XMUT315 Control Systems Engineering

Note 2: Laplace Transforms

Topic

- Differential equations and Laplace transforms.
- Transfer functions, poles and zeros, and modes.
- Modal decomposition and expansion method.
- Cover up (Heaviside) method.
- Complex factors.
- Repeated factors.
- Partial fractions.
- The s-plane and poles and zeros.
- Final value theorem.

1. Differential Equations and Laplace Transform

This section covers the relationship between differential equation and Laplace transform.

1.1. Solving DEs with the Laplace Transform

The Laplace transform is useful because it allows us to convert linear, constant-coefficient differential equations into algebraic equations.

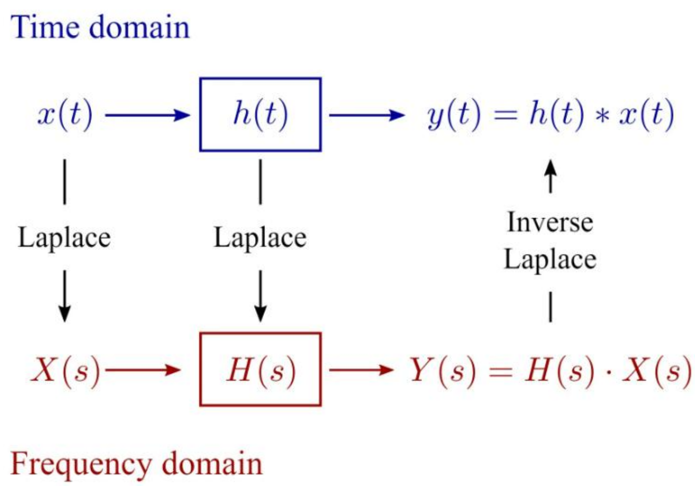

Figure 1: Time domain vs Laplace domain in the Laplace transform

This results from the differentiation in time property of the Laplace transform.

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0^-)$$

$$\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0^-) - y'(0^-)$$

…

$$\mathcal{L}\{y^{(n)}(t)\} = s^n\mathcal{L}\{y(t)\} - s^{(n-1)}y(0^-) \dots y^{(n-1)}(0^-)$$

Recall that $y(0^-)$, $y^j(0^-)$ and so forth are initial conditions.

For an n-th order DE, we need to know the initial values of the first n derivatives to solve a differential equation uniquely using the Laplace transform.

Example for Tutorial 1: Equation of Differential Equation

Find $y(t)$ in a system described by the differential equation (DE) as follows. [10 marks]

$$y''(t) + 4y'(t) + 3y(t) = 0$$

With initial conditions:

$$y(0) = 3, y'(0) = 1$$

Answer

We start by taking the Laplace transform of the entire differential equation.

Using the differentiation in time formula, we can write the transforms of each of the derivatives of y.

$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 3$$

And

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - 3s - 1$$

We can therefore write the complete Laplace transform.

$$(s^2Y(s) - 3s - 1) + 4(sY(s) - 3) + 3Y(s) = 0$$

Gather all coefficients of $Y(s)$ to the left and the rest of other coefficients to the right:

$$(s^2 + 4s + 3)Y(s) = 3s + 13$$

Rearrange the equation and factorise the roots.

$$Y(s) = \frac{3s + 13}{(s^2 + 4s + 3)} = \frac{3s + 13}{(s + 1)(s + 3)}$$

Apply partial fraction expansion to simplify the form of the equation.

$$Y(s) = \frac{5}{(s + 1)} + \frac{-2}{(s + 3)}$$

We have solved our DE by Laplace transforming it, solving an algebraic equation.

Then, using inverse Laplace transform, transform the equation back to the time domain (see table of Laplace transform).

$$y(t) = (5e^{-t} - 2e^{-3t})u(t)$$

2. The Transfer Function, Poles and Zeros and Modes

When characterising a system, we are interested in what the system does to an arbitrary input signal. We typically assume that any initial transients have been given time to die away, which is equivalent to assuming zero initial conditions.

