XMUT315 Control Systems Engineering

Note 12a: Introduction to Nyquist Diagram

Topic

- Fundamentals of Nyquist diagram.
- From Bode to Nyquist diagram.
- Examples of Bode to Nyquist diagram.
- From Pole-Zero to Nyquist diagram.
- Examples of Pole-Zero to Nyquist diagram.
- Nichols chart.

1 Introduction to Nyquist Diagram

We have been using the Bode and Root Locus plots of an open loop system to determine stability. However, as we discussed earlier, this method is only reliable for simple cases.

The Nyquist diagram provides a simple, universal method for assessing the stability of SISO systems. It works for simple systems that are manageable with a Bode plot, but also for more complicated systems.

The root locus and Routh-Hurwitz techniques also provide methods for determining stability, but the Nyquist diagram has the advantage of being applicable when you don't have a mathematical description of your system (you can use it on experimental data).

1.1. The Nyquist Diagram

A Nyquist diagram is a plot of the real part vs the imaginary part of an open-loop transfer function. Equivalently, you can think of it as a polar plot of a transfer function.

Freq $\boldsymbol{\omega}$	**Gain $\lvert\boldsymbol{G(\omega)}\rvert$**	**Phase $\angle\boldsymbol{G(\omega)}$**
ω_1	$\lvert G(\omega_1)\rvert$	$\angle G(\omega_1)$

ω_2	$\|G(\omega_2)\|$	$\angle G(\omega_2)$

Table 1: Gain and phase values of frequency response

The graph's axes are linear (not logarithmic), which makes the plot awkward for visualising the entire behaviour of a system that has high gain regions. The Nyquist diagram is specialised for considering system stability, as it focuses on the low gain region (i.e.: the region near unity gain).

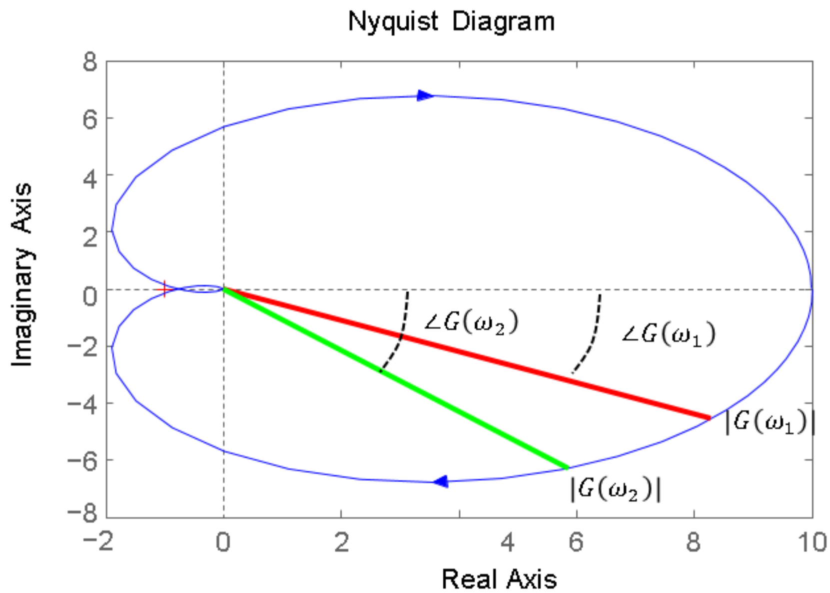


Figure 1: Nyquist diagram of a given control system

We will examine three ways to construct a Nyquist diagram:

- Based on a given root locus diagram.
- Using a Bode plot or frequency response.
- Directly on the diagram.

1.2. Stability from the Nyquist Diagram

Recall that a system is stable if and only if it has no poles in the right half of the s-plane. We seek a method that will tell us whether a system will be stable, once we enclose it in a feedback loop - we want to know about the stability of a closed-loop system.

The Nyquist plot is a graphical technique that enables us to determine whether a system has closed-loop poles in the right half of the s-plane by examining its open-loop poles. In fact, it does more e.g. it counts the number of closed-loop poles in the right-half plane.

2. Construction of Nyquist Diagram

In this section, we cover the topics on how to create Nyquist diagram. We will have a look first on how to construct it from the Root Locus, and later we will see how to do that from Bode plots.

2.1. From Root Locus to Nyquist

For the system in the figure above, the Nyquist criterion can tell us how many closed-loop poles are in the right half-plane.

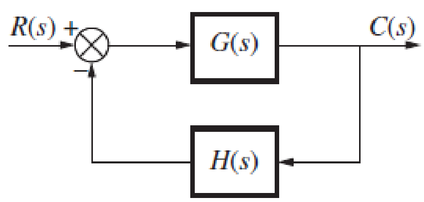


Figure 2: Root locus of systems and their relevant Nyquist diagrams

Four important concepts that will be used during the derivation of criteria:

- the relationship between the poles of $1 + G(s)H(s)$ and the poles of $G(s)H(s)$.
- the relationship between the zeros of $1 + G(s)H(s)$ and the poles of the closed-loop transfer function, $T(s)$.
- The concept of *mapping* points.
- The concept of mapping *contours*.

Given the transfer functions of feedback control system:

$$G(s) = \frac{N_G}{D_G} \qquad \text{and} \qquad H(s) = \frac{N_H}{D_H}$$

Thus

$$G(s)H(s) = \frac{N_G N_H}{D_G D_H}$$

Then

$$1 + G(s)H(s) = 1 + \frac{N_G N_H}{D_G D_H} = \frac{D_G D_H + N_G N_H}{D_G D_H}$$

As a result, the transfer function of closed loop system is:

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_G D_H}{D_G D_H + N_G N_H}$$

We conclude that:

(1) *Poles of* $1 + G(s)H(s)$ *are the same as the poles of* $G(s)H(s)$*, the open-loop system.*

(2) *Zeros of* $1 + G(s)H(s)$ *are the same as the poles of* $T(s)$*, the closed-loop system*.

