XMUT315 Control Systems Engineering

Note 6: Stability Analysis

Topic

- Stability and system responses.
- Stability of the systems and stability analysis.
- Methods of stability analysis.
- Routh-Hurwitz criterion.
- Construction of the criterion.
- Special cases of Routh-Hurwitz criterion.
    - Zero in a single column.
    - Zeros in a row.

1. Introduction to Stability

Stability is one of the most important design requirement and performance of the control system. We have very limited option to analyse and design an unstable control system.

1.1. Stability

The type of stability analysis to be conducted is limited for LTI system and is not based on non-linear system. A linear system is where the principles of supposition do apply (e.g. no saturation or hysteresis effects). A time-invariant system is where its characteristics do not vary with respect to time (e.g. no ageing). In LTI systems, we often approximate systems over a specific range or time period.

We want to build up a relationship between the response of the system and stability:

- if input is bounded and output ($c(t)$) does not approach ∞ as t approaches ∞ e.g. natural response is not approaching ∞.

- if input is unbounded, we can't conclude stability.

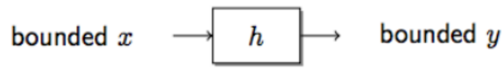


Figure 1: Bounded input and bounded output

A system is stable if every bounded input yields a bounded output or it is bounded-input bounded-output (BIBO). We want to build up a relationship between the total response and instability:

- If the input is bounded, but the output ($c(t)$) is unbounded, the system is unstable.
- If input is unbounded, we cannot conclude instability.

A system is unstable if any bounded input yields an unbounded output.

1.2. Stability vs. Instability

In terms of stability of the system, we might see several stability conditions:

- A linear, time-invariant system is stable if the natural response approaches zero as time approaches infinity.
- A linear, time-invariant system is unstable if the natural response grows without bound as time approaches infinity.
- A linear, time-invariant system is marginally stable if the natural response neither decays, nor grows, but remains constant or oscillates as time approaches infinity.

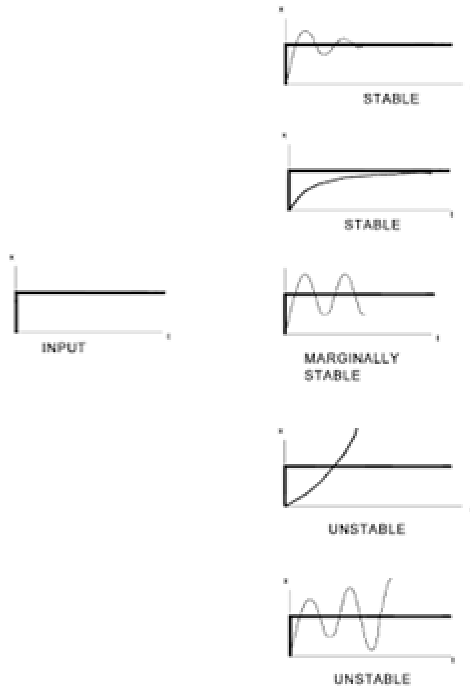


Figure 2: Input vs. stable and unstable output of a system

A control system is stable, e.g. a linear and time-invariant system is stable if the natural response approaches zero as time approaches infinity.

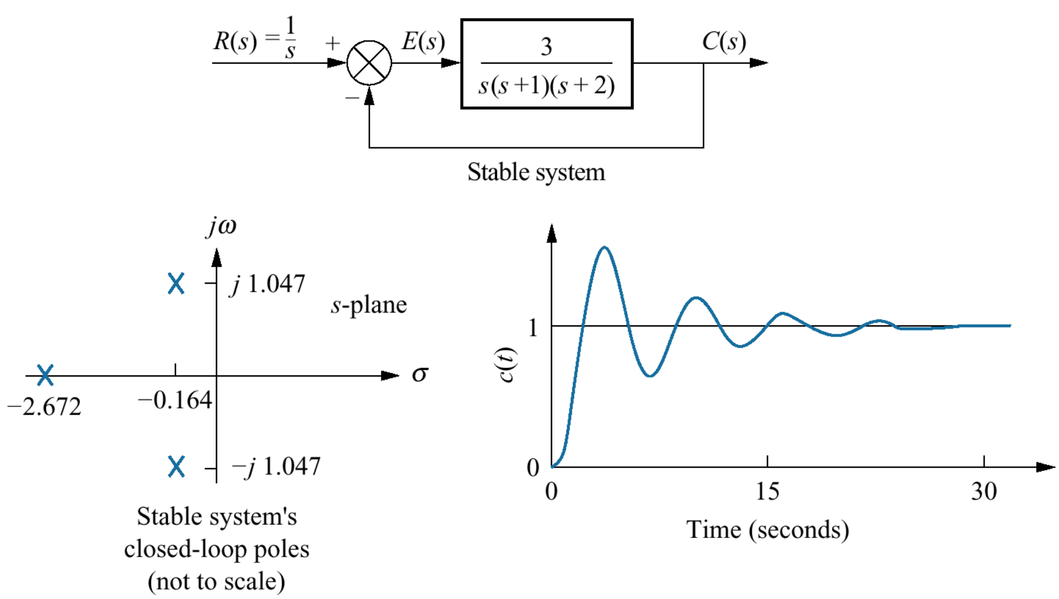


Figure 3: Root locus diagram and transient response of a stable system

A control system is unstable if the natural response grows without bound as time approaches infinity.

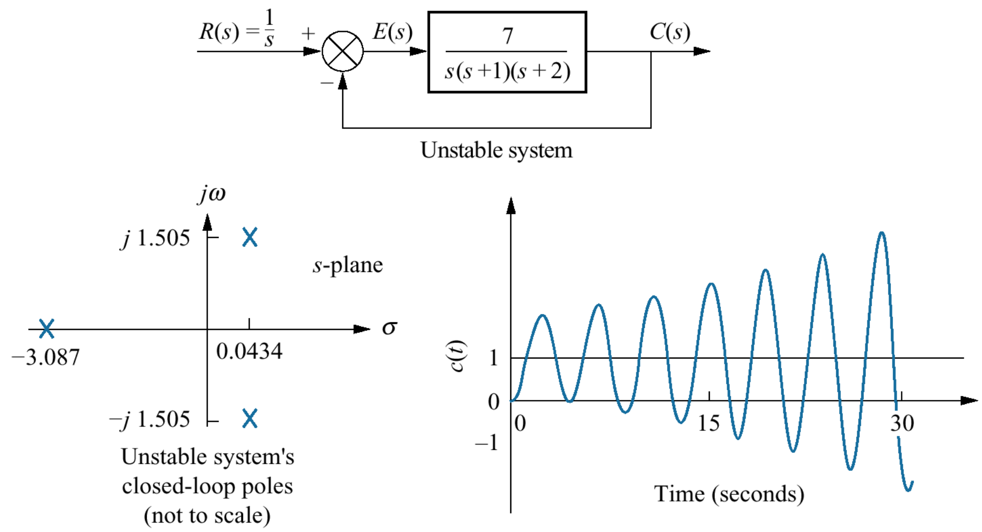


Figure 4: Root locus diagram and transient response of an unstable system

1.3. System Damping (Indicator of Stability)

Consider a control system described as:

$$as^2 + bs + c$$

Find roots of a quadratic equation:

$$\text{Root}_{1,2} = -\frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$

The following table shows different transient responses of the system based on roots of equation.

Equation	**Roots**	**Transient Response**
$b^2 - 4ac > 0$	Real, different	Overdamped
$b^2 - 4ac = 0$	Real, same	Critically damped
$b^2 - 4ac < 0$	Complex, different	Underdamped
$b = 0$	Complex, same	Undamped

Table 1: Various systems with different system damping

2. Laplace transform and its Operators

Laplace transform and its operator are tools that enable you to analyse and design control system.

2.1. 's' Variable

s' Laplace operator or Laplace transform variable. it can be considered as a differentiator ($df(t)/dt$) and it can be considered as a gradient.

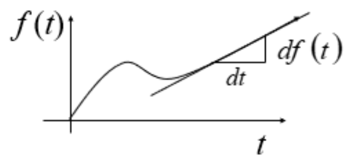


Figure 5: Slope of a graph

The Laplace operator's variable at an instance is a number:

- Numbers can be real or imaginary.

$$s = a \qquad \text{or} \qquad s = a + bj$$

- Usually given in control systems these are defined as:

$$s = \sigma \qquad \text{or} \qquad s = \sigma + j\omega$$

2.2. 's' Domain

We can plot the s variable on a s-domain diagram as shown in the figure below.

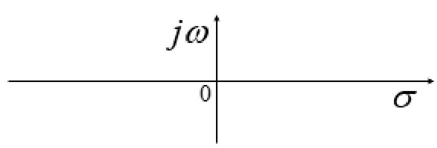


Figure 6: s-domain diagram

Consider the systems with the poles and zeros as given in the figure below.

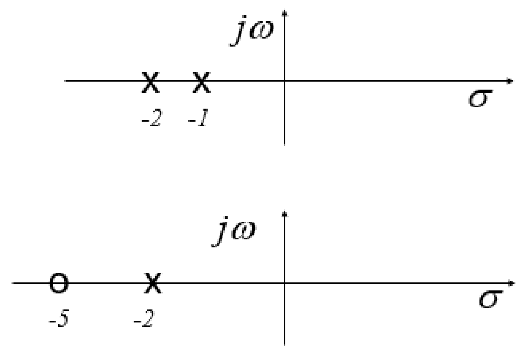


Figure 7: Poles and zeros of a system in the s-domain

Poles (x) cause system to be infinity in the s-domain and zeros (o) cause system to be zero in the s-domain.

3. Stability and System Response

Stability and system response are related very closely when we perform analysis and design of control system.

3.1. System Response

Depending on the location of the poles of the given control system in the s-plane, we can observe various responses of the system.

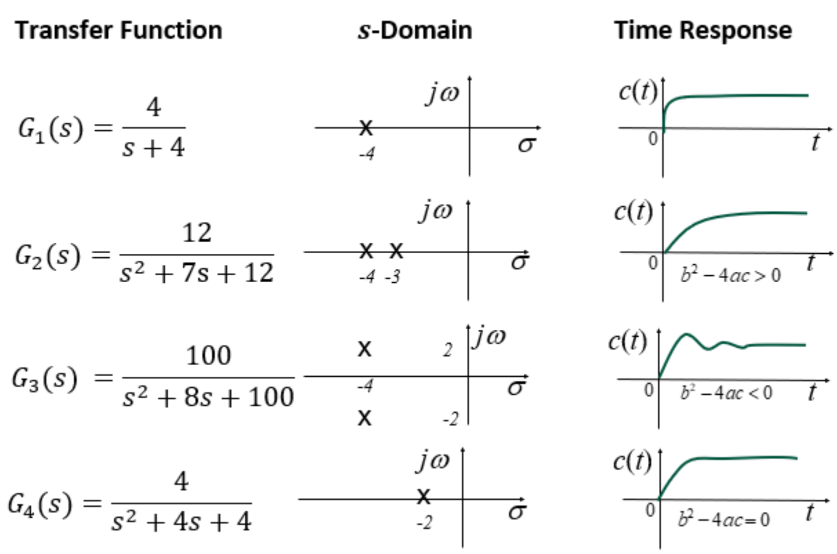


Figure 8: Various systems with their poles and zeros in s-domain and time response

Example for Tutorial 1: Stability of Control Systems

Consider the following two control system examples:

i. System 1:

$$G_1(s) = \frac{1}{(s+1)(s+2)}$$

ii. System 2:

$$G_2(s) = \frac{s+5}{s+2}$$

What happens when the following conditions exist?

a. Root s = -1? [4 marks]

b. Root s = -2? [4 marks]

c. Root s = -5? [4 marks]

Answer

As given in the previous section, what happens when the following conditions exist?

a. When s = -1?

$$G_1(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{(1-1)(2-1)} = \infty$$

And

$$G_2(s) = \frac{s+5}{s+2} = \frac{5-1}{2-1} = \frac{4}{1} = 1$$

Notice that the output of system 1, $G_1(s)$ is ∞ when s = -1 that indicates its response is a growing. The system is unstable. On the other hand, the output of system 2, $G_2(s)$ is 1 when s = -1 that indicates its response settles at 1. The system is stable.

b. When s = -2?

$$G_1(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{(-1)(0)} = \infty$$

And

$$G_2(s) = \frac{s+5}{s+2} = \frac{3}{0} = \infty$$

Both systems 1 and 2 are growing. Both systems are unstable.

c. When s = -5?

$$G_1(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{(-4)(-3)} = \frac{1}{12}$$

And

$$G_2(s) = \frac{s+5}{s+2} = \frac{0}{-3} = 0$$

Both systems settle to 1/12 and 0. Both systems are stable.

