XMUT315 Control Systems Engineering

Note 10b: Analysis with Bode Plots

Topic

- Closed loop stability.
- Gain and phase margin.
- Determining gain and phase margin in Bode plots.
- Damping and phase margin.
- Transient response parameters from Bode plots.
- System types.
- Steady-state errors.
- System errors and inputs.
- Determining steady-state errors in Bode plots.

1. Introduction to Analysis with Bode Plots

With Bode plots, we can perform stability analysis, transient response analysis, and steady-state analysis of the control systems.

1.1. Closed Loop Stability

Imagine a situation where we have a system described by a transfer function $G(s)$. We now enclose the system in a unity gain feedback loop.

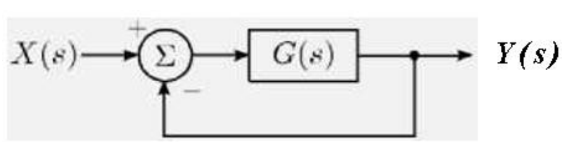


Figure 1: Closed loop feedback system

We know that negative feedback is useful in stabilising a system. However, instability results when the feedback is positive. The feedback in the system shown becomes positive when the plant transfer function $G(s)$ contributes 180° of phase shift to the overall system.

System stability is one of the basic concerns when designing a control system. We would like to be able to meaningfully talk about how close a system is to instability, not just whether it is stable or not.

For many systems, we can assess the stability by finding the frequency at which the phase curve crosses -180° and reading the gain at that point.

If the gain > 1, then the system will be unstable. If the gain < 1 at the frequency where the phase crosses -180°, then there is not enough gain to sustain the oscillations. This approach leads to a metric known as the gain margin.

1.2. Gain Margin

The gain margin is the amount by which we can increase the gain of a stable system before it becomes unstable.

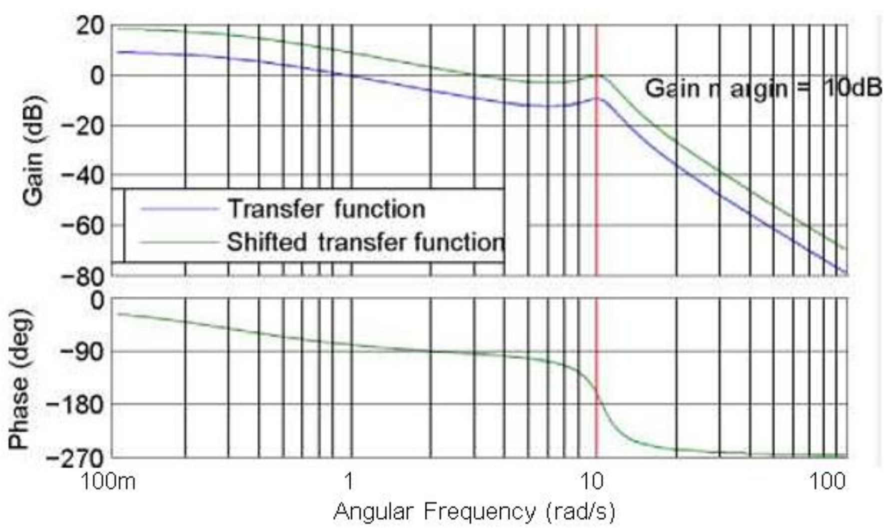


Figure 2: Gain margin in the Bode plots

To determine the gain margin of a system, read the gain at the frequency where the phase curve crosses 180°. Note that the gain margin must be positive for the system to be stable!

1.3. Unity Gain

In control applications we often use the Unity Gain Frequency, which is the frequency at which the system's gain has dropped to one (0 dB). We can use the Bode plot to simply read off the frequency where the gain plot crosses the 0 dB line. Note that some systems have multiple unity gain frequencies because their gain curves cross and recross the 0 dB line. In crude terms, the unity gain frequency of a control system is the highest frequency at which the control is doing anything useful. Beyond this point, the gain is too small to improve the system.

1.4. Phase Margin

The phase margin is the amount by which we can decrease the phase of a stable system before it become unstable.

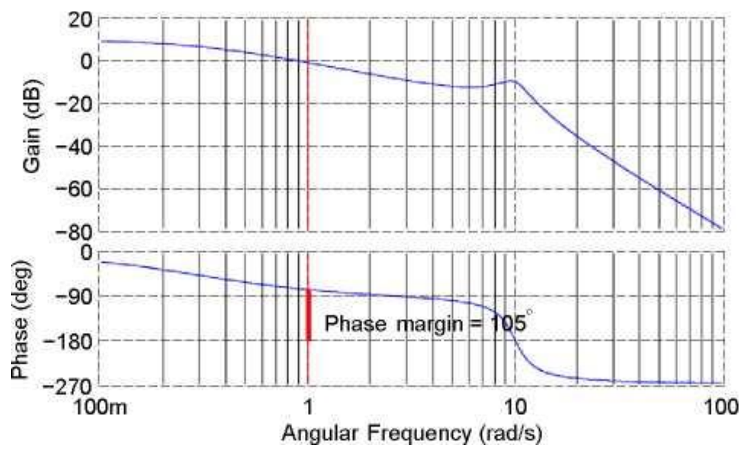


Figure 3: Phase margin in the Bode plots

To determine the phase margin of a system, find the unity gain frequency and read the system phase at that point. This reveals how much extra phase lag we could tolerate before instability sets in.

1.5. Gain and Phase Margins with MATLAB

The `margin()` command in MATLAB will tell you the gain and phase margins and the frequencies at which they occur. If you call it without any return arguments, it will draw a plot displaying the same information.

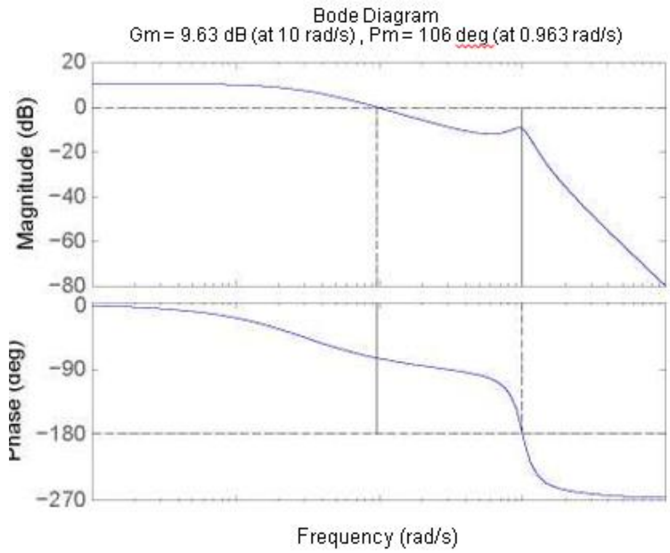
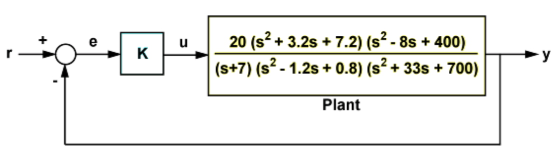
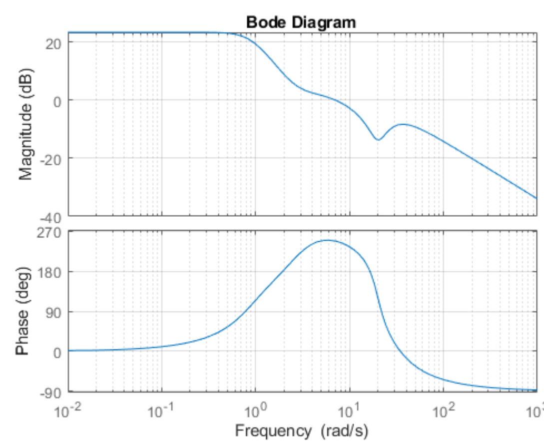


Figure 4: Gain and phase margins in Bode plot with MATLAB

1.6. Phase Margin with Multiple Unity Gain

The following Bode plot shows a higher order system that has multiple crossing of the 0 dB gain curve with unity gain line. There are two 180° phase crossings with corresponding gain margins of -9.35 dB and +10.6 dB.





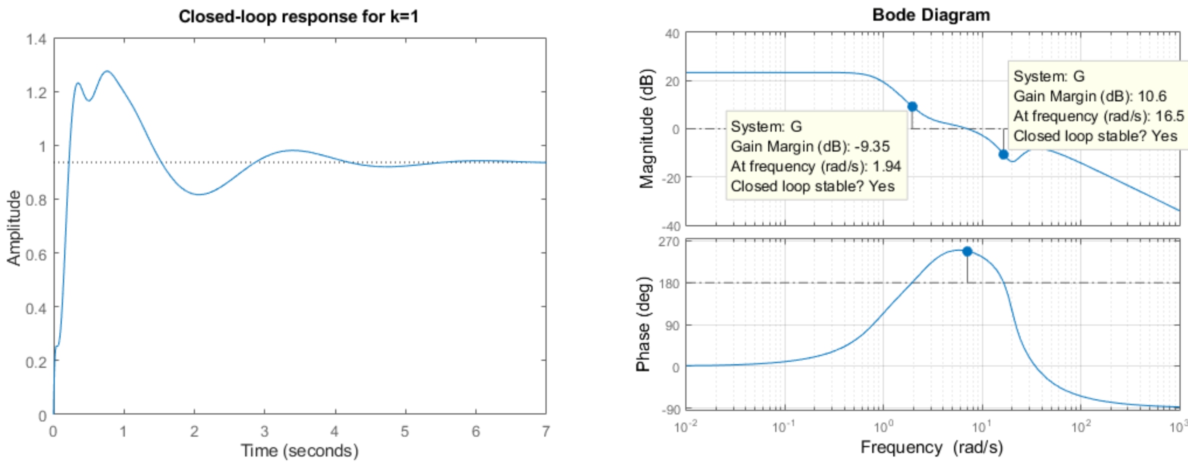


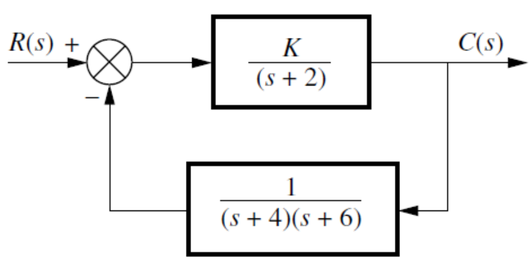
Figure 5: Systems with multiple phase and gain margin crossing

For some systems cross the 0 dB gain curve more than once, in general, there will be a different phase margin associated with each of these crossings. It is possible to define the system phase margin as the worst (smallest) of the individual phase margins.

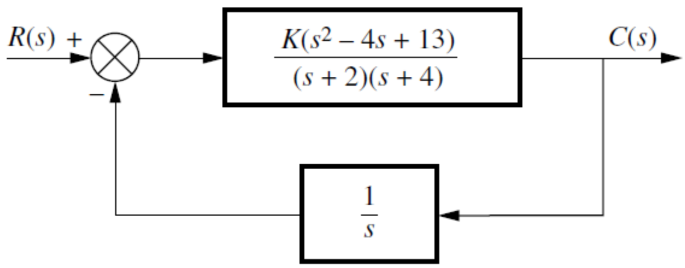
However, this is dangerous as there are some systems like this appear to be stable but are not. When you see a system with multiple crossings of the 0 dB line, you should double check the system stability with another method, such as a root locus diagram or (more traditionally) a Nyquist plot.

