VICTORIA UNIVERSITY OF WELLINGTON TE HERENGA WAKA

XMUT315 Control Systems Engineering

Note 11a: Introduction to Root Locus

Topic

- Pole- Zero Diagrams
- Poles and Zeros
- System Performance
- Closed-Loop Poles and Zeros
- Examples
- Design using Root Locus Diagram
- Formalisations of Root Locus diagram
- Examples of Plots

1. Introduction to Root Locus

Root locus is one of methods in control system that you can use to analyse and design the control systems.

1.1. Location of Poles in S-Plane

We know that the location of the system poles and zeros determine the response of a LTI system.

- Poles on the y-axis -> undamped response.
- Complex poles -> underdamped response.
- Single double poles on the x-axis -> critical damped response.
- Poles on the x-axis -> overdamped response.

As illustrated in the following diagram, location of the poles in the s-plane will determine the characteristics and behaviour of the control systems.

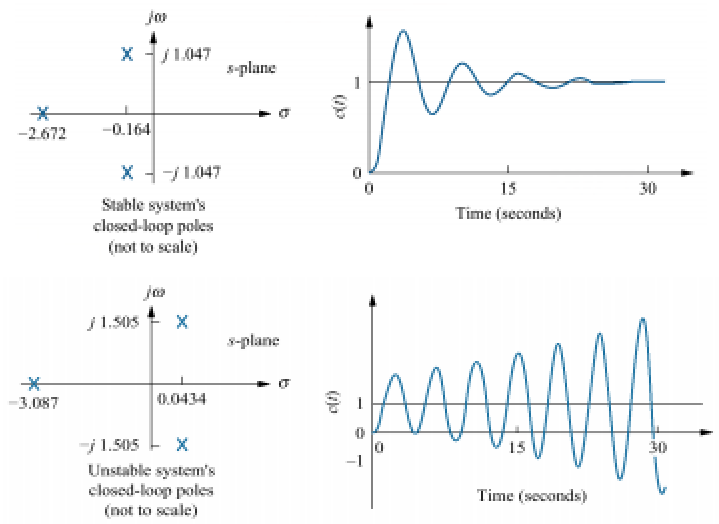


Figure 1: Locations of poles and zeros in s-plane and system responses

To change the system response, we therefore need to change these roots' (pole and zero) location(s). We will do this by applying feedback and including a compensator (or controller) into the system. Application of feedback enables us to manage the system. Compensators (or controllers) enable us to determine the specific response of the system.

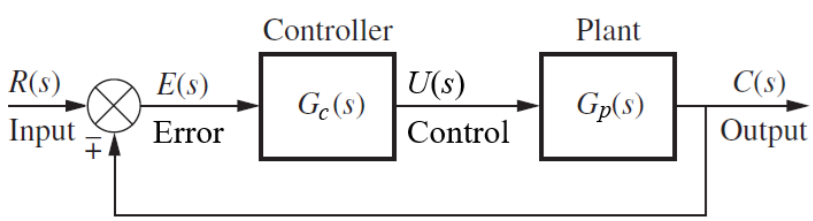


Figure 2: Feedback control system with compensator

The root locus is a visual presentation of the manner in which the roots of a system change as we vary some parameter of our compensator.

We will usually be interested in how altering the gain affects the system response, but we can use the root locus for other parameters too.

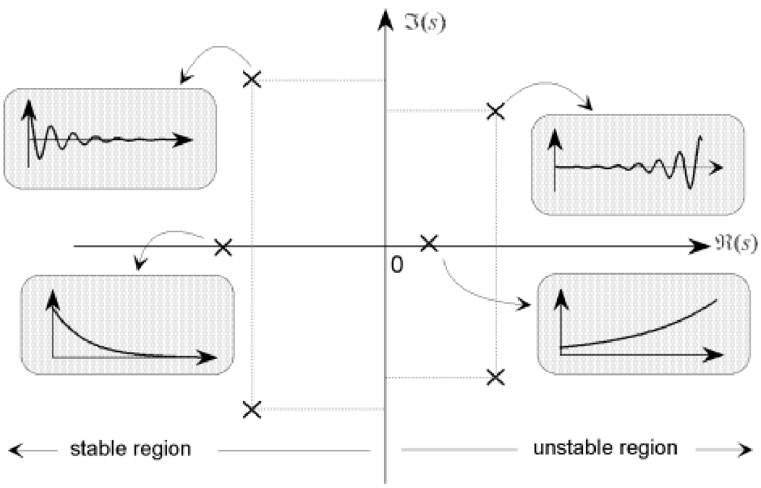


Figure 3: Locations of poles in the s-plane and system responses

1.2. Poles on the Pole-Zero Map

We know that pole location determines which modes will be present in a time response. The real part of a pole location determines the damping of a mode and the imaginary part determines the natural frequency of the mode.

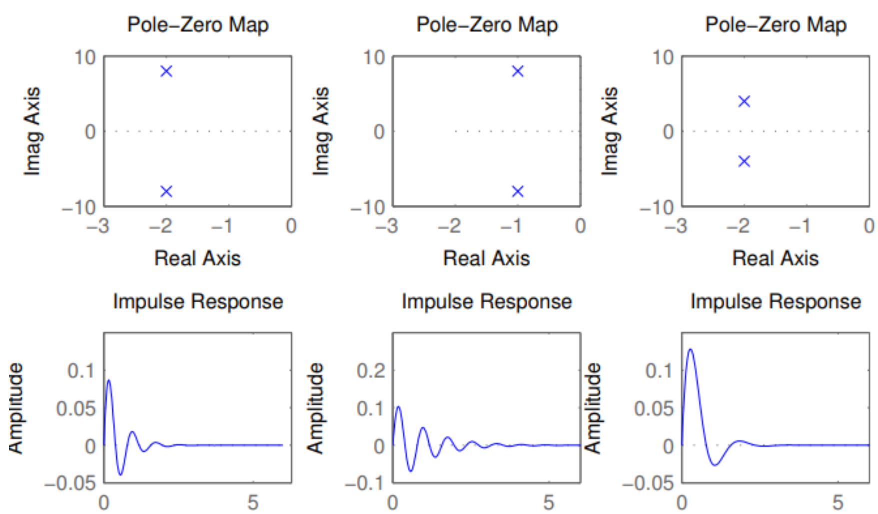


Figure 4: Poles on the Pole-Zero Map

1.3. Dominant Poles

The pole with the slowest response will dominate the response of any system.