3.4. Time Domain

We are worried when system is infinite or zero in TIME domain. System response often contains an exponential component:

$$G(t) = ke^{at}$$

When $a > 0$, the output will reach infinity. We MUST avoid right half of plane poles.

$$G(s) = \frac{(s+5)}{(s+2)(s-2)}$$

All poles must be in left-hand plane for stability and any poles in right-half plane will cause system to be unstable.

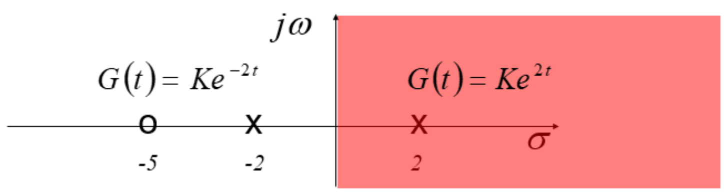


Figure 9: Location of poles and zeros in the s-domain

3.5. Stability

If the poles of a system transfer function all lie in the left half of the s-plane, then that system is stable.

$$G(s) = \frac{(s+5)}{(s+2)(s-1)(s-2)}$$

The poles and zeros in the s-domain of the system given above are shown in the figure below.

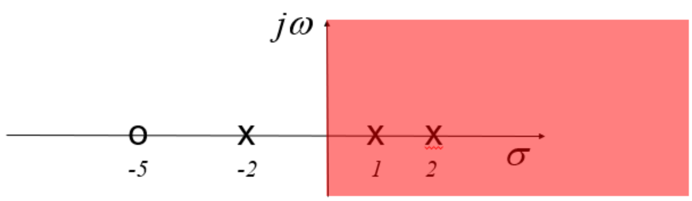


Figure 10: Poles in the right-hand side of the s-domain

It is only the poles of a system transfer function which are important as far as stability is concerned (non-cancelling zeros can be ignored).

$$G(s) = \frac{1}{(s+2)(s-1)(s-2)}$$

The poles and zeros in the s-domain of the system given above are shown in the figure below. Notice the two poles in the right-half plane cause the system to be unstable.

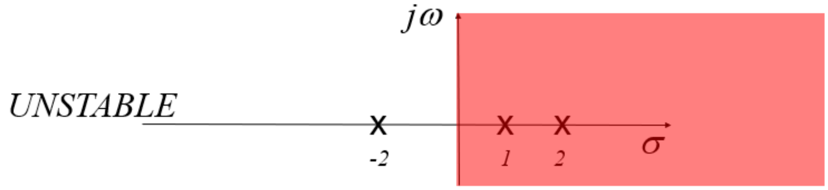


Figure 11: Poles in the right-hand side of the s-domain

The poles of a system are the polynomial roots obtained when the system denominator is equated with zero.

$$G(s) = \frac{(s+5)}{(s+2)(s-1)(s-2)}$$

The system denominator is known as the characteristic polynomial.

$$G(s) = \frac{(s+5)}{(s+2)(s-1)(s-2)}$$

The system denominator equated to zero is the characteristic equation.

$$(s+2)(s-1)(s-2)$$

When open-loop stable, may/may not be closed-loop stable and when open-loop unstable, may/may not be closed-loop unstable.

3.6. Unstable System Response

For a given unstable system with transfer function:

$$G(s) = \frac{(s+4)}{s^3 - 4s^2 + 21s - 34}$$

The pole and zero of the system are as shown in the following s-domain diagram. Notice that the poles are all in the right-hand side in the s-plane. This will cause to be unstable.

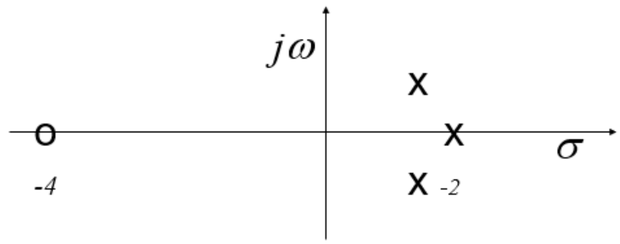


Figure 12: Unstable poles in the s-domain

The time response of the system as shown in the following transient response diagram. It can be seen from the plot that the transient response of the system is a growing response which is typical of an unstable system.

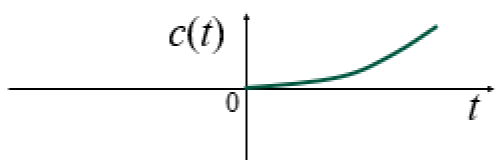


Figure 13: Time transient response of an unstable system

3.7. Forced and Natural Responses

There are differences between forced and natural responses of the control system:

- Natural response: output due to the response of the system itself, without external input.
- Forced response: output due to intentional input, external to the system.

Stability of the system mainly depends on the natural response of the system.

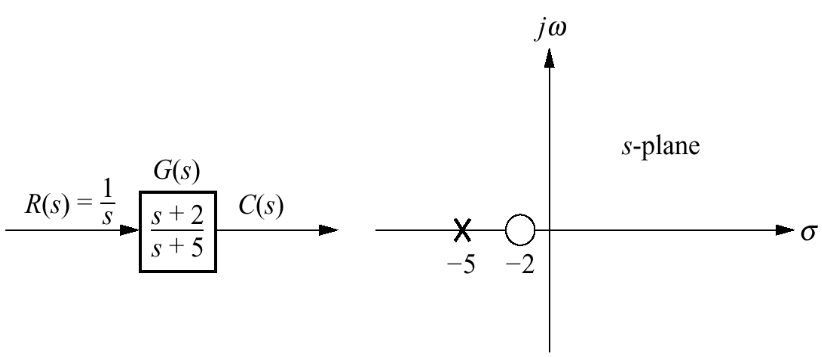


Figure 14: A given control system and its poles and zeros in s-plane

For the control system given above, we can split the overall response of the system into forced and natural responses as shown in the figure below.

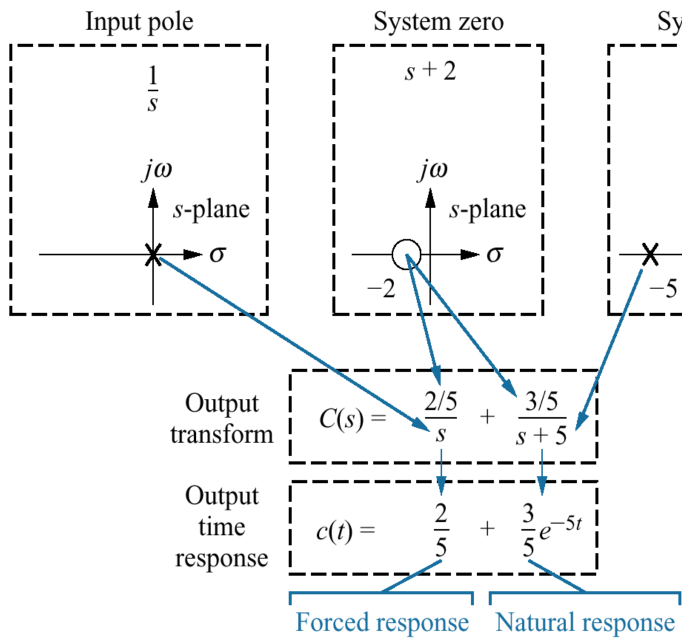


Figure 15: Forced and natural responses of the given control system

3.8. Forced Response

There are several standardised inputs that can be used for analysis and evaluation of the control system e.g. impulse, step, ramp, and sinusoid. If injected to the control system, any of these inputs contributes to the forced response of the control system.

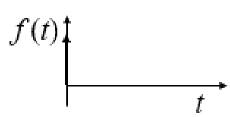
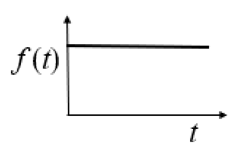
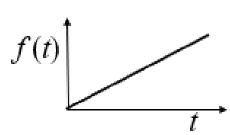
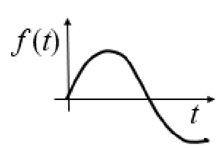
Input	**Function**	**Sketch**	**s-domain**
Impulse	$\delta(t)$		1
Step	$u(t)$		$\frac{1}{s}$
Ramp	$tu(t)$		$\frac{1}{s^2}$
Sinusoid	$\sin \omega t$		$\frac{\omega}{s^2+\omega^2}$

Figure 16: Inputs in control system and their details

4. Improvements to Stability of System

There are several approaches that could be implemented to improve the stability of the system e.g. applying negative feedback, adding controller/compensator, etc.

4.1. Negative Feedback

Negative feedback can reduce effects of disturbances and changes to input parameters. The application of negative feedback can also improve the stability of the system.

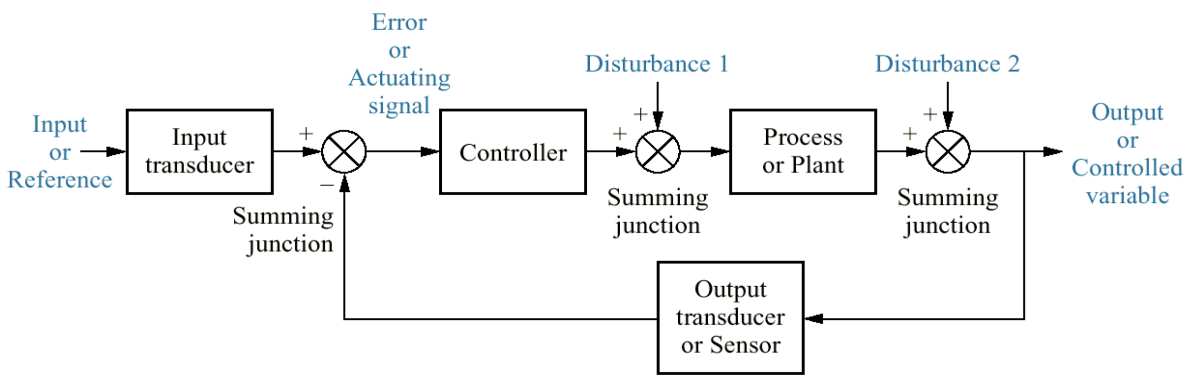


Figure 17: Negative feedback in the control system

Negative feedback is when:

$$|closed\ loop\ gain| < |open\ loop\ gain|$$

This reduces the steady-state error by making the output closer to the input. As a result, we have more robust system and hence more stable system.

4.2. Controllers or Compensator in Control Systems

To improve the stability of the system, we could also add a controller or compensator. Controllers or compensators change the natural response of the system. They adjust the poles of the system. They help achieve the desired output from a given input.