Example for Tutorial 1 – Stability Analysis with Bode Plots

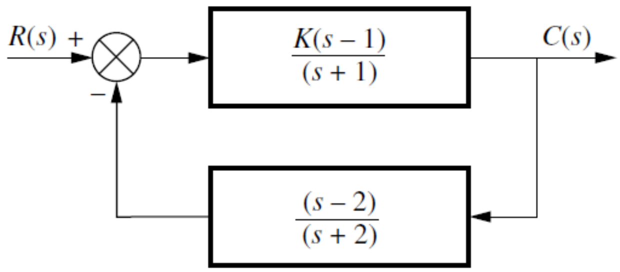
For each system given below, find the gain margin and phase margin if the value of gain K is 1, 100, 1000, and 0.1. Write a summary on the stability of each system. [30 marks]



System 1



System 2

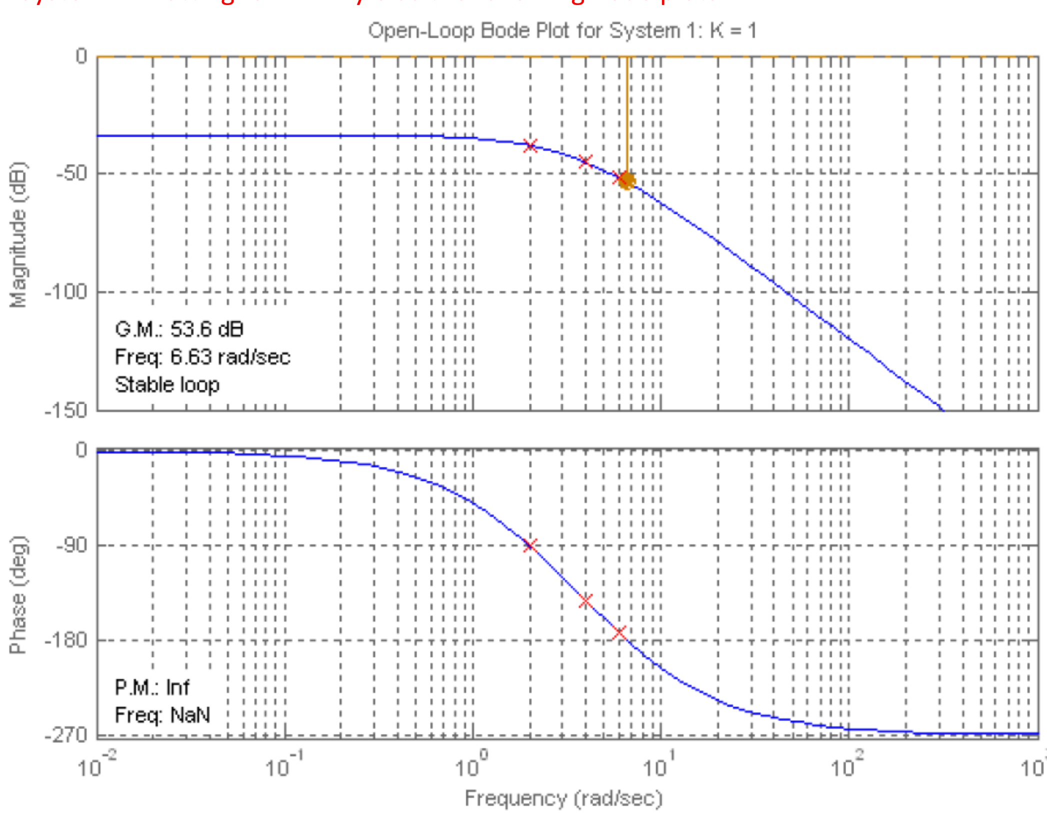


System 3

Answer

Note: All results for this problem are based upon a non-asymptotic frequency response.

a. System 1: Plotting for K = 1 yields the following Bode plots:



i. K = 1:

For K = 1, phase response is 180° at ω = 6.63 rad/s, the gain margin is -53.6 dB at this frequency. Phase margin is +∞ at any frequency.

ii. K = 100:

For K = 100, gain curve is raised by 40 dB yielding -13.6 dB at 6.63 rad/s. Thus, the gain margin is 13.6 dB.

Phase margin: Raising the gain curve by 40 dB yields 0 dB at 2.54 rad/s, where the phase curve is 107.3°. Hence, the phase margin is 180° - 107.3° = 72.7°.

iii. K = 1000:

For K = 1000, gain curve is raised by 60 dB yielding +6.4 dB at 6.63 rad/s. Thus, the gain margin is -6.4 dB.

Phase margin: Raising the gain curve by 60 dB yields 0 dB at 9.07 rad/s, where the phase curve is 200.3°. Hence, the phase margin is 180° - 200.3° = -20.3°.

iv. K = 0.1:

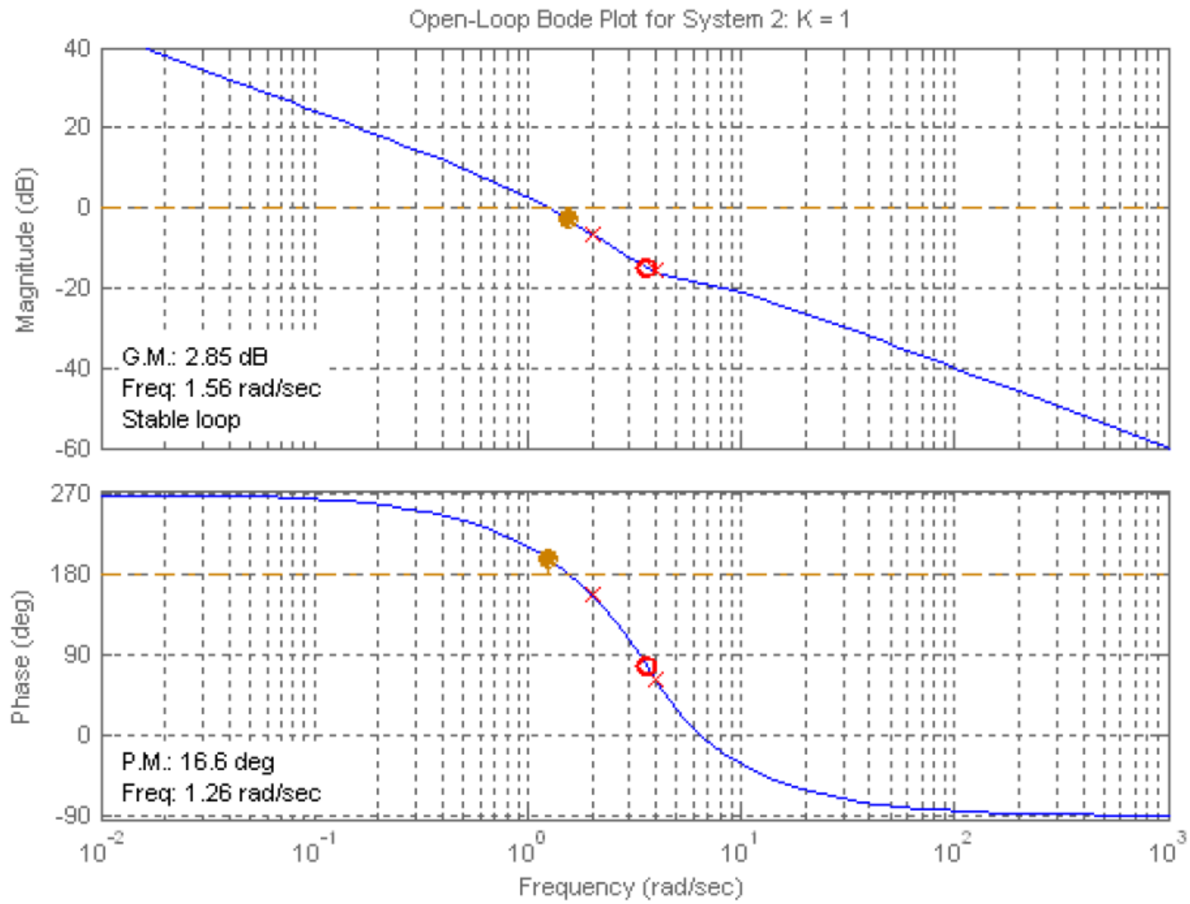
For K = 1, phase response is 180° at ω = 6.63 rad/s, the gain margin is increased to 53.6 dB at this frequency.

For K = 0.1, gain curve is lowered by 20 dB yielding -73.6 dB at 6.63 rad/s. Thus, the gain margin is increased to 73.6 dB.

Stability Summary of System 1:

- When $K = 1$, considering positive gain margin (GM = 53.6 dB at 6.63 rad/s) and phase margin (PM = ∞ at any frequency), the system is found to be stable.
- Any increase of the gain might reduce the gain margin of the system. If the increase is excessive, the system could be unstable.
- If the gain is lowered, the system stays stable with the margins are increased.

b. System 2: Plotting for K = 1 yields the following Bode plots:



i. $K = 1$:

For K = 1, when the phase response is 180° at ω = 1.56 rad/s, the gain margin is -2.85 dB and phase margin is -18.6° at 1.26 rad/s.

ii. K = 100:

For K = 100, gain curve is raised by 40 dB yielding +37.15 dB at 1.56 rad/s. Thus, the gain margin is -37.15 dB.

Phase margin: Raising the gain curve by 40 dB yields 0 dB at 99.8 rad/s, where the phase curve is -84.3°. Hence, the phase margin is 180° - 84.3° = 95.7°.

iii. K = 1000:

For K = 1000, gain curve is raised by 60 dB yielding +57.15 dB at 1.56 rad/s. Thus, the gain margin is -57.15 dB.

Phase margin: Raising the gain curve by 54 dB yields 0 dB at 500 rad/s, where the phase curve is -91.03°. Hence, the phase margin is 180° - 91.03° = 88.97°.

iv. K = 0.1:

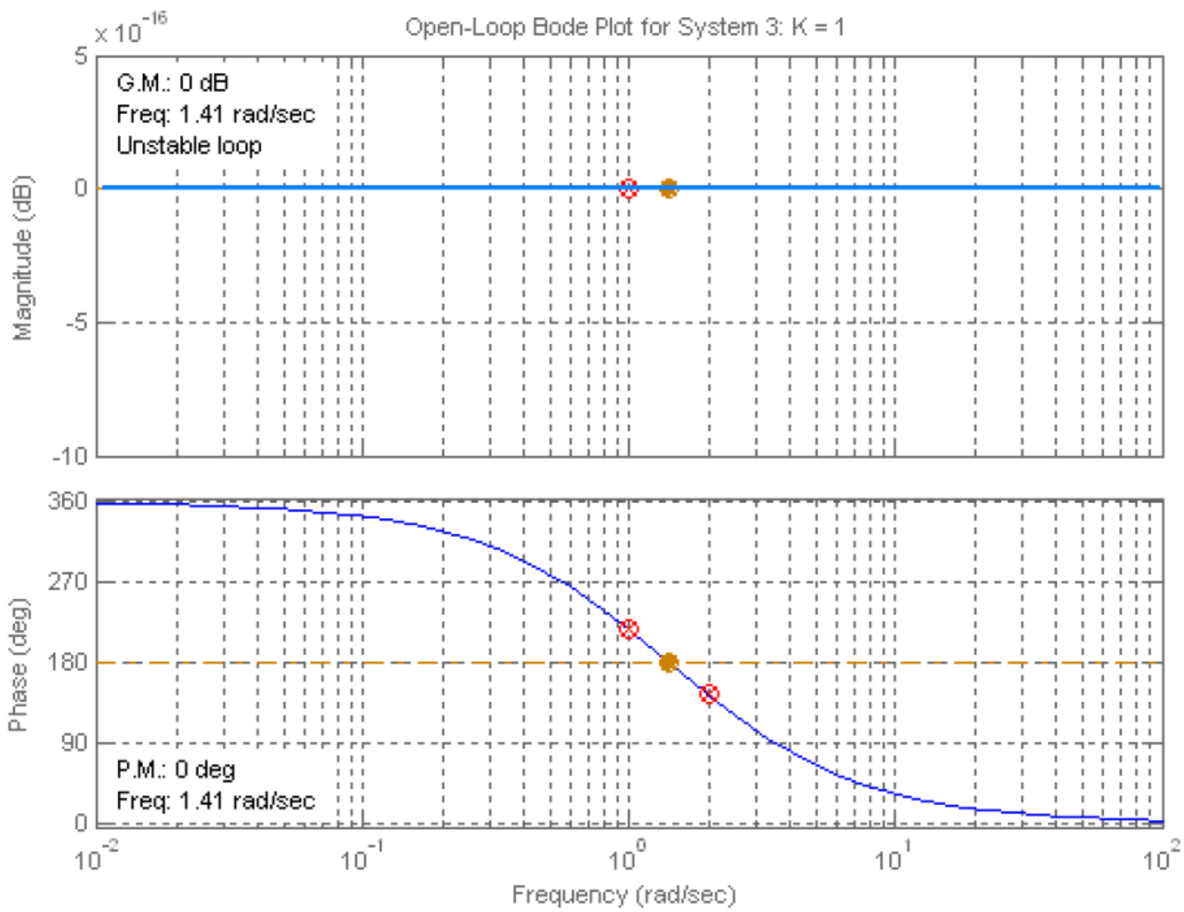
For K = 0.1, gain curve is lowered by 20 dB yielding -22.85 dB at 1.56 rad/s. Thus, the gain margin is -22.85 dB.

Phase margin: Lowering the gain curve by 20 dB yields 0 dB at 0.162 rad/s, where the phase curve is -99.8°. Hence, the phase margin is 180° - 99.86° = 80.2°.

Stability Summary of System 2:

- Both gain and phase margins of the system are negative i.e. -2.85 dB at 1.56 rad/s and -18.6° at 1.26 rad/s respectively. The system is unstable due to these negative margins.
- Increasing the gain reduces the gain margin further making the system to become more unstable.
- Decreasing the gain of the system might increase the margin and might turn the system to become stable.

c. System 3: Plotting for K = 1 yields the following Bode plots:



i. $K = 1$:

For K = 1, phase response is 180° at ω = 1.41 rad/s, the gain margin is 0 dB at this frequency. Phase margin is 0° at 1.41 rad/s.

ii. K = 100:

For K = 100, gain curve is raised by 40 dB yielding 40 dB at 1.41 rad/s. Thus, the gain margin is -40 dB.

Phase margin: Raising the gain curve by 40 dB yields no frequency where the gain curve is 0 dB. Hence, the phase margin is infinite.

iii. K = 1000:

For K = 1000, gain curve is raised by 60 dB yielding 60 dB at 1.41 rad/s. Thus, the gain margin is -60 dB.

Phase margin: Raising the gain curve by 60 dB yields no frequency where the gain curve is 0 dB. Hence, the phase margin is infinite.

iv. K = 0.1:

For K = 0.1, gain curve is lowered by 20 dB yielding -20 dB at 1.41 rad/s. Thus, the gain margin is 20 dB.

Phase margin: Lowering the gain curve by 20 dB yields no frequency where the gain curve is 0 dB. Hence, the phase margin is infinite.

Stability Summary of System 3:

- Both gain and phase margins are zero at 1.41 rad/s and 1.41 rad/s respectively. The system is critically stable.
- Increasing the gain might turn the system to become unstable.
- Reducing the gain increases the margins and these make the system to become stable.

2. Transient Response in Bode Plots

From the given Bode plots, we can determine a variety of transient response parameters. Transient response parameters that can be derived and approximated from the Bode diagrams are damping ratio, settling time, and peak time.

For transient response analysis, we need to know several of the values of these parameters to work out the transient response parameters: knowing phase margin will give you the damping ratio and knowing closed-loop bandwidth (+ damping ratio) will provide you the settling time and peak time.

2.1. Phase Margin and Damping Ratio

Phase margin is particularly useful because there is a direct link between a system's phase margin and its damping in the closed loop case. The smaller the phase margin, the more badly the system will ring.

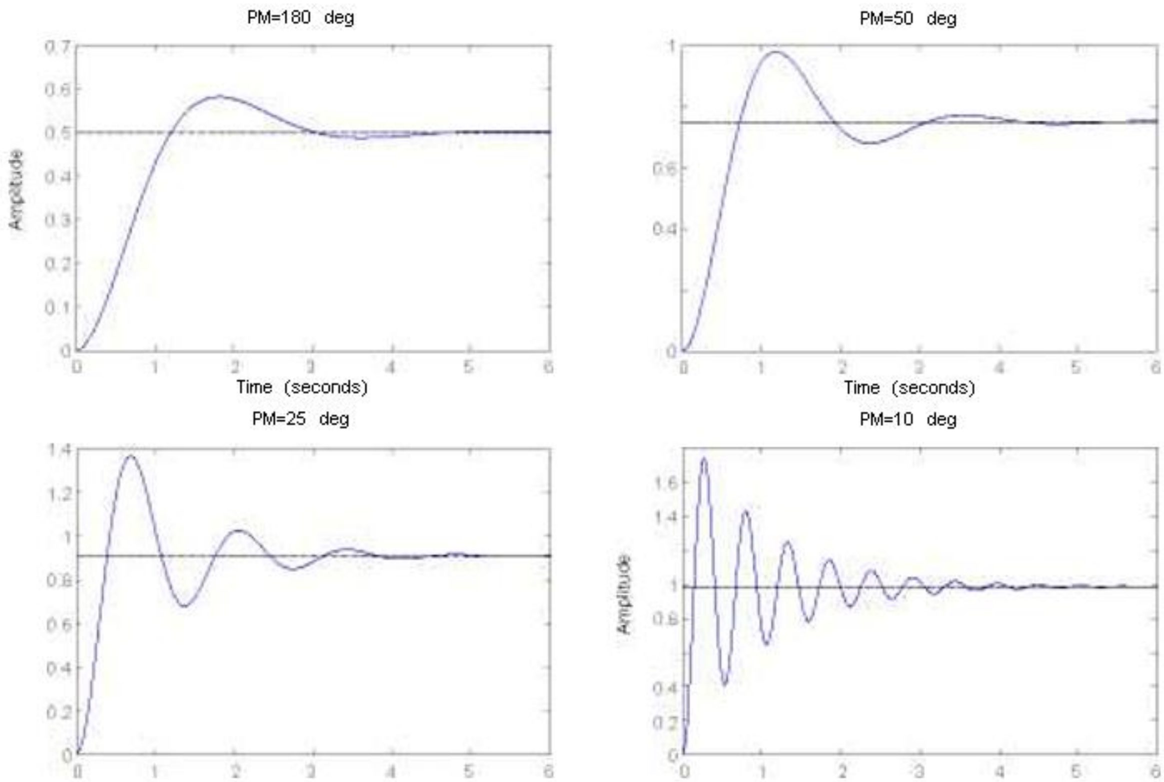


Figure 5: Transient response of systems with various phase margins

2.2. Phase Margin and Damping Ratio

It can be shown that there is a relationship between the phase margin the damping ratio of the closed loop response. For a standardised second order equation as shown in the figure below, the open-loop transfer function of the plant is:

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

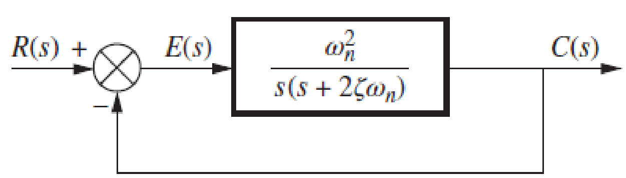


Figure 7: Standard second-order system

The closed-loop transfer function of the system is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

To evaluate the phase margin, find the frequency for which $|G(j\omega)| = 1$.

$$|G(j\omega)| = \frac{\omega_n^2}{-\omega^2 + j2\zeta\omega_n\omega} = 1$$

The frequency, ω_1, that satisfies the equation above is:

$$\omega_1 = \omega_n\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}$$

The phase angle of $G(j\omega)$ at this frequency is:

$$\angle G(j\omega) = -90° - \tan^{-1}\left(\frac{\omega_1}{2\zeta\omega_n}\right)$$

Substitute the equation for ω_1 into the equation above.

$$\angle G(j\omega) = -90° - \tan^{-1}\left(\frac{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}{2\zeta}\right)$$

The difference between the angle of the equation above and -180° is the phase margin, ϕ_m.

$$\phi_m = 90° - \tan^{-1}\left(\frac{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}}{2\zeta}\right)$$

The accurate relation of damping ratio with the phase margin of the system over the full range is:

$$\phi_m = \tan^{-1}\frac{2\zeta}{\sqrt{2\zeta^2 + \sqrt{1 + 4\zeta^4}}}$$

To keep the damping reasonable, we generally try to preserve a phase margin of about 60°.

Rearrange the equation given above, the damping ratio is:

$$\zeta = \sqrt[4]{\frac{1}{\left(\frac{4}{\tan^2\phi_m} + 2\right)^2 - 4}}$$

The relationship between phase margin and damping ratio is as shown in the graph below.

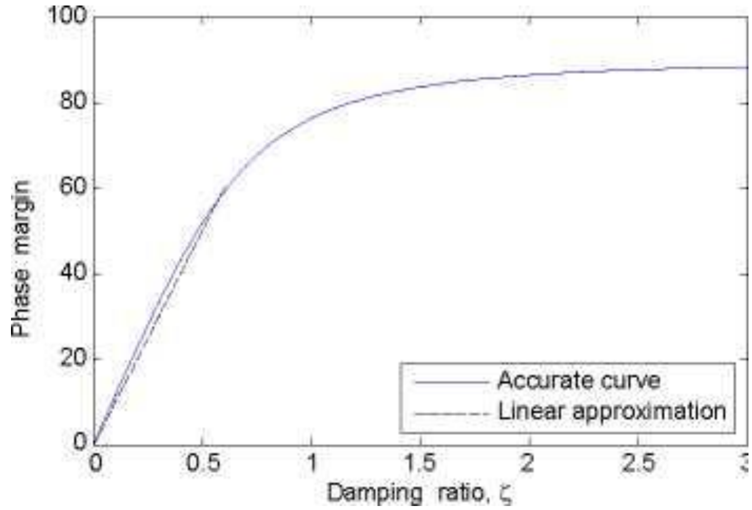


Figure 6: Graph of phase margin vs. damping ratio

For damping ratios less than 0.65, we can use the approximate relation ϕ_m = 100ζ as shown in the graph above.

2.3. Bandwidth of Control Systems

The magnitude or gain of the frequency response of the given control system is given as:

$$|T(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

To determine the transient response of the control system, we need to find the closed-loop bandwidth from the Bode plots.

For an open-loop system, the bandwidth of the control systems (ω_{BW}) is the width of frequency of gain of the system from DC (0 rad/s) to the half-power point (i.e. -3 dB).

For a typical second-order system, the gain plot of the equation given above is shown in the figure below. The bandwidth is located at ω_{BW}, or in log frequency scale, it is $\log \omega_{BW}$.