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$$y''(t) + 4y'(t) + 3y(t) = x(t)$$

Take Laplace transform of the equation.

$$s^2Y(s) + \ 4sY(s) + 3Y(s) = X(s)$$

Rearrange the equation.

$$(s^2 + 4s + 3)Y(s) = X(s)$$

Rearrange the equation in terms of the ratio of the parameters that we are interested e.g. $Y(s)$ and $X(s)$.

$$G(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 4s + 3}$$

This is the so-called transfer function (TF). It tells us what the system does to an arbitrary $X(s)$.

Example for Tutorial 2: Transfer Function of Differential Equation

Find the transfer function equation for the following differential equation. [4 marks]

$$y''(t) + 4y'(t) + 3y(t) = x(t)$$

Answer

Take Laplace transform of the equation.

$$s^2Y(s) + \ 4sY(s) + 3Y(s) = X(s)$$

Gather all coefficients of $Y(s)$ to the left and the rest of other coefficients to the right.

$$(s^2 + 4s + 3)Y(s) = X(s)$$

Form the equation of $Y(s)/X(s)$:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 4s + 3}$$

2.1. Poles and Modes

The poles of the transfer function are important (e.g. the values of s that make the denominator of the TF zero), as they allow us to find the modes of the system.

The modes are simply the characteristic responses that the system will exhibit when excited by a signal, or by initial conditions.

$$G(s) = \frac{1}{(s + a)(s + b)}$$

Poles at $s = -a$ and $s = -b$. The modes will be e^{-at} and e^{-bt}

$$y(t) = [Ae^{-at} + Be^{-bt}]u(t)$$

Where: A and B depend on the input and the initial conditions.

Example for Tutorial 3: Mode of Response

Find the mode of the characteristic response of the system as given below. [4 marks]

$$G(s) = \frac{1}{s^2 + 4s + 3}$$

Answer

For the given system, factorise its transfer function equation as shown below.

$$G(s) = \frac{1}{s^2 + 4s + 3} = \frac{1}{(s+1)(s+3)}$$

With the given system, the poles at $s = -1$ and $s = -3$.

The modes will be e^{-t} and e^{-3t} which are exponential responses as shown below.

$$g(t) = [Ae^{-t} + Be^{-3t}]u(t)$$

Where: A and B depend on the input and the initial conditions.

2.2. Zeros

In general, we can also have a polynomial of *s* in the numerator of the transfer function. The values of *s* that make the numerator zero are called zeros of the transfer function. The system will exhibit no output when driven by a signal having these values of *s*.

The zeros do not produce modes, but they play an important role in setting the relative magnitude of the various modes. A pole that has an *s* value that is close to that of a zero will have a "small" mode.

3. Modal Decomposition

We use partial fraction expansion to simplify expressions in the *s*-domain. Otherwise, we must begin with a proper rational polynomial, the form of the system modes will be obscured.

- Not a rational polynomial:

$$Y_1(s) = \frac{(1/s) + 1}{(s+2)(s+3)}$$

- Not strictly proper:

$$Y_2(s) = \frac{s^2}{(s+2)(s+3)}$$

Example for Tutorial 4: Simplification of Transfer Function Equation

Given the transfer function equation of a system, is it possible to simplify it further? [4 marks]

- System 1:

$$Y_1(s) = \frac{(1/s) + 1}{(s+2)(s+3)}$$

- System 2:

$$Y_2(s) = \frac{s^2}{(s+2)(s+3)}$$

Answer

If the transfer function of the system is not a rational polynomial, perform modal decomposition of the equation as shown below:

$$Y(s) = \frac{(1/s) + 1}{(s+2)(s+3)} = \left(\frac{s}{s}\right)\frac{(1/s + 1)}{(s+2)(s+3)} = \frac{(s+1)}{s(s+2)(s+3)}$$

If the transfer function of the system is not strictly proper, we can simplify the equation as below:

$$Y(s) = \frac{s^2}{(s+2)(s+3)} = \frac{s^2 + 5s + 6}{s^2 + 5s + 6} - \frac{5s + 6}{s^2 + 5s + 6} = 1 - \frac{5s + 6}{(s+2)(s+3)}$$