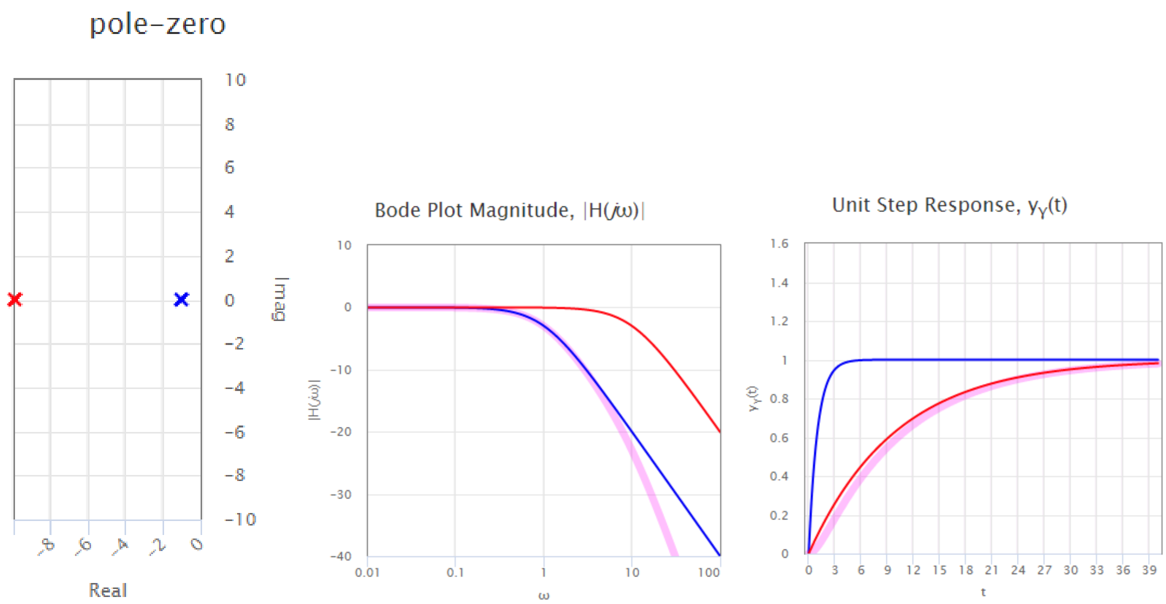


Figure 5: Location of dominant poles in the s-plane, Bode plot, and step response

The pole near the origin (blue – with the slowest response) compared with the pole further from the origin (red). The pole with the slowest response will dominate the response of any system.

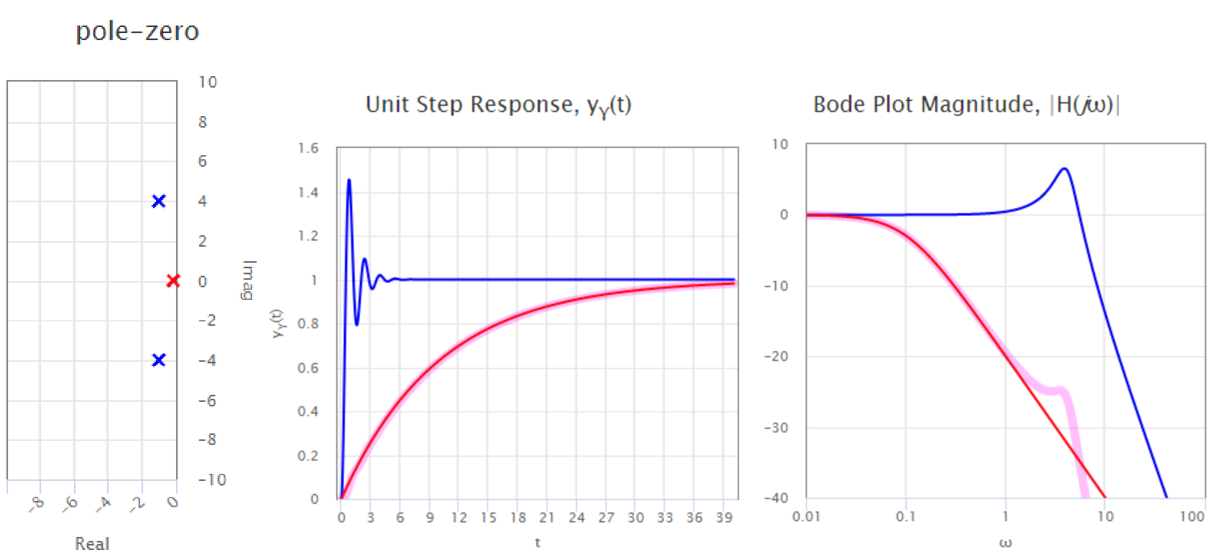


Figure 6: Location of dominant poles in the s-plane, Bode plot, and step response

The pole at the origin (red – with the slowest response) compared with complex poles (blue) further from origin. We can therefore often approximate the response of a system as either a first order, or a second order system.

We can then use the location of the dominant poles to determine the system's settling time, overshoot, damping, natural frequency etc. So, if the dominant pole (pair) is sufficiently far away from any other poles then we can ignore the other poles when designing our control system. This approximation is normally safe if other poles are at least a factor of three further to the left of the *s*-plane.

1.4. Zeros on the Pole-Zero Map

Zeros near a pole suppress the mode corresponding to the nearby pole. In the extreme case where the zeros and poles are collocated, we get complete cancellation of the corresponding mode.

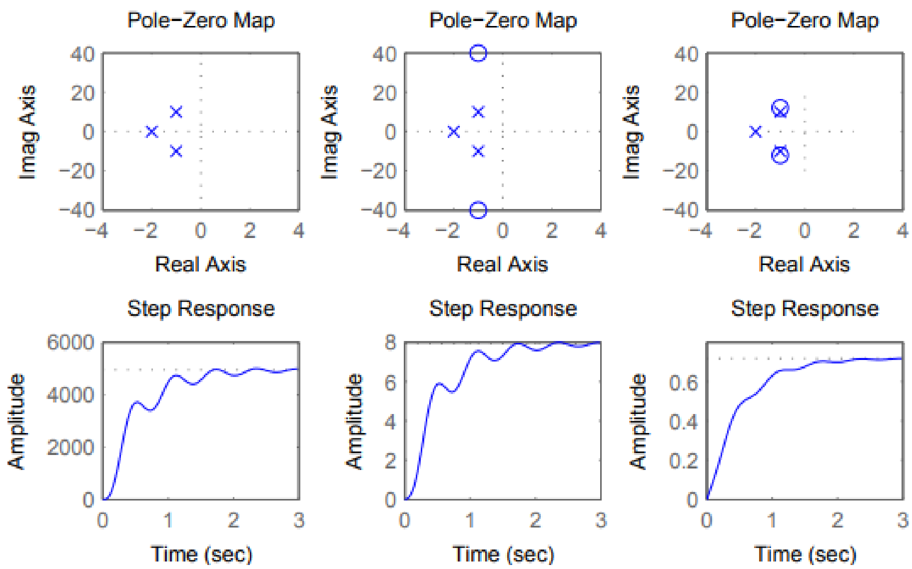


Figure 7: Zeros on the Pole-Zero Map

2. Poles Location in S-Plane and System Performance

Location of poles in the s-plane will determine the performance of the control systems. Notably, the stability and transient response of the control system. We will look closely the system performance in terms of settling time and damping using root locus diagram.

