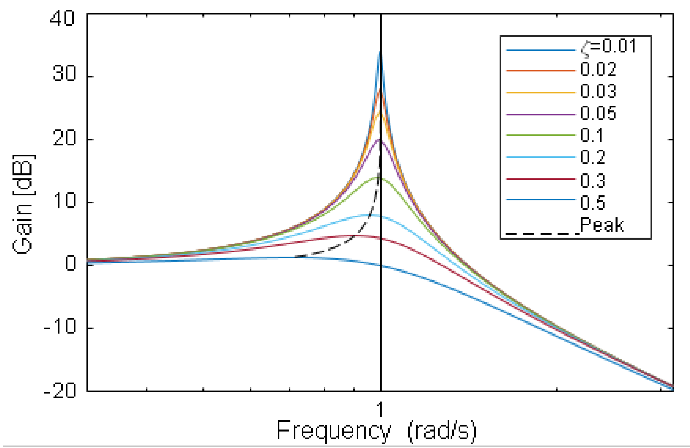


Figure 5: Resonant peak of second order system with various damping ratios

4.4. Systems with Form $\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1$

As you might expect the behaviour of a system with second-order zeros is opposite that with second-order poles.

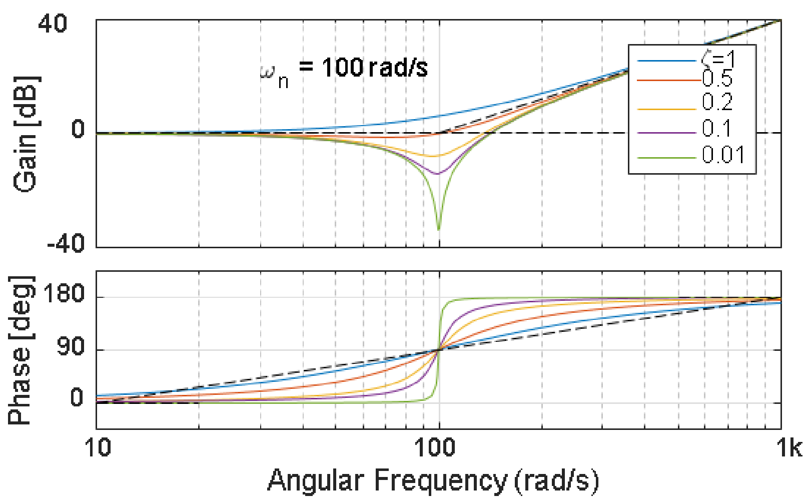


Figure 6: Bode plots of a system $\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1$

4.5. Building An Arbitrary Bode Plot

The following list outlines the steps required for creating an arbitrary Bode plot:

1. Arrange the transfer function into a convenient form.
2. Plot the straight-line approximations for each term in the transfer function.
3. If required, make corrections to the approximations for complex pairs of poles.
4. Gain peaks are approximate.
5. Add the various curves graphically and draw in the final response curves.

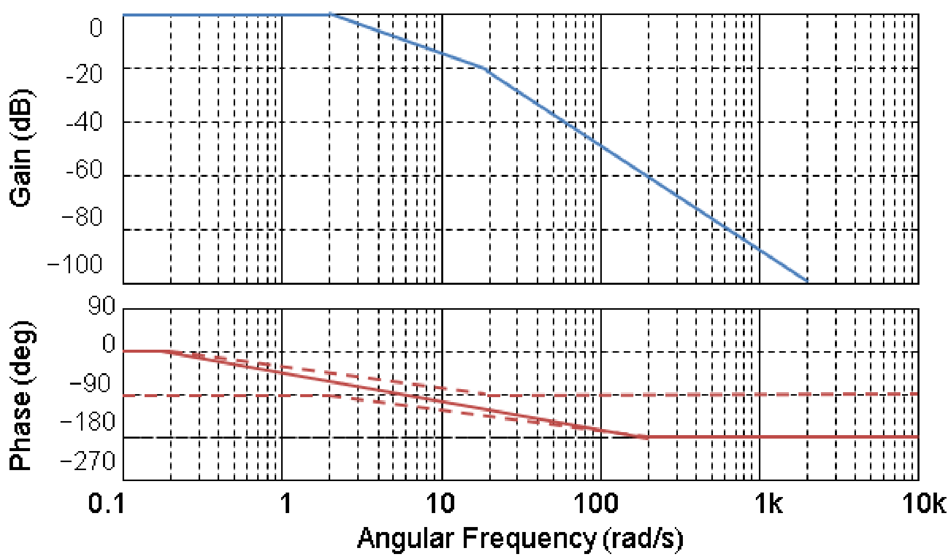
Example for Tutorial 8 – System with Double Poles