Controllers or compensators can be mechanical, natural, or electrical (used in industry). Three main types of controllers or compensators are:

- Gain (proportional).
- Lead or lag (lead, lag, or lead-lag).
- PID (proportional, integral, derivative, or any of their combinations).

5. Stability Analysis

There are various types of method to study stability analysis in control systems:

- Analytical (i.e. requiring model of the physical system and maths to solve differential equations – mostly approximation for complex system).
- Experimentation (i.e. running a set of trials and observation or measurement).

Approaches used in the course:

- Routh-Hurwitz criterion – mathematical process.
- Nyquist plot – graphical tool.
- Nichol's chart – graphical tool.
- Bode plot – graphical tool.
- Root locus diagram – graphical tool.

5.1. Routh-Hurwitz Criterion

It analyses stability of a system through mathematical analysis of characteristic equation.

$$G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s^1 + a_0 s^0}$$

The Routh table for analysing the stability of the system is constructed as shown below.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Figure 18: Routh table based on transfer function equation of a given system

5.2. Nyquist Plot

This method analyses stability of a system through use of polar plot of system equation.

In Cartesian coordinates, the real part of the transfer function is plotted on the x-axis. The imaginary part is plotted on the y-axis. The frequency is swept as a parameter, resulting in a plot per frequency.

Alternatively, in polar coordinates, the gain of the transfer function is plotted as the radial coordinate, while the phase of the transfer function is plotted as the angular coordinate.

System is stable if the plot does not encircle the unity gain point (-1, 0) in the Nyquist plot.

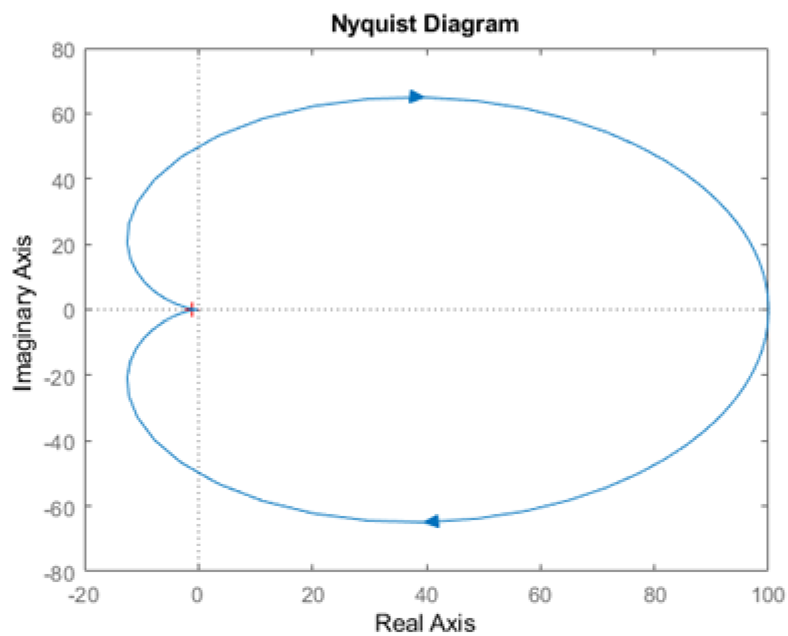


Figure 19: Nyquist diagram of a stable system

The above system on the left is deemed to be stable as the plot is not encircling the Nyquist stability node (-1, 0). On the other hand, the system on the figure below is not stable as the plot is encircling the Nyquist stability node (-1, 0).

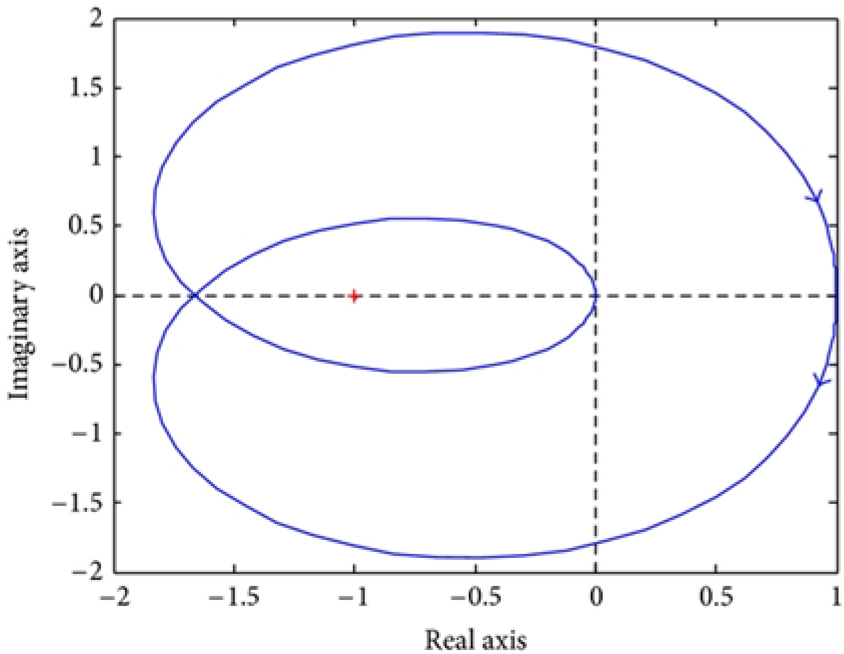


Figure 20: Nyquist diagram of an unstable system

5.3. Nichols Chart

By using this method, we can analyse stability of a system through use of gain and phase plot of the system. A Nichols chart displays the magnitude (in dB) plotted against the phase (in degrees) of the system response.

It is used to further analyse stability of a system beyond the Nyquist plot. A system is stable if the value of gain at 180° in the Nichols chart is positive.

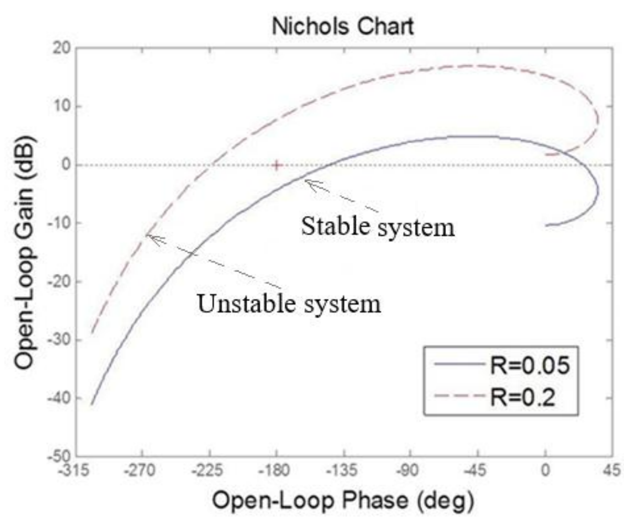
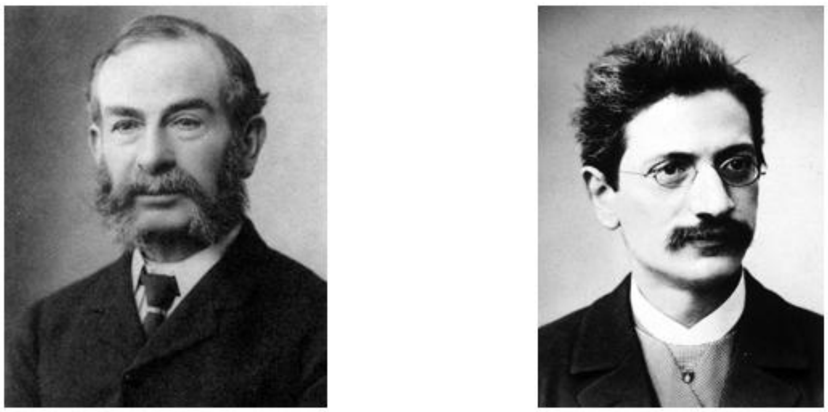


Figure 21: Nichol's chart of stable and unstable systems

The above system is deemed to be stable as the curve of the plot is at the gain > 0 dB whenever phase = 180° in the Nichols chart. Contrast with the unstable system when its gain < 0 dB at 180°.

6. Stability Analysis with Routh-Hurwitz Criterion

Any system that has closed-loop poles in the right-half of the s-plane will be unstable. The Routh Hurwitz table is a simple method that allows us to determine the number of poles in the left- and right-halves of the s-plane and the number of poles on the imaginary axis.



Edward Routh   Adolf Hurwitz

Figure 22: Inventors of Routh-Hurwitz stability analysis

It tells us nothing about where those poles are, so it is not generally sufficient for designing a control system. It is just a specialised tool to tell us about system stability.

6.1. Routh-Hurwitz Criterion

This stability analysis technique is based on the works of Edward Routh with his algorithm proposed in 1876 and Adolf Hurwitz who independently proposed contribution in 1895.

Using this method, we can tell how many closed-loop system poles are in the left half-plane, in the right half-plane, and on the jω-axis. We cannot tell where, but only how many are in each plane determining the system’s stability. The method requires two steps:

- Generate a data table called a Routh table.
- Interpret the Routh table to tell how many closed-loop system poles are in the left half-plane, the right half-plane, and on the jω -axis.

6.2. Construction of the Routh Array

Consider a closed loop transfer function having form:

$$G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s^1 + a_0 s^0}$$

When considering stability, we are interested only in the poles of $G(s)$, therefore we examine the denominator polynomial of G.

To form the Routh array, start by writing powers of s down the left-hand side, and then fill in the coefficients of the denominator polynomial. Note that in the first row, we start with the coefficient of the highest power of s and then write every second coefficient.

In the second row, we write in the coefficients that were not included in the top row. Include a zero at the end of row two if necessary to fill out the array. To complete the array, we fill out successive rows as a function of the two rows above. See the table for the required pattern.

Filling in the Array

To construct the Routh table, we need transfer function equation of the control system.

$$G(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s^1 + a_0 s^0}$$

From the transfer function equation given above, the following figure is the Routh table of the control system.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Figure 23: Routh table based on transfer function equation of a given system

The Routh criterion states that the number of closed loop poles in the right half of the s-plane is equal to the number of sign changes in the first column of the Routh table. For stability, we therefore require no sign changes in the first column. Any change in the first column indicates that the system is unstable.

s^4	a_4	a_2	a_0	+/-
s^3	a_3	a_1	0	+/-
s^2	b_1	b_2	0	+/-
s^1	c_1	0	0	+/-
s^0	d_1	0	0	+/-

Figure 24: Evaluation of the sign change in the first column

If the closed-loop transfer function has all poles in the left-half of the s-plane, the system is stable. Thus, a system is stable if there are no sign changes in the first column of the Routh table. The following table shows any sign change in Routh table and its interpretation.

Sign change	**Interpretation**
0	No right-hand side pole
1	1 right-hand side pole
2	2 right-hand side poles
N	N right-hand side poles

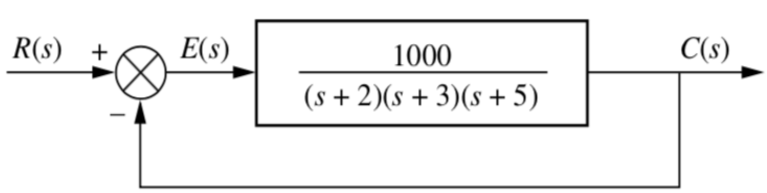
Figure 25: Sign change in the first column of the Routh table interpretation

Example for Tutorial 2: Stability Analysis of Control System

Consider a system with a transfer function of:

$$G(s) = \frac{1000}{(s+2)(s+3)(s+5)}$$

Determine whether this system is stable when closed in a unity gain feedback loop. [14 marks]



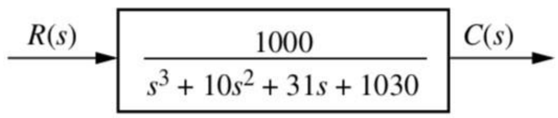
Answer

Find the Closed Loop Transfer Function

We first need to find the closed loop transfer function:

$$T(s) = \frac{G(s)}{1 + G(s)} = \frac{1000}{s^3 + 10s^2 + 31s + 1030}$$

Notice that we have completely expanded the denominator rather than leaving it as the product of poles as we normally do. The following figure shows the closed loop transfer function of example control system.