2.1. System Performance - Settling Time

The further a pole is to the left of the s-plane, the faster the corresponding mode decays. Therefore, if we are given a specification for system response time, we can convert this to a requirement on pole position.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Roots of the characteristic equation:

$$s = -\frac{2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\sigma \pm \omega_d$$

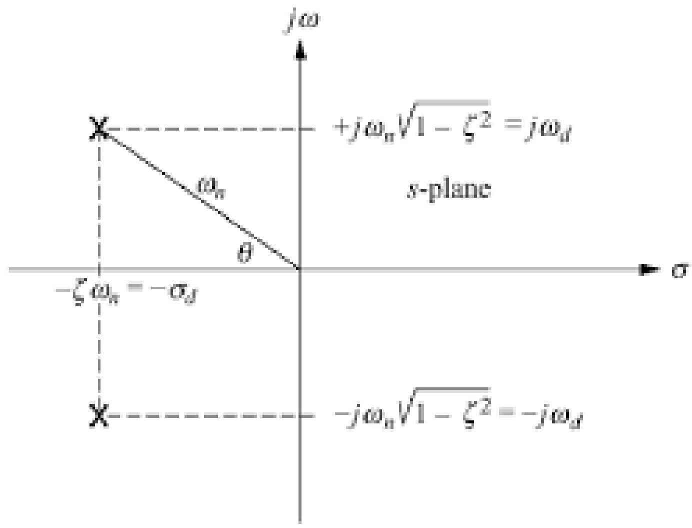


Figure 8: System Performance - Settling Time

For the given roots of characteristic equation:

- The two roots are imaginary when $\zeta = 0$.
- The two roots are complex conjugate when $0 < \zeta < 1$.
- The two roots are real and equal when $\zeta = 1$.
- The two roots are real but not equal when $\zeta > 1$.

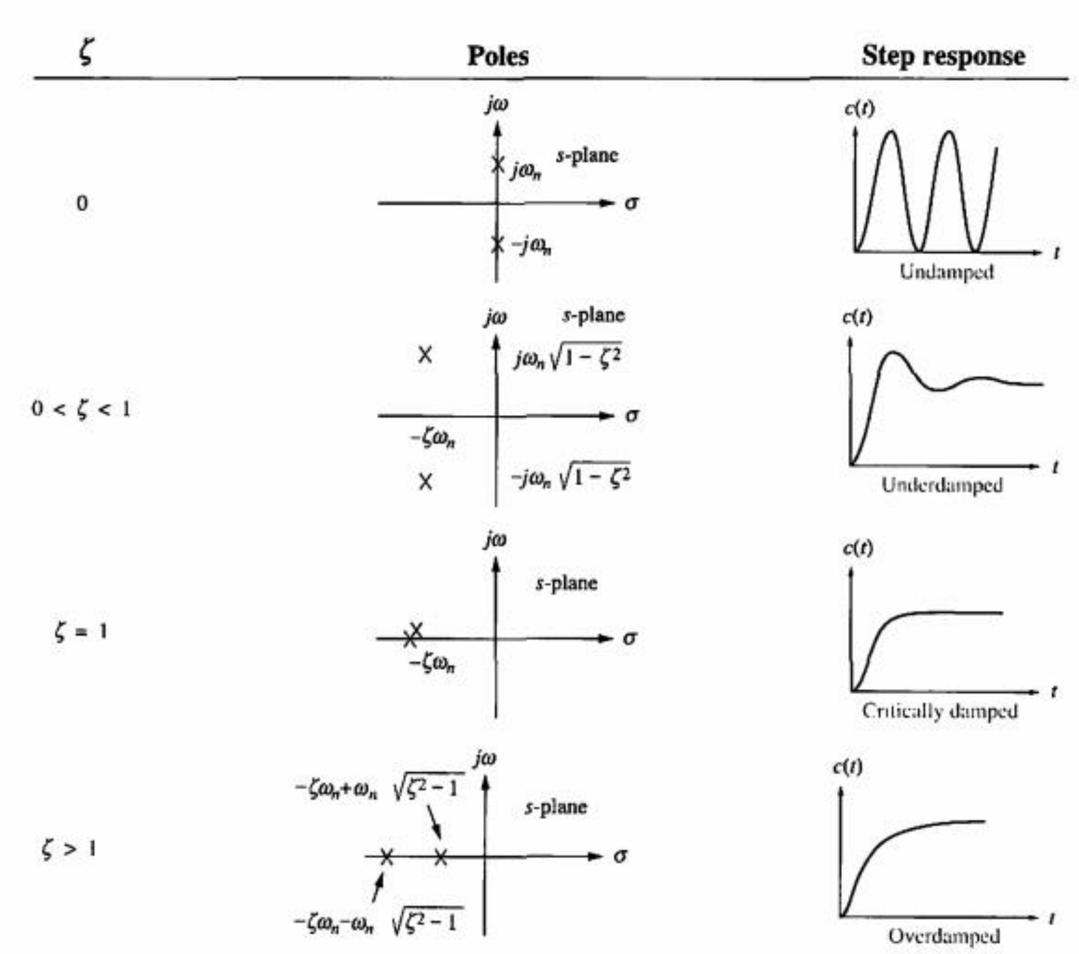


Table 1: Damping ratio, poles and step responses

The settling time of a second order system (T_s) is approximately (e.g.: for the 2% steady-state settling time standard):

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma}$$

Where: σ is the real part of the pole pair.

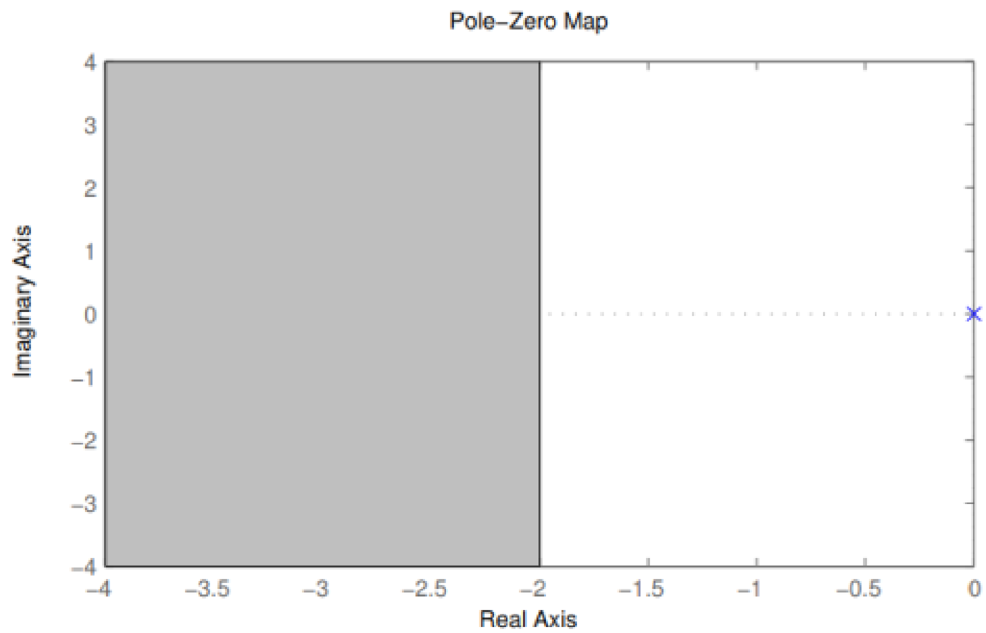


Figure 9: System with settling time, T_s < 2 second

So, for example if we have a requirement that settling time, T_s < 2 second, then we must have the dominant pole further left than s = - 2.

2.2. System Performance - Damping

We are often also given a requirement on damping ratio. Recall that a damping ratio ζ occurs on the pole-zero diagram as a straight line with an angle of $\theta = \cos^{-1}\zeta$ to the negative real axis. We can thus similarly construct an allowed region for the poles if we are given a damping specification. For example, if we require ζ > 0.707, then the poles must lie less than $\theta = \cos^{-1} 0.707 \approx 45°$ from the negative real axis.