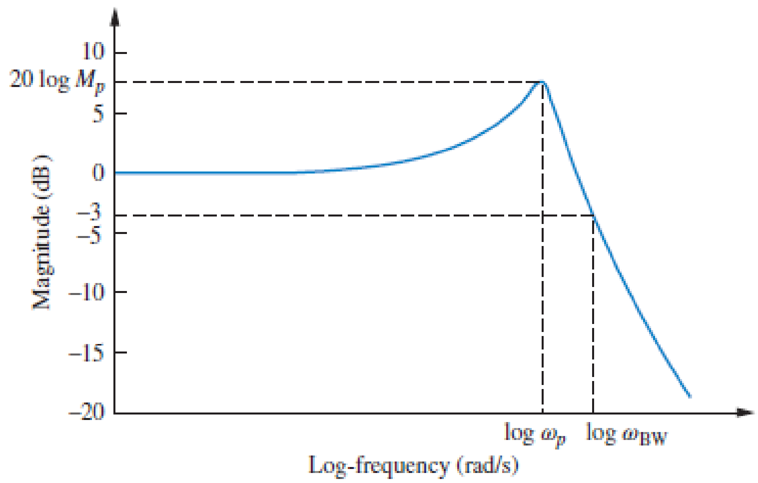


Figure 8: Peak gain and bandwidth of the second order system in the Bode plot

2.4. Closed-Loop Bandwidth

The bandwidth of the standardised control systems (ω_{BW}) is determined by finding the frequency for which $M = 1/\sqrt{2}$ (that is -3 dB).

$$M = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

Equate the equation above to be equal to $1/\sqrt{2}$ that happens when $\omega = \omega_{BW}$:

$$\frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega_{BW}^2)^2 + 4\zeta^2\omega_n^2\omega_{BW}^2}} = \frac{1}{\sqrt{2}}$$

Rearranging the equation above, the bandwidth of the control system is:

$$\omega_{BW} = \omega_n\sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

The closed-loop bandwidth, ω_{BW} is the frequency at which the closed-loop gain response is -3 dB. As shown in the figure below, if the open-loop phase response is between -135° and -225°, this equals the frequency at which the open-loop gain response is approximated to be between -6 and -7.5 dB.

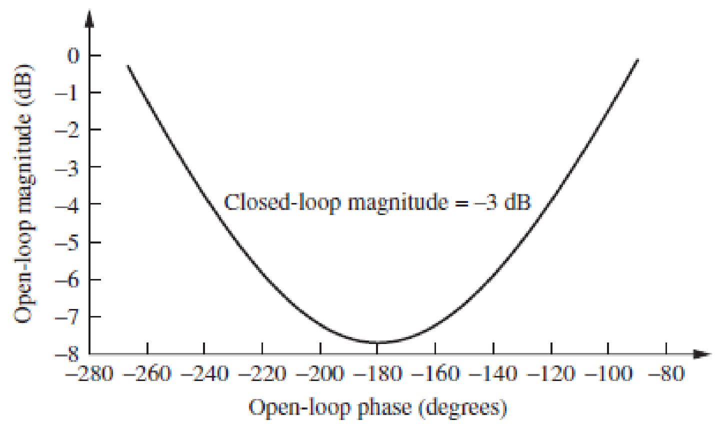


Figure 9: Graph of open-loop gain (in dB) vs. open loop phase (in degree)

Given the Bode plots of a control system as shown in the figure below, we can determine the phase margin and bandwidth of the system. From the plots, phase margin (PM) is 180° - 150° = 30°. Also, we found that gain margin (GM) is 10 dB at 4 rad/s. Considering the open-loop system and looking at the frequency when the gain of the system is -7.5 dB, the bandwidth (ω_{BW}) is approximately 3.5 rad/s.

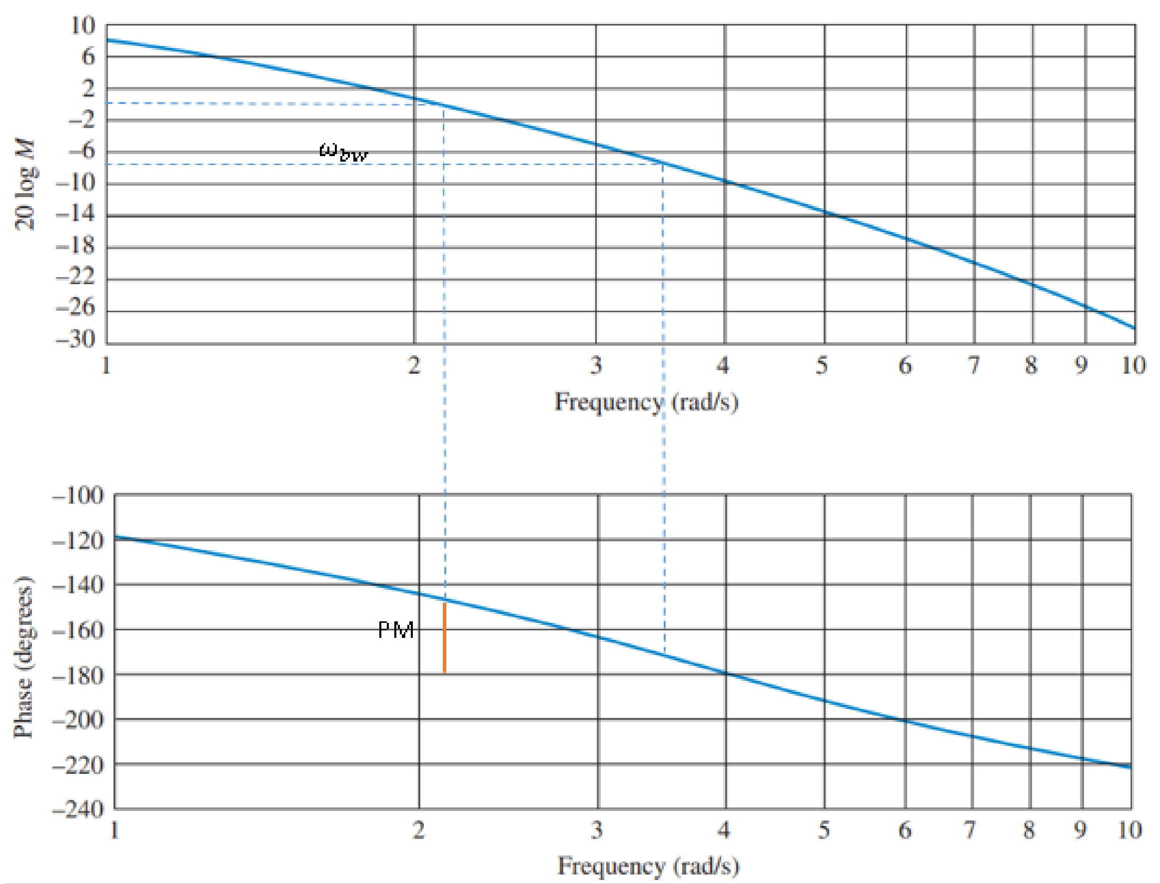


Figure 10: Bandwidth in the Bode plots

2.5. Transient Response Parameters in Bode Plots

We could determine the transient response parameters of the system from the normalised bandwidth vs. damping ratio for settling time, peak time, and rise time.

2.5.1. Settling Time in Bode Plots

The settling time of a second order system (T_s) is:

$$T_s = \frac{4}{\omega_n \zeta}$$

Rearranging the equation above

$$\omega_n = \frac{4}{T_s \zeta}$$

Substituting the ω_n in the bandwidth equation, the bandwidth equation becomes:

$$\omega_{BW} = \frac{4}{T_s \zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^{\,2}+2}}$$

Where: ζ is the damping ratio.

Hence, the settling time is:

$$T_s = \frac{4}{\omega_{BW} \zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

The following figure shows the damping ratio vs. normalised settling time (T_s) of the system.

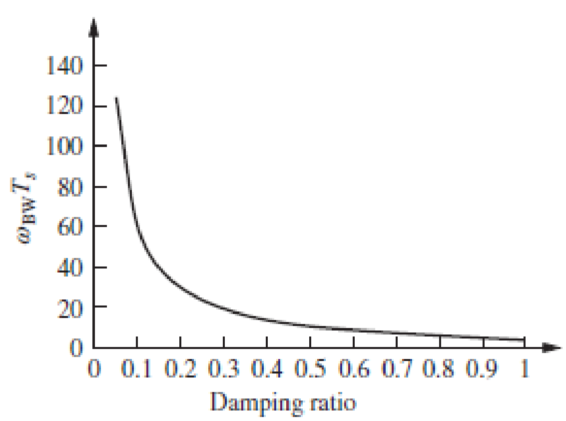


Figure 11: Graphs of settling time (T_s) vs. damping ratio

2.5.2. Peak Time in Bode Plots

Like settling time, we can determine also the time-to-peak (T_p) from the Bode plots through the bandwidth of the closed loop system (ω_{BW}). For a given second order system, the peak time is:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Rearranging the equation above

$$\omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}}$$

Substituting ω_n in the bandwidth equation, the bandwidth equation becomes:

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Where: ζ is the damping ratio.

Hence, rearranging the equation, the peak time is:

$$T_p = \frac{\pi}{\omega_{BW}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

The following figure shows the damping ratio vs. normalised peak time (T_p) of the system.

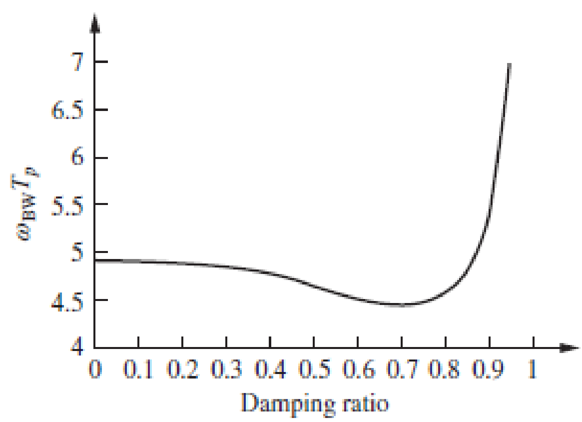


Figure 12: Graph of peak time (T_p) vs. damping ratio

2.5.3. Rise Time in Bode Plots

To relate the bandwidth to rise time (T_r), knowing the desired ζ, we can calculate it from:

$$T_r = \frac{\pi-\phi}{\omega_n\sqrt{1-\zeta^2}}$$

Where:

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Rearranging the equation above

$$\omega_n = \frac{\pi - \phi}{T_r\sqrt{1-\zeta^2}}$$

Thus, substituting ω_n into the bandwidth equation

$$\omega_{BW} = \frac{\pi - \phi}{T_r\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

Or

$$T_r = \frac{\pi - \phi}{\omega_{BW}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

The following figure shows the damping ratio vs. normalised rise time (T_r) of the system.