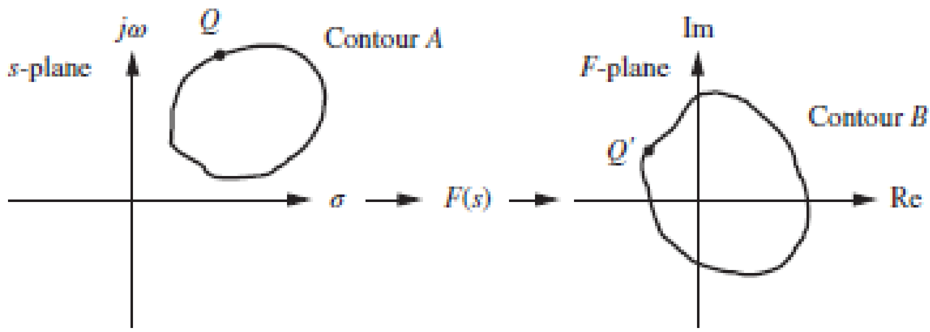


Figure 3: Mapping contour from s-plane to F-plane

Finally, we discuss the concept of mapping *contours*. Consider the collection of points, called a *contour*, shown in Figure above as contour A. Also, assume that

$$F(s) = \frac{(s - z_1)(s - z_2) \ldots}{(s - p_1)(s - p_2) \ldots}$$

Contour *A* can be mapped through $F(s)$ into contour *B* by substituting each point of contour *A* into the function $F(s)$ and plotting the resulting complex numbers. For example, point *Q* in the figure above maps into point *Q´* through the function $F(s)$.

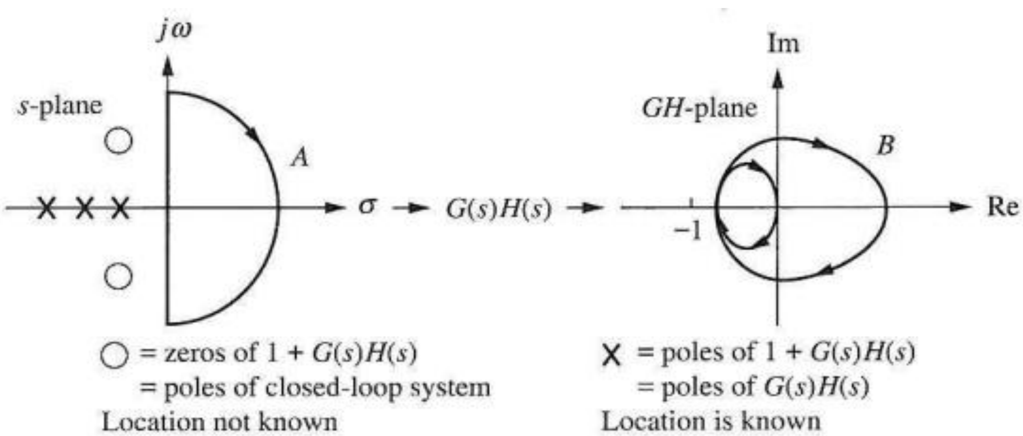


Figure 4: Stable system contour mapping from s-plane to GH-plane

Figure above shows a contour *A* that does not enclose closed-loop poles, that is, the zeros of $1 + G(s)H(s)$. The contour thus maps through $G(s)H(s)$ into a Nyquist diagram that does not encircle the test point (-1,0). Hence, $P = 0$; $N = 0$, and $Z = P - N = 0$. Since Z is the number of closed-loop poles inside contour *A*, which encircles the right half-plane, this system has no right-half-plane poles and is stable.

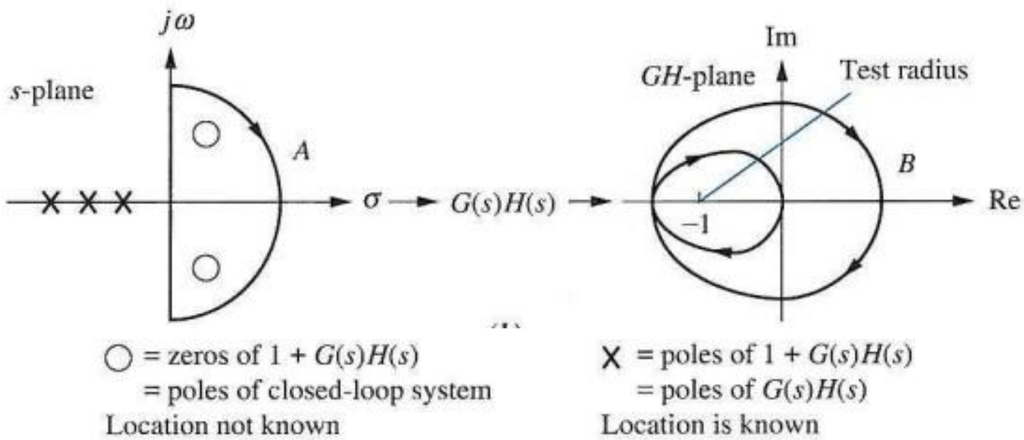


Figure 5: Unstable system contour mapping from s-plane to GH-plane

Figure above shows a contour *A* that, while it does not enclose open-loop poles, does generate two clockwise encirclements of the test point (-1,0). Thus, $P = 0$; $N = 2$, and the system is unstable; it has two closed-loop poles in the right half-plane, since $Z = P - N = 2$. The two closed-loop poles are shown inside contour *A* in the figure above as zeros of $1 + G(s)H(s)$.