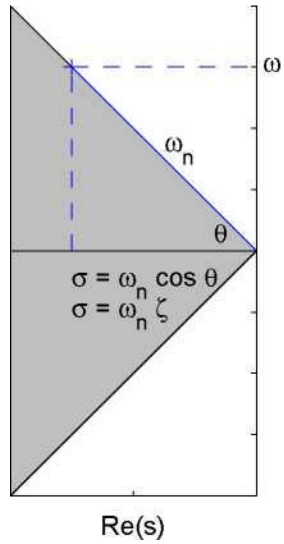


Figure 10: System with damping ratio ζ > 0.707

2.3. System Performance - Allowed Pole Region

We are often given specifications on both settling time and damping, so we must combine the constraints imposed by the two. We therefore obtain a composite region in which we can place the poles.

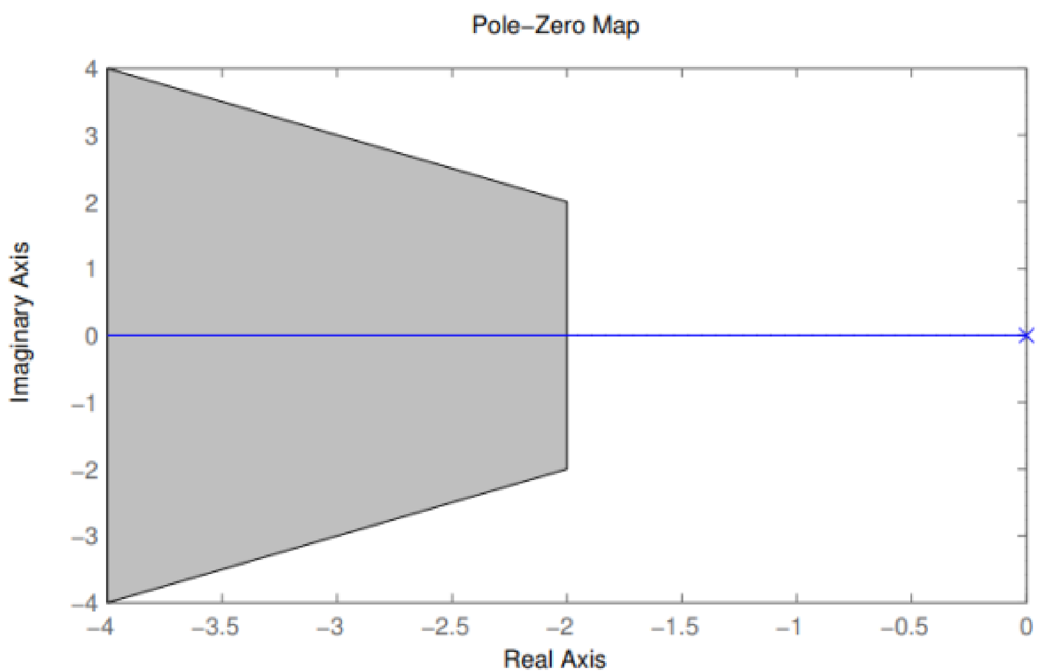


Figure 11: System with settling time, T_s < 2 second and damping ratio ζ > 0.707

So, if the closed-loop system poles lie within the allowed region then we have satisfied the design specification.

3. Closed-Loop System

The characteristics and behavior of the control systems in open loop system might not be necessarily the same as the closed-loop system. We will see differences between these two as follows.

3.1. Closed-Loop Poles

If we have a plant described by an open-loop transfer function $G(s)$, it will have a certain set of open-loop poles and zeros.

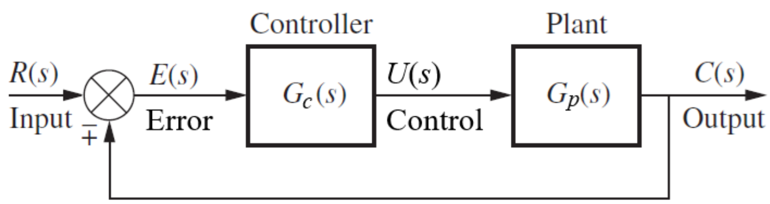


Figure 12: Closed-loop feedback control system with compensator

We now enclose the plant within a unity-gain feedback loop including a compensator with a transfer function $G_C(s)$, which results in a closed-loop transfer function:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G_C(s)G_P(s)}{1 + G_C(s)G_P(s)}$$

Notice that the closed-loop transfer function will not have the same pole locations as the open-loop transfer function. There is no reason that the denominator of $G(s)$ and the denominator of $T(s)$ would have the same roots. Closing the loop has moved the poles, but we do not yet know where to?

Open-loop transfer function:

$$G(s) = \frac{1}{(s+10)}$$

Root of characteristic equation:

$$s = -10$$

Closed-loop transfer function:

$$T(s) = \frac{\frac{1}{(s+10)}}{1 + \frac{1}{(s+10)}} = \frac{1}{s+11}$$

Root of characteristic equation:

$$s = -11$$

We will find is that the location of the closed-loop poles will depend on the DC gain K or the system. In this case:

$$K = G_C(s)G_P(s)|_{s=0}$$

As we can change $G_C(s)$, we can use K as a tuning parameter to move the closed-loop poles to a desired location. The desired location for the poles will, in turn, be determined by the performance specification for our control system.

3.2. Closed-Loop Zeros

Now, consider the effect that feedback has on zeros. Again, consider a plant with a transfer function $G_P(s)$ that we have controlled by adding a series compensator $G_C(s)$. Again, the closed-loop transfer function is:

$$T(s) = \frac{G_C(s)G_P(s)}{1 + G_C(s)G_P(s)}$$

The only way that $T(s)$ can be zero is if either $G_C(s)$ or $G_P(s)$ is zero (or both). Thus, the set of zeros of the closed-loop system is the combination of the zeros of the plant and the compensator.

It is not possible to move the zeros of a plant using feedback.

Example for Tutorial 1: Root Locus Diagram of First-Order System

Consider a first-order system that has a single open-loop pole at s = -1.

$$G(s) = \frac{1}{s+1}$$

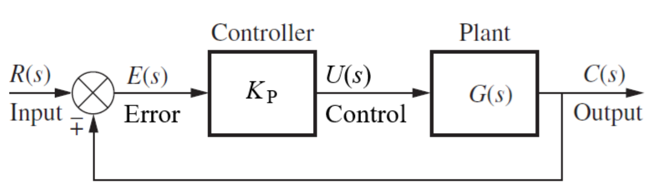
Let us use a proportional compensator $G_C(s) = K_p$ and see what effect this has on the pole.

a. Derive the transfer function of the closed-loop system. [4 marks]

b. Comment on the location of the poles in the closed-loop system. [2 marks]

c. Calculate the possible locations of the poles in the closed-loop system given the value of K_p = 0, 1, 2, and 4. [4 marks]

d. Simulate using MATLAB the loci of these poles in the s-plane and the transient response of the system when K_p = 0, 1, 2, and 4. Comment on the results of the simulations. [10 marks]



Answer

a. The closed-loop transfer function of the system is:

$$T(s) = \frac{G_C(s)G_P(s)}{1 + G_C(s)G_P(s)} = \frac{\left(\frac{K_p}{s+1}\right)}{1 + \left(\frac{K_p}{s+1}\right)} = \frac{K_p}{s + (1 + K_p)}$$

b. Thus, the closed-loop pole is at $s = -1 - K_p$, compared to the open-loop pole at s = -1.