The characteristic equation of the system is:

$$s^3 + 10s^2 + 31s + 1030$$

Scaling in the Rows in the Table

Now write the powers of *s* down the left side and fill in the coefficients from the denominator polynomial of the closed loop transfer function. The following figure shows a partial filling of the array into Routh table.

s^3	1	31
s^2	10	1030
s^1		
s^0		

We are allowed to divide any row of the array through by a positive number if it will simplify the calculation. We will divide the second row though by ten. The following figure shows scaling of rows in the table.

s^3	1	31
s^2	10/10 = 1	1030/10 = 103
s^1		
s^0		

Filling in the Array

We see that the first column changes from positive to negative after the second row and then from negative to positive after the third row. We therefore have two sign changes. The Routh criterion tells us that the transfer function has two poles in the right half of the *s*-plane. The following figure shows a complete filling in the array in the Routh table.

s^3	1	31	0
s^2	1	103	0
s^1	$\frac{-\begin{vmatrix}1 & 31\\1 & 103\end{vmatrix}}{1}=-72$	$\frac{-\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1}=0$	$\frac{-\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1}=0$
s^0	$\frac{-\begin{vmatrix}1 & 103\\-72 & 0\end{vmatrix}}{-72}=103$	$\frac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72}=0$	$\frac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72}=0$

Interpreting a Routh Table

Simply stated, the Routh-Hurwitz criterion declares that the number of roots of the polynomial that are in the right hand-plane is equal to the number of sign changes in the first column. The following figure shows how you perform interpreting a Routh table.

s^3	1	31	0	+
s^2	1	103	0	+
s^1	$\frac{-\begin{vmatrix}1 & 31\\1 & 103\end{vmatrix}}{1}=-72$	$\frac{-\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1}=0$	$\frac{-\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1}=0$	-
s^0	$\frac{-\begin{vmatrix}1 & 103\\-72 & 0\end{vmatrix}}{-72}=103$	$\frac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72}=0$	$\frac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72}=0$	+

There are two sign changes e.g. 1 to -72 and -72 to 103. So, there are two poles in the right-hand side of the s-plane, the system is unstable.

6.3. Special Cases of the Routh Array

A couple of special situations can arise when constructing a Routh array. While constructing the array we might find that we could end up with:

- A zero in the first column.
- A row of zeros.

Either of these occurrences will "break" the procedure outlines above. However, there are simple tricks to deal with such situations - consult any control text for further information.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix}a_4 & a_2\\a_3 & a_1\end{vmatrix}}{a_3}=b_1=(\mathbf{0})$	$\frac{-\begin{vmatrix}a_4 & a_0\\a_3 & 0\end{vmatrix}}{a_3}=b_2$	$\frac{-\begin{vmatrix}a_4 & 0\\a_3 & 0\end{vmatrix}}{a_3}=0$
s^1	$\frac{-\begin{vmatrix}a_3 & a_1\\b_1 & b_2\end{vmatrix}}{b_1}=c_1=\infty$	$\frac{-\begin{vmatrix}a_3 & 0\\b_1 & 0\end{vmatrix}}{b_1}=\infty$	$\frac{-\begin{vmatrix}a_3 & 0\\b_1 & 0\end{vmatrix}}{b_1}=\infty$

s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Thus, the Routh table becomes as shown below:

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 = (\mathbf{0})$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 = (\mathbf{0})$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = (\mathbf{0})$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Figure 26: Special cases: a zero in the first column (top) and a row of zeros (bottom)

6.3.1. A Zero in the First Column

Consider for a Routh table with zero in the first row or column as shown below.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 = (\mathbf{0})$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Figure 27: Special case: Routh table with a zero in the first column

If we encountered zero in the first row or column, in this case, zero is replaced with epsilon (ϵ) and will tend to zero.

s^4	a_4	a_2	a_0	+/-
s^3	a_3	a_1	0	+/-

s^2	$b_1 = (\epsilon)$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$	+/-
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	+/-
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	+/-

Figure 28: Working with a zero in the first column of the Routh table

Finally, evaluate the sign change in the first column and interpret stability of the system.

Example for Tutorial 3: Stability Analysis of Special Control System 1

Given the following transfer function of a control system, we perform stability analysis of the system using Routh-Hurwitz method. [6 marks]

$$G(s) = \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$$

Answer

The Routh-Hurwitz table of the control system given above is shown in the figure below. As illustrated in the table below, the zero in first row/column that is replaced with epsilon (ϵ).

s^5	2	2	2
s^4	3	3	1
s^3	$\frac{-\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}}{3} = 0$	$\frac{-\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}}{3} = 4/3$	$\frac{-\begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix}}{3} = 4/3$
s^2	$\frac{-\begin{vmatrix} 3 & 3 \\ 0 & 4/3 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} 2 & 1 \\ 0 & 4/3 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} 2 & 1 \\ 0 & 4/3 \end{vmatrix}}{0} = \infty$
s^1	$\frac{-\begin{vmatrix} 0 & 4/3 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$	$\frac{-\begin{vmatrix} 0 & 4/3 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$	$\frac{-\begin{vmatrix} 0 & 4/3 \\ \infty & \infty \end{vmatrix}}{\infty} = 0$
s^0	$\frac{-\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix} \infty & \infty \\ 0 & 0 \end{vmatrix}}{0} = \infty$

Or

s^5	2	2	2
s^4	3	3	1

s^3	**(0)**	4/3	4/3
s^2	∞	∞	∞
s^1	0	0	0
s^0	∞	∞	∞

Notice that in the table the row that has zero (row with the s^3), zero in the column is replaced with ϵ. Then, the rest of the equations in the columns underneath are revised appropriately.

s^5	2	2	2	+
s^4	3	3	1	+
s^3	(ϵ)	4/3	0	+
s^2	$\frac{3\epsilon - 4}{\epsilon}$	1	0	-
s^1	$\frac{12\epsilon - 16 - 3\epsilon^2}{9\epsilon - 12}$	0	0	+
s^0	1	0	0	+

The first column in the table above has two sign changes e.g. row of s^3 to row of s^2 and row of s^2 to row of s^1. As a result, the system is unstable due to these poles at the right-hand side in the s-plane.

6.3.2. A Row of Zeros

On the other hand, consider a Routh table with all zeros in a row as shown below.

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1 = (\mathbf{0})$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2 = (\mathbf{0})$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = (\mathbf{0})$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1 = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = \infty$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1 = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Figure 29: Special case: Routh table with all zeros in a given row

Entire row of zeros means that there is an even polynomial that is a factor of the original polynomial and has to be handled differently from case with a zero in the first column of a row.

An entire row of zeros will appear in the Routh table when a purely even or purely odd polynomial is a factor of the original polynomial. For example, the equation below is an even polynomial. It has only even powers of s and even polynomials only have roots that are symmetrical about the origin.

$$s^4 + 5s^2 + 7$$

This symmetry can occur under three conditions of root position:

(1) The roots are symmetrical and real,

(2) the roots are symmetrical and imaginary, or

(3) the roots are quadrantal.

Figure below shows examples of these cases. Each case or combination of these cases will generate an even polynomial. It is this even polynomial that causes the row of zeros to appear. Thus, the row of zeros tells us of the existence of an even polynomial whose roots are symmetric about the origin.

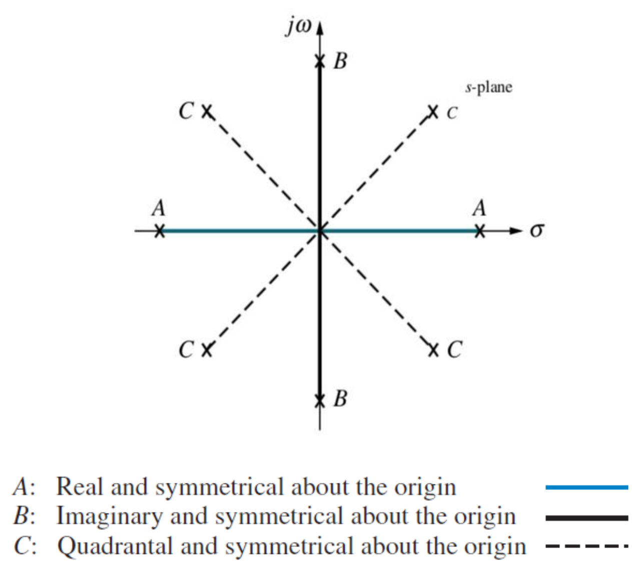


Figure 30: Poles and zeros of entire row of zeros in the s-plane

Some of these roots could be on the $j\omega$-axis. On the other hand, since $j\omega$ roots are symmetric about the origin, if we do not have a row of zeros, we cannot possibly have $j\omega$ roots.

Another characteristic of the Routh table for the case in question is that the row before the row of zeros contains the even polynomial that is a factor of the original polynomial.

Finally, everything from the row containing the even polynomial down to the end of the Routh table is a test of only the even polynomial.

To work around entire zero in a row, you need to differentiate the equation in the row above the row that has zeros in its entire row.

$$P = a_3 s^3 + a_1 s^1 + 0$$

Thus

$$\frac{dP}{ds} = 3a_3 s^2 + a_1$$

Then, the coefficients of the equation resulted from the differentiation is substituted to the row with zeros in its entire row. As a result, the Routh table becomes as shown below.

s^4	a_4	a_2	a_0	+/-
s^3	a_3	a_1	0	+/-
s^2	$b_1 = (\mathbf{3a_3s^2})$	$b_2 = (\mathbf{a_1})$		+/-
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	+/-
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	+/-

Figure 31: Working with entire row of zeros in the Routh table

Evaluate the sign change in the first column of the table. In the end, interpret the stability of the system.