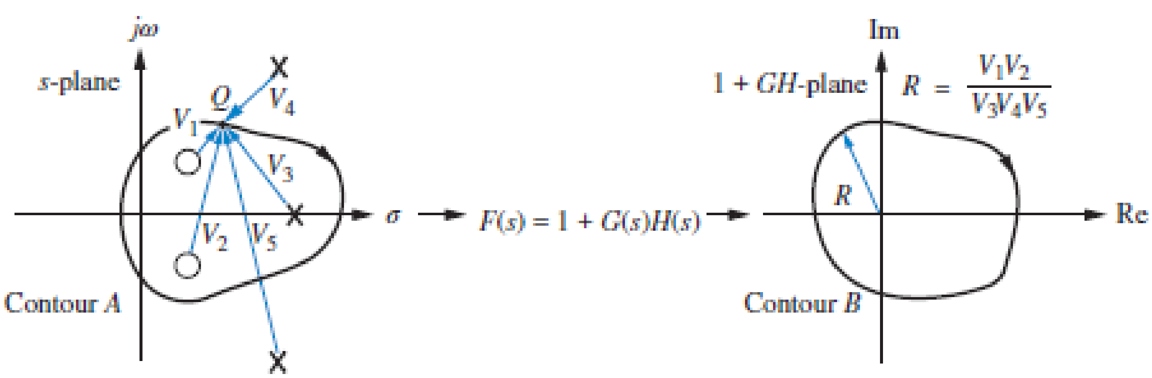


Figure 6: Vector mapping from s-plane to 1+GH-plane

Thus, number of counterclockwise rotations of contour B about the origin is:

$$N = P - Z$$

Where:

P - Number of poles of $1 + G(s)H(s)$ inside contour *A*.

Z - Number of zeros of $1 + G(s)H(s)$ inside contour *A*.

Nyquist stability criterion is as follows:

*If a contour, A, that encircles the entire right half-plane is mapped through $G(s)H(s)$, then the number of closed-loop poles, Z, in the right half-plane equals the number of open-loop poles, P, that are in the right half-plane minus the number of counterclockwise revolutions, N, around the testing point (*1,0*) of the mapping; that is, $Z = P - N$. The mapping is called the Nyquist diagram, or Nyquist plot, of $G(s)H(s)$.*

Example for Tutorial 1: Constructing Root Locus Diagram

Given a first-order system with its transfer function equation:

$$G(s) = \frac{1}{s + 0.1}$$

a. Derive the real and imaginary equations needed for sketching the Nyquist diagram. [4 marks]

b. Using equations derived in part (a), calculate the points required for sketching the Nyquist diagram. [4 marks]

c. Sketch the Nyquist diagram of the system. [6 marks]

d. Simulate the root locus diagram of the system in MATLAB. By determining and obtaining values of the points in the diagram required for sketching Nyquist diagram, convert the root locus diagram to Nyquist diagram. [12 marks]

Answer

a. Substituting $s = j\omega$, the transfer function equation of the system becomes:

$$G(j\omega) = \frac{1}{j\omega + 0.1} = \left(\frac{1}{j\omega + 0.1}\right)\left(\frac{j\omega - 0.1}{j\omega - 0.1}\right) = -\left(\frac{j\omega - 0.1}{\omega^2 + 0.01}\right)$$

For sketching the Nyquist diagram, we need the following equations for determining the points in the Nyquist diagram:

- The real part of the complex equation:

$$\text{Re}\{G(j\omega)\} = \frac{0.1}{\omega^2 + 0.01}$$

- The imaginary part of the complex equation:

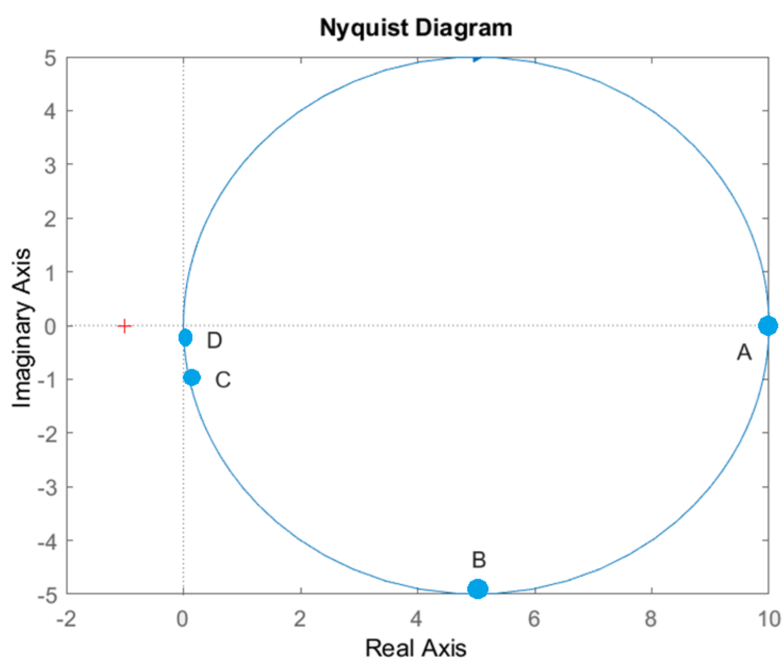
$$\text{Im}\{G(j\omega)\} = -\left(\frac{j\omega}{\omega^2 + 0.01}\right)$$

b. Choose the frequencies from 0 to $+\infty$ along the y-axis (imaginary axis) e.g. A = 0 rad/s, B = 0.1 rad/s, C = 1 rad/s, and D = 10 rad/s.