3.1. Modal Expansion

We then write our expression as the sum of a set of appropriate terms, each of which corresponds to a particular mode and has an unknown amplitude. We write the denominator as a combination of three types of poles:

- Simple real poles: $(s + a) \Leftrightarrow Ae^{-at}$
- Complex pole pairs: $(s + a)^2 + \omega_d^2 \Leftrightarrow Be^{-at}\cos(\omega_d t + \phi)$

or

Control Systems: $s^2 + 2\zeta\omega_n s + \omega_n^2 \Leftrightarrow Be^{-\zeta t}\cos(\omega_n t + \phi)$

- Repeated real poles: $(s + a)^n \Leftrightarrow C_n t^{n-1} e^{-at} + \ \ldots$

You only need to use partial fraction expansion when you need to write an equation for the output of a system.

If you need to know the amplitudes of the modes, then use partial fractions. If you do not need the amplitudes, just wish to find out the mode of the system, then stop!

3.2. Simple Real Poles

For real poles:

$$Y(s) = \frac{n(s)}{(s+a)d(s)} = Y_1(s) + \frac{A}{s+a}$$

Thus

$$y(t) = Y_1(t) + Ae^{-at}u(t)$$

We can solve this system with simple real poles using ordinary partial-fraction expansion method or using Heaviside or cover-up method.

Example for Tutorial 5: Partial Fraction Expansion

Find the time-domain equation of a system described as the transfer-function equation below using ordinary partial-fraction expansion method. [8 marks]

$$Y(s) = \frac{s+1}{s^3+s^2-6s}$$

Answer

For the given system, factorise the transfer function equation.

$$Y(s) = \frac{s+1}{s^3+s^2-6s} = \frac{s+1}{s(s^2+s-6)} = \frac{s+1}{s(s-2)(s+3)}$$

Then, perform partial fraction expansion.

$$Y(s) = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

The first method we will use is multiplying out the new form of the equation and equating it with the original form.

$$\frac{s+1}{s^3+s^2-6s} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

$$= \frac{A(s-2)(s+3) + Bs(s+3) + Cs(s-2)}{s^3+s^2-6s}$$

$$= \frac{(A+B+C)s^2 + (A+3B-2C)s - 6A}{s^3+s^2-6s}$$

The denominators of these expressions are identical, so the numerators must be equivalent. We therefore equate coefficients of the various powers of s in the numerator polynomials of the two sides.

$$s^2:\ 0 = A + B + C$$

$$s^1:\ 1 = A + 3B - 2C$$

$$s^0:\ 1 = -6A$$

From the last of these equations, we know that $A = -1/6$.

Substituting into the other two equations we find:

$$B + C = \frac{1}{6} \qquad \text{and} \qquad 3B - 2C = \frac{7}{6}$$

Solving these two equations simultaneously we find:

$$B = \frac{3}{10},\ C = -\frac{2}{15}$$

Entering the values of the coefficient back into the equation.

$$Y(s) = \frac{s+1}{s(s^2+s-6)} = -\frac{1}{6}\left(\frac{1}{s}\right) + \frac{3}{10}\left(\frac{1}{s-2}\right) - \frac{2}{15}\left(\frac{1}{s+3}\right)$$

Taking the inverse Laplace transform (from the table), we therefore find:

$$y(t) = \left[-\frac{1}{6} + \frac{3}{10}e^{2t} - \frac{2}{15}e^{-3t}\right]u(t)$$

3.3. The Heaviside or Cover-up Method

There is a quicker method for finding the partial fraction expansion, known as the Heaviside or cover-up method.

The "Cover" the term in the denominator for which you are trying to find the coefficient and then calculate the value of the remaining fraction at the value that would cause the covered term to be zero.