Example for Tutorial 4: Stability Analysis of Special Control System 2

For the transfer function equation of a fifth-order system, perform stability analysis using Routh-Hurwitz method. [10 marks]

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

Answer

The Routh-Hurwitz table of the control system given above is shown in the figure below. At s^4 row, we multiply through by 1/7 for convenience. Note that row with s^3 is all zero. The following table shows how all zeros exist in a row.

s^5	1	6	8
s^4	$\left(\frac{7}{7}\right) = 1$	$\left(\frac{42}{7}\right) = 6$	$\left(\frac{56}{7}\right) = 8$

s^3	$\frac{-\begin{vmatrix}1 & 6\\1 & 6\end{vmatrix}}{1} = 0$	$\frac{-\begin{vmatrix}1 & 8\\1 & 8\end{vmatrix}}{1} = 0$	$\frac{-\begin{vmatrix}1 & 8\\1 & 8\end{vmatrix}}{1} = 0$
s^2	$\frac{-\begin{vmatrix}1 & 6\\0 & 0\end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix}1 & 8\\0 & 0\end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix}1 & 8\\0 & 0\end{vmatrix}}{0} = \infty$
s^1	$\frac{-\begin{vmatrix}0 & 0\\\infty & \infty\end{vmatrix}}{\infty} = 0$	$\frac{-\begin{vmatrix}0 & 0\\\infty & \infty\end{vmatrix}}{\infty} = 0$	$\frac{-\begin{vmatrix}0 & 0\\\infty & \infty\end{vmatrix}}{\infty} = 0$
s^0	$\frac{-\begin{vmatrix}\infty & \infty\\0 & 0\end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix}\infty & \infty\\0 & 0\end{vmatrix}}{0} = \infty$	$\frac{-\begin{vmatrix}\infty & \infty\\0 & 0\end{vmatrix}}{0} = \infty$

Or

s^5	1	6	8
s^4	1	6	8
s^3	**(0)**	**(0)**	**(0)**
s^2	∞	∞	∞
s^1	0	0	0
s^0	∞	∞	∞

In the Routh-Hurwitz table, go up a row above the row with all zero coefficients. Create an equation that is made up of the coefficients of the given row. At the row of s^4, the equation below is constructed from the coefficients in this row.

$$P = s^4 + 6s^2 + 8$$

Differentiate the respected equation about the 's' function. The first derivate of the equation above is calculated as follow:

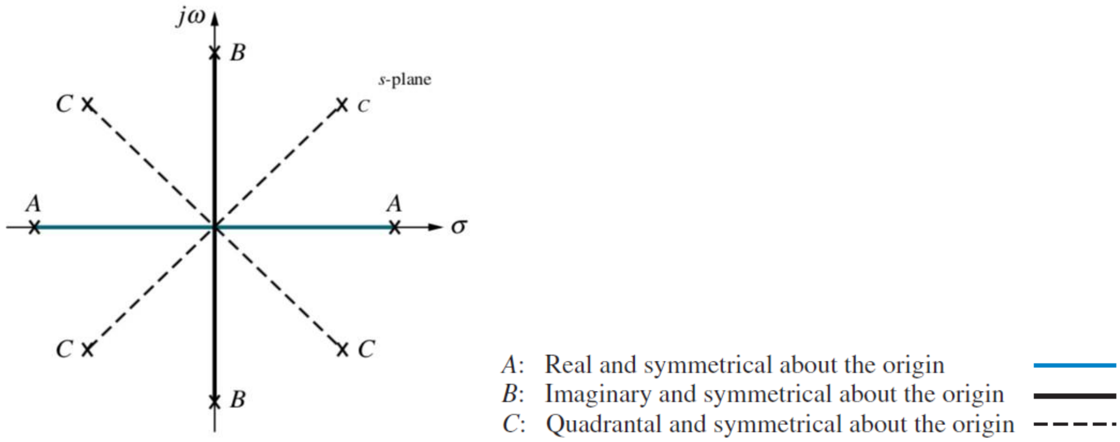
$$\frac{dP}{ds} = 4s^3 + 12s + 0$$

Replace the row with all zero coefficients with the coefficients of the resulting function after the differentiation. As a result, the Routh table becomes as shown in the table below.

s^5	1	6	8	+
s^4	1	6	8	+
s^3	**(4)**	**(12)**	**(0)**	**+**
s^2	3	8	0	+
s^1	1/3	0	0	+
s^0	8	0	0	+

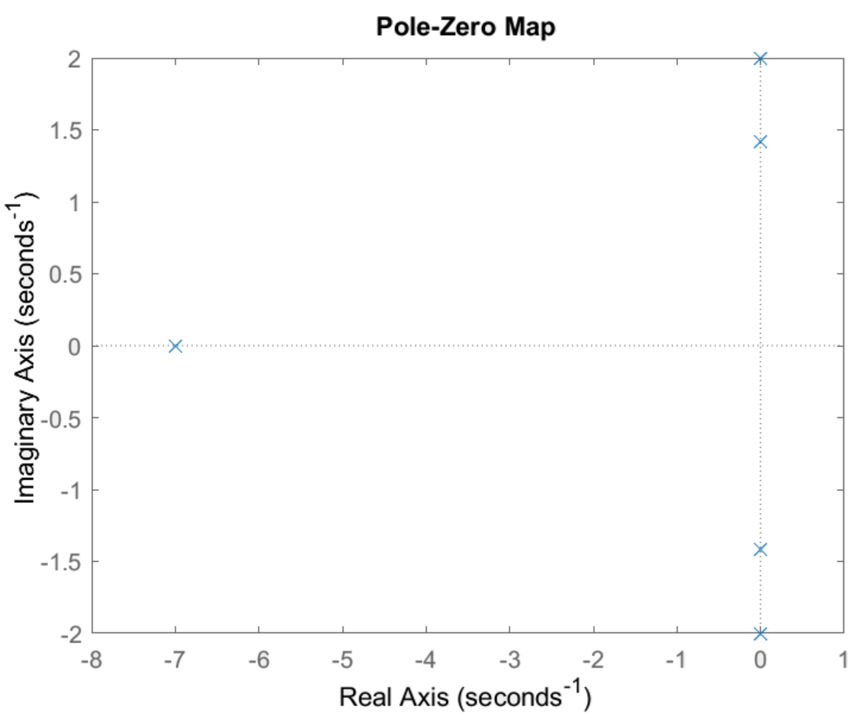
From the result of the analysis, with all zeros in a row, it seems that there is no change of sign in the first column of the Routh table. As a result, the given system has real and symmetrical poles about the origin.

As indicted in the s-plane diagram of the system as shown in the figure below, the poles for this system are real and symmetrical about the origin.

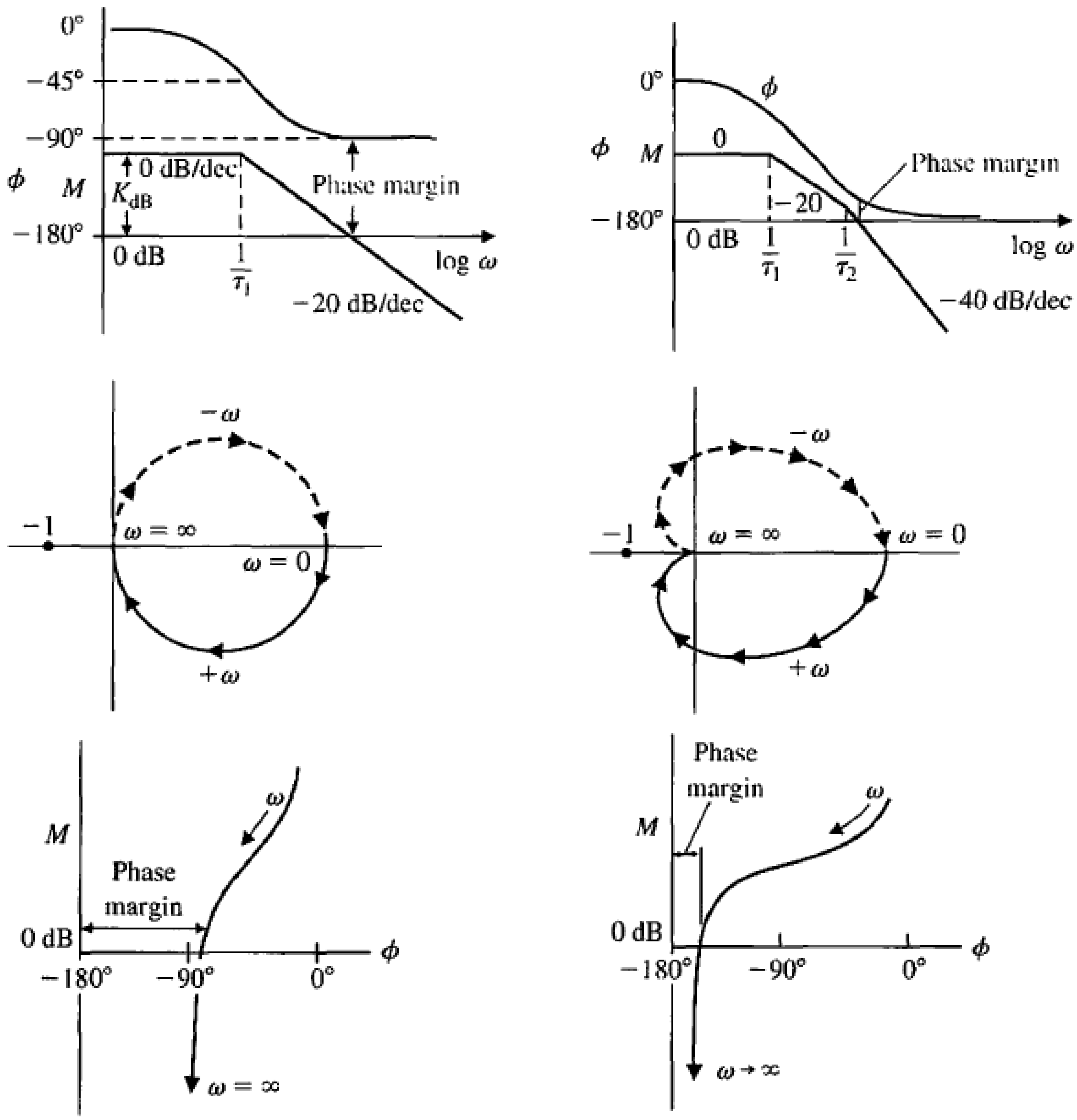


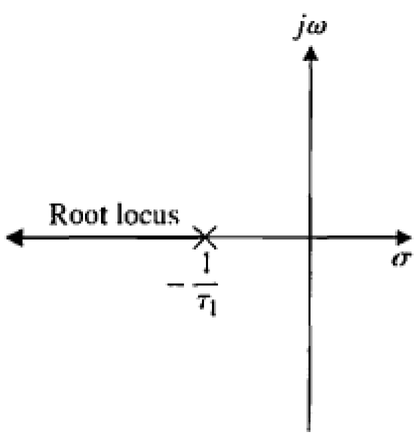
Evaluating the even polynomial $P = s^4 + 6s^2 + 8$, no change of sign in the first column from row with s^4 to the last row. So, the four poles of the polynomial are located on the y-axis.

No change of sign from row with s^5 to row with s^4, so the remaining fifth pole is located on the left half-side of the s-plane.



Appendix – List of Systems for Stability Analysis



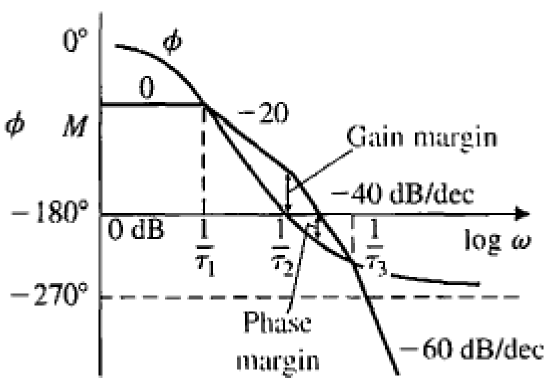


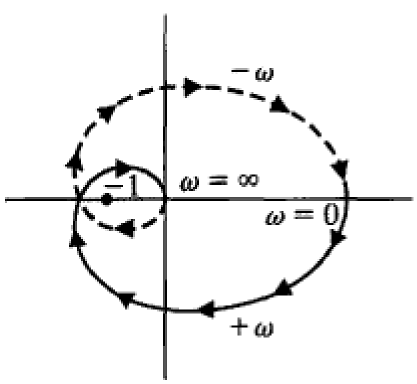
(a)

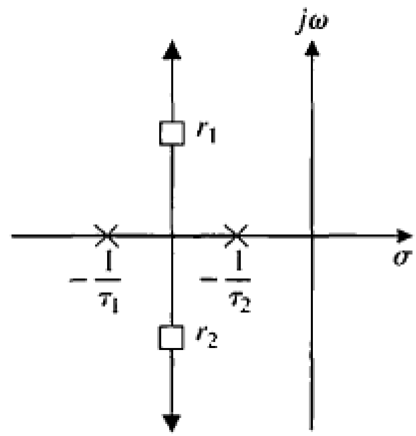
System (a)

$$G(s) = \frac{K}{s\tau_1 + 1}$$

Stable, gain margin = ∞.