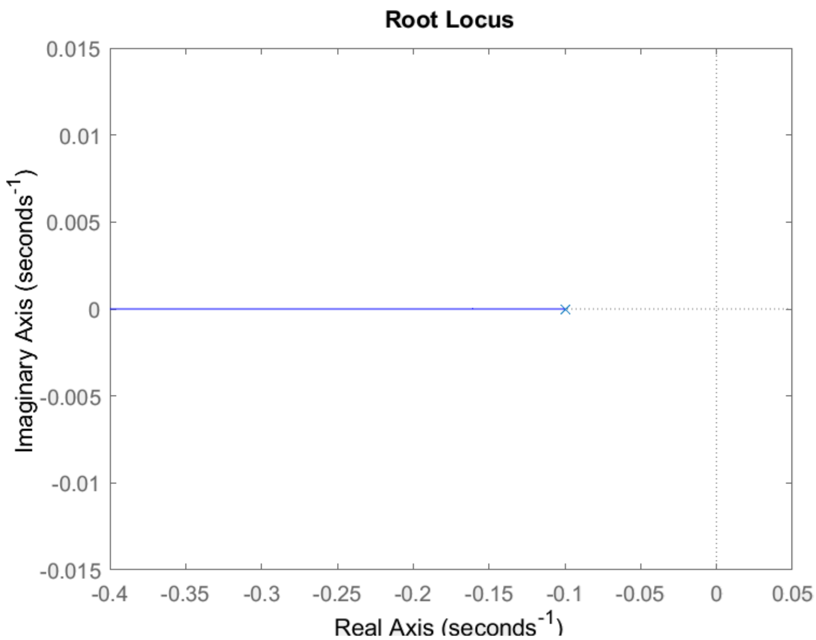
The points for sketching the Nyquist diagram are calculated and tabulated in the following table.

Points	$\boldsymbol{\omega}$	$\textbf{Re}\{\boldsymbol{G(j\omega)}\}$	$\textbf{Im}\{\boldsymbol{G(j\omega)}\}$
A	0	$\frac{0.1}{(0)^2 + 0.01} = 10$	$-\left(\frac{j(0)}{(0)^2 + 0.01}\right) = 0$
B	0.1	$\frac{0.1}{(0.1)^2 + 0.01} = 5$	$-\left(\frac{j(0.1)}{(0.1)^2 + 0.01}\right) = -5$
C	1	$\frac{0.1}{(1)^2 + 0.01} = 0.099$	$-\left(\frac{j(1)}{(1)^2 + 0.01}\right) = -0.99$
D	10	$\frac{0.1}{(10)^2 + 0.01} = 0.001$	$-\left(\frac{j(10)}{(10)^2 + 0.01}\right) = -0.099$

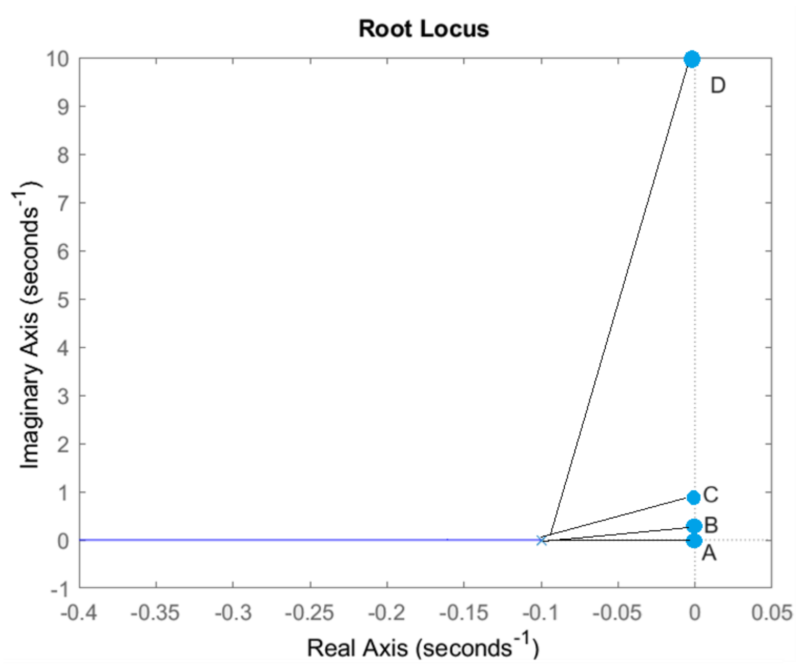
c. Based on the points listed in the table in part (b), the following diagram shows the sketched Nyquist diagram.



d. The result of root locus diagram simulation of the system in MATLAB is shown in the figure below.



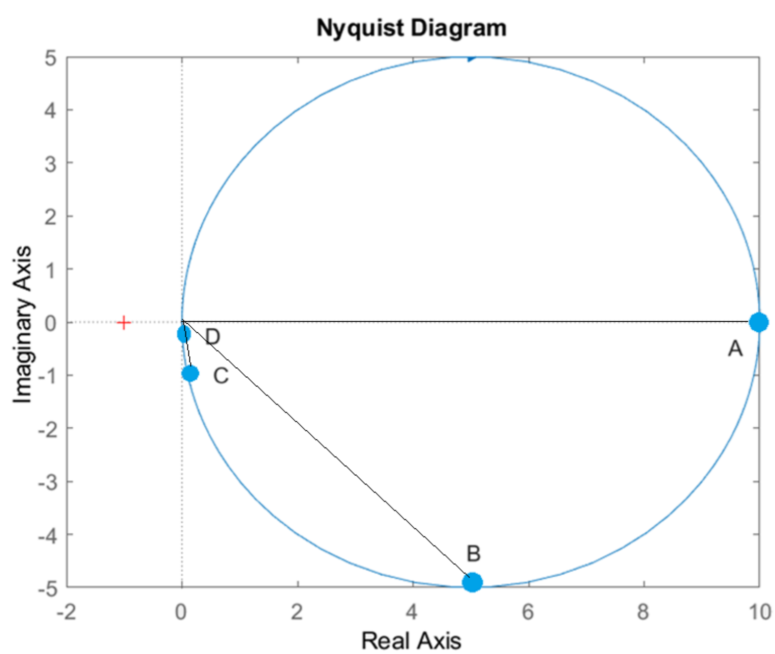
Choose the frequencies from 0 to $+\infty$ along the y-axis (imaginary axis) e.g. A = 0j, B = 0.1j, C = 1j, and D = 10j.



Calculate the magnitudes and angles formed by the zero at (-1, 0) with the chosen points in the diagram.