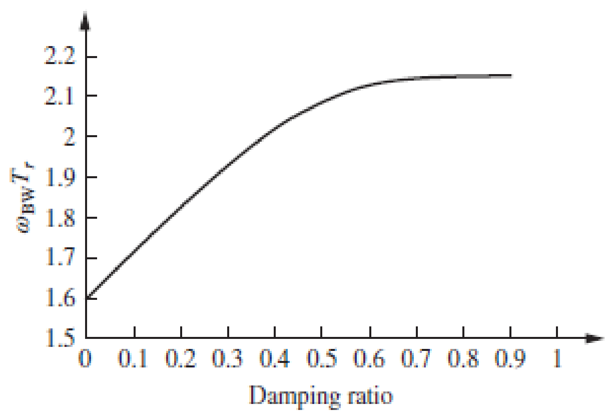
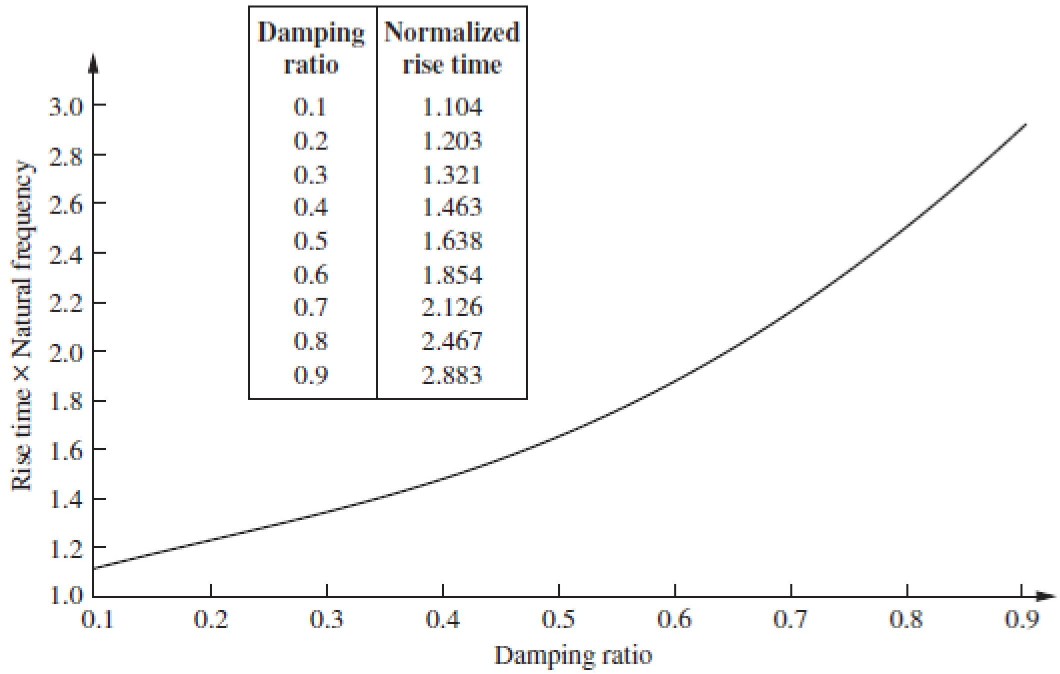


Figure 13: Graph of rise time (T_r) vs. damping ratio

Alternatively, the following figure shows the damping ratio vs. normalised rise time (T_r) of the system. Using this graph, we could determine the bandwidth of the system graphically.



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Figure 14: Normalised rise time versus damping ratio for a second-order underdamped response

For example, assume $\zeta = 0.4$ and $T_r = 0.2$ second.

Using the graph given above, the ordinate $T_r\omega_n$ = 1.463, from which ω_n = 1.463/0.2 = 7.315 rad/s.

$$\omega_{BW} = \omega_n\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

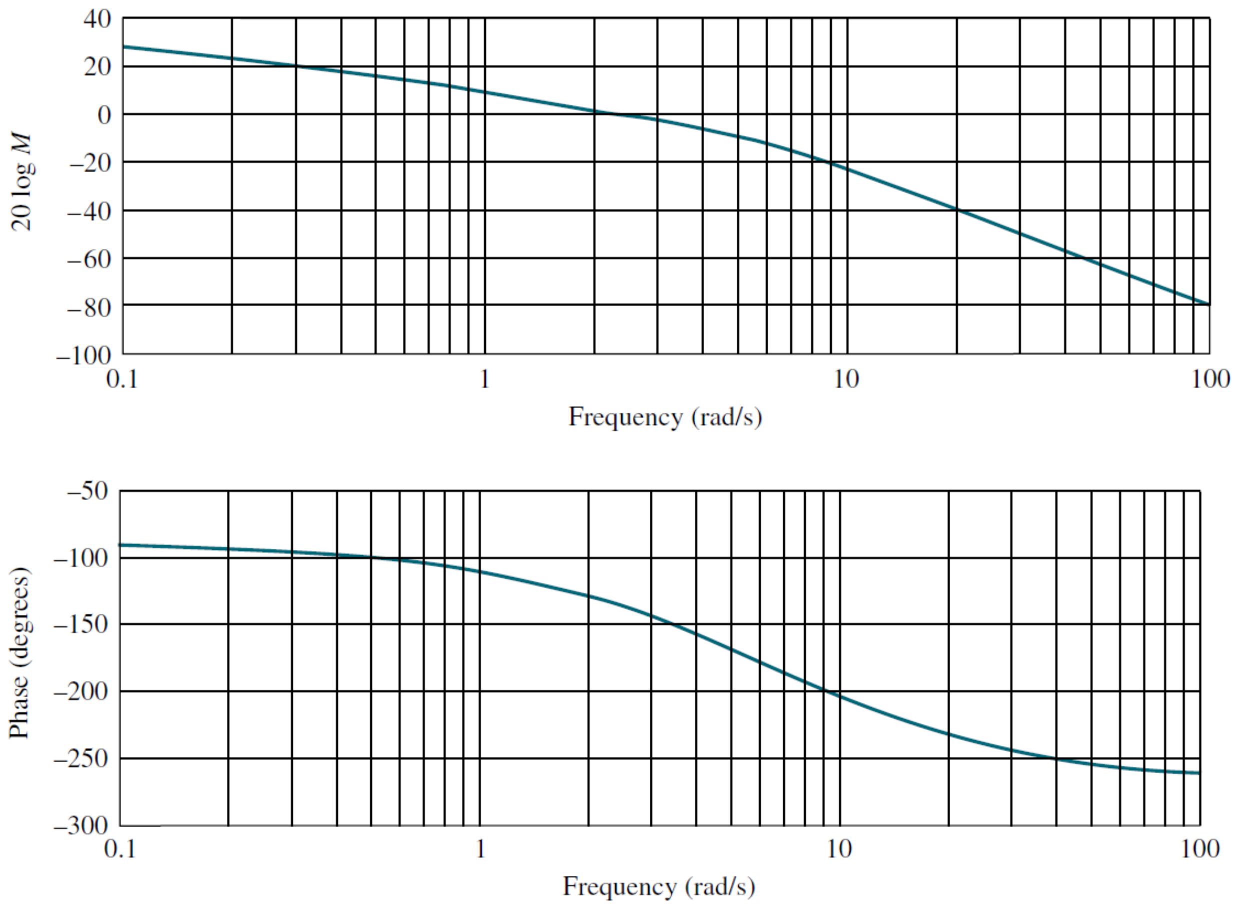
Using the above given equation, ω_{BW} = 10.05 rad/s.

Example for Tutorial 2 – Transient Response Analysis with Bode Plots

The Bode plots for a plant, $G(s)$, used in a unity feedback system are shown in the figure below. Do the following:

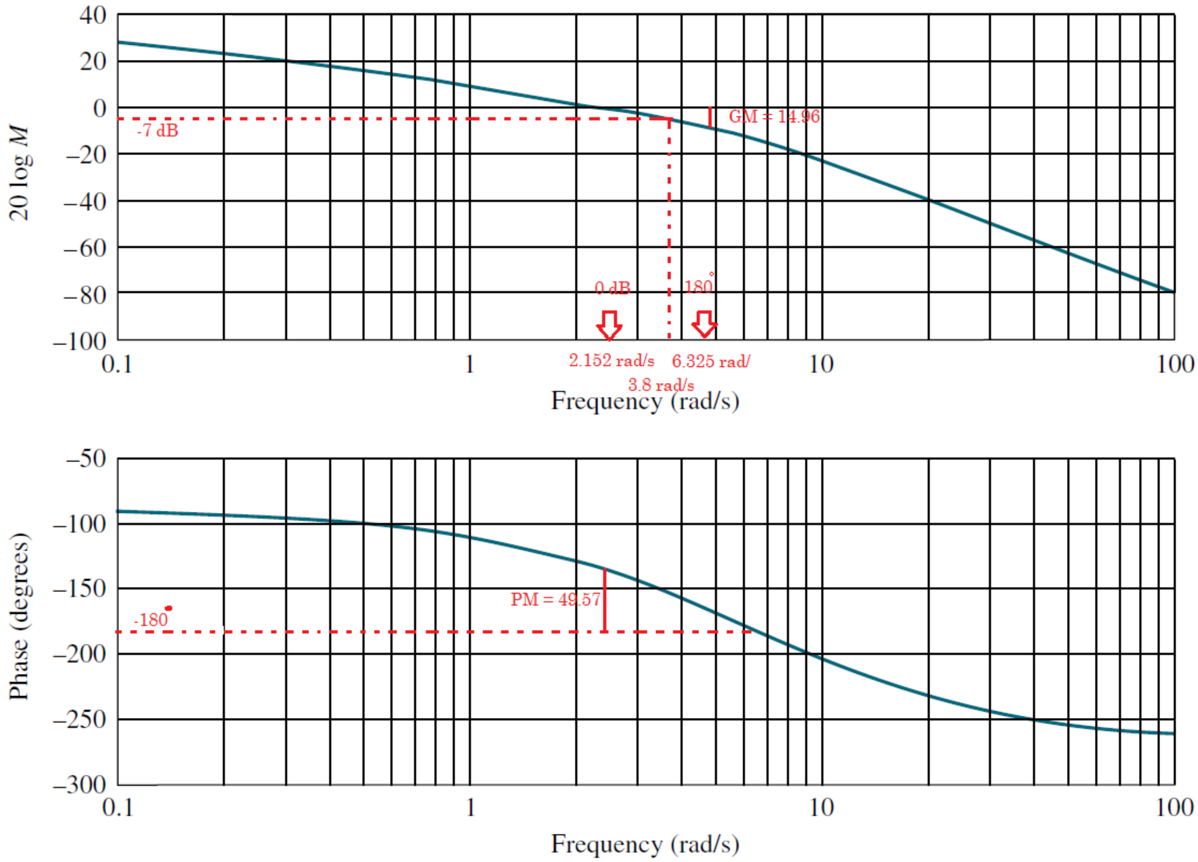
a. Find the gain margin, phase margin, 0 dB frequency (unity gain), 180° frequency, and the closed-loop bandwidth. [10 marks]

b. Use your results in part (a) to estimate the damping ratio, percent overshoot, settling time, and peak time. [10 marks]



Answer

From the Bode plots given below, the gain margin, phase margin, 0 dB frequency (unity gain), 180° frequency, and the closed-loop bandwidth are determined from the plots.