We can choose the closed-loop pole position by choosing K_p, though only along a constrained path (e.g. a locus). Depending on the value of proportional compensator, K_p, the locations of the closed-loop poles are along a constraint path (e.g.: a locus).

c. Give the transfer function equation for the closed loop system, these locations of poles are calculated from:

$$T(s) = \frac{K_p}{s + (1 + K_p)}$$

When value of K_p = 0, 1, 2, 3 and 4, these loci are:

$$K_p = 0 \quad \rightarrow \quad s = -1$$

$$K_p = 1 \quad \rightarrow \quad s = -2$$

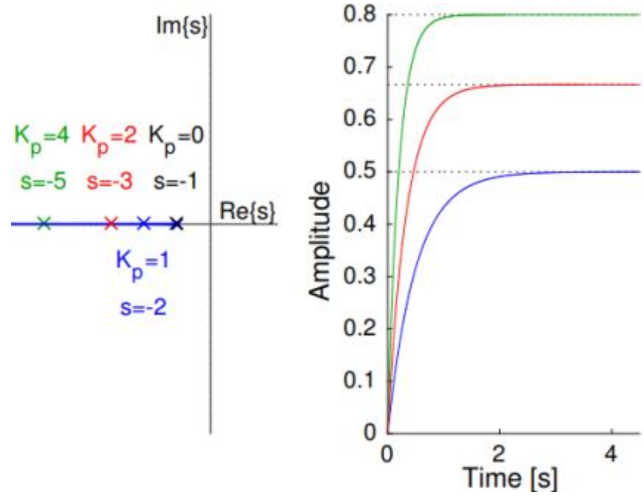
$$K_p = 2 \quad \rightarrow \quad s = -3$$

$$K_p = 4 \quad \rightarrow \quad s = -5$$

d. The following figures show the root locus diagram (left) and the transient response plot (right) of system $G(s) = 1/(s+1)$.

In the root locus diagram, notice that the pole of the system is moving around several locations in the s-plane along a series of loci in the diagram depending value of K_p e.g. (-1, 0) when K_p is 0, (-2, 0) when K_p is 1, (-3,0) when K_p is 2, and so forth.

As shown in the transient response plot, increasing the value of K_p makes the amplitude in the response to settle at higher values. It is shown in the diagram that when K_p is 1, the response settles at 0.5, when K_p is 2, the response settles at 0.67, and when K_p is 4, the response settles at 0.8.



Example for Tutorial 2: Root Locus Diagram of Second-Order System

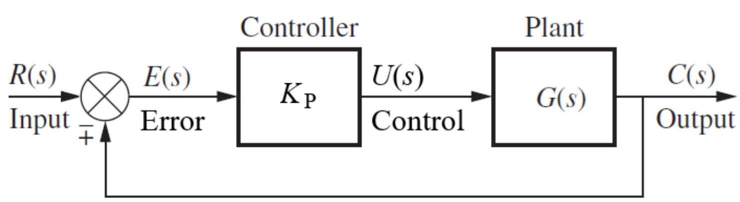
Considering a DC motor described as a second-order transfer-function equation shown below, let us add a proportional controller $G_C(s) = K_p$ to this plant.

$$G(s) = \frac{1}{s(s+1)}$$

a. Determine the transfer function equation of the closed-loop system. [4 marks]

b. Derive the expression that could be used for determining the location of the poles in the closed-loop system. [6 marks]

c. Simulate the system in MATLAB. Comment on the results of the simulation. [8 marks]



Answer

a. The transfer function equation of the closed-loop system is derived as follows.

$$T(s) = \frac{\left[\frac{K_p}{s(s+1)}\right]}{1 + \left[\frac{K_p}{s(s+1)}\right]} = \frac{K_p}{s^2 + s + K_p}$$

b. Now, the closed-loop poles are located where the denominator is equal to zero. Thus, we find them by solving $s^2 + s + K_p = 0$, or (more generally) by solving the characteristic equation, $1 + G_C G_P = 0$. So, let us find the roots of $s^2 + s + K_p = 0$ using the quadratic equation.

$$s = \frac{-1 \pm \sqrt{1 - 4K_p}}{2}$$

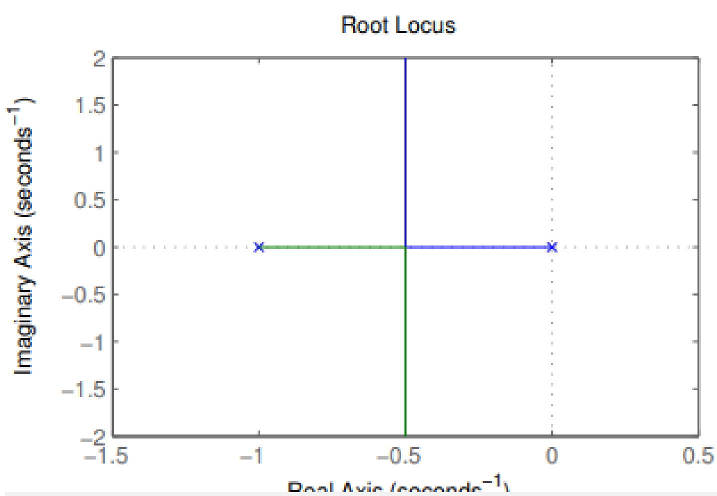
For K_p < 1/4, we get two real roots poles located at:

$$s = -\frac{1}{2} \pm \frac{\sqrt{1 - 4K_p}}{2}$$

For K_p > 1/4, we get a pair of complex roots poles located at:

$$s = -\frac{1}{2} \pm j\frac{\sqrt{1 - 4K_p}}{2}$$

c. The following figure shows the root-locus diagram of system $G(s) = 1/s(s+1)$.



Notice that there are two loci of the poles of the system as shown in the diagram. One locus starts from 0 and settles down to +∞ (e.g.: blue line) and the other starts from -1 and settles down to -∞ (e.g. green line).

- For K_p < 1/4, we have the system as a first order as the poles moves along the x-axis.
- For K_p > 1/4, we change from a first order to a second order (oscillatory) response as the poles become complex.

4. Design using Root Locus

We have seen the use of root locus for analysing the control systems. With its rich features for evaluating the control systems, this method is also used for designing the control systems.

4.1. Introduction to Design with Root Locus

Let us imagine that we have been asked to design a controller for the dc motor so that it has a settling time of less than 20 seconds and a damping ratio better than ζ = 0.707.

As we need $T_s = 4/\sigma < 20$ the dominant pole must be further left than s = -0.2 (e.g.: σ = 4/20 = 0.2).

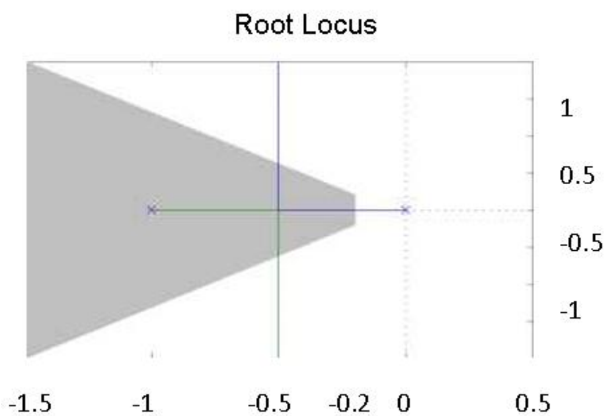


Figure 13: System with settling time $T_s < 20$ and damping ratio ζ > 0.707

Damping ratio, ζ > 0.707 requires that the angle from the negative real axis be no greater than $\theta = \cos^{-1} 0.707 = 45°$. In this example there are a range of possible K values that would satisfy the design specification.