Points	ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
A	0	$\frac{1}{\sqrt{(0.1)^2+(0)^2}} = 10$	$-\tan(0/0.1) = 0°$
B	0.1	$\frac{1}{\sqrt{(0.1)^2+(0.1)^2}} = 7.07$	$-\tan\left(\frac{0.1}{0.1}\right) = -45°$
C	1	$\frac{1}{\sqrt{(0.1)^2+(1)^2}} = 0.99$	$-\tan\left(\frac{1}{0.1}\right) = -84.29°$
D	10	$\frac{1}{\sqrt{(0.1)^2+(10)^2}} = 0.01$	$-\tan(10/0.1) = -89.43°$

Sketch the Nyquist diagram based on the magnitudes and angles obtained above.



2.2. From Bode to Nyquist

The curve on a Nyquist diagram can be determined by choosing gain-phase points from a Bode plot at multiple frequencies. Thus, the Nyquist diagram is the locus traced out by the transfer function as we vary frequency. Thus, the frequency varies along the Nyquist curve, but not in a regular way. There is no way to use a Nyquist diagram to determine the frequency at which something happens.

To construct the Nyquist diagram, choose the points where something “interesting” happens on the Bode plot and transfer them to the Nyquist diagram. Join the points with sensible curves. This produces a plot of gain vs. phase as the frequency varies from 0 to ∞. Drawing a Nyquist diagram is slightly more complex than plotting the gain vs phase curve for f = 0 → ∞.

Nyquist's criterion (see later) requires that we evaluate the transfer function as we traverse a clockwise path that completely encloses the right half of the s-plane. Evaluating the transfer function for f = 0 → ∞ corresponds to traversing the upper straight part of the semicircle shown in the diagram. As the pole-zero diagram must be symmetric we know that the Nyquist diagram must be symmetric in the section from f = −∞ → 0.

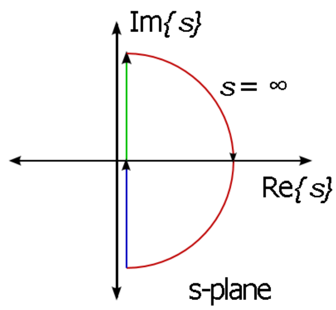


Figure 7: Plot of gain and phase as the frequency is varied in the s-plane

To draw the complete Nyquist diagram, we need to add to the plot that you produced by transferring data from the Bode plot. The section from -∞ to 0 is straightforward, as it is just the mirror image of the 0 to ∞ section that you have already drawn.

Most transfer functions that you will meet have the degree of the denominator larger than the numerator (they are strictly proper). A consequence of this is that the gain is infinitesimally small at infinite frequency. Thus, the response for the circular part of the contour from $\omega = \infty$ to $\omega = -\infty$ is always zero. The whole sweep is therefore mapped to the origin of the Nyquist diagram.

2.3. From Bode to Nyquist Example

In this section, we cover several case-study examples of the construction of Nyquist diagrams from given Bode plots.

Example for Tutorial 2: Nyquist Diagram of First-Order System

Consider a first-order system with the transfer function:

$$G(s) = \frac{100}{s + 10}$$

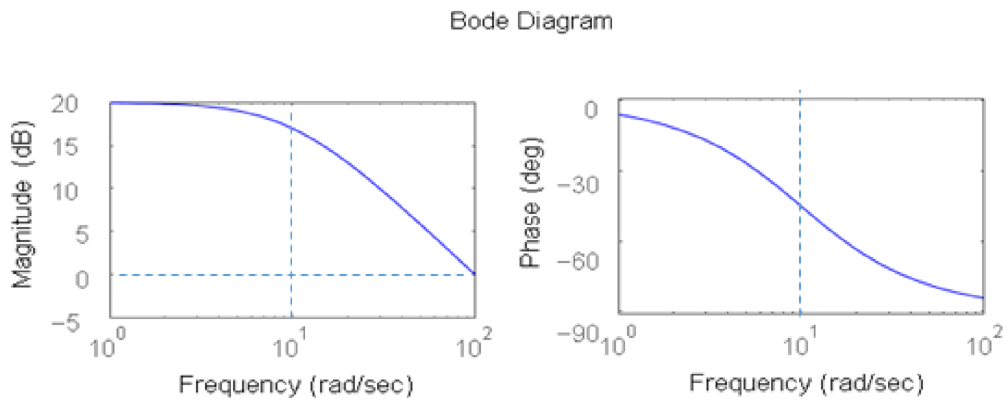
a. Simulate the Bode plots of the system the in MATLAB. [5 marks]

b. Determine the gain and phase of the frequency response of the system from the Bode plots for ω = 1, 10, and 100 rad/s. [6 marks]

c. Based on the results obtained in part (b), construct Nyquist diagram of the system. [5 marks]

Answer

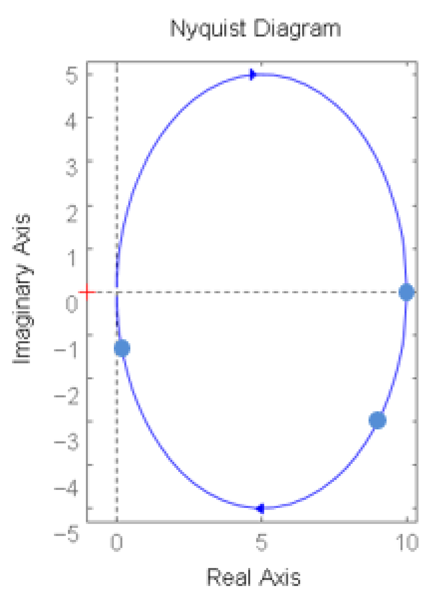
a. The Bode plots of the given example system is given in the following figure.



b. Selective gain and phase values from the Bode plots of the example system are listed in the table below.

$\boldsymbol{\omega}$	$\lvert \boldsymbol{G(\omega)} \rvert$	$\angle \boldsymbol{G(\omega)}$
1	20 dB (10)	0°
10	17 dB (7)	-45°
100	0 dB (1)	−90°

c. The resulting Nyquist diagram of the example system is given in the figure below.