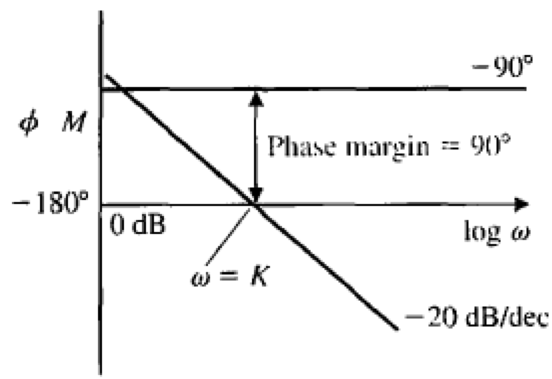


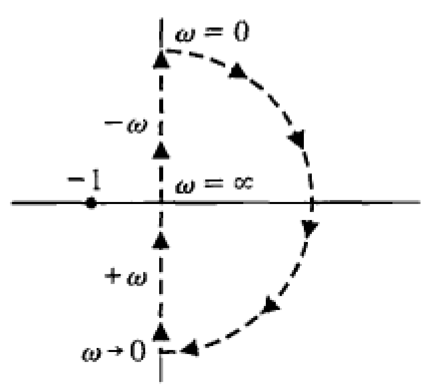
(b)

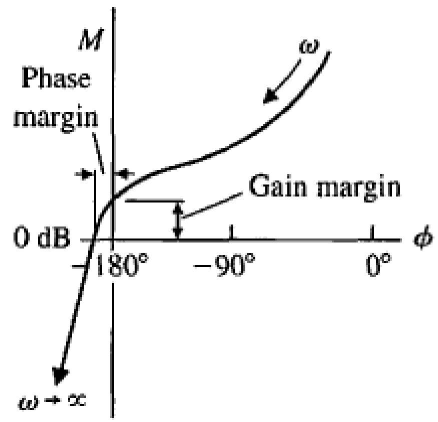
System (b)

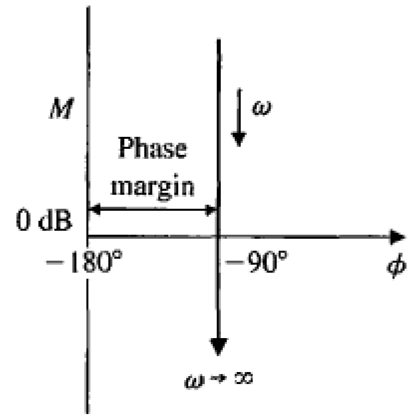
$$G(s) = \frac{K}{(s\tau_1 + 1)(s\tau_2 + 1)}$$

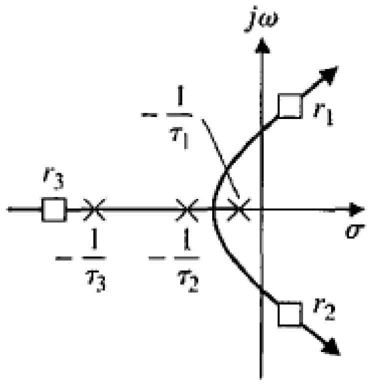
Elementary regulator; stable; gain margin = ∞.









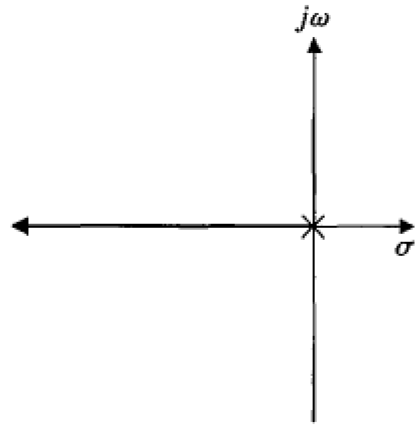


(c)

System (c)

$$G(s) = \frac{K}{(s\tau_1 + 1)(s\tau_2 + 1)(s\tau_3 + 1)}$$

Regulator with additional energy-storage component; unstable, but can be made stable by reducing gain.

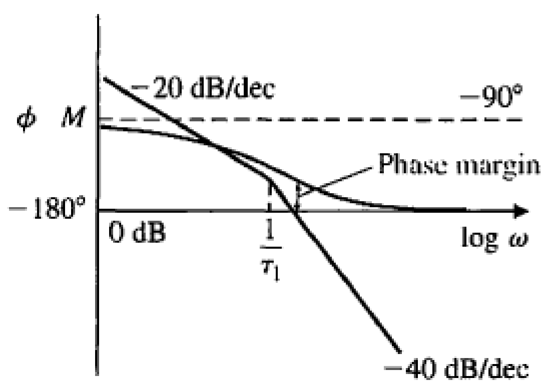


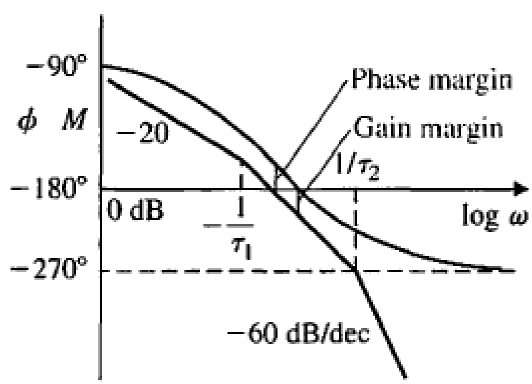
(d)

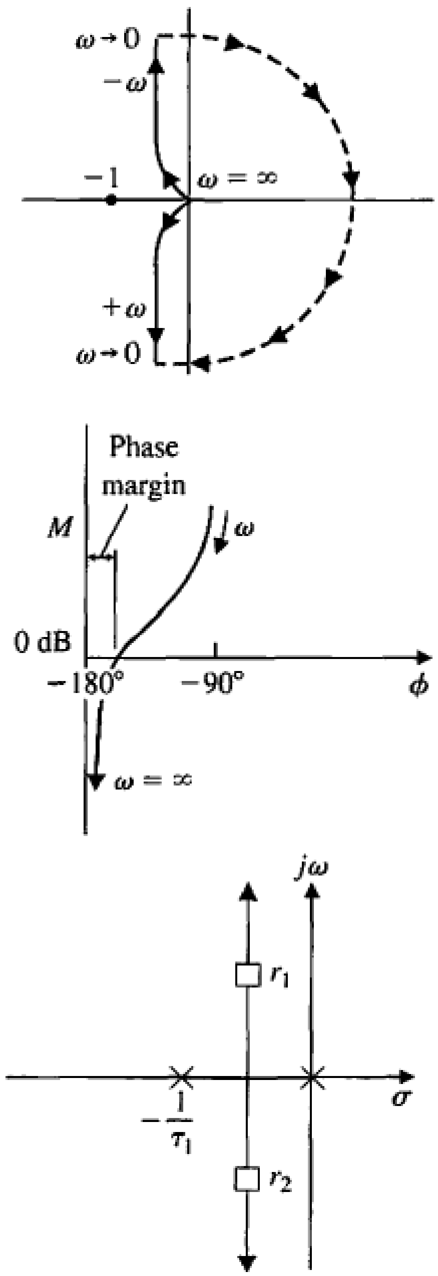
System (d)

$$G(s) = \frac{K}{s}$$

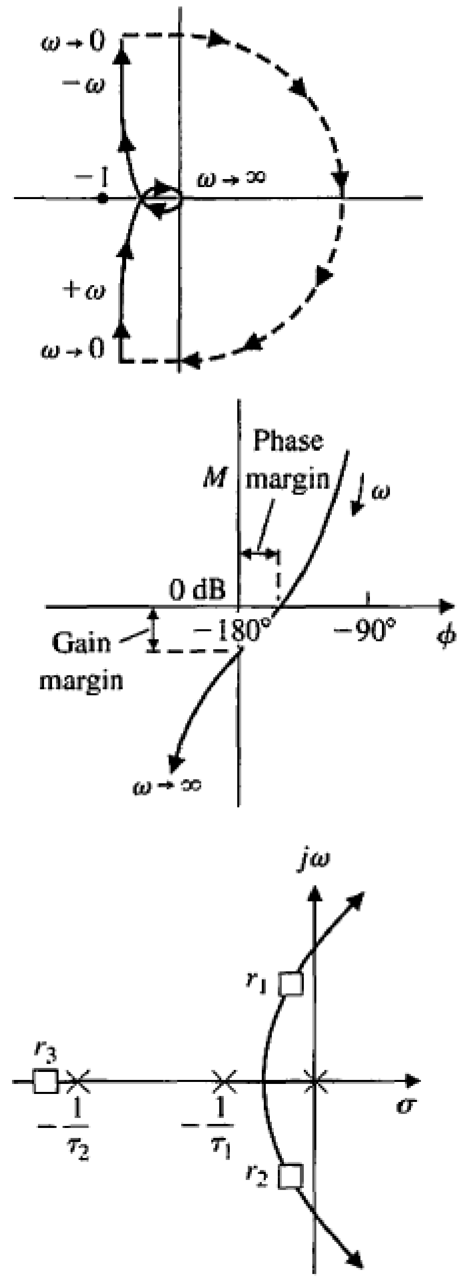
Ideal integrator; stable.







(e)



(f)

System (e)

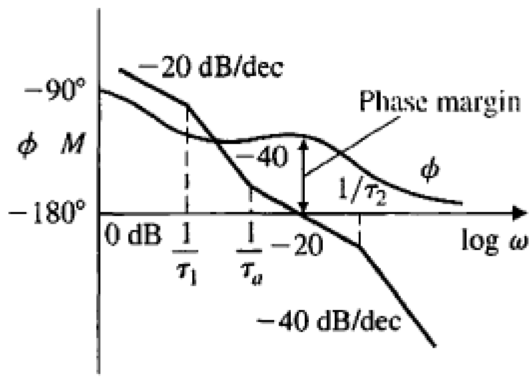
$$G(s) = \frac{K}{s(s\tau_1 + 1)}$$

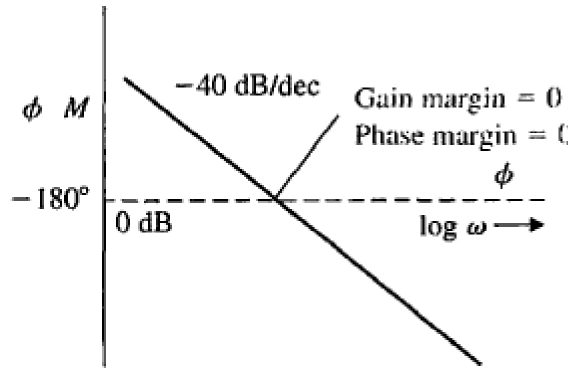
Elementary instrument servo; inherently stable; gain margin = ∞.