Say for example that you have the following function to be decomposed into partial fractions:

$$\frac{x-7}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

On the left-hand side, we mentally remove (or cover up with a finger) the factor $x - 1$ associated with A, and substitute $x = 1$ into what's left; this gives A:

$$\left.\frac{x-7}{x+2}\right|_{x=1} = \frac{1-7}{1+2} = -2 = A$$

Similarly, B is found by covering up the factor $x + 2$ on the left and substituting $x = -2$ into what's left. This gives:

$$\left.\frac{x-7}{x-1}\right|_{x=-2} = \frac{-2-7}{-2-1} = 3 = B$$

Thus, the partial fraction of the function is:

$$\frac{x-7}{(x-1)(x+2)} = \frac{-2}{x-1} + \frac{3}{x+2}$$

Example for Tutorial 6: Partial Fraction with Cover-up Method

Find the time domain equation of a control system given as the following transfer function equation below using the cover-up method. [5 marks]

$$Y(s) = \frac{s+1}{s^3+s^2-6s}$$

Answer

For the given system, factorise the transfer function equation.

$$\frac{s+1}{s^3+s^2-6s} = \frac{s+1}{s(s-2)(s+3)}$$

Then, perform partial fraction expansion using cover-up method.

$$Y(s) = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+3}$$

The coefficients A, B, and C in the equation above are calculated from:

$$A = \left.\frac{s+1}{(s-2)(s+3)}\right|_{s=0} = -\frac{1}{6}$$

$$B = \left.\frac{s+1}{s\,(s+3)}\right|_{s=2} = \frac{3}{2(2+3)} = \frac{3}{10}$$

$$C = \left.\frac{s+1}{s(s-2)}\right|_{s=-3} = \frac{-2}{-3(-3-2)} = -\frac{2}{15}$$

Thus

$$\frac{s+1}{s^3+s^2-6s} = -\frac{1}{6s} + \frac{3}{10(s-2)} - \frac{2}{15(s+3)}$$

3.4. Unique Complex Factors

For complex pole pair:

$$Y(s) = \frac{n(s)}{[(s+a)^2 + \omega^2]d(s)} = Y_1(s) + \frac{As + B}{(s+a)^2 + \omega^2}$$

Thus

$$y(t) = y_1(t) + [Ae^{-at}\cos(\omega t + \phi)]u(t)$$

Where: ϕ depends on A and B.

The coefficients of complex factors must be found by the cross-multiplication method. Find the residuals of other factors first.

Example for Tutorial 7: Partial Fraction of Complex Factors

For a given system described as the following transfer function equation with a pair of complex factors, find its time-domain equation. [8 marks]

$$Y(s) = \frac{1}{s(s^2 + s + 1)}$$

Answer

The coefficients of complex factors must be found by the cross-multiplication method. Find the residuals of other factors first.

$$Y(s) = \frac{1}{s(s^2 + s + 1)}$$

$$= \frac{1}{s} + \frac{A_2 s + A_3}{s^2 + s + 1}$$

$$= \frac{(s^2 + s + 1) + A_2 s^2 + A_3 s}{s(s^2 + s + 1)}$$

$$= \frac{(A_2 + 1)s^2 + (A_3 + 1)s + 1}{s(s^2 + s + 1)}$$

We equate the coefficients of the powers of s in the numerators of the two sides.

s^2: $0 = A_2 + 1 \Longrightarrow A_2 = -1$

s^1: $0 = A_3 + 1 \Longrightarrow A_3 = -1$

s^0: $1 = 1$

Entering the values of the coefficient back into the equation.

$$Y(s) = \frac{1}{s} + \frac{-s - 1}{s^2 + s + 1}$$

$$= \frac{1}{s} - \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{1}{s} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{1}{s} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

Thus, taking inverse transform of the equation given above

$$y(t) = \left[1 - e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{\sqrt{3}}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]u(t)$$

3.5. Repeated Real Factors

Repeated real poles:

$$Y(s) = \frac{n(s)}{(s+a)^k d(s)}$$

Or

$$Y(s) = Y_1(s) + \frac{A_k}{(s+a)^k} + \frac{A_{k-1}}{(s+a)^{k-1}} + \cdots + \frac{A_0}{s+a}$$

Thus

$$y(t) = y_1(t) + [A_k t^{k-1}e^{-at} + A_{k-1}t^{k-2}e^{-at} + \cdots + A_0 e^{-at}]u(t)$$

Repeated poles lead to a set of partial fractions, with decreasing multiplicity of the pole.