Example for Tutorial 3: Nyquist Diagram of Second Order System

Consider a second-order system with the transfer function:

$$G(s) = \frac{100}{s^2 + 2s + 26}$$

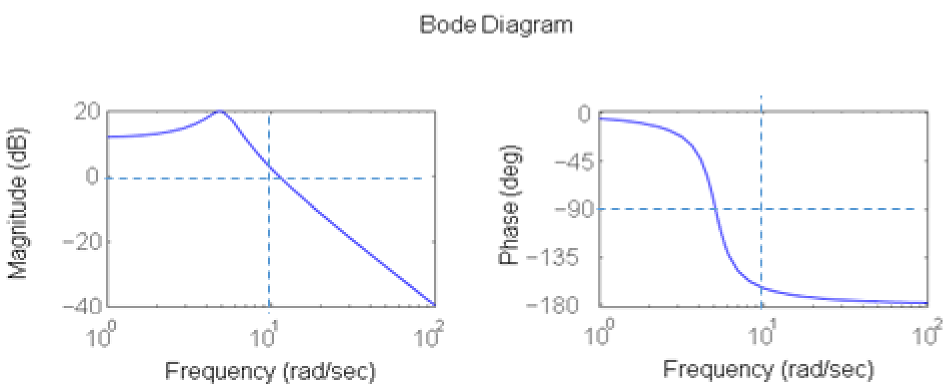
a. Simulate the Bode plots of the system the in MATLAB. [5 marks]

b. Determine the gain and phase of the frequency response of the system from the Bode plots for ω = 1, 5, 10, 50, and 100 rad/s. [10 marks]

c. Based on the results obtained in part (b), construct Nyquist diagram of the system. [5 marks]

Answer

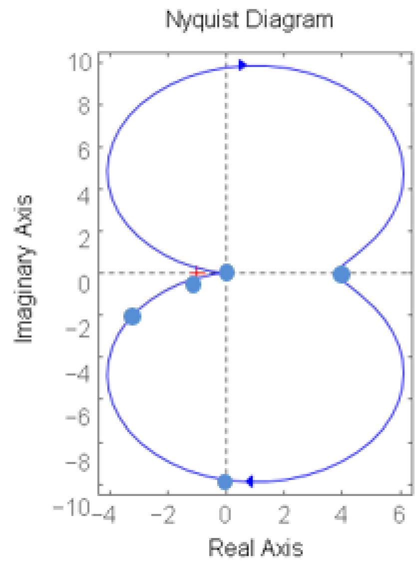
a. The Bode plots of the given example system $G(s)$ is given in the following figure.



b. Selective gain and phase values from the Bode plots of the example system $G(s)$ are listed in the table below.

ω	$\|G(\omega)\|$	$\angle G(\omega)$
1	12 dB (4)	5°
5	20 dB (10)	-90°
10	5 dB (1.77)	-165°
50	-30 dB (0.03)	-175°
100	-40 dB (0.01)	-180°

c. The resulting Nyquist diagram of the example system is given in the figure below.



Example for Tutorial 4 - Nyquist Diagram Analysis

Apply Nyquist criterion to determine the stability of the following feedback systems:

a. System (i): [5 marks]

$$G(s) = \frac{s + 20}{(s + 2)(s + 7)(s + 50)}$$

b. System (ii): [5 marks]

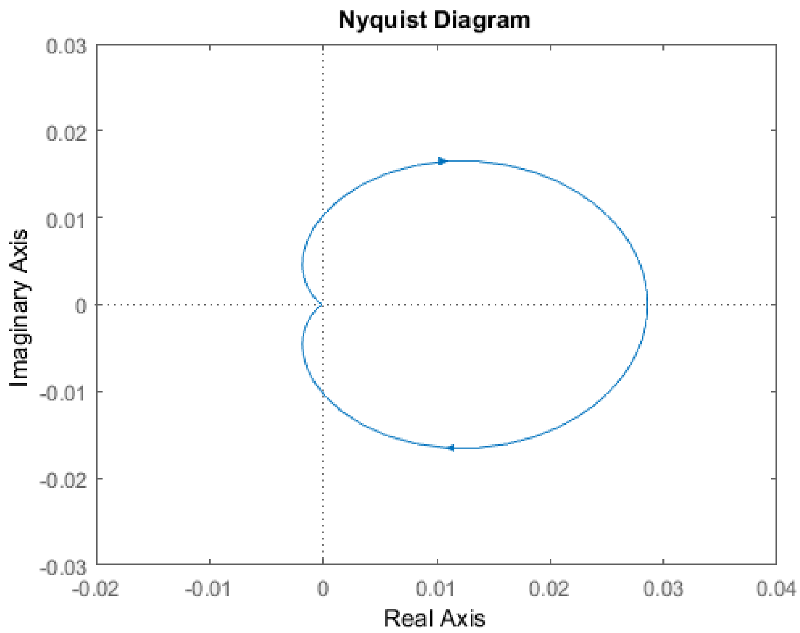
$$G(s) = \frac{s + 3}{(s + 2)(s^2 + 2s + 25)}$$

c. System (iii): [5 marks]

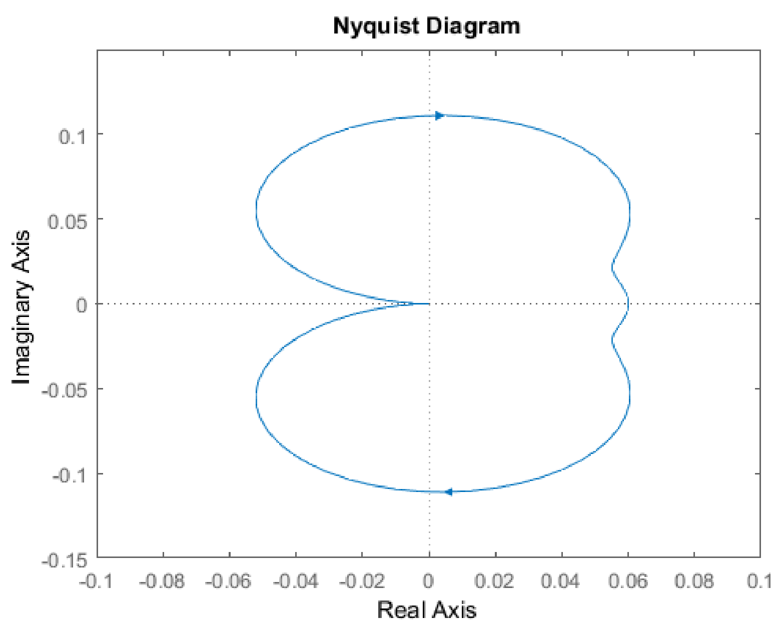
$$G(s) = \frac{500(s - 2)}{(s + 2)(s + 7)(s + 50)}$$