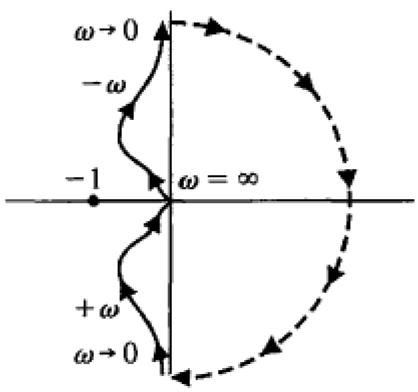
System (f)

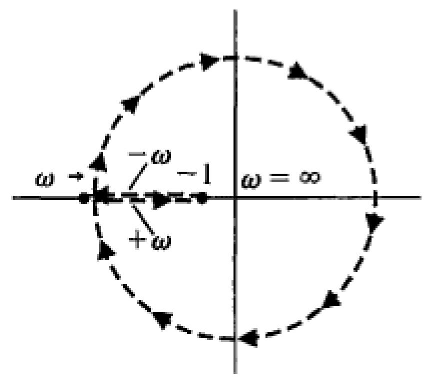
$$G(s) = \frac{K}{s(s\tau_1 + 1)(s\tau_2 + 1)}$$

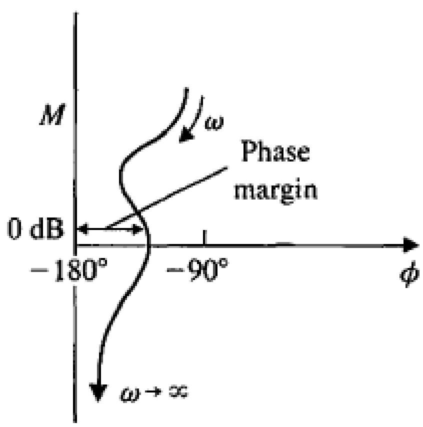
Instrument servo with field control motor or power servo with elementary Ward-Leonard drive; stable as shown, but may become unstable with increased gain.

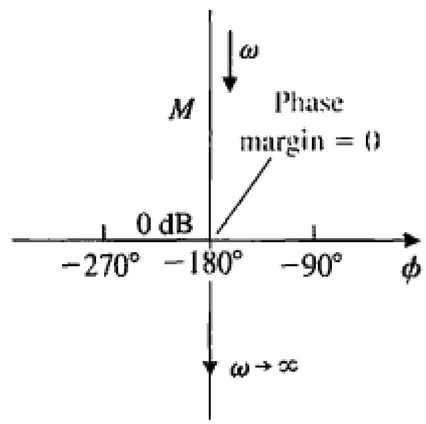

−20 dB/dec
−90°
Phase margin
ϕ M
−40
ϕ
1/τ2
−180°
0 dB
1/τ1
1/τa
−20
log ω
−40 dB/dec

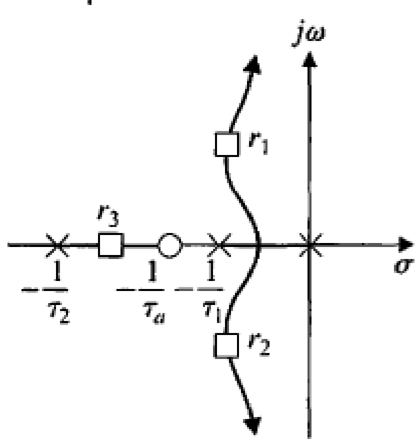

−40 dB/dec
Gain margin = 0
Phase margin = 0
ϕ M
ϕ
−180°
0 dB
log ω

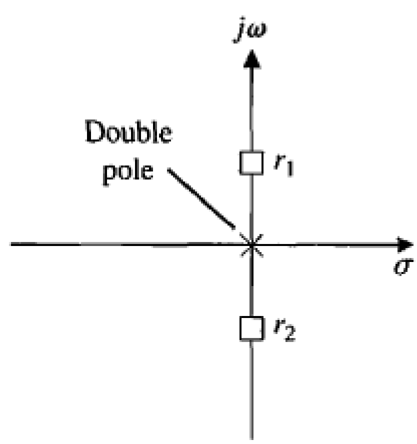

ω → 0
−ω
−1
ω = ∞
+ω
ω → 0


−ω
ω
−1
ω = ∞
+ω


M
ω
Phase margin
0 dB
−180°
−90°
ϕ
ω → ∞


ω
M
Phase margin = 0
0 dB
−270°
−180°
−90°
ϕ
ω → ∞


jω
r1
r3
σ
−1/τ2
−1/τa
−1/τ1
r2

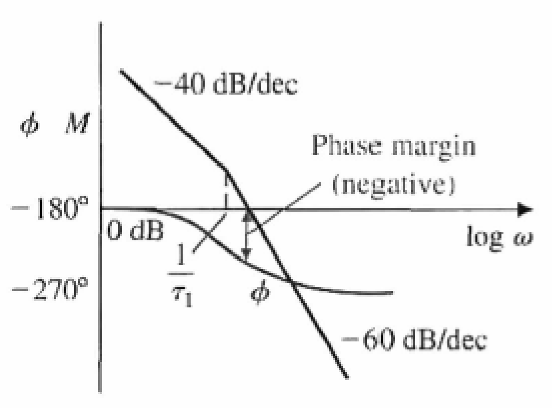

jω
Double pole
r1
σ
r2

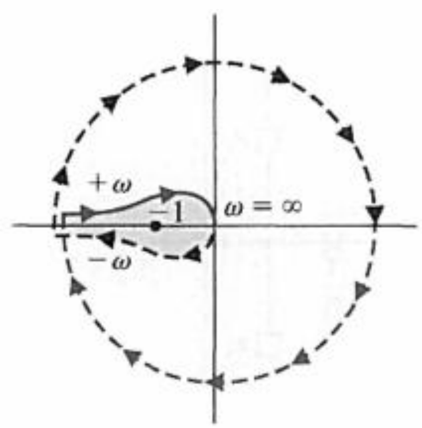
(g)

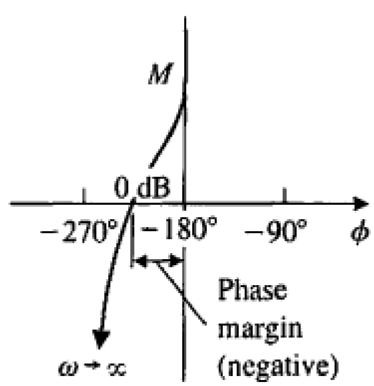
System (g)

$$G(s) = \frac{K(s\tau_a + 1)}{(s\tau_1 + 1)(s\tau_2 + 1)}$$

Elementary instrument servo with phase-lead (derivative) compensator; stable.





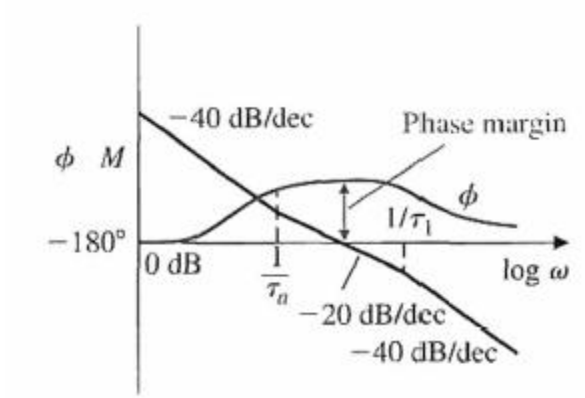


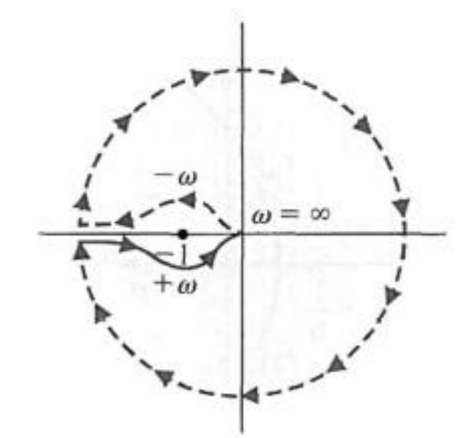
(h)

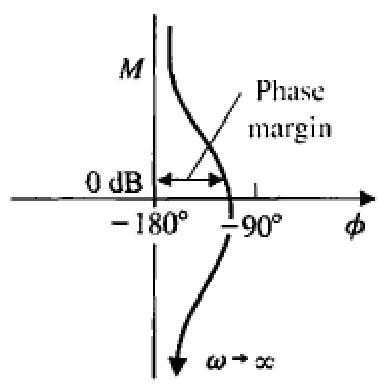
System (h)

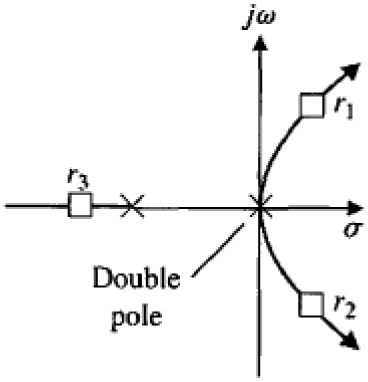
$$G(s) = \frac{K}{s^2}$$

Inherently marginally stable; must be compensated.







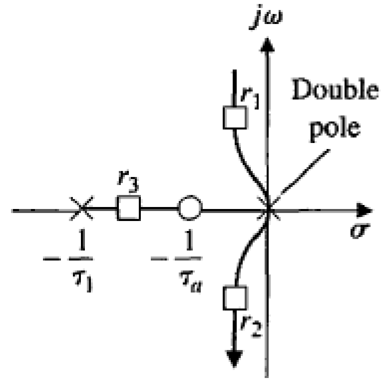


(i)

System (i)

$$G(s) = \frac{K}{s^2(s\tau_1 + 1)}$$

Inherently unstable; must be compensated.

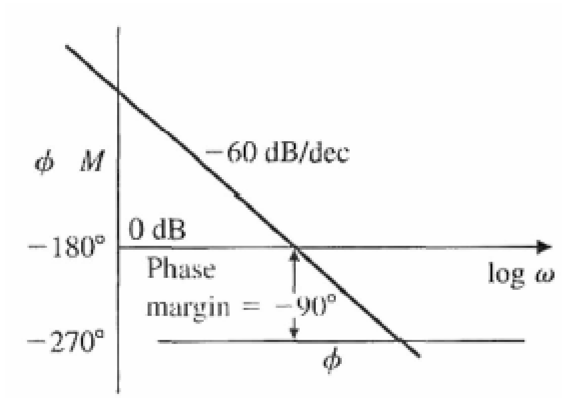


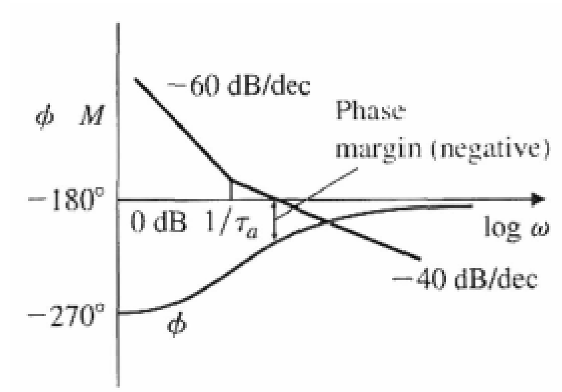
(j)

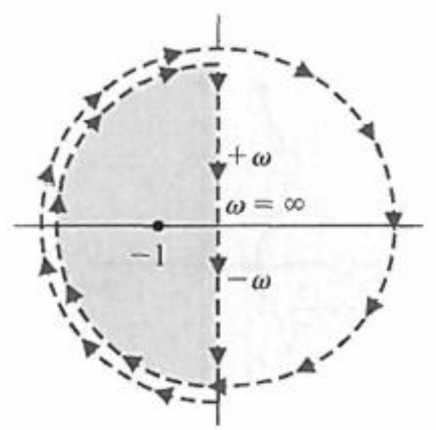
System (j)

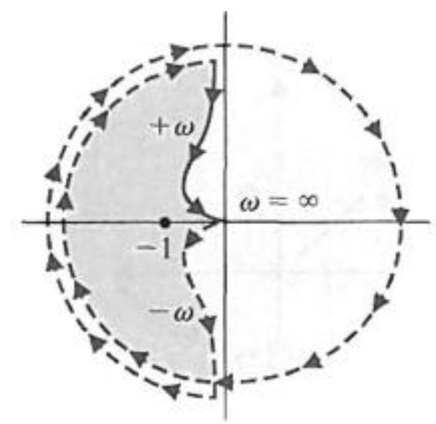
$$G(s) = \frac{K(s\tau_a + 1)}{s^2(s\tau_1 + 1)} \qquad \tau_a > \tau_1$$

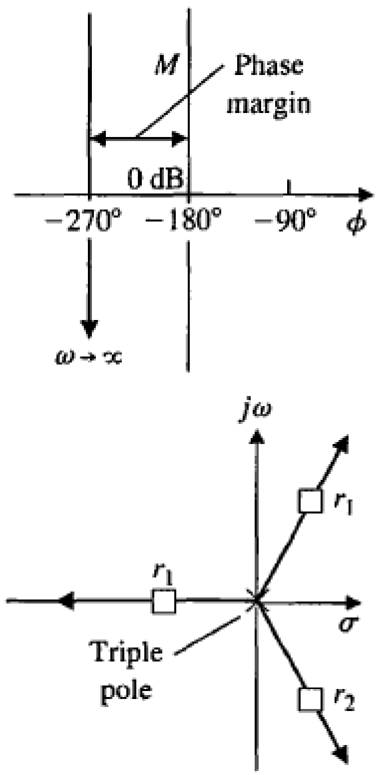
Stable for all gains.