Let us work out the minimum and maximum gains that would be acceptable. The dominant pole crosses into the allowed region at s = -0.2. We know that while the roots are real, they are located at:

$$s = -\frac{1}{2} \pm \frac{\sqrt{1 - 4K_p}}{2}$$

So, in this case:

$$s = -0.5 \pm \frac{\sqrt{1 - 4K_p}}{2} = -0.2$$

Solving the equation given above:

$$\frac{\sqrt{1 - 4K_p}}{2} = 0.3$$

As a result, the position error constant of the system is:

$$K_p = 0.16$$

By inspection of the diagram, we can see that the poles leave the allowed region at $s = -0.5 \pm j0.5$. These points are the maximum values before that meet the design specification of the system.

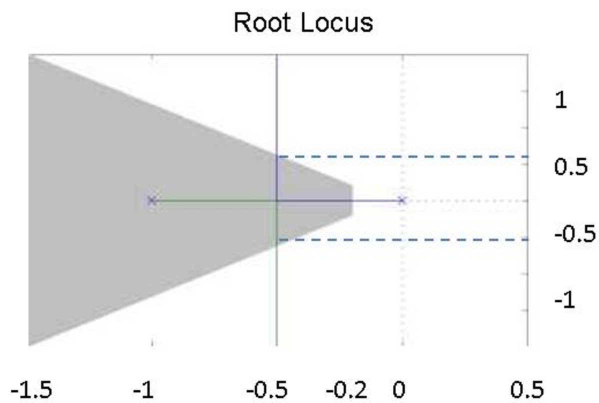


Figure 14: Maximum values of system with $T_s < 20$ and ζ > 0.707

We previously calculated that the complex poles are located at:

$$s = -\frac{1}{2} \pm j\frac{\sqrt{4K_p - 1}}{2}$$

So, σ = -0.5 and again we can equate to find the value of K_p on the boundary:

$$\frac{\sqrt{4K_p - 1}}{2} = 0.5$$

So, the position error constant of the system is:

$$K_p = 0.5$$

So, we have found that the design specification will be met for 0.16 < K_p < 0.5.

The graph shows the step responses of the system when $K_p = 0.16$ and $K_p = 0.5$.

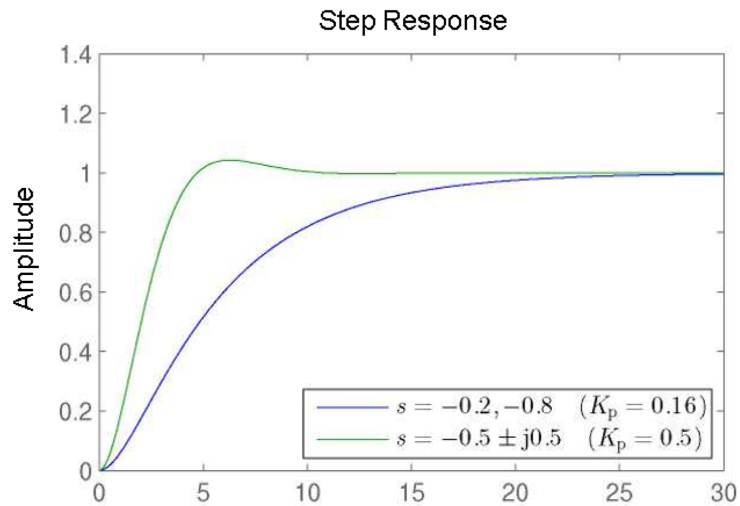


Figure 15: Step response of the given example system

4.2. The Evan's Root Locus

We can always determine the root locus using mathematical analysis as in the examples above.

However, this becomes tedious as the number of poles increases.

The Evan's root locus is a graphical technique that automates the mathematics to provide a method to draw the locus directly.

Examination of the root locus allows us to:

a. Determine the stability of a system as gain changes.

b. Choose an appropriate gain to produce a desired closed-loop response.

c. Modify the form of $C(s)$ if an adequate closed-loop response cannot be achieved.

4.3. Root Locus Formalities

The root locus is a pole zero diagram that shows the "tracks" taken by the system poles as some parameter (gain in our case) is varied.

Formally, the root locus shows the locus traced out by the roots of the characteristic equation, $1 + G_C(s)G_P(s) = 0$, as the gain is varied.

To find a root locus, we are therefore searching for values of s that satisfy the characteristic equation:

$$1 + G_C(s)G_P(s) = 0$$

Rearrange the equation:

$$G_C(s)G_P(s) = -1$$

Thus

$$G_C(s)G_P(s) = 1\angle(2k+1)180° \quad \text{for} \quad k \in \mathbb{Z}$$

4.4. Examples of Root Locus Diagrams

In this section, we will cover some root locus diagram simulations of various systems in MATLAB and their analysis.

Example for Tutorial 3: Root Locus Diagram 1

For a system with the following transfer function:

$$G(s) = \frac{(s+3)(s+4)}{(s+1)(s+2)}$$

a. Determine the poles and zeros of the system. [2 marks]

b. Simulate the root locus of the system in MATLAB. [6 marks]

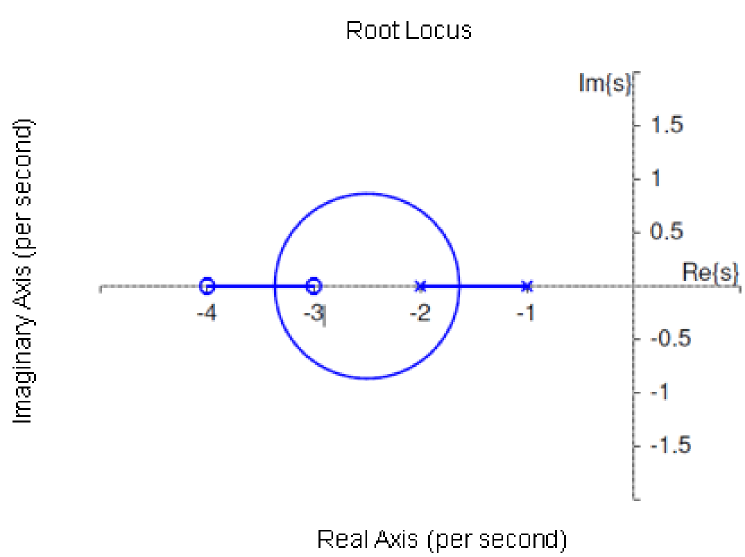
c. Describe the characteristics of the root locus in the s-plane diagram. [4 marks]

Answer

a. The system has the following poles and zeros:

- Zeros: -3 and -4.
- Poles: -1 and -2.

b. The following figure shows the root locus diagram of the system as simulated in MATLAB.



c. There are two root loci in the s-plane diagram:

- From the pole (-1, 0), going up to quadrant 2 in the s-plane, and settles down to the zero (-3, 0).
- From the pole (-2, 0), going down to quadrant 3 in the s-plane, and settles down to the zero (-4, 0).