Answer

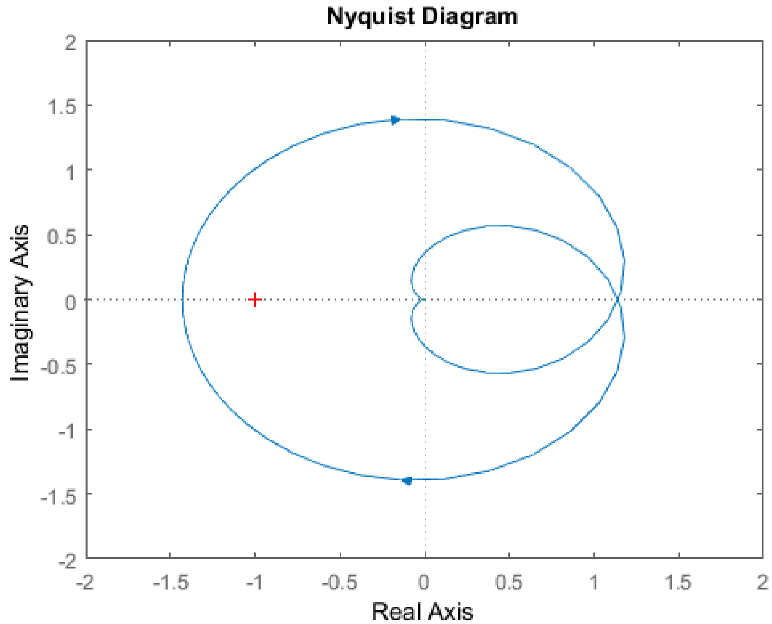
a. System (i) - stable system. For System (i), we have P = 0 (open loop stable system). The Nyquist diagram does not enclose (-1, j0), (N = 0). Thus, $Z = P - N = 0$. Systems (i) is stable since there are no closed loop poles in the right half plane. The following figure shows the simulation result using MATLAB of Nyquist diagram of system (i).



b. System (ii) - stable system with a pair of complex poles. For System (ii), we have P = 0 (open loop stable system). The Nyquist diagram does not enclose (-1, j0), (N = 0). Thus, $Z = P - N = 0$. Systems (ii) is stable since there are no closed loop poles in the right half plane. The following figure shows the simulation result using MATLAB of Nyquist diagram of system (ii).



c. System (iii) – Zero in the RHS of s-plane. For System (iii), we have P = 0 (open loop stable system), but N = -1. System (iii) is unstable with one closed-loop pole in the right-half plane. The following figure shows the simulation result using MATLAB of Nyquist diagram of system (iii).



4. Nichols Chart

Nichol's chart is a variant of the Nyquist diagram, and it is often used also for stability analysis of the control system.

4.1. Introduction to Nichols Chart

A Nichols chart displays the gain (in dB) plotted against the phase (in degrees) of the system response. Nichols charts are useful to analyse open- and closed-loop properties of single-input-single -output (SISO) systems, but offer little insight into multi-input-multi-output (MIMO) control loops.

As shown in the figure below, we can determine the stability of the system by evaluating the contour around the test point (-180°, 0) whether it encircles the test point or not, similar to the Nyquist diagram.

We can also determine the gain margin (GM) and phase margin (PM) of the system in the Nichols chart similar to the steps in Nyquist diagram.

The phase margin is determined as the phase difference from the test point to the intersection between the contour with x-axis in the Nichols chart. The example Nichols chart shows a gain margin of 35°.

The gain margin is calculated as the difference between the gain at the point of the contour at 180° and the x-axis. The gain margin in the Nichols chart is calculated as 15 dB.

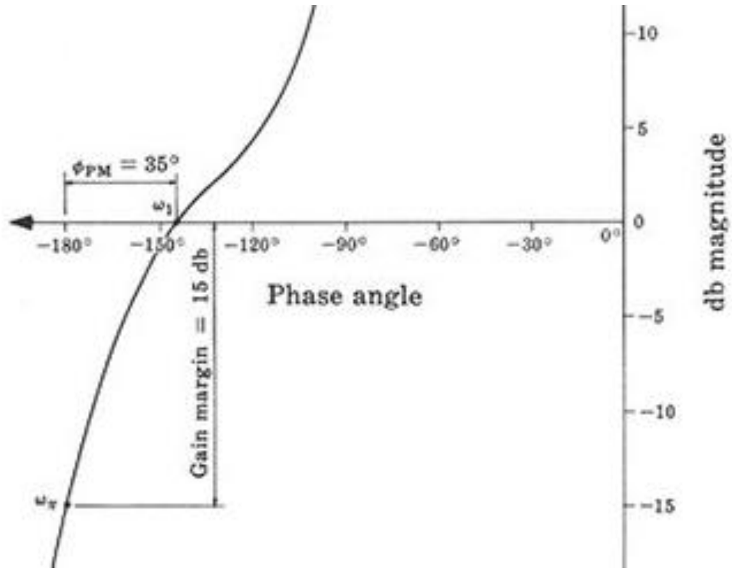


Figure 8: Nichol's chart of a control system

4.2. Nyquist Diagram vs. Nichols Chart

Comparing with the Nyquist chart, these are the pitfalls of the Nyquist diagram:

- Becomes messy for systems with multiple crossover frequencies.
- Crossover region is imperceptible for systems with large resonant peaks.
- Lacks system composition (superposition) properties of Bode plots.