For a second-order system with real poles at -2 and -20 as shown in the following transfer function, sketch its Bode plots. [5 marks]

$$G(s) = \frac{1}{\left(\frac{s}{2}+1\right)\left(\frac{s}{20}+1\right)}$$

Answer

The following figure shows the Bode plots of system $G(s) = \frac{1}{\left(\frac{s}{2}+1\right)\left(\frac{s}{20}+1\right)}$



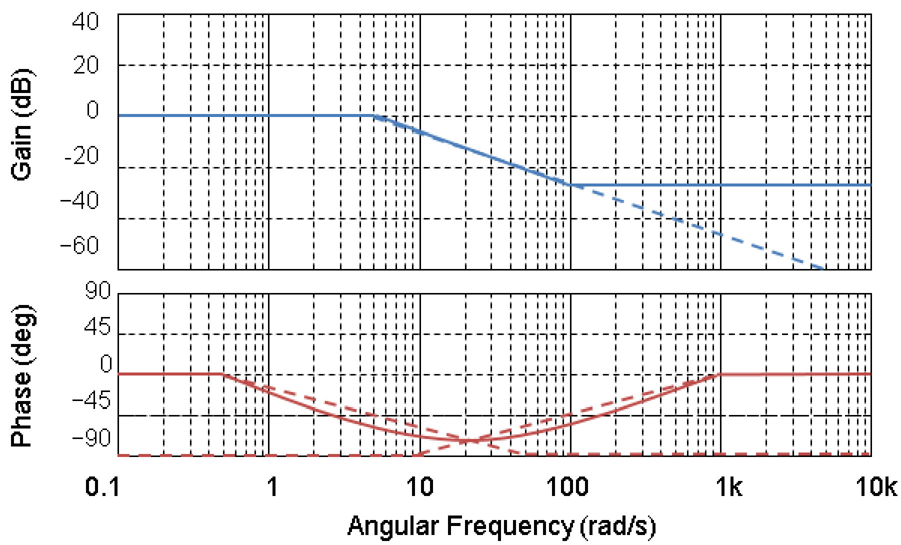
Example for Tutorial 9 – System with Pole and Zero

For a system with a real pole at -5 and a real zero at -100 as in the following transfer function, sketch its Bode plots. [5 marks]

$$G(s) = \frac{\left(\frac{s}{100} + 1\right)}{\left(\frac{s}{5} + 1\right)}$$

Answer

The following figure shows the Bode plots of system $G(s) = \frac{\left(\frac{s}{100}+1\right)}{\left(\frac{s}{5}+1\right)}$



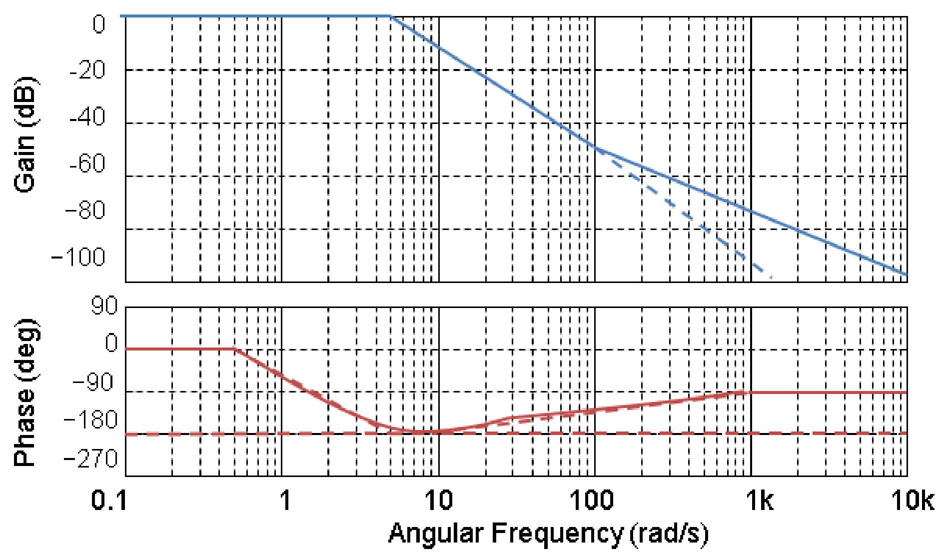
Example for Tutorial 10 – System with Zero and Double Poles