Example for Tutorial 8: Partial Fraction of Repeated Real Factors

Find the time domain equation of a system expressed as the following transfer function equation with a pair of repeated poles. [5 marks]

$$Y(s) = \frac{3s+8}{(s+2)^2}$$

Answer

With the given transfer function equation, factorise and perform partial fraction expansion.

$$Y(s) = \frac{3s+8}{(s+2)^2} = \frac{A_2}{(s+2)^2} + \frac{A_1}{s+2}$$

The coefficients A_1 and A_2 are found from:

$$A_2 = \lim_{s\to -2} \frac{(s+2)^2(3s+8)}{(s+2)^2}$$

$$= 3s + 8|_{s\to -2} = 2$$

(Just the Heaviside technique)

$$A_1 = \lim_{s\to -2} \frac{d}{ds} \frac{(s+2)^2(3s+8)}{(s+2)^2}$$

$$= \lim_{s\to -2} \frac{d}{ds}(3s+8)$$

$$= (3)|_{s\to -2} = 3$$

Thus, the transfer function equation is:

$$Y(s) = \frac{2}{(s+2)^2} + \frac{3}{s+2}$$

Taking inverse Laplace transform of the transfer function, the equation in the time domain is:

$$y(t) = (2te^{-2t}+3e^{-2t})u(t)$$

3.6. Summary of Partial Fractions

- Real poles:

$$Y(s) = \frac{n(s)}{(s+a)d(s)} = Y_1(s) + \frac{A}{s+a}$$

Thus:

$$y(t) = y_1(t) + Ae^{-at}u(t)$$

- Complex pole pair:

$$Y(s) = \frac{n(s)}{[(s+a)^2+\omega^2]d(s)} = Y_1(s) + \frac{A+B}{(s+a)^2+\omega^2}$$

Thus:

$$y(t) = y_1(t) + [Ae^{-at}\cos(\omega t + \phi)]u(t)$$

Where: ϕ depends on A and B.

- Repeated Real poles:

$$Y(s) = \frac{n(s)}{(s+a)^k d(s)} = Y_1(s) + \frac{A_k}{(s+a)^k} + \frac{A_{k-1}}{(s+a)^{k-1}} + \cdots + \frac{A_0}{s+a}$$

Thus:

$$y(t) = y_1(t) + [A_k t^{k-1} e^{-at} + A_{k-1} t^{k-2} e^{-at} + \cdots + A_0 e^{-at}]u(t)$$

4. The s-Plane and Poles and Zeros

We often do not care about the precise amplitude of modes but are instead content to talk about the modes themselves. Plotting pole zero diagrams lets us visualise what is happening as shown in the figure below.