Example for Tutorial 4: Root Locus Diagram 2

For a system with the following transfer function:

$$G(s) = \frac{s+3}{s(s+1)(s+2)(s+4)}$$

a. Determine the poles and zeros of the system. [2 marks]

b. Simulate the root locus of the system in MATLAB. [6 marks]

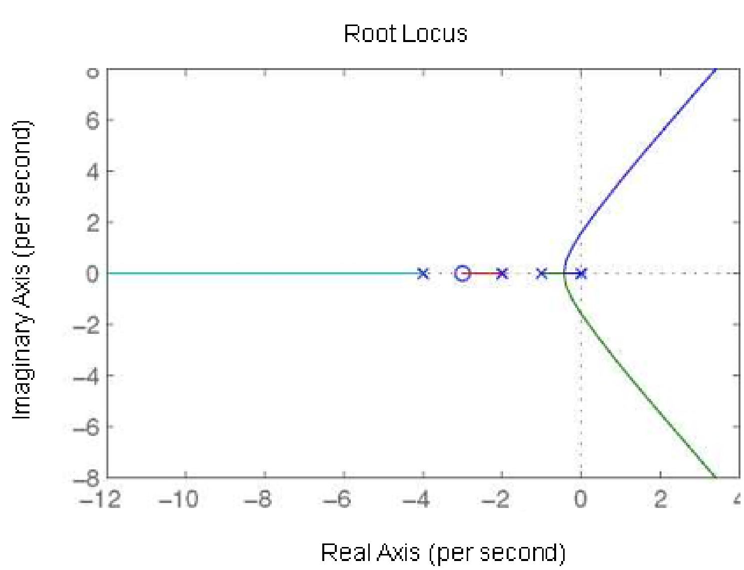
c. Describe the characteristics of the root locus in the s-plane diagram. [4 marks]

Answer

a. The system has the following poles and zeros:

- Zeros: -3.
- Poles: 0, -1, -2, and -4.

b. The following figure shows the root locus diagram of the given system as simulated in MATLAB.



c. There are four root loci in the s-plane diagram:

- Blue line: From the pole at the origin (0, 0), going up to quadrant 1 in the s-plane, and settles down to the first asymptote (+∞,+∞).
- Green line: From the pole (-1, 0), going down to quadrant 4 in the s-plane, and settles down to the second asymptote (+∞, -∞).
- Brown line: From the pole (-2, 0) and settles down along the x-axis in the s-plane to the zero (-3, 0).
- Light blue line: From the pole (-4, 0) and settles down to the third asymptote along the x-axis in the s-plane (-∞, 0).

Example for Tutorial 5: Root Locus Diagram 3

For a system with the following transfer function:

$$G(s) = \frac{s^2 - 4s + 24}{(s+2)(s+4)}$$

a. Determine the poles and zeros of the system. [2 marks]

b. Simulate the root locus of the system in MATLAB. [6 marks]

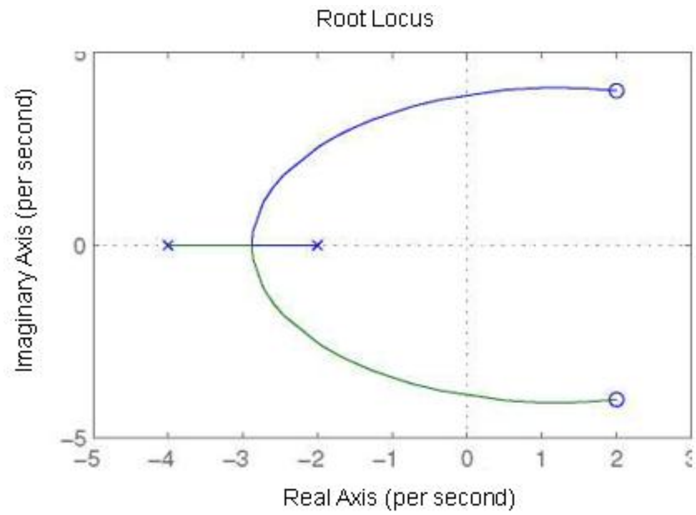
c. Describe the characteristics of the root locus in the s-plane diagram. [4 marks]

Answer

a. The system has the following poles and zeros.

- Zeros: $2 \pm j4$.
- Poles: -2 and -4.

b. The following diagram shows the root locus diagram of the system as simulated in MATLAB.



c. There are two root loci in the s-plane diagram:

- Blue line: From the pole (-2, 0), going up to quadrant 2 in the s-plane, and settles down to the zero (2, 4).
- Green line: From the pole (-4, 0), going down to quadrant 3 in the s-plane, and settles down to the zero (2, -4).

Example for Tutorial 6: Root Locus Diagram 4

For a system with the following transfer function:

$$G(s) = \frac{1}{s[(s+4)^2 + 16]}$$

a. Determine the poles and zeros of the system. [2 marks]

b. Simulate the root locus of the system in MATLAB. [6 marks]

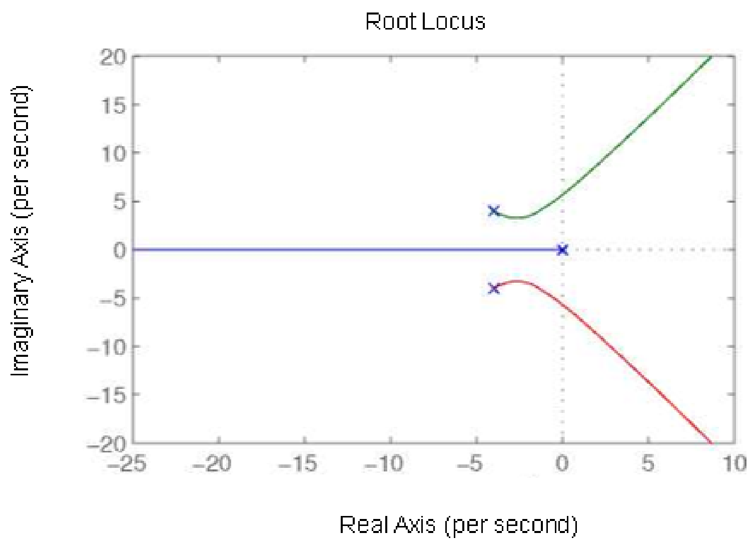
c. Describe the characteristics of the root locus in the s-plane diagram. [4 marks]

Answer

a. The system has the following poles and zeros:

- Zeros:
- Poles: 0 and -4 $\pm$ j4.

b. The following diagram shows the root locus diagram of the system as simulated in MATLAB.



c. There are three root loci in the s-plane diagram:

- Green line: From the pole (-4, 4), it goes up to quadrant 2 in the s-plane and settles down to the first asymptote (+∞, +∞).
- Brown line: From the pole (-4, -4), it goes down to quadrant 3 on the s-plane and settles down to the second asymptote (+∞, -∞).
- Blue line: From the pole at the origin (0, 0), it goes along the x-axis in the s-plane and settles down to third asymptote (-∞, 0).