The results estimated from the graphs given above:

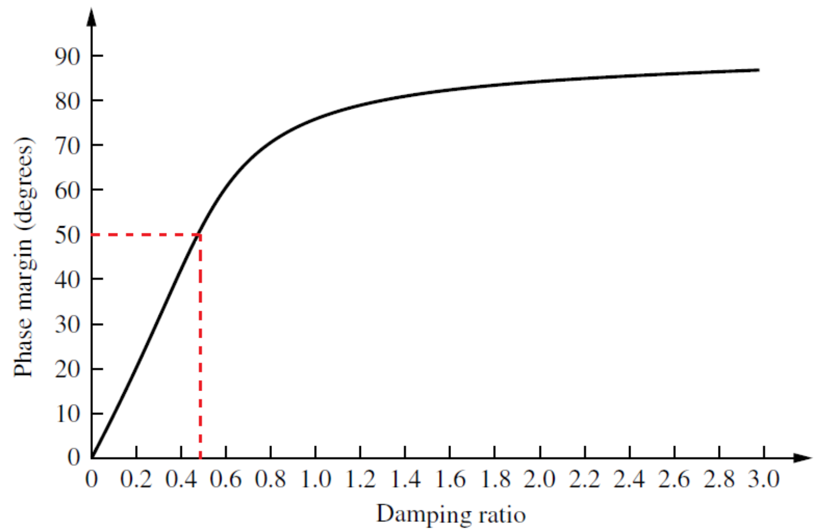
- Gain margin = 14.96 dB.
- Phase margin = 49.57°.
- 0 dB frequency = 2.152 rad/s.
- 180° frequency = 6.325 rad/s.
- Bandwidth (@-7 dB point) = 3.8 rad/s.

From the equation given below, the damping ratio of the system can be calculated.

$$\zeta = \sqrt[4]{\frac{1}{\left(\frac{4}{\tan^2 \theta_M} + 2\right)^2 - 4}} = \sqrt[4]{\frac{1}{\left(\frac{4}{\tan^2 49.57} + 2\right)^2 - 4}} = 0.48$$

The damping ratio of the system, ζ is 0.48.

Or, from the graph given below, the damping ratio of the system, ζ is estimated to be 0.5.



From the equation given below, the percentage overshoot of the system can be calculated from:

$$\%OS = e^{\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\% = e^{\frac{\pi(0.48)}{\sqrt{1-(0.48)^2}}} \times 100\% = 17.93\%$$

The percentage overshoot of the system, $\%OS$ is 17.93%.

From the equation given below, the settling time of the system (2% standard) can be calculated from:

$$T_s = \frac{4}{\omega_{BW}\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

$$= \frac{4}{(3.8)(0.48)}\sqrt{(1-2(0.48)^2)+\sqrt{4(0.48)^4-4(0.48)^2+2}} = 2.84 \text{ s}$$

The settling time of the system, T_s is 2.84 s.

From the equation given below, the time-to-peak (n = 1) can be calculated from:

$$T_p = \frac{\pi}{\omega_{BW}\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

$$= \frac{\pi}{(3.8)\sqrt{1-(0.48)^2}}\sqrt{(1-2(0.48)^2)+\sqrt{4(0.48)^4-4(0.48)^2+2}}$$

$$= 1.22 \text{ s}$$

The time-to-peak of the system, T_p is 1.22 s.

3. Steady-State Characteristics in Bode Plots

From the given Bode plots, we can determine a variety of steady-state parameters. Steady-state parameters that can be derived and approximated from the Bode diagrams are:

- System type.
- Steady-state static error constants (K_p, K_v, and K_a).
- Steady-state errors.

With these parameters, we could analyse the characteristics and behaviour of the control system at steady-state conditions:

Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

Table 1: Steady-state analysis of control system

3.1. System Type on a Bode Plot

The type of a system is defined to be equal to the number of integrators in the open loop transfer function. We can find the type of a system by examining its Bode plot.

a. A type 0 system has a slope of 0 and a phase of 0° at low frequencies.

b. A type 1 system has a slope of -20 dB/decade and a phase of -90° at low frequencies.

c. A type 2 system has a slope of -40 dB/decade and a phase of -180° at low frequencies.

"Low frequencies" in this context means in the frequency range below any of the system zeros or poles. Note that examination of an experimental frequency response allows you to determine the system type without needing a transfer function.

As illustrated in the diagram below, the type of the system can be determined as follows:

- For the first system $1/(s+10)$ with blue line, the gain of the frequency response at low frequency is 0 dB and the phase shift at low frequency is 0 degree. So, the system is a type 0.

- For the second system $1/s$ with orange line, the gain of the frequency response at low frequency is a slope with -20 dB/decade and the phase shift at low frequency is -90 degree. So, based on these results, the system is defined as a type 1 system.

- For the second system $1/s^2$ with yellow line, the gain of the frequency response at low frequency is a -40 dB/decade slope and the phase shift at low frequency is -180 degree. As a result, the system is a type 2.

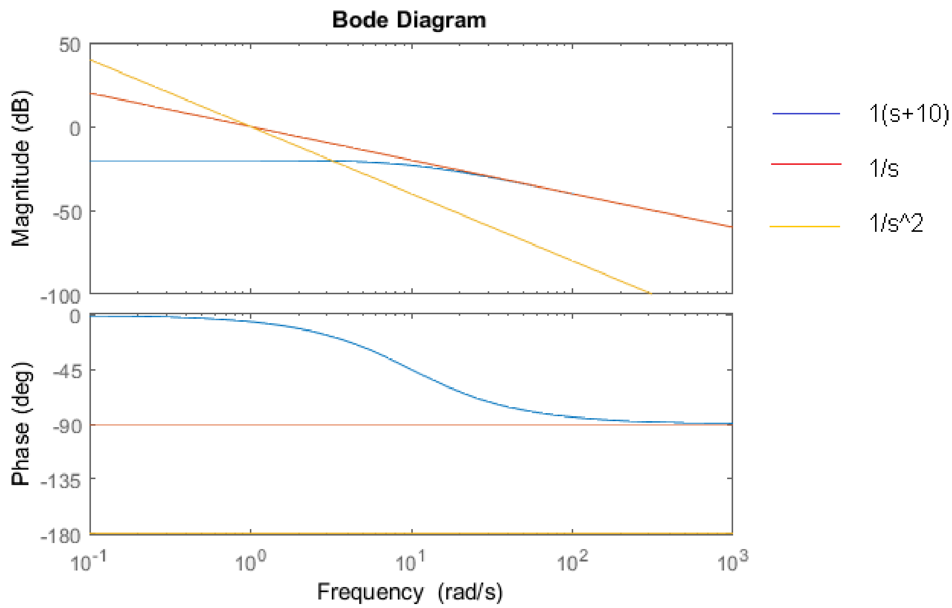


Figure 15: System's types as determined in Bode plot

3.2. Steady-State Errors from a Bode Plot

The system type is related to the error that a closed loop system will exhibit when attempting to follow a reference signal. Reminder:

a. A type 0 system will have an error $1/(1+K_p)$ for a step input and infinite error for ramps and paraboloids.

b. A type 1 system will have zero error for a step, an error of $1/K_v$ for a ramp and an infinite error for an input paraboloid.

c. A type 2 system will have zero error when tracking input steps or ramps, but an error $1/K_a$ when tracking a command paraboloid.

3.2.1. Steady-State Error with a Step Input

The error to a steady-state unity gain step is given by:

$$e(\infty) = \frac{1}{1+K_p}$$

Where: K_p is the position-error constant.

For a type 0 system, K_p is equal to the value of the open loop gain of the system. Thus, if our Bode plot indicates a type 0 system (zero slope at low frequency), we can directly read off the K_p value.

For systems of higher type, the DC gain of the system is infinite, so the value of K_p is also infinite. This corresponds to zero static error to a step function for systems including one or more integrators in the forward path. The position error constant is the DC gain of the system.

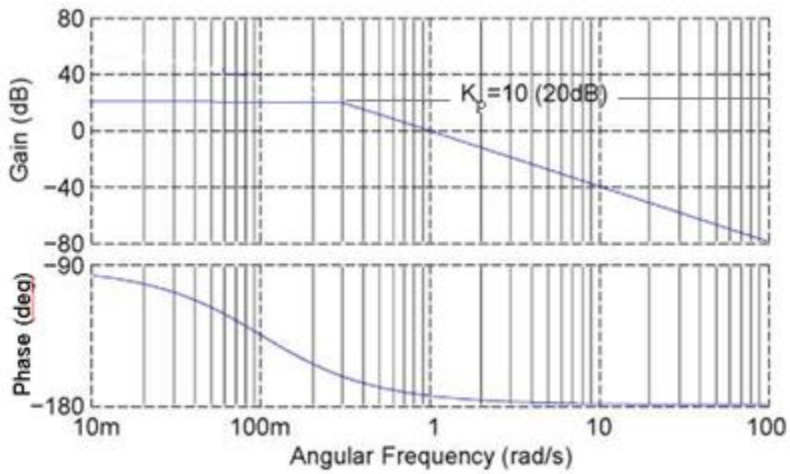


Figure 16: Position-error constant in Bode plots.

For the system given in the figure above, the magnitude of the low-frequency gain extended to 1 rad/s is 20 dB. So, the position error constant (K_p) is found to be $K_p = \log^{-1}(|G(s)|/20) = \log^{-1}(20/20)$ = 10.

Alternatively, we could find the position error constant from the intersection of the low-frequency slope with the frequency axis. For a given type 0 system, the transfer function of the system is:

$$G(s) = K\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)}$$

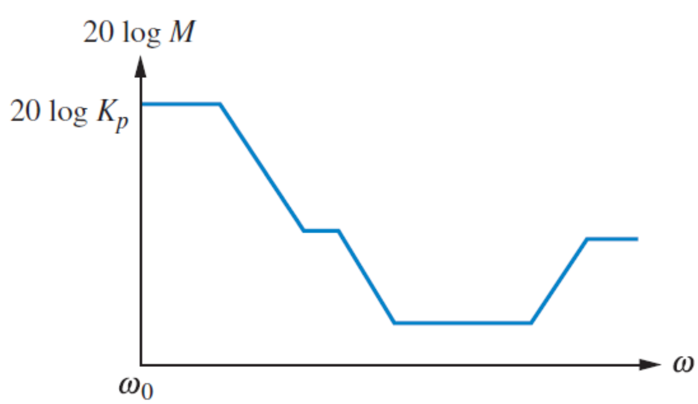


Figure 17: Determining the position-error constant (K_p) from Bode plot

The initial value of the gain plot of the frequency response is:

$$20\log|G(s)| = 20\log K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i}$$

The value of position-error constant is:

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} K\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)} = K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i}$$

This is the same as the value of the low-frequency gain of the system for type 0 system.

$$20\log K_p = |G(s)|$$

Thus

$$K_p = \log^{-1}\left[\frac{|G(s)|}{20}\right]$$

3.2.2. Steady-State Error with a Ramp Input

The error in the presence of a unit ramp input is specified as:

$$e(\infty) = K_v$$

Where: K_v is the velocity error constant.

Recall that $K_v = \lim_{s\to 0} sG(s)$. For a type 1 system, the multiplication by s would result in a level gain curve at low frequencies. If we were to plot a gain plot of $sG(s)$, then K_v would be the low frequency gain.

Rather than plot this explicitly, we can instead examine the gain that the $1/s$ part of the transfer function has at a frequency of 1 rad/s.

Similarly, we can find the velocity error constant by determining the gain of the $1/s$ part of the transfer function if extended to ω = 1.