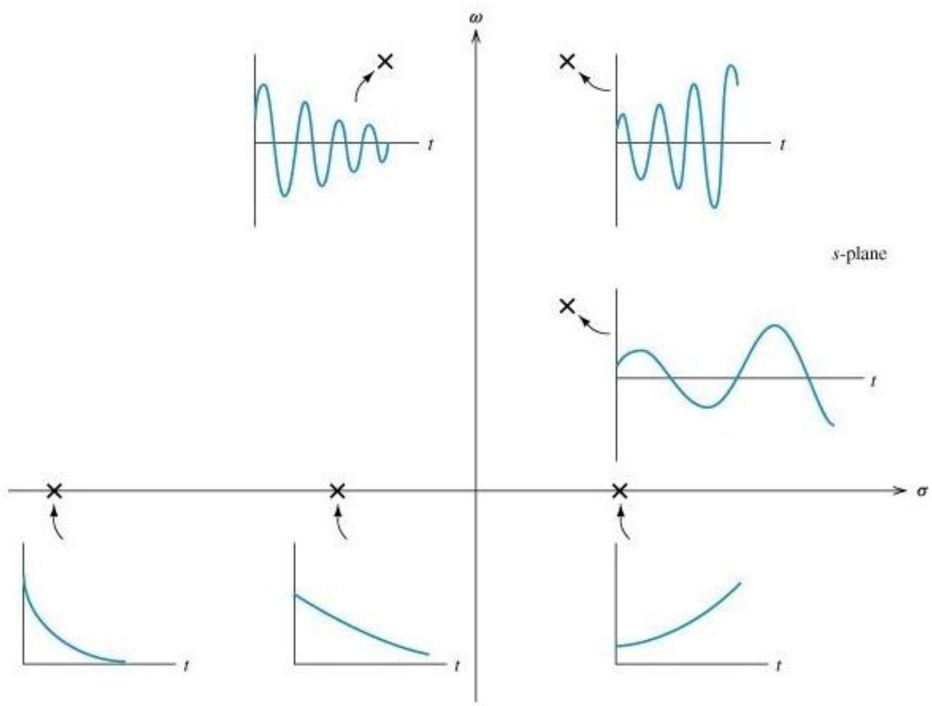


Figure 2: Location of the poles of the system in the s-plane

Remember that any system having poles only on the left-half side of the *s*-plane will be stable. Its modes will (eventually) decay to zero.

Conversely, a system having one or more poles in the right half of the *s*-plane will be unstable, and its output will tend to infinity with increasing time.

Example for Tutorial 9: Poles and Zeros in s-Plane

For each of the given control systems below, determine the location of poles and/or zeros in the s-plane and predict its transient response. [9 marks]

a. System 1

$$Y_1(s) = \frac{s + 20}{s^2 + 101s + 100}$$

b. System 2

$$Y_2(s) = \frac{0.5s + 2.5}{s^2 + 2s + 10}$$

c. System 3

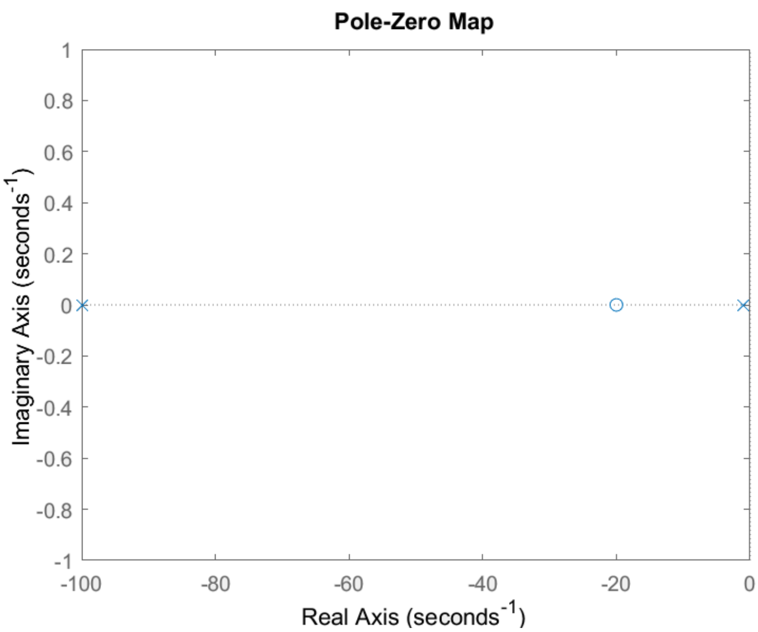
$$Y_3(s) = \frac{5s - 500}{s^3 - 3s - 2}$$

Answer

a. System 1

$$Y_1(s) = \frac{(s + 20)}{(s + 1)(s + 100)}$$

The s-plane diagram of the system is as shown in the figure below.