Example for Tutorial 7: Root Locus Diagram 5

For a system with the following transfer function:

$$G(s) = \frac{1+s}{s^2}$$

a. Determine the poles and zeros of the system. [2 marks]

b. Simulate the root locus of the system in MATLAB. [6 marks]

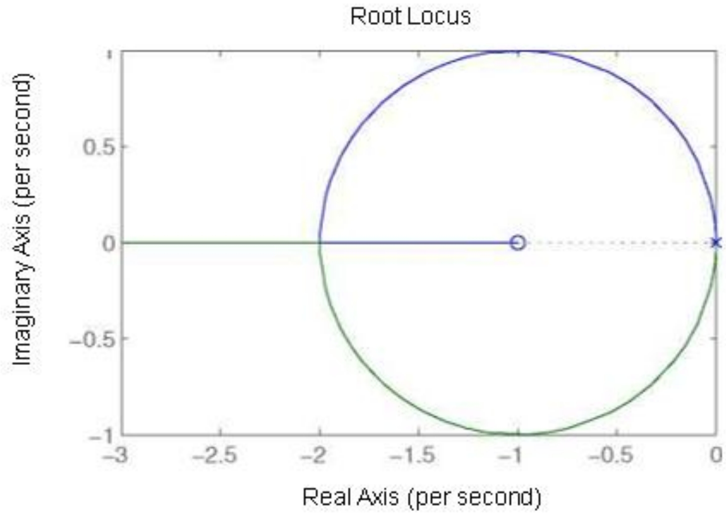
c. Describe the characteristics of the root locus in the s-plane diagram. [4 marks]

Answer

a. The system has the following poles and zeros:

- Zeros: -1.
- Poles: -0 (double poles at origin).

b. The following diagram shows the root locus diagram of the system as simulated in MATLAB.



c. There are two root loci in the s-plane diagram:

- Blue line: From the pole at the origin (0, 0), going up to quadrant 2 in the s-plane, and settles down to the zero (-1, 0).
- Green line: From the pole at the origin (0, 0), going down to quadrant 3 in the s-plane, and settles down along x-axis to an asymptote (-∞, 0).

Example for Tutorial 8: Root Locus Diagram 6

For a system with the following transfer function:

$$G(s) = \frac{1}{s(s+2)[(s+1)^2+4]}$$

a. Determine the poles and zeros of the system. [2 marks]

b. Simulate the root locus of the system in MATLAB. [6 marks]

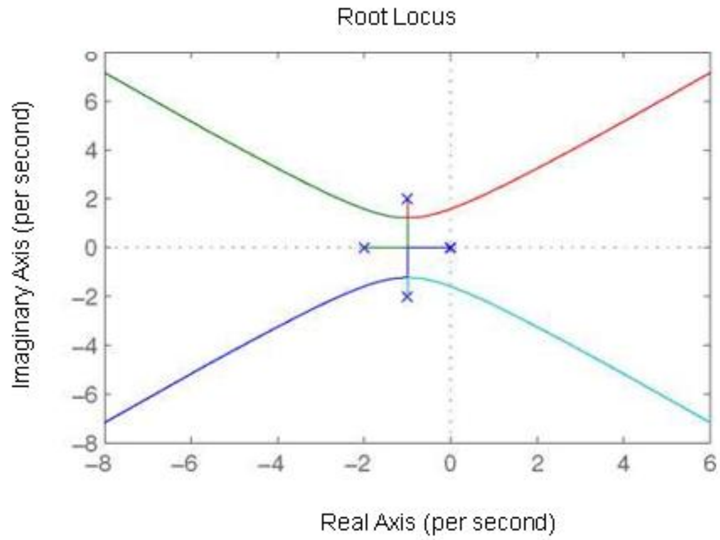
c. Describe the characteristics of the root locus in the s-plane diagram. [4 marks]

Answer

a. The system has the following poles and zeros:

- Zeros: None
- Poles: 0, -2, and -1 ± j2.

b. The following diagram shows the root locus diagram of the system as simulated in MATLAB.



c. There are four root loci in the s-plane diagram:

- Blue line: From the pole at the origin (0, 0), going down to quadrant 3 in the s-plane, and settles down to the first asymptote (-∞, -∞).
- Brown line: From the pole (-1, 2), going down in quadrant 2 in the s-plane, and settles down to the second asymptote (+∞, +∞).
- Light blue line: From the pole (-1, -2), going up in quadrant 3 in the s-plane, and settles down to the third asymptote (+∞, -∞).
- Green line: From the pole (-2, 0), going up to quadrant 2 in the s-plane, and settles down to the fourth asymptote (-∞, +∞).

Example for Tutorial 9: Root Locus Diagram 7

For a system with the following transfer function:

$$G(s) = \frac{s^2 + 2s + 4}{s(s+4)(s+6)(s^2 + 1.4s + 1)}$$

a. Determine the poles and zeros of the system. [2 marks]

b. Simulate the root locus of the system in MATLAB. [6 marks]

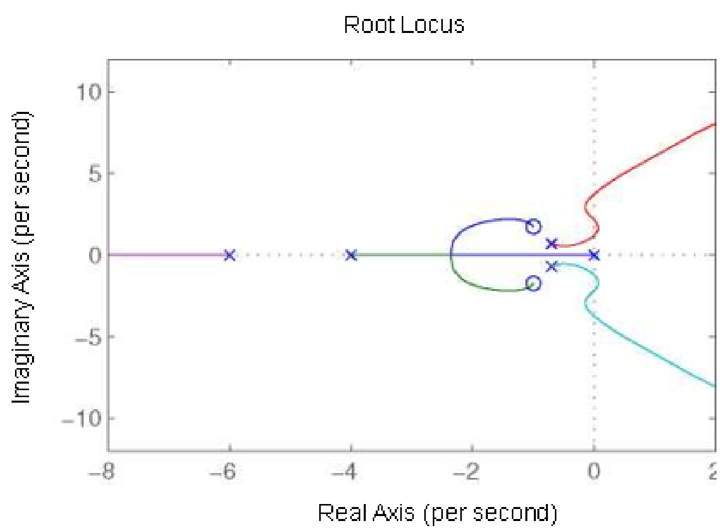
c. Describe the characteristics of the root locus in the s-plane diagram. [4 marks]

Answer

a. The system has the following poles and zeros:

- Zeros: -1 ± j$\sqrt{3}$.
- Poles: 0, -4, -6, and -0.7 ± j$\sqrt{0.51}$.

b. The following diagram shows the root locus diagram of the system as simulated in MATLAB.



c. There are five root loci in the s-plane diagram:

- Blue line: From the pole at the origin (0, 0), going up to quadrant 2 in the s-plane, and settles down to the zero (-1, +j$\sqrt{3}$).
- Green line: From the pole (-4, 0), going down to quadrant 3 in the s-plane, and settles down to the zero (-1, -j$\sqrt{3}$).
- Brown line: From the pole (-0.7 + j$\sqrt{0.51}$), going up in quadrant 2 in the s-plane, and settles down to the first asymptote (+∞, +∞).
- Light blue line: From the pole (-0.7 - j$\sqrt{0.51}$), going down in quadrant 3 in the s-plane, and settles down to second asymptote (+∞, -∞).
- Purple line: From the pole (-6, 0), going along the x-axis in the s-plane, and settles down to the third asymptote (-∞, 0).

Things to Notice

These are the guidelines for construction of root locus diagram:

1. Each branch of the root locus begins at an open-loop pole.

2. Each branch of the root locus either terminates at a zero or goes to (complex) infinity.
3. One and only one branch leaves each pole.
4. One and only one branch enters each zero.
5. Like any pole-zero diagram, the root locus is always symmetric about the real axis (and complex poles always come in conjugate pairs).