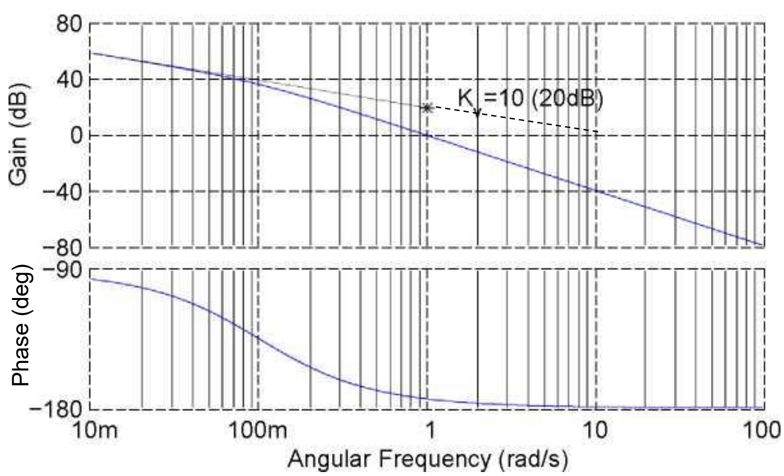


Figure 18: Velocity-error constant in Bode plots

For the system given in the figure above, the magnitude of the low-frequency gain extended to 1 rad/s is 20 dB. So, the velocity error constant (K_v) is found to be $K_v = \log^{-1}(|G(s)|/20) = \log^{-1}(20/20)$ = 10.

Alternatively, we could find the velocity error constant from the intersection of the low-frequency slope with the frequency axis. For a given type 1 system, the transfer function of the system is:

$$G(s) = K\frac{\prod_{i=1}^{n}(s+z_i)}{s\prod_{i=1}^{m}(s+p_i)}$$

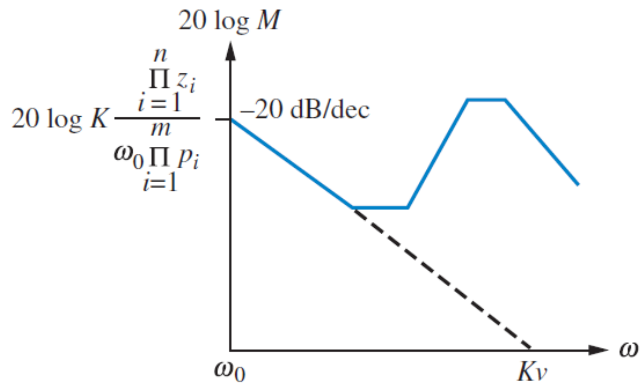


Figure 19: Determining the velocity-error constant (K_v) from Bode plot

The initial value of the gain plot of the frequency response is:

$$20\log|G(s)| = 20\log K\frac{\prod_{i=1}^{n}z_i}{\omega_0\prod_{i=1}^{m}p_i}$$

With a type 1 system, the -20 dB/decade slope of the frequency response can be considered as originated from a function:

$$G'(s) = K\frac{\prod_{i=1}^{n}z_i}{s\prod_{i=1}^{m}p_i}$$

The $G'(s)$ intersects the frequency axis when the frequency of the frequency response is:

$$\omega = K\frac{\prod_{i=1}^{n}z_i}{\prod_{i=1}^{m}p_i}$$

Thus, the velocity-error constant of the system is:

$$K_v = \lim_{s\to 0} sG(s) = s\left(K\frac{\prod_{i=1}^{n}z_i}{s\prod_{i=1}^{m}p_i}\right) = K\frac{\prod_{i=1}^{n}z_i}{\prod_{i=1}^{m}p_i}$$

Hence

$$\omega = K_v \qquad \text{or} \qquad K_v = \omega$$

This is actually the same as the frequency-axis intercept. Extending the initial -20 dB/decade slope to the frequency axis will give you the velocity-error constant.

3.2.3. Steady-State Error with a Parabolic Input

The error in the presence of a unit parabolic input is specified as:

$$e(\infty) = K_a$$

Where: K_a is the parabolic error constant.

Recall that $K_a = \lim_{s \to 0} s^2 G(s)$. For a type 2 system, the multiplication by s^2 would result in a level gain curve at low frequencies. If we were to plot a gain plot of $s^2 G(s)$, then K_a would be the low frequency gain.

Rather than plot this explicitly, we can instead examine the gain that the $1/s^2$ part of the transfer function has at a frequency of 1 rad/s.

Similarly, we can find the acceleration error constant by determining the gain of the $1/s^2$ part of the transfer function if extended to ω = 1.

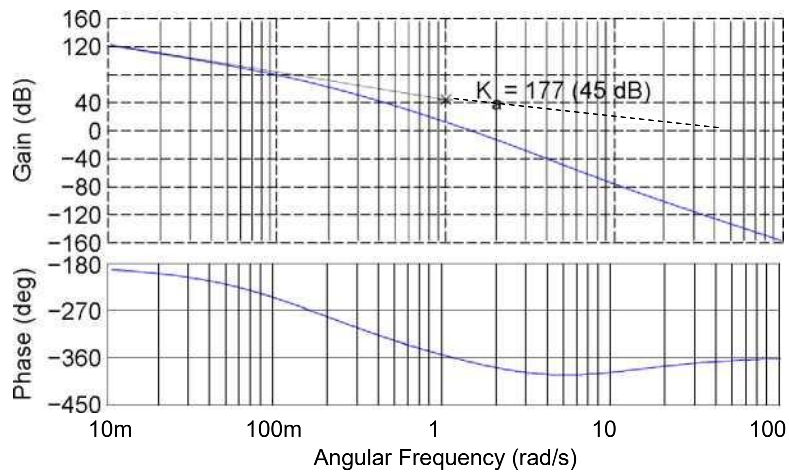


Figure 20: Parabolic-error constant in Bode plots

For the system given in the figure above, the magnitude of the low-frequency gain extended to 1 rad/s is 45 dB. So, the parabolic error constant (K_a) is found to be $K_a = \log^{-1}(|G(s)|/20) = \log^{-1}(45/20)$ = 177.

Alternatively, we could find the parabolic error constant from the intersection of the low-frequency slope with the frequency axis. For a type 2 system, the transfer function of the system is:

$$G(s) = K \frac{\prod_{i=1}^{n}(s + z_i)}{s^2 \prod_{i=1}^{m}(s + p_i)}$$

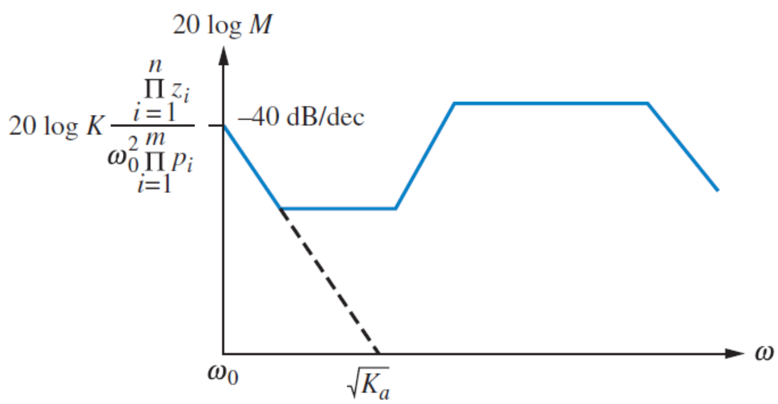


Figure 21: Determining the parabolic-error constant (K_a) from Bode plot

The initial value of the gain plot of the frequency response is:

$$20\log|G(s)| = 20\log K\frac{\prod_{i=1}^{n} z_i}{\omega_0^2\prod_{i=1}^{m} p_i}$$

With a type 1 system, the -20 dB/decade slope of the frequency response can be considered as originated from a function:

$$G'(s) = K\frac{\prod_{i=1}^{n} z_i}{s^2\prod_{i=1}^{m} p_i}$$

The $G'(s)$ has an intersection with the frequency axis when the frequency of the frequency response is:

$$\omega = \sqrt{K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i}}$$

But, since the acceleration-error constant of the system is:

$$K_a = \lim_{s\to 0} s^2 G(s) = s^2\left[K\frac{\prod_{i=1}^{n} z_i}{s^2\prod_{i=1}^{m} p_i}\right] = K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i}$$

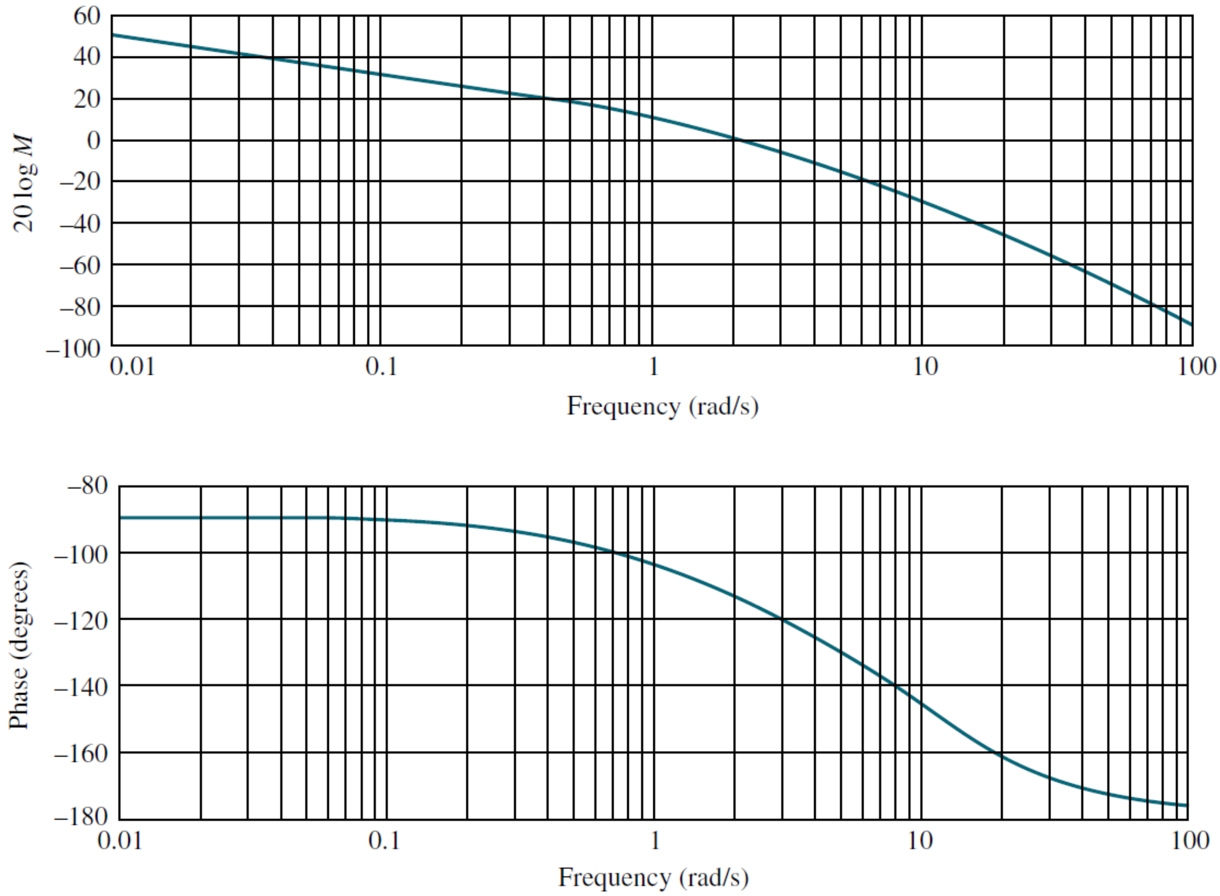
Hence

$$\omega = \sqrt{K_a} \qquad \text{or} \qquad K_a = \omega^2$$

Consider the above two equations, extending the initial -40 dB/decade slope to the frequency axis will give you the velocity-error constant at $\sqrt{K_a}$.

Example for Tutorial 3 – Steady-State Analysis with Bode Plots

The open-loop frequency response shown in the figure below was experimentally obtained from a unity feedback system.