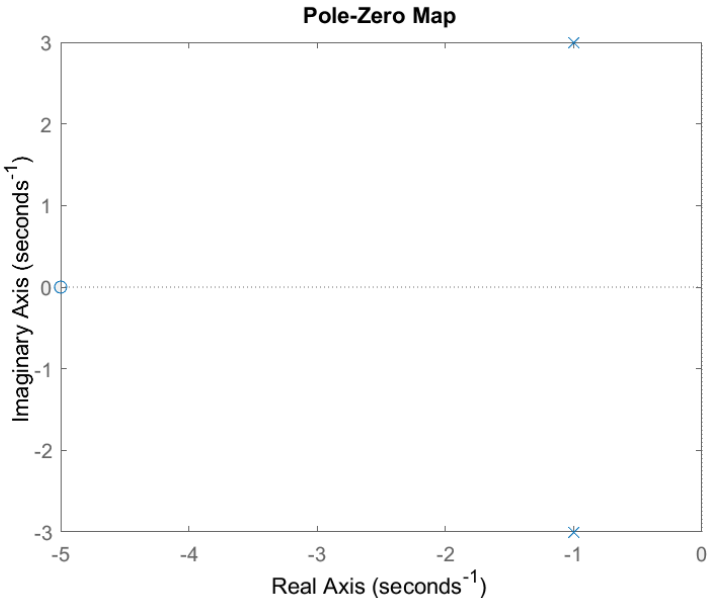


Since all the poles and zero are located at the left-hand side of the diagram, the system is stable. As the poles are all real, then the transient response of the system is overdamped.

b. System 2

$$Y_2(s) = \frac{0.5(s+5)}{s^2+2s+10} = \frac{0.5(s+5)}{(s+1)^2+(3)^3}$$

The s-plane diagram of the system is as shown in the figure below.

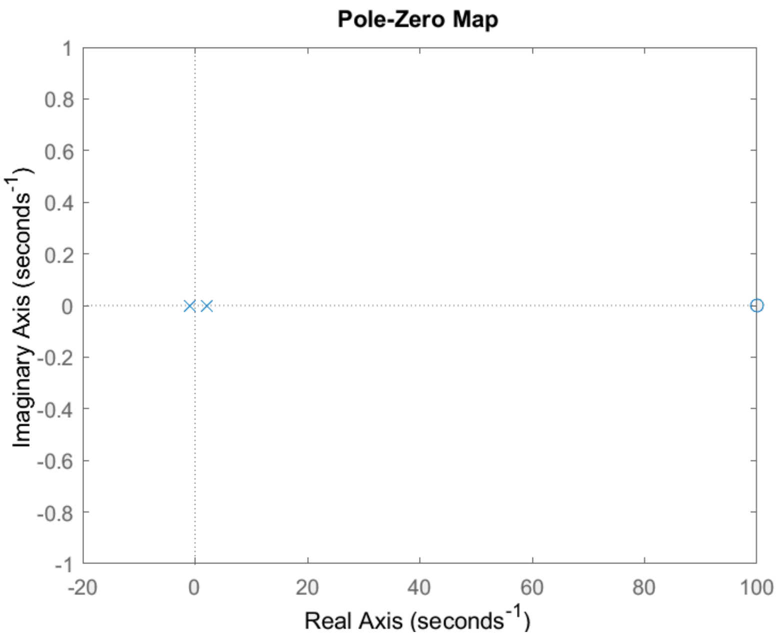


Since all the poles are a pair of complex poles at the left-hand side of the diagram, the system is stable. Because of these complex poles, the transient response of the system is underdamped.

c. System 3

$$Y_3(s) = \frac{5(s-100)}{(s-2)(s+1)^2}$$

The s-plane diagram of the system is as shown in the figure below.



Since there is a pole at the right-hand side of the diagram, the system is found to be unstable.

5. Final Value Theorem

The final value theorem allows us to calculate the final value that a system output will take, without needing to do partial fractions expansion and inverting the Laplace transform.

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

We will find this particularly useful in finding the response of a system to a step input, which makes the equation particularly simple.

$$\lim_{t\to\infty} g(t) = \lim_{s\to 0} sG(s)\frac{1}{s} = \lim_{s\to 0} G(s)$$

The final theorem is only held if the system is stable. Be careful!