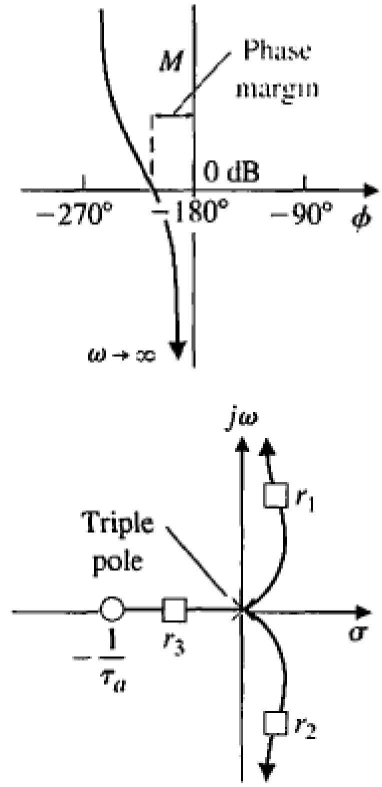








(k)



(l)

System (k)

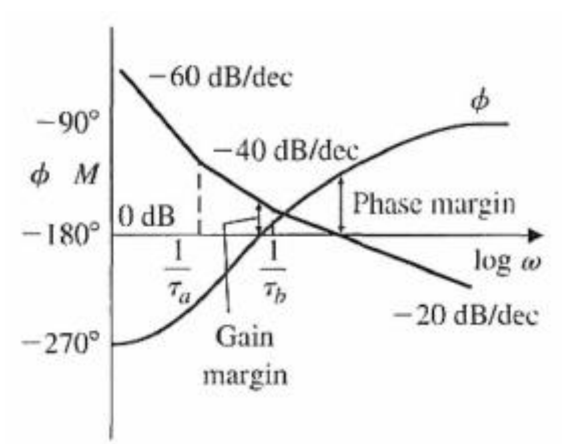
$$G(s) = \frac{K}{s^3}$$

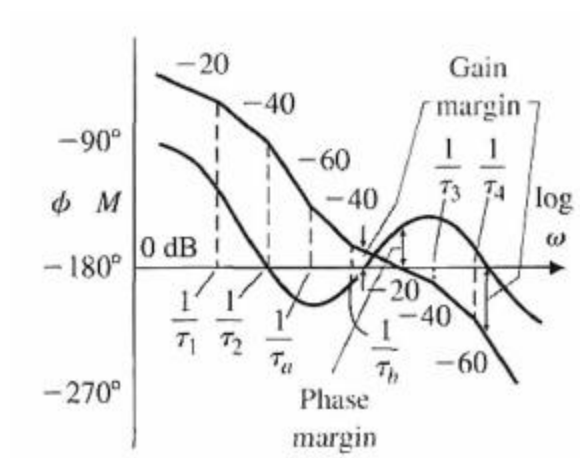
Inherently unstable.

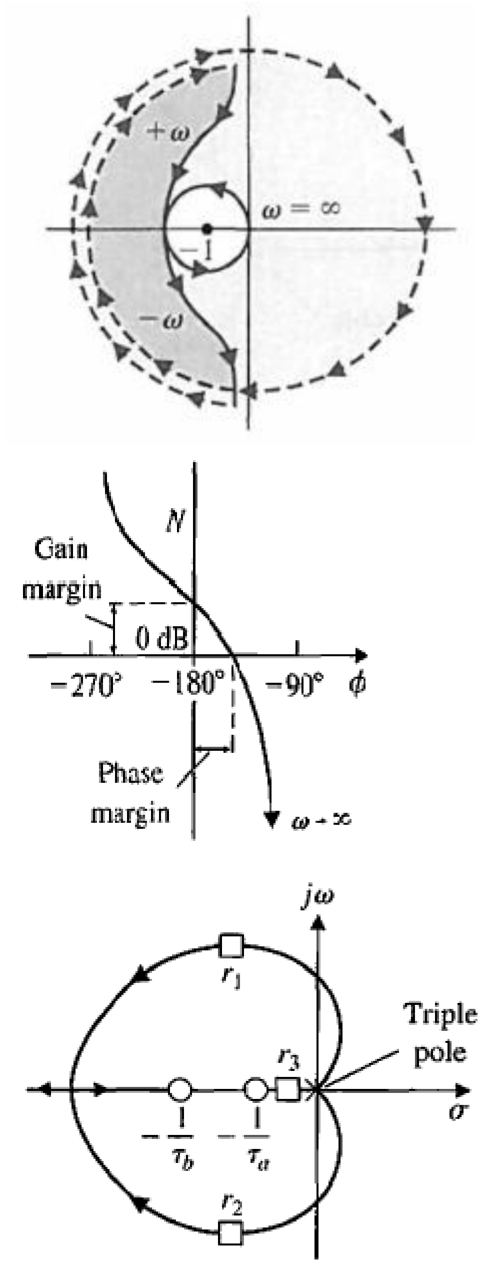
System (l)

$$G(s) = \frac{K(s\tau_a + 1)}{s^3}$$

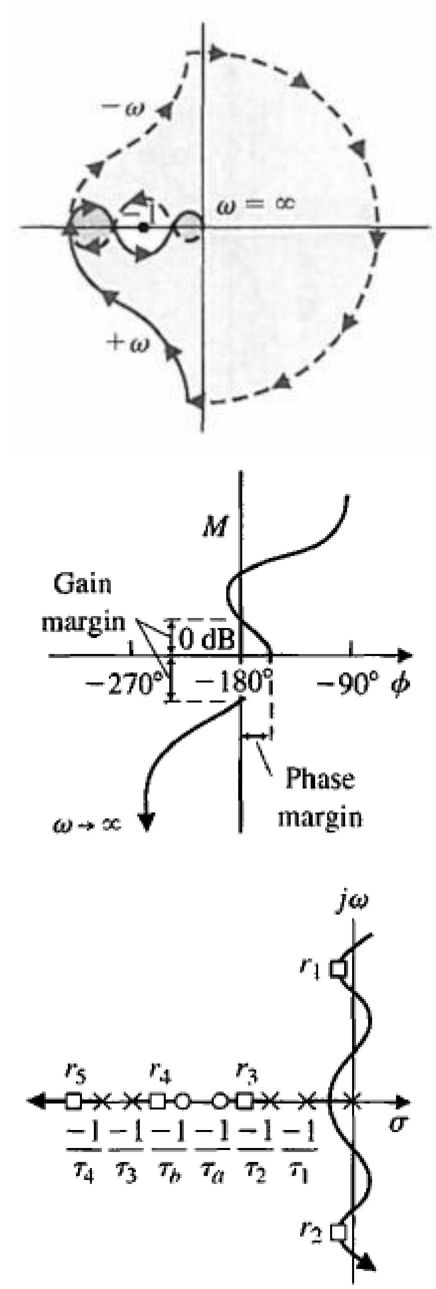
Inherently unstable.







(m)



(n)

System (m)

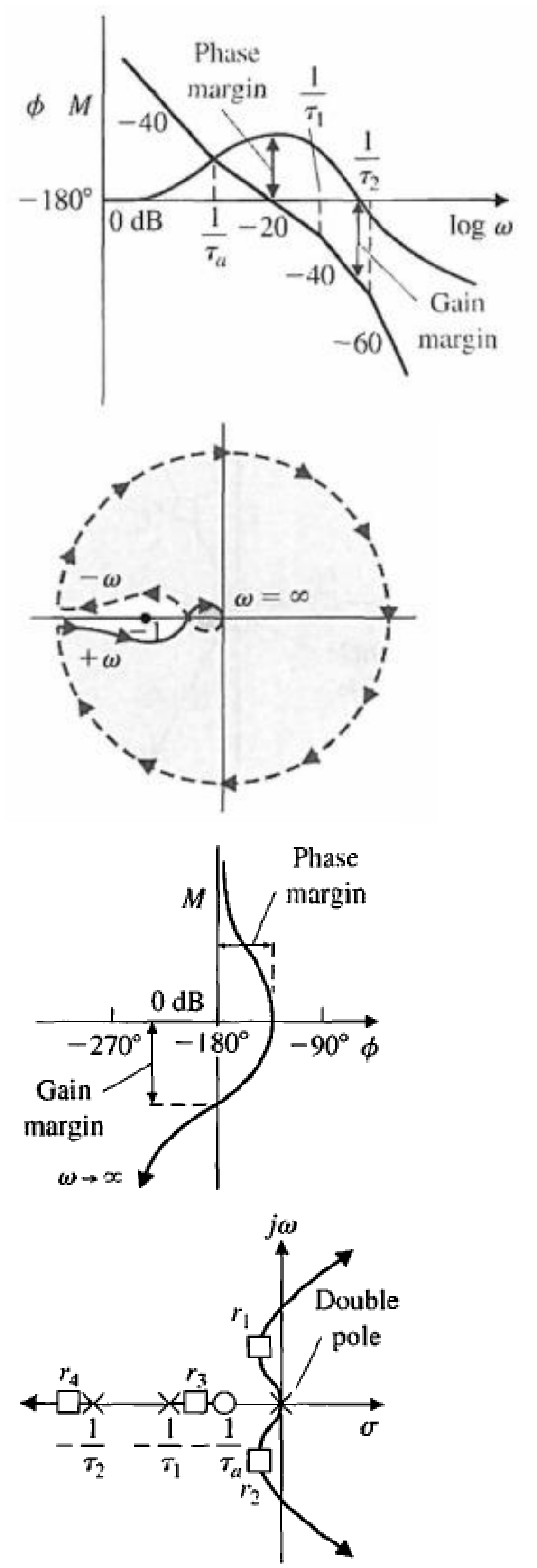
$$G(s) = \frac{K(s\tau_a + 1)(s\tau_b + 1)}{s^3}$$

Conditionally stable; becomes unstable if gain is too low.

System (n)

$$G(s) = \frac{K(s\tau_a + 1)(s\tau_b + 1)}{s(s\tau_1 + 1)(s\tau_2 + 1)(s\tau_3 + 1)(s\tau_4 + 1)}$$

Conditionally stable; stable at low gain, becomes unstable as gain is raised, again becomes stable as gain is further increased, and becomes unstable for very high gains.



(o)

System (o)

$$G(s) = \frac{K(s\tau_a + 1)}{s^2(s\tau_1 + 1)(s\tau_2 + 1)}$$

Conditionally stable; becomes unstable at high gain.