Since phase scale is linear rather than polar, comparing Nichol's chart with the Nyquist chart:

- Nichol's chart is typically cleaner than Nyquist diagram especially for systems with large phase lags, like time-delay systems.
- As gain scale is in dB, regions with large gain don't dominate, hence the crossover region is more visible.
- Also, the consequence of the logarithmic scale of $|\log(j\omega)|$ is that multiplication of systems results in superposition on Nichol's chart, almost as easy as on the Bode plots.

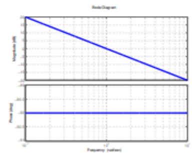
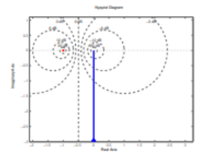
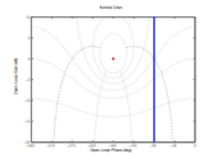
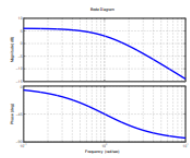
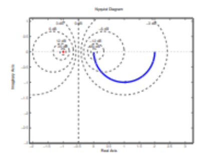
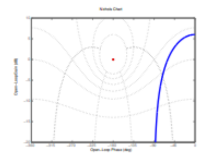
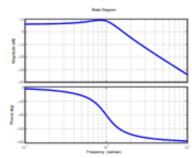
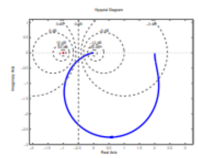
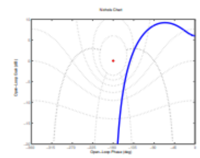
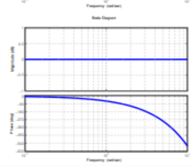
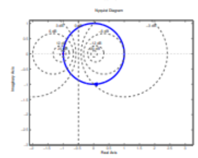
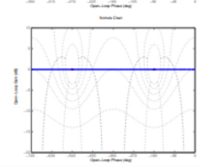
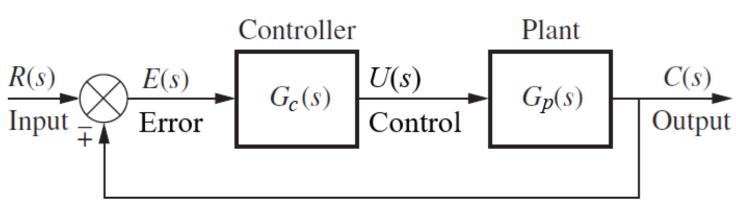
$L(s)$	Bode	Nyquist	Nichols
$\frac{k}{s}$			
$\frac{k}{\tau s+1}$			
$\frac{k\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$			
e^{-sh}			

Table 2: Comparison of diagrams of Bode, Nyquist and Nichols

Example for Tutorial 5 - Nichols Chart

For a feedback control system as shown in the figure below, attempt the following tasks.



The transfer function equation of the plant is:

$$G_P(s) = \frac{10^5}{(s+1)(s^2+4s+1.639\times10^4)}$$

The transfer function equation of the controller is:

$$G_C(s) = \frac{25}{s+25}$$

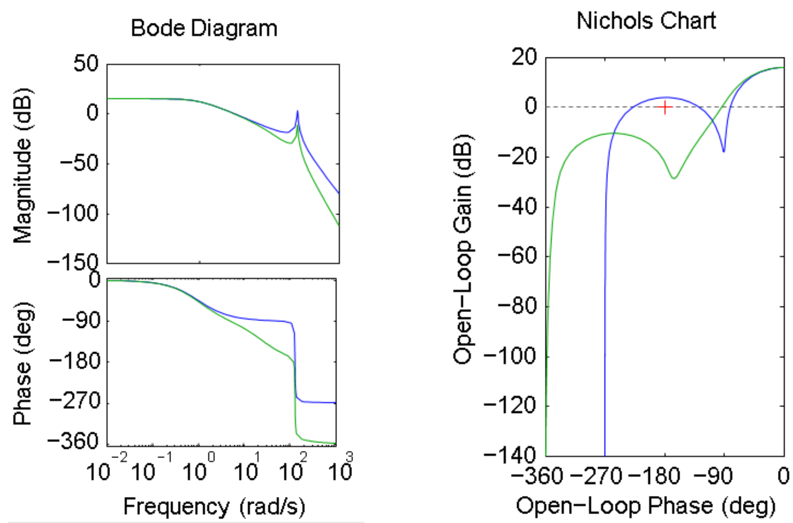
a. Simulate Bode plots and Nichols charts for an uncompensated system ($G_P(s)$) and compensated system ($G_P(s)G_C(s)$). [10 marks]

b. Compared the stability characteristics of the system based on the results of the simulation obtained in part (a). [4 marks]

c. For determining the stability of the system, is Nyquist diagram is easier to use than Bode plot?
[2 marks]

Answer

a. The following figures show the Bode diagram and Nichol's chart of a control system.



Note: blue line is uncompensated system, and green line is compensated system.

b. For the Bode diagram, notice that the gain margin (GM) and phase margin (PM) of the uncompensated system (blue line) are smaller compared with those of the compensated system (green line). We need to determine the exact values of GM and PM of the system to find out if it is stable or not

For the Nichol's chart, notice that the contour of the uncompensated system (blue line) encircles the test point (-180°, 0), but the contour of the compensated system is underneath the test point. From these results, compensated system is stable whereas uncompensated system is unstable.

c. From the results of part (b), it seems that Nyquist diagram is easier to use than Bode plots for analysing stability of the system.

We need only to evaluate the contour in the Nyquist diagram whether it is encircling the test point (-180°, 0) to determine if the system is stable or not. On the other hand, we need to find out the values of GM and PM of the system using Bode plots.