Example for Tutorial 10: Final Value Theorem of Systems

Determine the steady-state characteristics of the following control systems given as the following transfer function equations. [6 marks]

a. System 1 when it is subjected to a step input ($1/s$):

$$F_1(s) = \frac{s(s+10)}{(s+2)(s+50)}$$

b. System 2 when it is subjected to a ramp input ($1/s^2$):

$$F_2(s) = \frac{10(s+5)}{s(s^2+s+10)}$$

c. System 3 when it is subjected to a parabolic input ($1/s^3$):

$$F_3(s) = \frac{s^2(s+2)}{(s+15)(s+100)}$$

Answer

The steady-state characteristics of the following control systems are as outlined below.

a. System 1 with a step input:

$$\begin{aligned}\lim_{t\to\infty} f_1(t) &= \lim_{s\to 0} sF_1(s)\left(\frac{1}{s}\right)\\ &= \lim_{s\to 0} s\left[\frac{s(s+10)}{(s+2)(s+50)}\right]\left(\frac{1}{s}\right) = 0\end{aligned}$$

It looks like the system settles down to 0 at steady-state condition.

b. System 2 with a ramp input:

$$\begin{aligned}\lim_{t\to\infty} f_2(t) &= \lim_{s\to 0} sF_2(s)\left(\frac{1}{s^2}\right)\\ &= \lim_{s\to 0} s\left[\frac{10(s+5)}{s(s^2+s+10)}\right]\left(\frac{1}{s^2}\right) = \infty\end{aligned}$$

It appears that the system is unstable at steady-state condition as the gain is approaching infinity.

This also shows that the type of input has significant effect on the response of the system at steady-state condition.

c. System 3 with a parabolic input:

$$\begin{aligned}\lim_{t\to\infty]} f_3(t) &= \lim_{s\to 0} sF_3(s)\left(\frac{1}{s^3}\right)\\ &= \lim_{s\to 0} s\left[\frac{s^2(s+2)}{(s+15)(s+100)}\right]\left(\frac{1}{s^3}\right) = \frac{2}{(15)(100)} = \frac{1}{750}\end{aligned}$$

It seems that the system settles down to a constant value e.g. 1/750 at steady-state condition and this results in a big steady-state error.

Appendix – Laplace Transform Table and Properties

Laplace Transforms of Elementary Functions

Signal	Transform	ROC
1. $\delta(t)$	1	All s
2. $u(t)$	$\frac{1}{s}$	$\Re e\{s\} > 0$
3. $-u(-t)$	$\frac{1}{s}$	$\Re e\{s\} < 0$
4. $\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$
5. $-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re e\{s\} < 0$
6. $e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} > -\Re e\{\alpha\}$
7. $-e^{-\alpha t}u(-t)$	$\frac{1}{s+\alpha}$	$\Re e\{s\} < -\Re e\{\alpha\}$
8. $\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} > -\Re e\{\alpha\}$
9. $-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s+\alpha)^n}$	$\Re e\{s\} < -\Re e\{\alpha\}$
10. $\delta(t-T)$	e^{-sT}	All s
11. $[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
12. $[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re e\{s\} > 0$
13. $[e^{-\alpha t}\cos\omega_0 t]u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\Re e\{\alpha\}$
14. $[e^{-\alpha t}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\Re e\{s\} > -\Re e\{\alpha\}$
15. $u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16. $u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re e\{s\} > 0$

Properties of the Laplace Transform

Property	Signal	Transform	ROC
	$x(t)$	$X(s)$	R
	$x_1(t)$	$X_1(s)$	R_1
	$x_2(t)$	$X_2(s)$	R_2
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in the s-Domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R [i.e., s is in the ROC if $(s - s_0)$ is in R]
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	"Scaled" ROC (i.e., s is in the ROC if (s/a) is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the Time Domain	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re e\{s\} > 0\}$

Initial- and Final Value Theorems

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$