For a system with double poles at -5 and a zero at -100 as shown in the following transfer function, sketch its Bode plots: [5 marks]

$$G(s) = \frac{\left(\frac{s}{100} + 1\right)}{\left(\frac{s}{5} + 1\right)^2}$$

Answer

The following figure shows the Bode plots of system $G(s) = \frac{\left(\frac{s}{100}+1\right)}{\left(\frac{s}{5}+1\right)^2}$



4.6. Checking the Bode Plot

You should always make sure that your final plot makes sense at both low and high frequencies.

Low Frequency:

At low frequencies, the response is determined by only the differentiators/integrators in the system. If the overall transfer function includes a factor of s^n, then the slope of the gain curve should be $20n$ dB/decade and the phase at low frequency should be $90n$ degrees.

High frequency:

The high-frequency behaviour is determined by the number of poles, P, and zeros, Z. At high frequency, the slope of the gain should be $-20(P-Z)$ dB/decade and the phase should be at $-90(P-Z)$ degrees.

Note: As we will see next, the phase checks only work when all of the system poles and zeros are in the left half of the s-plane. The roots in the discussion have all been in the left half of the s-plane.

4.7. Right-Half Plane Roots

Having right-half plane poles will make the system to be unstable. The transient response of the system with right-hand plane poles is an increasing amplitude function.

4.7.1. Bode Plots for Roots in the Right-Half Plane

Let's first consider poles in the right-half plane. Consider the transfer functions as follow:

$$G_1(s) = \frac{1}{s-a} \qquad \text{and} \qquad G_2(s) = \frac{1}{s+a}$$

The gain of these systems:

$$|G_1(j\omega)| = \frac{1}{j\omega - a} = \frac{1}{\sqrt{a^2 + \omega^2}}$$

And

$$|G_2(j\omega)| = \frac{1}{j\omega + a} = \frac{1}{\sqrt{a^2 + \omega^2}}$$

The two transfer functions have identical gains.

4.7.2. Phase Plot for Roots in the Right-half Plane Roots

Now, consider the phase responses of the two systems.

$$G_1(s) = \frac{1}{s - a} = \frac{-a - j\omega}{\omega^2 + a^2}$$

And

$$G_2(s) = \frac{1}{s + a} = \frac{a - j\omega}{\omega^2 + a^2}$$

The phase shifts of these systems are:

$$\angle G_1(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\{G_1(s)\}}{\mathrm{Re}\{G_2(s)\}}\right) = \tan^{-1}\left(\frac{\omega}{a}\right)$$

$$\angle G_2(j\omega) = \tan^{-1}\left(\frac{\mathrm{Im}\{G_2(s)\}}{\mathrm{Re}\{G_2(s)\}}\right) = \tan^{-1}\left(\frac{-\omega}{a}\right) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

The phase response is opposite to that we expect for a pole in the left-half side of the s-plane.

$$\angle G_1(j\omega) = -\angle G_2(j\omega)$$

Example for Tutorial 11 - Right-hand Plane Pole

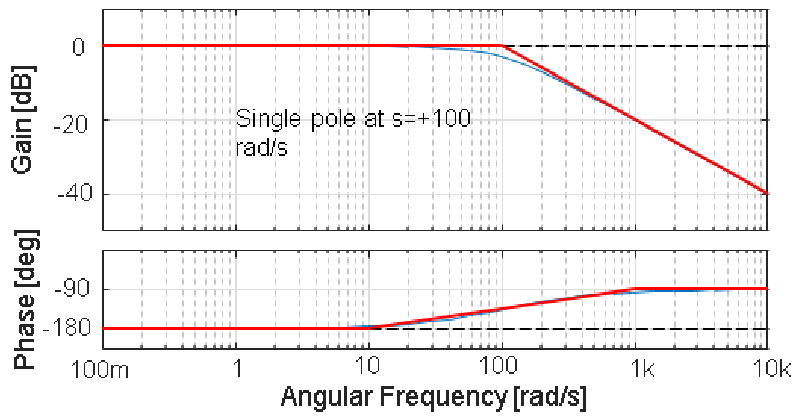
For the system with right-half plane pole at 100 and a gain of 100 as shown below, sketch its Bode plots:
[5 marks]

$$G(s) = \frac{100}{s - 100}$$

Answer

A quick examination of a Bode plot is a good check whenever you enter a system in MATLAB, as it is easy to put a root in the right half plane unintentionally (e.g.: particularly with the `zpk()` function).