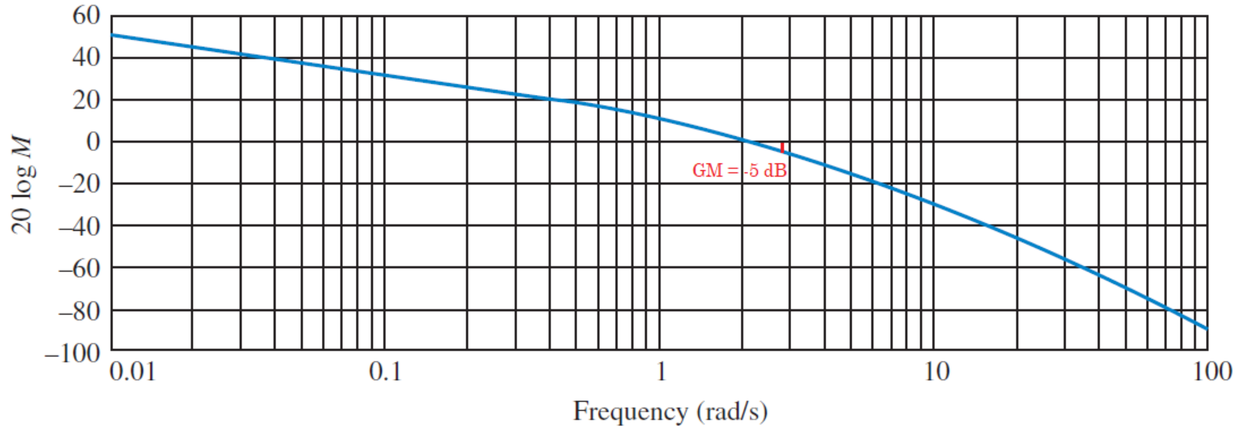


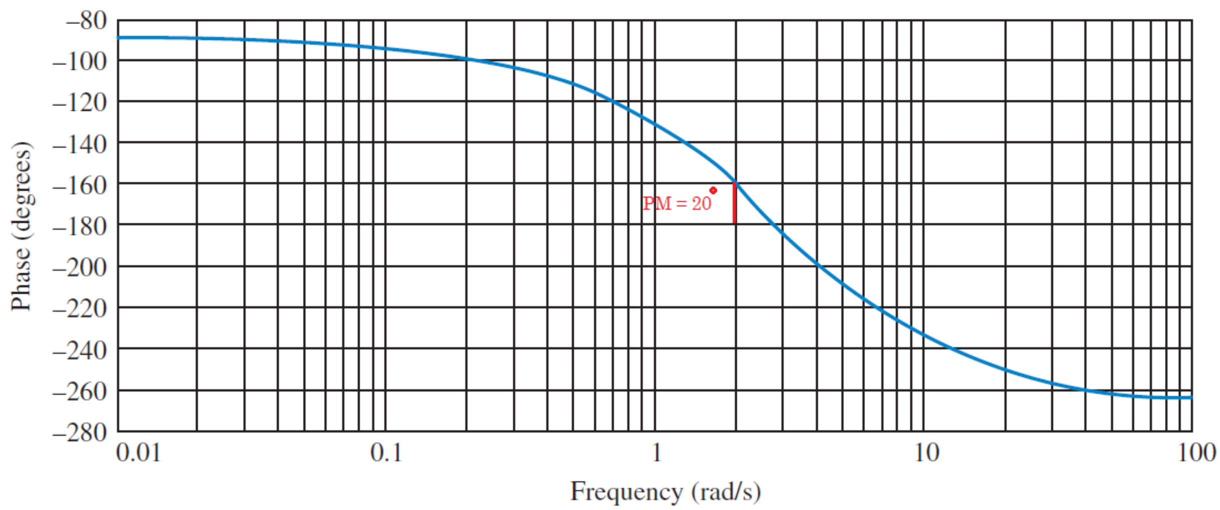
a. Estimate the percent overshoot of the closed-loop system. [20 marks]

b. Estimate the steady-state error of the closed-loop system. [20 marks]

Answer

a. The phase margin of the closed-loop system is determined from following frequency response diagram.



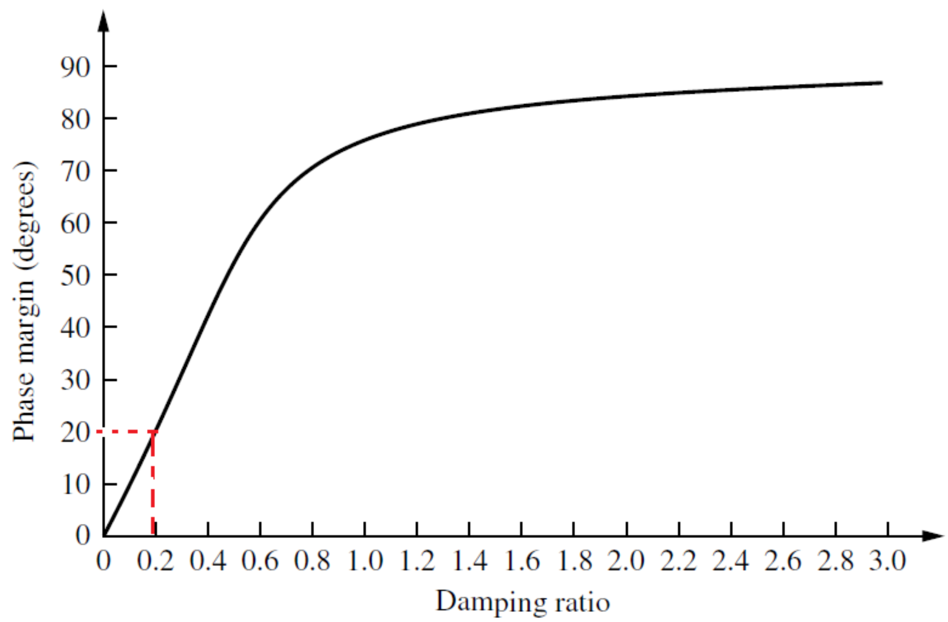


From the Bode plots given above, the phase margin (PM) of the given closed-loop system is 20° and the gain margin is 5 dB.

The damping ratio of the system is calculated from the following equation:

$$\zeta = \sqrt[4]{\frac{1}{\left(\frac{4}{\tan^2 \phi_M} + 2\right)^2 - 4}} = \sqrt[4]{\frac{1}{\left(\frac{4}{\tan^2 20} + 2\right)^2 - 4}} = 0.176$$

Or using the graph below, the damping ratio can be estimated from the system phase margin.



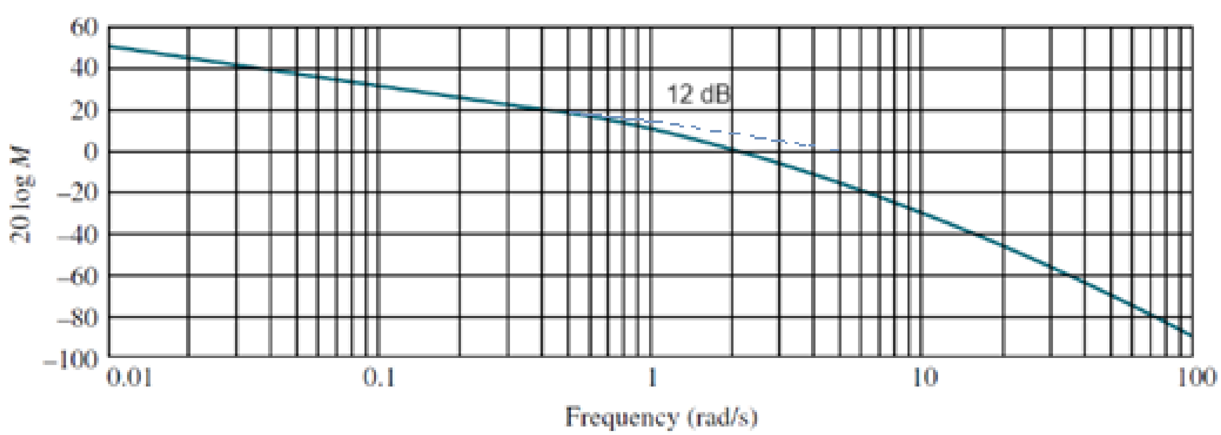
Using the equation or the graph given above, the damping ratio, ζ is 0.176 or 0.18 respectively.

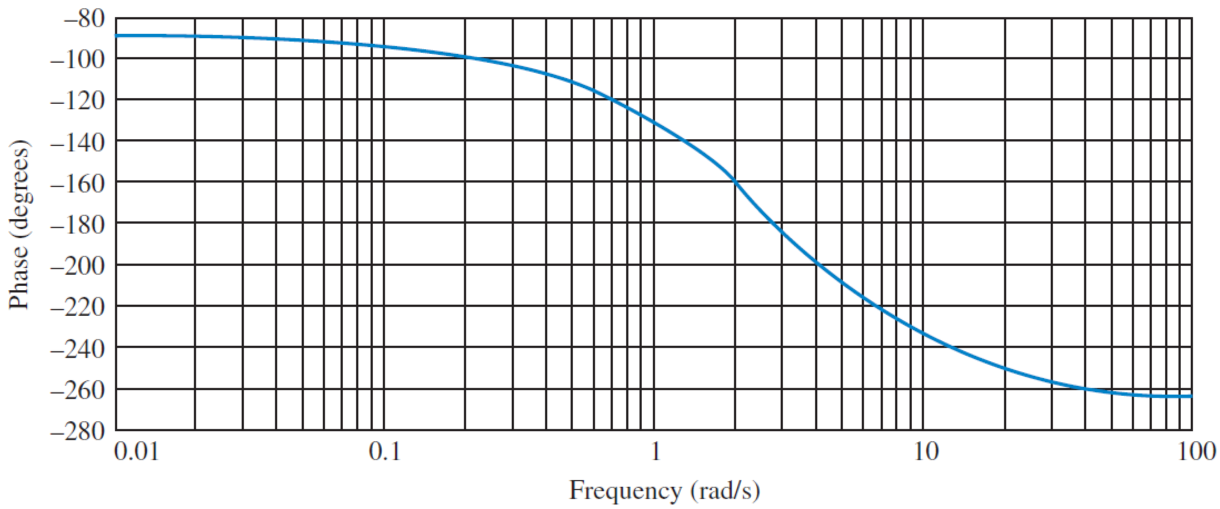
The percentage overshoot of the system is calculated from the following equation:

$$\%OS = e^{\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\% = e^{\frac{\pi(0.176)}{\sqrt{1-(0.176)^2}}} \times 100\% = 57\%$$

The equation given above yields 57% overshoot.

b. The system is Type 1 since the initial slope is - 20 dB/dec and extending this slope intersection with the gain at 1 rad/s is 12 dB. Continuing the low frequency slope down to the 0 dB line yields 4 rad/s.





Knowing that we found the intersection of low frequency slope with 1 rad/s is 12 dB. So, the velocity-error constant is:

$$K_v = \log^{-1}\left[\frac{|G(s)|}{20}\right] = \log^{-1}\left(\frac{12}{20}\right) = 4$$

Or, from the intersection with the frequency axis (i.e. zero dB line), the velocity error constant is:

$$K_v = 4$$

As a result, for $K_v = 4$ and given relevant inputs, the steady-state errors of the system are:

- For a unit step input is zero.
- For a unit ramp input is:

$$e(\infty)_{ramp} = \frac{1}{K_v} = 0.25$$

- For a parabolic input is infinite.