The following figure shows the Bode plots of system $G(s) = \frac{100}{s-100}$



4.8. Systems with Non-Minimum Phase

Non-minimum phase systems are causal and stable systems whose inverses are causal but unstable. Having a delay in a given system or a zero on the right half of the s-plane may lead to a non-minimum phase system.

4.8.1. Non-minimum Phase Systems

The same analysis can be performed on systems having zeros in the right-half plane. Perhaps unsurprisingly, we find that these too have their gain response unchanged, but their phase response reversed from the left-half plane analogue.

Systems containing at least one right-half plane zero are called non-minimum phase systems. Non-minimum phase systems tend to be harder to control than minimum phase systems, but easier than open-loop unstable systems (those with right half-plane poles).

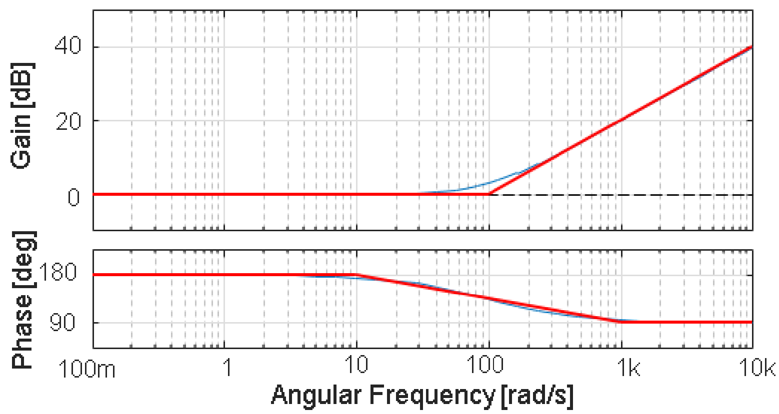
Example for Tutorial 12 – Non-minimum Phase

For a non-minimum phase system with a right-hand side pole at 100 and a gain of 1/100 as given below, sketch its Bode plots. [5 marks]

$$G(s) = \frac{s-100}{100}$$

Answer

The following figure shows the Bode plots of system $G(s) = \frac{s-100}{100}$



4.8.2. Response of Non-minimum Phase Systems

Non-minimum phase systems are troublesome because their initial response is “the wrong way” when driven by an input.

Example for Tutorial 13 – Comparison of Non-minimum Phase

Sketch and compare the step responses and frequency response of two second-order systems having the following transfer functions. [10 marks]

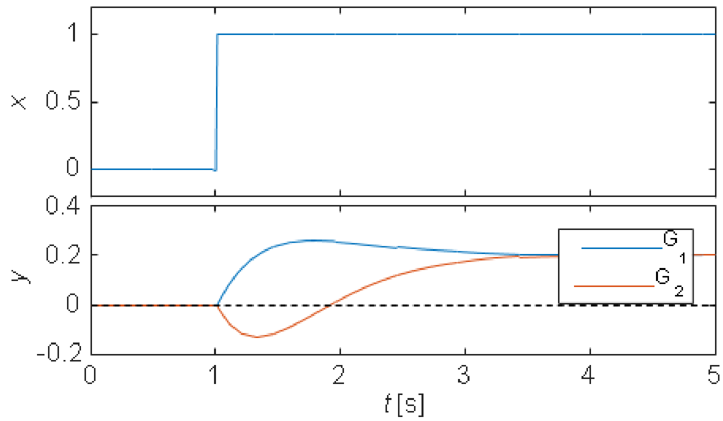
$$G_1(s) = \frac{s+1}{s^2+4s+5}$$

And

$$G_2(s) = \frac{-(s-1)}{s^2+4s+5}$$

Answer

The following figure shows the transient response of systems $G_1(s) = \frac{s+1}{s^2+4s+5}$ and $G_2(s) = \frac{-(s-1)}{s^2+4s+5}$



We can also compare their Bode plots. The greater change in the phase for G_2 is what leads to the name "non-minimum phase".

The following figure shows the Bode plots of systems:

$$G_1(s) = \frac{s+1}{s^2+4s+5} \qquad \text{and} \qquad G_2(s) = \frac{-(s-1)}{s^2+4s+5}$$